Isomorphism Between Algebraic Cobordism and K-theory Over Singular Schemes

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Abstract

The theory of algebraic cobordism, denoted by Ω_*, was created by Levine and Morel as a universal oriented cohomology theory on the category of smoooth quasi-projective schemes over a field k. One of the important facts about algebraic cobordism is the existence of a natural isomorphism between $\Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}]$ and $K_0 \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$ over smooth schemes, where K_0 is the Grothendieck group of algebraic vector bundles.

In this dissertation, we give an extension of this natural isomorphism to the category of quasi-projective k-schemes when $char(k) = 0$. Among the main tools we used are Hironaka's resolution of singualarities and properties of algebraic cobordism proved by Levine and Morel, especially the localization property.

Analogous to the fact that the natural isomorphism over smooth k-schemes implies the Grothendieck-Riemann-Roch theorem, the extension of this isomorphism to singular k-schemes implies the singular Riemann-Roch theorem of Baum-Fulton-MacPherson. Moreover, we obtain a new type of Riemann-Roch theorem with respect to pull-backs of locally complete morphisms.

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Contents

Introduction

This thesis is dedicated to the study of the relations between algebraic cobordism and Grothendieck groups over singular schemes.

For a field k, let Sch_k denote the category of finite type separated k schemes, $qSch_k$ the full subcategory of quasi-projective k-schemes and Sm_k the full subcategory of quasi-projective smooth k-schemes.

Let k be a field of characteristic zero.

Background and Motivation

An oriented cohomology theory A^* on Sm_k is a functor

$$X \mapsto A^*(X)$$

sending $X \in Sm_k$ to the category of graded commutative rings with functorial pull-back maps for arbitrary morphisms, and push-forward for projective morphisms, satisfying the following properties: functoriality for push-forwards, projection formula over a cartesian square, projective bundle formula, and homotopy (c.f. *Definition* 1.3.2).

An oriented cohomology theory A^* on Sm_k has a formal group law $F_A(u, v)$ that describes how the first Chern class behaves with respect to the tensor product of line bundles: for two line bundles $L, M \to X$ over a k-scheme X, we have $F_A(c_1(L), c_1(M)) = c_1(L \otimes M)$.

In [8], Levine and Morel construct a universal oriented cohomology theory on Sm_k, called *Algebraic Cobordism* written as Ω^*, which is the algebro-geometric version of

Quillen's complex cobordism. They showed that Ω^* has the universal formal group law. That is to say, given a formal group law (F_R, R), there is a unique homomorphism $\Omega^*(k) \to R$ sending F_Ω to F_R, which allows one to construct the universal theory with formal group law (F_R, R) as

$$\Omega_F^*(X) := \Omega(X) \otimes_{\Omega(k)} R.$$

Similar to an oriented cohomology theory on Sm_k, an oriented Borel-Moore homology theory A_* on Sch_k is a functor

$$X \mapsto A_*(X)$$

sending X in Sch_k to the category of graded abelian groups with functorial push-forward for projective morphisms, and pull-back maps for locally complete intersection (l.c.i.) morphisms, satisfying some natural axioms. See *Definition* 1.3.4 for details.

We are especially interested in the following two examples of oriented Borel-Moore homology theories: the Grothendieck group theory $G_0[\beta, \beta^{-1}]$ and the twisted Chow theory $CH[\beta, \beta^{-1}]_{\mathbb{Q}}^{(td)}$, where (td) indicates the Todd twisting (see §1.5).

Levine and Morel prove the following isomorphism:

$$K_0[\beta, \beta^{-1}] \cong \Omega_\times^*,$$

where × indicates the group law $u + v - \beta uv$ over $\mathbb{Z}[\beta, \beta^{-1}]$.

Thesis

We are interested in the problem that whether or not we can extend the above isomorphism to singular schemes. It turns out that this extension exists if we work over quasi-projective schemes over a field k of characteristic 0. In this setting, we have the following main result of this dissertation:

Theorem 0.0.1. *On the category $qSch_k$, with char(k)=0, the natural transformation $\Omega_*^\times \to G_0[\beta, \beta^{-1}]$ descends to an isomorphism of oriented Borel-Moore homology theories.*

Outline

Chapter 1 consists of the background material needed for the main results of this dissertation. We begin with Levine-Morel's construction of algebraic cobordism. The definitions of oriented cohomology theory and oriented Borel-Moore homology theory are recalled. We also collect some important properties and facts of algebraic cobordism proved by Levine-Morel, among which are the universality property and localization property. We demonstrate how to create a new theory by imposing a twisting to an already existing theory. In the end, we list some important examples that are of use in what follows in this dissertation.

Chapter 2 deals with the proof of the main result stated above. We use Hironaka's resolution of singularities to obtain a sequence of blowups along smooth centers. For each single blowup, we analyze its behavior using mainly the localization property of algebraic cobordism and some facts on algebraic K-theory. This results in a sequence of lemmas which lead to the proof of our main theorem.

In *Chapter 3*, a new type of Riemann-Roch theorem is proved, i.e., the Riemann-Roch theorem with respect to l.c.i. pull-backs. We are also able to recover the singular Riemann-Roch by Baum-Fulton-MacPherson. A new proof of the module structure compatibility between G_0-theory and Chow group theory, another result of Baum-Fulton-MacPherson, is obtained as a consequence of our main theorem.

Notations and Conventions

Throughout this thesis, k will denote an arbitrary field, unless otherwise stated.

- Sch_k will denote the category of separated finite type schemes over k.
- $qSch_k$ will denote the full subcategory of Sch_k consisting of only quasi-projective schemes.
- Sm_k will denote the category of smooth quasi-projective k-schemes.
- K_0 and G_0 will denote the Grothendieck group functors. In other words, for

$X \in Sch_k$, $K_0(X)$ will be the Grothendieck group of algebraic vector bundles on X, and $G_0(X)$ the Grothendieck group of coherent sheaves on X.

- CH^* and CH_* will denote the usual Chow group functors with CH^n the Chow group of codimension n cycles and CH_n the Chow group of dimension n cycles, both modulo rational equivalence.
- Ab_* will denote the category of graded abelian groups.
- $Rings^*$ will denote the category of graded commutative rings with units.
- We define the following notations:

$$G_0[\beta, \beta^{-1}] := G_0 \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}] \text{ with } \beta \text{ an indeterminate of degree 1}$$
$$K_0[\beta, \beta^{-1}] := K_0 \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}] \text{ with } \beta \text{ an indeterminate of degree -1.}$$

By a smooth morphism in Sch_k, we will always mean a smooth and quasi-projective morphism. In particular, a smooth k-scheme will always be assumed to be quasi-projective over k.

For a locally free coherent sheaf $\mathcal{E}$ on a scheme X, we let $q : \mathbb{P}(\mathcal{E}) \to X$ denote the projective bundle $Proj_{\mathcal{O}_X}(Sym^*_{\mathcal{O}_X}\mathcal{E})$, and $q^*\mathcal{E} \to \mathcal{O}(1)_{\mathcal{E}}$ the canonical quotient. For a vector bundle $E \to X$ with sheaf of sections $\mathcal{E}$, we write $\mathbb{P}(E)$ for $\mathbb{P}(\mathcal{E})$, and $q^*E \to \mathcal{O}(1)_E$ for the canonical quotient. For $n > 0$, $\mathcal{O}^n_X$ will denote the trivial vector bundle of rank n over X, and we write γ_n for the line bundle $\emptyset(1)_{\emptyset^{n+1}_X}$ on $\mathbb{P}^n_X$.

In this thesis, we use *OBM* to stand for *oriented Borel-Moore.*

Chapter 1

Algebraic Cobordism and OBM Theories

In this chapter, we will recall the definition of oriented Borel-Moore functors and variants, including the notion of oriented Borel-Moore $\mathbb{L}_*$-functors of geometric type. We then recall Levine and Morel's construction of algebraic cobordism groups.

In [9], Levine and Morel create and develop the theory of algebraic cobordism over the category of schemes of finite type over a field k, which associates to every X in Sch_k a graded abelian group $\Omega_*(X)$. In addition, they show that the theory Ω_* is the universal oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type over Sch_k, where $\mathbb{L}$ is the *Lazard ring*, cf. [6].

1.1 Cobordism cycles

For a full subcategory $\mathcal{V}$ of Sch_k, we let $\mathcal{V}'$ denote the subcategory of Sch_k with the same objects as $\mathcal{V}$, but with only the projective morphisms.

Definition 1.1.1. Let $\mathcal{V}$ be a full subcategory of Sch_k, closed under finite disjoint union. A functor $H_* : \mathcal{V}' \to \mathbf{Ab}_*$ is called *additive* if for any finite family $(X_1, \ldots, X_r)$

of finite type k-schemes, the homomorphism

$$\oplus_{i=1}^{r} H_*(X_i) \to H_*(\amalg_{i=1}^{r} X_i)$$

induced by the (projective) morphisms $X_i \subset \amalg_{i=1}^{r} X_i$ is an isomorphism.

Definition 1.1.2. An *admissible* subcategory $\mathcal{V}$ of Sch_k is a full subcategory of Sch_k satisfying the following conditions:

1. Spec$k$ and the empty scheme are in $\mathcal{V}$.
2. If $Y \to X$ is a smooth quasi-projective morphism with $X \in \mathcal{V}$, then $Y \in \mathcal{V}$.
3. If X and Y are in $\mathcal{V}$, then so is the product $X \times_k Y$.
4. If X and Y are in $\mathcal{V}$, so is $X \amalg Y$.

For us, the admissible subcategories of interest are $\mathcal{V} = Sch_k, qSch_k$ and Sm_k.

Definition 1.1.3. An *oriented Borel-Moore functor* on an admissible subcategory $\mathcal{V}$ of Sch_k is given by:

(D1). An additive functor $H_* : \mathcal{V}' \to \mathbf{Ab}_*$.

(D2). For each smooth equidimensional morphism $f : Y \to X$ in $\mathcal{V}$ of relative dimension d a morphism of graded groups

$$f^* : H_*(X) \to H_{*+d}(Y).$$

(D3). For each line bundle L on X a homomorphism of graded abelian groups:

$$\tilde{c}_1(\mathcal{L}) : H_*(X) \to H_{*-1}(X).$$

These data satisfy the following axioms:

(A1). For any pair of composable smooth equidimensional morphisms $(f : Y \to X,\ g : Z \to Y)$ respectively of dimension d and e, one has $(f \circ g)^* = g^* \circ f^* : H_*(X) \to H_{*+d+e}(Z)$, and $Id_X^* = Id_{H_*(X)}$ for any $X \in Sch_k$.

(A2). Let $f : X \to Z,\ g : Y \to Z$ be morphisms in $\mathcal{V}$, giving the cartesian square

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

Suppose that f is projective and g is smooth equidimensional, then

$$g^* f_* = f'_* g'*$$

.

(A3). Given a projective morphism $f : Y \to X$ in $\mathcal{V}$ and a line bundle L over X, one has

$$f_* \circ \tilde{c}_1(f^* L) = \tilde{c}_1(L) \circ f_*$$

(A4). Given a smooth equidimensional morphism $f : Y \to X$ in $\mathcal{V}$ and a line bundle L over X, one has

$$\tilde{c}_1(f^* L) \circ f^* = f^* \circ \tilde{c}_1(L)$$

(A5). Given line bundles L and M on $X \in \mathcal{V}$, one has

$$\tilde{c}_1(L) \circ \tilde{c}_1(M) = \tilde{c}_1(M) \circ \tilde{c}_1(L)$$

.

Definition 1.1.4. Let X be a k-scheme of finite type.

1). A *cobordism cycle* over X is a family $(f : Y \to X, L_1, \ldots, L_r)$ consisting of:

1. a projective morphism $f : Y \to X$ with Y integral, smooth over k.

2. a finite sequence $(L_1, \ldots, L_r)$ of r line bundles over Y that is interpreted as empty if $r = 0$.

The dimension of $(f : Y \to X, L_1, \ldots, L_r)$ is $dim_k(Y) - r \in \mathbb{Z}$.

2) An *isomorphism* Φ of cobordism cycles $(Y \to X, L_1, \dots, L_r) \cong (Y^{'} \to X, L_1^{'}, \dots, L_{r'}^{'})$ is a triple $\Phi = (\phi : Y \to Y^{'}, \sigma, (\psi_1, \dots, \psi_r))$ consisting of:

1. an isomorphism $\phi : Y \to Y^{'}$ of X-schemes.
2. a bijection $\sigma : \{1, \dots, r\} \cong \{1, \dots, r^{'}\}$.
3. for each $i \in \{1, \dots, r\}$ an isomorphism of line bundles over Y:

$$\psi_i : L_i \cong \phi^*(L_{\sigma(i)}^{'})$$

.

3) We let $\mathcal{Z}_*(X)$ denote the free abelian group on the set of isomorphism classes of cobordism cycles over X that is graded by the dimension of cobordism cycles. We call $\mathcal{Z}_*(X)$ the group of cobordism cycles on X. The image of a cobordism cycle $(f : Y \to X, L_1, \dots, L_r)$ in this group is denoted $[f : Y \to X, L_1, \dots, L_r]$, or $[Y \to X, L_1, \dots, L_r]$.

Let $g : X \to X^{'}$ be a projective morphism in Sch_k, composing with g defines the map of graded groups $g_* : \mathcal{Z}_*(X) \to \mathcal{Z}_*(X^{'})$ by

$$[f : Y \to X, L_1, \dots, L_r] \mapsto [g \circ f : Y \to X^{'}, L_1, \dots, L_r]$$

called the push-forward of g.

If $g : X^{'} \to X$ is a smooth equidimensional morphism of relative dimension d, sending $[f : Y \to X, L_1, \dots, L_r]$ to $[p_2 : (Y \times_X X^{'}) \to X^{'}, p_1^*(L_1), \dots, p_1^*(L_r)]$ defines the homomorphism

$$g^* : \mathcal{Z}_*(X) \to \mathcal{Z}_{*+d}(X^{'})$$

called pull-back of g.

Let L be a line bundle over a k-scheme X of finite type, we call the homomorphism $\tilde{c}_1(L) : \mathcal{Z}_*(X) \to \mathcal{Z}_{*-1}$ sending $[f : Y \to X, L_1, \dots, L_r]$ to $[Y \to X, L_1, \dots, L_r, f^*(L)]$ the *first Chern class homomorphism* of L.

Notation 1.1.5. When Y is smooth over k, we denote by 1_Y the cobordism cycle $[id : Y \to Y]$, and if L is a line bundle on Y, we denote by $[L]$ the element $\tilde{c}_1(L)(1_Y)$.

Definition 1.1.6. Let $\mathcal{V}$ be an admissible subcategory. An *oriented Borel-Moore functor on $\mathcal{V}$ with product* is an oriented Borel-Moore functor on $\mathcal{V}$, H_*, together with:

(D4). An element $1 \in H_0(k)$ and, for each pair (X, Y) of k-schemes in $\mathcal{V}$, a bilinear graded pairing (called the external product)

$$\begin{gathered} \times : H_*(X) \times H_*(Y) \to H_*(X \times Y) \\ (\alpha, \beta) \mapsto \alpha \times \beta \end{gathered}$$

which is commutative, associative, and admits 1 as unit.

These satisfy

(A6). Given projective morphisms f and g in $\mathcal{V}$, one has $f_* \times g_* = (f \times g)^*$.

(A7). Given smooth equidimensional morphisms f and g in $\mathcal{V}$, one has

$$f^* \times g^* = (f \times g)^*$$

(A8). Given k-schemes X and Y in $\mathcal{V}$ and a line bundle L on X, one has for any classes $\alpha \in H_*(X)$ and $\beta \in H_*(Y)$,

$$(\tilde{c}_1(L)(\alpha)) \times \beta = \tilde{c}_1(p_1^*(L))(\alpha \times \beta)$$

.

Theorem 1.1.7 (Levine-Morel)**.** *Let $\mathcal{V}$ be an admissible subcategory of Sch_k. The functor $\mathcal{Z}_* : \mathcal{V}' \to \mathbf{Ab}_*$ by $X \mapsto \mathcal{Z}_*(X)$ endowed with the above operations of smooth pull-backs, first Chern classes, and its natural external product is an oriented Borel-Moore functor on Sch_k with product. Moreover, it is the universal one in the sense that given any oriented Borel-Moore functor on Sch_k with product H_*, and an element $a \in H_0(k)$, there is one and only one morphism of oriented Borel-Moore functors with product $\mathcal{Z}_* \to H_*$ that sends 1 to a.*

Proof: Cf. [9].

Definition 1.1.8. Let R_* be a commutative graded ring with unit, $\mathcal{V}$ an admissible subcategory. An *oriented Borel-Moore R_*-functor on $\mathcal{V}$, A_**, is an oriented Borel-Moore functor on $\mathcal{V}$ with product, together with a graded ring homomorphism $\Phi : R_* \to A_*(k)$.

1.2 Algebraic cobordism

Recall from [6] that a commutative formal group law of rank one with coefficients in A is a pair (A, F) consisting of a commutative ring A and a formal power series

$$F(u,v) = \sum_{i,j} a_{i,j} u^i v^j \in A[[u,v]]$$

such that the following hold:

1. $F(u,0) = F(0,u) = u \in A[[u]]$;
2. $F(u,v) = F(v,u) \in A[[u,v]]$;
3. $F(F(u,v),w) = F(u,F(v,w)) \in A[[u,v,w]]$.

Let us also recall from [6] that there is a universal formal group law, denoted by $(F_{\mathbb{L}}, \mathbb{L}^*)$, where $\mathbb{L}^*$ is the ring freely generated by A_{ij} for i, $j \in \mathbb{N}$ over $\mathbb{Z}$ modulo the three relations above. We write $\mathbb{L}^* = \mathbb{Z}[a_{ij}]$, and $F_{\mathbb{L}}(u,v) = \sum_{i,j} a_{ij} u^i v^j$. We observe that the degree of a_{ij} is $1 - i - j$. We also observe that $a_{ij} = 0$ when $ij = 0$ unless $(i,j) = (1,0)$ or $(0,1)$ in which case $a_{1,0} = a_{0,1} = 1$. The ring $\mathbb{L}^*$ is called *the Lazard ring.* The Lazard ring has an equivalent grading defined by $\mathbb{L}_* := \mathbb{L}^{-*}$.

Following Levine and Morel in [9], we call a formal group law (F_R, R) *multiplicative* if $F_R(u,v) = u + v - buv$ for some element $b \in R$, *additive* if $F_R(u,v) = u + v$. A multiplicative formal group law on R is said to be *periodic* if $b \in R$ is invertible.

Definition 1.2.1. An oriented Borel-Moore $\mathbb{L}_*$-functor A_* is said to be *of geometric type* if the following three axioms hold:

(Dim). For any smooth scheme Y and any family $(L_1, \ldots, L_r)$ of line bundles on Y with $r > dim_k(Y)$, one has

$$\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)(1_Y) = 0 \in A_*(Y)$$

.

(Sect). For any smooth k-scheme Y, any line bundle L on Y, any section s of L which is transverse to the zero section of L, one has $\tilde{c}_1(L)(1_Y) = i_*(1_Z)$, where $i : Z \to Y$ is the closed immersion of the zeros of s.

(FGL). Let $F_A \in A_*(k)[[u, v]]$ be the image by the homomorphism $\mathbb{L}_* \to A_*(k)$ of the power series $F_{\mathbb{L}}$. Then for any smooth k-scheme Y and any pair (L, M) of line bundles on Y, one has

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y) = \tilde{c}_1(L \otimes M)(1_Y) \in A_*(Y)$$

.

Definition 1.2.2. Let X be a smooth and irreducible k-scheme. We let $R_*^{Dim}(X) \subset \mathcal{Z}_*(X)$ denote the subset generated by all elements of the form

$$[Y \to X, L_1, \ldots, L_r], \text{ where } dim_k(Y) < r.$$

.

We denote by $\underline{\mathcal{Z}}_*$ the quotient O.B.M. functor $\mathcal{Z}_*/R_*^{Dim}$.

Remark 1.2.3. In the above definition, and throughout the construction of Ω, we follow Levine-Morel in defining the quotient O.B.M. functor in the category of O.B.M. functors. See [9] for details.

Definition 1.2.4. Let Y be a smooth and irreducible k-scheme. We let $R_*^{Sect}(Y) \subset \underline{\mathcal{Z}}_*(Y)$ denote the subset composed by all elements of the form

$$[id_Y, L] - [Z \to Y],$$

where L is a line bundle on Y, $s : Y \to L$ is a section transverse to the zero section and $Z \to Y$ is the closed subscheme of zeroes of s. We denote by $\underline{\Omega}_*$ the oriented Borel-Moore functor $\underline{\mathcal{Z}}_*/R_*^{Sect}$.

Definition 1.2.5. Let Y be a smooth irreducible k-scheme. We let $R_*^{FGL}(Y) \subset \mathbb{L}_* \otimes \underline{\Omega}_*(X)$ be the subset of elements of the form

$$[F_L(L, M)] - [L \otimes M]$$

for any pair (L, M) of line bundles over Y.

Definition 1.2.6. We recall from Levine-Morel the definition of *algebraic cobordism*:

$$\Omega_* : X \mapsto \Omega_*(X),$$

to be the oriented Borel-Moore $\mathbb{L}_*$-functor on Sch_k which is the quotient of $\mathbb{L}_* \otimes \underline{\Omega}_*$ by the relations $\mathbb{L}_* R_*^{FGL}$, i.e.,

$$\Omega := \mathbb{L}_* \otimes \underline{\Omega}_* / < \mathbb{L}_* R_*^{FGL} >.$$

Theorem 1.2.7 (Levine-Morel). *Let $\mathcal{V}$ be an admissible subcategory. Algebraic cobordism is the universal oriented Borel-Moore $\mathbb{L}_*$-functor on $\mathcal{V}$ of geometric type. More precisely, given an oriented Borel-Moore $\mathbb{L}_*$-functor on $\mathcal{V}$ of geometric type A_*, there is a unique morphism of oriented Borel-Moore $\mathbb{L}_*$-functors*

$$\vartheta_A : \Omega_* \to A_*.$$

Proof: Cf. [9].

1.3 O.B.M. theory

Let A_* be an O.B.M. functor on an admissible sbucategory $\mathcal{V}$.

We say that A_* satisfies the projective bundle formula (PB) if the following axiom holds:

(PB). Given a rank n vector bundle $E \to X$ on $X \in Sch_k$ with sheaf of sections ε, let $q : \mathbb{P}(\varepsilon) \to X$ denote the projective bundle, and let $O(1)_E \to \mathbb{P}(\varepsilon)$ be the canonical quotient line bundle of q^*E. For any $i \in \{0, \ldots, n\}$, let

$$\xi^{(i)} : A_{*+i-n}(X) \to A_*(\mathbb{P}(\varepsilon))$$

be the composition of $q^* : A_{*+i-n}(X) \to A_{*+i}(\mathbb{P}(\varepsilon))$ followed by $\tilde{c}_1(O(1)_E)^i : A_{*+i}(\mathbb{P}(\varepsilon)) \to A_*(\mathbb{P}(\varepsilon))$. Then the homomorphism

$$\textstyle\sum_{i=0}^{n-1} \xi^{(i)} : \oplus_{i=0}^{n-1} A_{*+1-n}(X) \to A_*(\mathbb{P}(\varepsilon))$$

is an isomorphism.

We say that A_* satisfies the homotopy property (H) if the following axiom holds:

(H). Let $E \to X$ be a vector bundle of rank r over $X \in \mathcal{V}$, and let $p : V \to X$ be an E-torsor. Then $p^* : A_*(X) \to A_{*+r}(V)$ is an isomorphism.

Theorem 1.3.1 (Levine-Morel). *Let A_* be an O.B.M. functor on $\mathcal{V}$ with product which satifies axioms (PB), (H), and (Sect).Then there is a unique power series*

$$F_A(u, v) = \textstyle\sum_{i,j} a_{i,j} u^i v^j \in A_*(k)[[u, v]]$$

with $a_{i,j} \in A_{i+j-1}(k)$, such that, for any integers $n, m > 0$ we have in the endomorphism ring of $A_*(\mathbb{P}^n \times \mathbb{P}^m)$:

$$F_A(\tilde{c}_1(pr_1^*(\gamma_n)), \tilde{c}_1(pr_2^*(\gamma_m))) = \tilde{c}_1(pr_1^*(\gamma_n) \otimes pr_2^*(\gamma_m)).$$

Moreover, $(A_*(k), F_A(u, v))$ is a commutative formal group law.

Definition 1.3.2. An *oriented cohomology theory* on Sm_k is given by

(D1). An additive functor $A^* : (Sm_k)^{op} \to Rings^*$.

(D2). For each projective morphism $f : Y \to X$ in Sm_k of relative codimension d, a homomorphism of graded $A^*(X)$-modules:

$$f_* : A^*(Y) \to A^{*+d}(X)$$

where $A^*(Y)$ is an $A^*(X)$-module via the ring homomorphism
$f^* : A^*(X) \to A^*(Y)$.

These satisfy

(A1). One has $(Id_X)_* = Id_{A^*(X)}$ for any $X \in Sm_k$. Moreover, given projective morphisms $f : Y \to X$ and $g : Z \to Y$ in Sm_k, with f of relative codimension d and g of relative codimension e, one has

$$(f \circ g)_* = f_* \circ g_* : A^*(Z) \to A^{*+d+e}(X).$$

(A2). Let $f : X \to Z$, $g : Y \to Z$ be transverse morphisms in Sm_k, giving the cartesian square in Sm_k

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

Suppose that f is projective of relative dimension d (thus so is f'). Then $g^* f_* = f'_* g'^*$.

(PB). Let $E \to X$ be a rank n vector bundle over some X in Sm_k, $O(1) \to \mathbb{P}(E)$ be the canonical quotient line bundle with zero section $s : \mathbb{P}(E) \to O(1)$. Let $1 \in A^0(\mathbb{P}(E))$ denote the multiplicative unit element. Define $\xi \in A^1(\mathbb{P}(E))$ by $\xi := s^*(s_*(1))$. Then $A^*(\mathbb{P}(E))$ is a free $A^*(X)$-module, with basis $(1, \xi, \ldots, \xi^{n-1})$.

(H). Let $E \to X$ be a vector bundle over some X in Sm_k, and let $p : V \to X$ be an E-torsor. Then $p^* : A^*(X) \to A^*(V)$ is an isomorphism.

A morphism of oriented cohomology theories is a natural transformation of functors $(Sm_k)^{op} \to Rings^*$ which commutes with the maps f_*.

If $f : X \to Z$ and $g : Y \to Z$ are morphisms in Sch_k, we say that f and g are *transverse in* Sch_k if f and g are Tor-independent, i.e., $Tor_i^{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{O}_Y) = 0$ for all $i > 0$.

Definition 1.3.3. An admissible subcategory $\mathcal{V}$ of Sch_k is *l.c.i. closed* if

1. if $f : Y \to X$ is an l.c.i. morphism in Sch_k and X is in $\mathcal{V}$, then Y is in $\mathcal{V}$
2. if $i : Z \to X$ is a regular embedding with X in $\mathcal{V}$ and $Y \to X$ is the blow-up of X along Z, then Y is in $\mathcal{V}$.

Sch_k, $qSch_k$ and Sm_k are all l.c.i. closed.

Definition 1.3.4. An *oriented Borel-Moore homology theory* A on an l.c.i. closed admissible subcategory $\mathcal{V}$ is given by

(D1). An additive functor $A_* : \mathcal{V}' \to Ab_*,\ X \mapsto A_*(X)$.

(D2). For each l.c.i. morphism $f : Y \to X$ in $\mathcal{V}$ of relative dimension d, a homomorphism of graded groups

$$f^* : A_*(X) \to A_{*+d}(Y).$$

(D3). An element $1 \in A_0(k)$ and, for each pair (X, Y) of k-schemes in Sch_k, a bilinear graded pairing:

$$A_*(X) \otimes A_*(Y) \to A_*(X \times_k Y)$$
$$u \otimes v \mapsto u \times v,$$

called the external product, which is associative, commutative and admits 1 as unit element.

They satisfy, besides (PB) and (H), the following axioms:

(BM1). One has $Id_X^* = Id_{A_*(X)}$ for any $X \in Sch_k$. Moreover, given composible l.c.i. morphisms $f : Y \to X$ and $g : Z \to Y$ in Sch_k of pure relative dimension, one has $(f \circ g)^* = g^* \circ f^*$.

(BM2). Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in Sch_k. Suppose that f and g are transverse, that f is projective and that g is an l.c.i. morphism, giving the cartesian square

$$\begin{tikzcd}
W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Note that f' is projective and g' is an l.c.i. morphism. Then $g^*f_* = f'_*{g'}^*$.

(BM3). Let $f : X' \to X$ and $g : Y' \to Y$ be morphisms in Sch_k. If f and g are projective, then for $u' \in A_*(X')$ and $v' \in A_*(Y')$ one has

$$(f \times g)_*(u' \times v') = f_*(u') \times g_*(v').$$

If f and g are l.c.i. morphisms, then for $u \in A_*(X)$ and $v \in A_*(Y)$ one has

$$(f \times g)^*(u \times v) = f^*(u) \times g^*(v)$$

(CD). For integers $r, N > 0$, let $W = \mathbb{P}^N \times_S \ldots \times_S \mathbb{P}^N$ (r factors), and let $p_i : W \to \mathbb{P}^N$ be the ith projection. Let $X_0, \ldots, X_N$ be the standard homogeneous coordinations on $\mathbb{P}^N$, let $n_1, \ldots, n_r$ be non-negative integers, and let $i : E \to W$ be the subscheme defined by $\prod_{i=1}^r p_i^*(X_N)^{n_i} = 0$. Then $i_* : A_*(E) \to A_*(W)$ is injective.

Let us recall from §1 of [7] the following results:

Theorem 1.3.5. *If A_* is an OBM homology theory on an l.c.i. closed admissible subcategory $\mathcal{V}$, define*

$$\tilde{c}_1(L)(x) := s^*s_*(x)$$

for $L \to X$ a line bundle with zero section $s : X \to L$ and $x \in A_*(X)$. Then A_* is an O.B.M. functor on $\mathcal{V}$ with product which satifies axioms (PB), (H), and (Sect). In addition, if F_A is the formal group law given by theorem 1.3.1, then

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M))(1_Y) = \tilde{c}_1(L \otimes M)(1_Y)$$

for all $Y \in Sm_k$.

We define

$$\Phi_A : \mathbb{L}_* \to A_*(k)$$

to be the ring homomorphism classifying the formal group law $(F_A, A_*(k))$.

Theorem 1.3.6. *Let k be a field admitting resolution of singularities. Then the O.B.M. $\mathbb{L}_*$-functor*

$$X \mapsto \Omega_*(X)$$

on Sch_k admits one and only one oriented Borel-Moore homology theory structure on Sch_k, which we still denote it by Ω_*.

Moreover, Ω_* is the universal such theory on Sch_k, and the restriction of Ω_* to an l.c.i. closed admissible subcategory $\mathcal{V}$ is the universal oriented Borel-Moore homology theory on $\mathcal{V}$.

Theorem 1.3.7. *Let A_* be an oriented Borel-Moore homology theory on an l.c.i. ! closed admissible $\mathcal{V}$. For $X \in Sm_k$ irreducible, define $A^n(X) := A_{dim_k X-n}(X)$, and extend to all $X \in Sm_k$ by additivity. Then $X \mapsto A^*(X)$ defines an oriented cohomology theory on Sm_k.*

In particular, the restriction of Ω_* on Sch_k to Sm_k is in a natural way an oriented cohomology theory Ω^*.

Moreover, assume k admits resolution of singularities, Ω^* is the universal oriented cohomology theory on Sm_k.

Let us also recall from §12 of [9] the following result:

Theorem 1.3.8 (Levine-Morel)**.** *Let k be a field of characteristic 0. Then the canonical homomorphism $\Phi_\Omega : \mathbb{L}_* \to \Omega_*(k)$ is an isomorphism.*

We will also need from §of [9]

Theorem 1.3.9 (Levine-Morel)**.** *Suppose k admits resolution of singularities. Let $i : Z \to X$ be a closed immersion in Sch_k with open complement $j : U \to X$. Then the sequence*

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0$$

is exact

1.4 Examples

In this section, we collect some examples that are in use in this thesis.

Example 1.4.1 (Chow group)**.** The *Chow group* functor on Sm_k given by

$$X \mapsto CH^*(X)$$

for $X \in Sm_k$ is an oriented cohomology theory with usual pullback for arbitrary morphisms in Sm_k, (see Fulton [3]) and usual pushforward for projective morphisms in Sm_k. The first chern class map is given by

$$\tilde{c}_1(L) : CH^*(X) \to CH^{*+1}(X)$$
$$\alpha \mapsto \alpha \cap [D]$$

where D is the associated divisor of the line bundle $L \to X$.
In particular, we have

$$\tilde{c}_1(L)(1_X) = c_1(L) = [D]$$

where 1_X is the fundamental class of $X \in Sm_k$. Let $L^{'} \to X$ be another line bundle over X with associated divisor $D^{'}$, we have

$$c_1(L \otimes L^{'}) = [D] + [D^{'}] = c_1(L) + c_1(L^{'}),$$

We deduce that the formal group law of *Chow* theory is additive as given by

$$F_{CH}(u,v) = u + v.$$

Example 1.4.2 (K-theory)**.** The theory $K_0[\beta, \beta^{-1}]$ with $\deg(\beta) = -1$ on Sm_k is another example of an oriented cohomology theory on Sm_k, which is the assignment

$$X \mapsto K_0(X)[\beta, \beta^{-1}]$$

for $X \in Sm_k$. It has the usual pullback for arbitrary morphisms in Sm_k and push-forward for a projective morphism $f : Y \to X$ in Sm_k of relative dimension d given by

$$f_*([E]\beta^n) = (\sum_i (-1)^i [R^i f_*(E)])\beta^{n-d}.$$

Here we use the facts that the sum is finite, that $R^i f_*(E)$ is a coherent sheaf and that $G_0(X) = K_0(X)$ since X is smooth. For a line bundle $L \to X$ over X, the first chern class map is given by

$$\begin{gathered} \tilde{c}_1(L) : K_0(X)[\beta, \beta^{-1}] \to K_0(X)[\beta, \beta^{-1}] \\ x \mapsto x \bullet (1 - [L^\vee])\beta^{-1}, \end{gathered}$$

where $\bullet$ is the multiplication in $K_0(X)$. Thus, we have

$$c_1(L) = (1 - [L^\vee])\beta^{-1}.$$

We can easily deduce that given another line bundle $M \to X$ over X,

$$c_1(L \otimes M) = c_1(L) + c_1(M) - \beta c_1(L)c_1(M).$$

We conclude that the formal group law of $K_0[\beta, \beta^{-1}]$ is multiplicative as given by

$$F_K(u,v) = u + v - \beta uv.$$

Example 1.4.3 (Homological Chow)**.** The *Chow group* functor on Sch_k given by

$$X \mapsto CH_*(X)$$

is an oriented Borel-Moore homology theory on Sch_k with l.c.i. pull-back given by Fulton [3]. After being restricted to Sm_k, the theory CH_* is equivalent to the theory CH^* as in example 1.4.1. Thus the OBM homology theory CH_* on Sch_k has the same additive formal group law as of the oriented cohomology theory CH^* on Sm_k.

Example 1.4.4 (G-theory)**.** The assignment

$$X \mapsto G_0(X)[\beta, \beta^{-1}]$$

for $X \in Sch_k$ defines another example of an oriented Borel-Moore homology on Sch_k, where $\deg(\beta) = 1$. The pushforward for projective f is given by

$$f_*([\mathcal{F}]\beta^n) := \sum_i (-1)^i [R^i f_*(\mathcal{F})]\beta^n$$

and the pull-back for l.c.i. $f : Y \to X$ of relative dimension d is

$$f^*([\mathcal{F}]\beta^n) := \sum_i (-1)^i [L_i f^*(\mathcal{F})]\beta^{n+d}.$$

Here $L_i f^*$ is the ith left derived functor of f^*, i.e.

$$L_i f^*(\mathcal{F}) := Tor_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{F}).$$

Since f is l.c.i. the sum is finite.

Since the restriction of $G_0[\beta, \beta^{-1}]$ to Sm_k is the same as $K_0[\beta, \beta^{-1}]$ on Sm_k, the OBM homology theory $G_0[\beta, \beta^{-1}]$ on Sch_k has the same multiplicative formal group law as of the oriented cohomology theory $K_0[\beta, \beta^{-1}]$ on Sm_k.

Example 1.4.5 (Universality)**.** Let (F_R, R) be a formal group law with classifying ring homomorphism $\mathbb{L} \to R$. We define on Sm_k

$$\Omega^*_F := \Omega^* \otimes_{\mathbb{L}} R,$$

i.e. $\Omega^*_F(X) := \Omega^*(X) \otimes_{\mathbb{L}} R$ with pullback and pushforward induced from Ω^*. One easily checks that this defines an oriented cohomology theory on Sm_k with formal group law

(F_R, R). In addition, it follows from the universality of Ω^* that Ω^*_F is the universal oriented cohomology theory on Sm_k with formal group law (F_R, R).

Similarly, we define on Sch_k

$$\Omega^F_* := \Omega_* \otimes_{\mathbb{L}} R.$$

The following result is thus a direct from the universality of Ω_* (theorem 1.3.6):

Theorem 1.4.6. *Let k be a field admitting resolution of singularities and let $\mathcal{V}$ be an l.c.i. closed admissible subcategory of Sch_k. Let (F, R) be a formal group law. Then the theory Ω^F_* is the universal oriented Borel-Moore homology theory on $\mathcal{V}$ with formal group law (F, R).*

Similarly, the universality of Ω^* as an oriented cohomology theory on Sm_k (theorem 1.3.7) gives

Theorem 1.4.7. *Let k be a field admitting resolution of singularities. Let (F, R) be a formal group law. Then the theory Ω^*_F is the universal oriented cohomology theory on Sm_k with formal group law (F, R).*

Example 1.4.8 (Additiive and multiplicative theories)**.** Consider the graded ring homomorphisms

$$\Phi_a : \mathbb{L} \to \mathbb{Z}$$

and

$$\Phi_m : \mathbb{L} \to \mathbb{Z}[\beta, \beta^{-1}]$$

classifying respectively the additive and the multiplicative formal group laws.

In view of the previous example 1.4.5 and the universality theoremes 1.4.6 and 1.4.7, we have four theories

- $\Omega^*_+ := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}$ on Sm_k,

- $\Omega^*_\times := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}]$ on Sm_k
- $\Omega^+_* := \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}$ on Sch_k,
- $\Omega^\times_* := \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}]$ on Sch_k,

which are respectively

- the universal additive oriented cohomology theory on Sm_k,
- the universal multiplicative oriented cohomology theory on Sm_k,
- the universal additive oriented Borel-Moore homology theory on Sch_k,
- the universal multiplicative oriented Borel-Moore homology theory on Sch_k.

For details, see §1.6 in this thesis.

Remark 1.4.9. Since CH^* and CH_* have the additive group law, the canonical natural transformations $\Omega^* \to CH^*$ (of oriented cohomology theories on Sm_k) and $\Omega_* \to CH_*$ (of O.B.M. homology theories on Sch_k) descend to

$$\Omega^*_+ \to CH^*$$
$$\Omega^+_* \to CH_*$$

Similarly, since $K_0[\beta, \beta^{-1}]$ and $G_0[\beta, \beta^{-1}]$ have the multiplicative group law the canonical natural transformations $\Omega^* \to K_0[\beta, \beta^{-1}]$ (of oriented cohomology theories on Sm_k) and $\Omega_* \to G_0[\beta, \beta^{-1}]$ (of O.B.M. homology theories on Sch_k) descend to

$$\Omega^*_\times \to K_0[\beta, \beta^{-1}]$$
$$\Omega^\times_* \to G_0[\beta, \beta^{-1}]$$

1.5 Twisting of O.B.M. Theories

Let us begin by recalling from [7] how to twist an O.B.M. homology theory. We need a lemma from §8 of [7]

Lemma 1.5.1 (Levine). *Let A_* be an O.B.M. homology theory on Sch_k and $\tau = (\tau_i) \in \prod_{i\geqslant 0} A_i(k)$, with $\tau_0 = 1$. Then one can define in a unique way for each $X \in Sch_k$ and each vector bundle E on X an endomorphism (of degee zero)*

$$\tilde{c}_\tau(E) : A_*(X) \to A_*(X)$$

such that the following hold:

(0) Given vector bundles $E \to X$ and $F \to X$ one has

$$\tilde{c}_\tau(E) \circ \tilde{c}_\tau(F) = \tilde{c}_\tau(F) \circ \tilde{c}_\tau(E).$$

(1) For a line bundle L one has

$$\tilde{c}_\tau(L) = \textstyle\sum_{i\geqslant 0} \tilde{c}_1(L)^i \tau_i.$$

(2) For any l.c.i. morphism $Y \to X \in Sch_k$, and any vector bundle $E \to X$ over X, one has

$$\tilde{c}_\tau(f^*E) \circ f^* = f^* \circ \tilde{c}_\tau(E).$$

(3) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles over X, then one has

$$\tilde{c}_\tau(E) = \tilde{c}_\tau(E') \circ \tilde{c}_\tau(E'').$$

(4) For any projecive morphism $Y \to X \in Sch_k$, and any vector bundle $E \to X$ over X, one has

$$f_* \circ \tilde{c}_\tau(f^*E) = \tilde{c}_\tau(E) \circ f_*.$$

Proof: Roughly speaking, one uses the splitting principle to reduce to the case of E being a direct sum of line bundles, and note that (0), (1) and (3) forces $\tilde{c}_\tau(\oplus_{j=1}^r L_j) = \prod_{j=1}^r \sum_{i\geqslant 0} \tilde{c}_1(L_j)^i \tau_i$. For details, see §10 of [9] and §8 of [7].

Let A_* be a Borel-Moore homology theory on some l.c.i. closed admissible $\mathcal{V} \subset Sch_k$ and $\tau = (\tau_i) \in \prod_{i=0}^{\infty} A_i(k)$, with $\tau_0 = 1$. Following Levine and Morel in [8], we can twist A_* by τ as follows:

The groups and push-forward maps are unchanged: $A_*^{(\tau)}(X) := A_*(X)$, $f_*^{(\tau)} = f_*$. Let $f : Y \to X$ be an l.c.i. morphism. In order to define the twisting of the pullback of f, we need to recall from [7] the construction of the virtual normal bundle $[N_f]$ of f.

Choose a factorization of f as $f = qi$, with $i : Y \to P$ a regular embedding and $q : P \to X$ a smooth morphism. We have the *relative tangent bundle* $T_q \to P$, defined as the vector bundle whose dual has sheaf of sections the relative differentials $\Omega^1_{Y/X}$. Letting $\mathcal{I}$ be the ideal sheaf of Y in P, we let $N_i \to Y$ be the bundle whose dual has sheaf of sections $\mathcal{I}/\mathcal{I}^2$. We let $[N_f] \in K^0(Y)$ be the class $[N_i] - [i^*T_q]$. We call $[N_f]$ *the virtual normal bundle* of $f : Y \to X$. It is easy to see that $[N_f]$ is independent of the choice of the factorization of f.

We define

$$f^*_{(\tau)} := \tilde{c}_\tau(N_f) \circ f^*,$$

and for any line bundle L over X, we set

$$\tilde{c}_1^{(\tau)}(L) := \tilde{c}_\tau(L) \circ \tilde{c}_1(L)$$

Remark 1.5.2. One easily checks this does define a new oriented Borel-Moore homology theory, denoted by $A_*^{(\tau)}$, on $\mathcal{V}$, with 1^{st} Chern class operator given by the above formula. And the definition of $\tilde{c}_1^{(\tau)}(L)$ can be rewritten as

$$\tilde{c}_1^{(\tau)}(L) = \lambda_{(\tau)}(\tilde{c}_1(L))$$

where $\lambda_{(\tau)}(u) = \sum_{i \geqslant 0} \tau_i \cdot u^{i+1} \in A_*(k)[[u]]$. Thus there is a unique power series $\lambda_{(\tau)}^{-1}(u)$ such that $\lambda_{(\tau)}^{-1}(\lambda_{(\tau)}(u)) = u$. This easily implies that the formal group law $F_A^{(\tau)}$ associated to $A_*^{(\tau)}$ is given by the following equation:

$$F_A^{(\tau)}(u, v) = \lambda_{(\tau)}(F_A(\lambda_{(\tau)}^{-1}(u), \lambda_{(\tau)}^{-1}(v))).$$

Moreover, it is clear that to give a twisting is equivalent to giving a formal series $\lambda_{(\tau)}(u)$ with leading term u.

Remark 1.5.3. By example 1.4.3, the Chow theory CH_* has the structure of O.B.M. homology theory on Sch_k. We give $CH_* \otimes \mathbb{Q}[\beta, \beta^{-1}]$ the structure of O.B.M. homology theory on Sch_k by taking the $\mathbb{Q}[\beta, \beta^{-1}]$-linear extension, i.e.,

$$f^*_{CH_* \otimes \mathbb{Q}[\beta, \beta^{-1}]} := f^*_{CH} \otimes id$$

and similarly for all other structures.

We can produce a new theory on Sch_k, denoted by $CH_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(td)}$, by applying our twisting for the family τ given by

$$\tau = \lambda_{(\tau)}(u) = (1 - e^{-\beta u})/\beta$$

In effect, the presence of the exponential term $e^{-\beta u}$ converts the additive OBM homology theory $CH_* \otimes \mathbb{Q}[\beta, \beta^{-1}]$ on Sch_k into a multiplicative one $CH_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(td)}$ on Sch_k with the multiplicative formal group law

$$F^{(td)}_{CH} = u + v - \beta uv.$$

This follows by a direct computation using the formal group law formula in the previous remark.

We define an oriented Borel-Moore homology theory on Sm_k, denoted by A^{τ}_*, by requiring that the pull-backs are unchanged ($f^*_{\tau} = f^*$), that for a projective morphism $f : Y \to X$, we have

$$f^{\tau}_* := f_* \circ \tilde{c}_{\tau}(T_f),$$

and that for a line bundle L over X we have

$$\tilde{c}^{\tau}_1(L) := \tilde{c}_1(L) \circ \tilde{c}_{\tau}(-L).$$

Remark 1.5.4. By §10 of [9] and §8 of [7], this defines an O.B.M. homology theory on Sm_k, denoted by A_*^τ.

Let $\lambda_\tau(u)$ be the power series such that $\tilde{c}_1^\tau(L) = \lambda_\tau(\tilde{c}_1(L))$. If we set $\vartheta_\tau(u) := \sum_{i\geqslant 0} \tau_i u^i$, then for a given family $\tau = (\tau_i)$, we define $\tau^{(-1)}$ to be such a family that

$$\vartheta_{\tau^{(-1)}}(u) = 1/(\vartheta_\tau(u)).$$

Let us end this section by recalling a lemma in [9].

Lemma 1.5.5 (Levine-Morel)**.** *Let X be in Sm_k, with tangent bundle T_X . Then the automorphism*

$$\tilde{c}_\tau(T_X) : A_*^{(\tau)}(X) \cong A_*^\tau(X)$$

determines an isomorphism of Borel-Moore weak homology theories on Sm_k. In particular, the two theories have the same formal group law.

Proof: Cf. Lemma 4.1.23 of [8].

Remark 1.5.6. By example 1.4.4, $G_0[\beta, \beta^{-1}]$ has the structure of an O.B.M. homology theory on Sch_k.

Moreover, the theory $G_0[\beta, \beta^{-1}]$ on Sch_k and the theory $CH_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(td)}$ on Sch_k both have the multiplicative formal group law $u + v - \beta uv$.

1.6 Main result

We recall the following universal property of K-theory from [9]:

Theorem 1.6.1 (Levine-Morel)**.** *Let A^* be an oriented cohomology theory on Sm_k. Assume that A^* is multiplicative and periodic. Then there exists one and only one morphism of oriented cohomology theories*

$$ch_A : K_0[\beta, \beta^{-1}] \to A^*.$$

Since $\Omega_*^\times$ is also the universal multiplicative periodic theory on Sm_k by theorem 1.4.7, this yields

Corollary 1.6.2 (Levine-Morel)**.** *Suppose k has characteristic zero. Then the canonical transformation* $\Omega^* \to K_0[\beta, \beta^{-1}]$ *descends to an isomorphism of oriented cohomology theories on* Sm_k

$$\Omega_*^\times \to K_0[\beta, \beta^{-1}].$$

Since G_0 and K_0 are equivalent on Sm_k, $G_0[\beta, \beta^{-1}]$ is naturally isomorphic to $\Omega_*^\times$ as O.M.B. homology theories on Sm_k. It is then natural to ask if we can extend this natural isomorphism over Sm_k to one of O.B.M. homology theories over Sch_k.

In this thesis, we will show that for any quasi-projective k-scheme X, we have a natural isomorphism between $G_0(X)[\beta, \beta^{-1}]$ and $\Omega_*^\times(X)$.

Recall that $qSch_k$ denotes the full subcategory of Sch_k consisting of all quasi-projective k-schemes.

We proved the following main result of this dissertation:

Theorem 1.6.3. *Let k be a field of characteristic zero. Let* $\mathcal{V}$ *be an l.c.i. closed admissible subcategory of* Sch_k*, contained in* $qSch_k$*. Then* $G_0[\beta, \beta^{-1}]$ *is the universal multiplicative oriented Borel-Moore homology theory on* $\mathcal{V}$*. That is to say, for any multiplicative oriented Borel-Moore homology theory* A_* *on* $\mathcal{V}$*, there is a unique natural transformation of O.B.M. homology theories*

$$\tau : G_0[\beta, \beta^{-1}] \to A_*.$$

As above, this yields

Corollary 1.6.4. *Let k be a field a characteristic zero. Then the canonical natural transformation of O.B.M. homology theories on* $qSch_k$*,* $\Omega_* \to G_0[\beta, \beta^{-1}]$ *descends to* *an isomorphism of O.B.M. homology theories on* $qSch_k$

$$\Omega_*^\times \to G_0[\beta, \beta^{-1}].$$

In fact, we prove directly that $\Omega_*^\times \to G_0[\beta, \beta^{-1}]$ is an isomorphism of O.B.M. homology theories on $qSch_k$, which yields Theorem 1.6.3 via the universality of $\Omega_*^\times$ (theorem 1.4.6).

Chapter 2

Main Theorem

2.1 Several lemmas

By the universality of the O.B.M. homology theory $\Omega_*^\times$ on Sch_k (example 1.4.8), and the fact that $G_0[\beta,\beta^{-1}]$ has multiplicative periodic group law (example 1.4.4), the canonical natural transformation $\Omega_* \to G_0[\beta,\beta^{-1}]$ descends to a natural transformation

$$\theta_G^\times : \Omega_*^\times \to G_0[\beta,\beta^{-1}] \tag{2.1}$$

of O.B.M. homology theories on Sch_k. To prove corollary 1.6.4, we must show that $\theta_G^\times(X)$ is an isomorphism for all quasi-projective k-schemes X. In this section, we prove some preliminarly results needed for this. We assume throughout that k admits resolution of singularities, in particular, k is a perfect field. We denote $G_0(X)[\beta,\beta^{-1}]$ by $G_0(X)_\beta$. We recall Quillen's localization theorem for G_0:

Theorem 2.1.1 (Quillen [12]). *Let X be a noetherian scheme, $i : Z \to X$ a closed immersion, $j : U \to X$ the open complement of Z. Then there is a natural long exact* *sequence*

$$\begin{aligned} \ldots \to G_n(Z) \xrightarrow{i_*} G_n(X) \xrightarrow{j^*} G_n(U) & \\ \xrightarrow{\delta} G_{n-1}(Z) \to \ldots \to G_1(U) & \\ \xrightarrow{\delta} G_0(Z) \xrightarrow{i_*} G_0(X) \xrightarrow{j^*} G_0(U) \to 0 & \end{aligned}$$

Similarly, the localization theorem 1.3.9 for Ω_* plus the right-exactness of tensor product yields

Theorem 2.1.2. *Let X be in Sch_k. Let $i : Z \to X$ be a closed immersion, $j : U \to X$ the open complement. Then the sequence*

$$\Omega_*^{\times}(Z) \xrightarrow{i_*} \Omega_*^{\times}(X) \xrightarrow{j^*} \Omega_*(U) \to 0$$

is exact.

Lemma 2.1.3. *Take X in Sch_k. Let $i : X_{red} \to X$ be the reduction of X. Then the maps*

$$i_* : \Omega_*^{\times}(X_{red}) \to \Omega_*^{\times}(X)$$

$$i_* : G_0(X_{red})_\beta \to G_0(X)_\beta$$

are isomorphisms.

Proof. The result for G_0 follows from theorem 2.1.1 applied to $i : X_{red} \to X$, since the complement is empty.

For $\Omega_*^{\times}$, this follows from the same result for Ω_*, which follows directly from the definition. □

Lemma 2.1.4. *For $X \in Sch_k$, the map*

$$\theta_G^{\times}(X) : \Omega_*^{\times}(X) \to G_0(X)_\beta$$

is surjective.

Proof. If X is in Sm_k, then we may use theorem 1.6.1 and the fact that $K_0[\beta, \beta^{-1}] = G_0[\beta, \beta^{-1}]^*$ on Sm_k.

In general, we may assume X is reduced. Then X admits a filtration by reduced closed subschemes

$$\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X_N = X$$

with $U_l := X_l \setminus X_{l-1}$ in Sm_k. In particular, X_0 is in Sm_k and the result is thus proven for X_0.

We have the commutative diagram

$$\begin{tikzcd}
\Omega^{\times}_*(X_{l-1}) \arrow[r, "i_*"] \arrow[d, "\theta"'] & \Omega^{\times}_*(X_l) \arrow[r, "j^*"] \arrow[d, "\theta"'] & \Omega^{\times}_*(U_l) \arrow[r] \arrow[d, "\theta"'] & 0 \\
G_0(X_{l-1})_\beta \arrow[r, "i_*"'] & G_0(X_l)_\beta \arrow[r, "j^*"'] & G_0(U_l)_\beta \arrow[r] & 0
\end{tikzcd}$$

The rows are exact by theorems 2.1.1 and 2.1.2. The result follows by induction on l and a diagram chase. □

Lemma 2.1.5. *Let $p : V \to X$ be a vector bundle of rank $n+1$ in Sch_k, and $q : P = P(V) \to X$ the associated projective bundle. Then*

$$q_* : \Omega_*(P) \to \Omega_*(X)$$

is surjective.

Proof. Special Case: Let us first prove the case where $V = X \times_k \mathbb{A}^{n+1}$, thus $P = X \times_k \mathbb{P}^n$. There is a closed immersion $i : X \to X \times_k \mathbb{P}^n$ such that $q \circ i = id_X$. The compostion of the induced morphisms

$$q_* \circ i_* : \Omega_*(X) \to \Omega_*(X \times_k \mathbb{P}^n) \to \Omega_*(X)$$

is the identity on $\Omega_*(X)$. It follows that q_* is surjective. The lemma holds for this case.

General Case: Now let $V \to X$ be a general vector bundle of rank $n+1$. Let Z be a proper closed subscheme of X such that the restriction of P to $U := X \setminus Z$, the complement of Z in X, is $U \times_k \mathbb{P}^n$. We denote by $P^{'}$ the restriction of P to Z. We have the following commutative diagram of morphisms of localization sequences

$$\begin{tikzcd}
\Omega_*(P^{'}) \arrow[r] \arrow[d] & \Omega_*(P) \arrow[r] \arrow[d, "q_*"'] & \Omega_*(U \times_k \mathbb{P}^n) \arrow[r] \arrow[d] & 0 \\
\Omega_*(Z) \arrow[r] & \Omega_*(X) \arrow[r] & \Omega_*(U) \arrow[r] & 0
\end{tikzcd}$$

The vertical map on the left is surjective by induction on dimension of X, and the vertical map on the right is surjective as shown in the special case, we thus conclude the map q_* is surjective by 5-lemma. The proof of the lemma is completed.

□

Lemma 2.1.6. *Let M be in Sm_k, $Z \subset M$ a reduced closed subscheme. Consider the following commutative diagram*

$$\begin{tikzcd}
D \arrow[r, hook] \arrow[d] & M' \arrow[d, "p"'] \\
Z \arrow[r, hook] & M
\end{tikzcd}$$

where p is a sequence of blowups along smooth centers lying over Z, $D = p^{-1}(Z)$, then both vertical maps in the following commutative diagram are surjective:

$$\begin{tikzcd}
\Omega_*^{\times}(D) \arrow[r, two heads] \arrow[d, "p_*"'] & G_0(D)_\beta \arrow[d, "p_*"] \\
\Omega_*^{\times}(Z) \arrow[r, two heads] & G_o(Z)_\beta
\end{tikzcd}$$

Proof. Since p is a sequence of blowups along smooth centers lying over Z, it suffices to show that the lemma holds for the case where M' is the blowup of M along some smooth subscheme F of Z, as displayed in the following diagram:

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d, "p"'] & D \arrow[r, hook] \arrow[d, "p"'] & M_F \arrow[d, "p"'] \\
F \arrow[r, hook] & Z \arrow[r, hook] & M
\end{tikzcd}$$

Let U denote the complement of F in Z, which is the same as the complement of E in D, then we have the following commutative diagram, with the rows being the respective exact localization sequences:

$$\begin{tikzcd}
\Omega_*(E) \arrow[r] \arrow[d, "p_*"'] & \Omega_*(D) \arrow[r] \arrow[d, dashed, "p_*"'] & \Omega_*(U) \arrow[r] \arrow[d, "id"] & 0 \\
\Omega_*(F) \arrow[r] & \Omega_*(Z) \arrow[r] & \Omega_*(U) \arrow[r] & 0
\end{tikzcd}$$

The map p_* on the left is surjective by lemma 2.1.5 as $p : E \to F$ is a projective bundle over F. The surjectivity of the dashed map p_* then follows by 5-lemma.

The surjectivity of $G_0(D)_\beta \to G_0(Z)_\beta$ follows from the commutivity of the diagram. The proof of lemma is completed. □

Lemma 2.1.7. *Let D be a reduced finite type k-scheme, D_2 an irreducible component of D and D_1 the union of the remaining irreducible components of D, so $D = D_1 \cup D_2$. Let $D_{12} = D_1 \cap D_2$ with inclusions $i_j : D_{12} \to D_j$, $\phi_j : D_j \to D$ for $j = 1, 2$. If we write $i_*^- = (i_{1*}, -i_{2*})$ and $\phi = \phi_{1*} + \phi_{2*}$, we have*

(1) the sequence

$$G_0(D_{12})_\beta \xrightarrow{i_*^-} G_0(D_1)_\beta \oplus G_0(D_2)_\beta \xrightarrow{\phi} G_0(D)_\beta \to 0$$

is exact.

(2) the map $\phi : \Omega_*^\times(D_1) \oplus \Omega_*^\times(D_2) \to \Omega_*^\times(D)$ is surjective.

Proof. (1). Consider the morphism $p : D_1 \amalg D_2 \to D_1 \cup D_2$ induced by closed embeddings $D_j \to D_1 \cup D_2$ for $j = 1,\ 2$. Let $U_j := D_j \setminus D_{12}$ with open immersions $\sigma_j : U_j \to D_j$ for $j = 1, 2$ and let $i : D_{12} \to D$ be the inclusion. We denote by σ_j' the open immersions $U_j \to D$ for $j = 1, 2$. Let $\sigma_* := \sigma_{1*} \oplus \sigma_{2*}$ and $\sigma_*' := (\sigma_{1*}', \sigma_{2*}')$.

Since

$$D_1 \amalg D_2 \setminus D_{12} \amalg D_{12} = D \setminus D_{12} = U_1 \amalg U_2,$$

we have the following morphism of localization sequences:

$$\begin{tikzcd}
G_1(U_1) \oplus G_1(U_2) \arrow[r, "id"] \arrow[d, "\partial_1 \oplus \partial_2"'] & G_1(U_1) \oplus G_1(U_2) \arrow[d, "\partial_1 + \partial_2"] \\
G_0(D_{12}) \oplus G_0(D_{12}) \arrow[r, "\Sigma"] \arrow[d, "i_{1*} \oplus i_{2*}"] & G_0(D_{12}) \arrow[d, "i_*"] \\
G_0(D_1) \oplus G_0(D_2) \arrow[r, "p_*"] \arrow[d, "\sigma^*"] & G_0(D) \arrow[d, "{\sigma'}^*"] \\
G_0(U_1) \oplus G_0(U_2) \arrow[r, "id"] \arrow[d] & G_0(U_1) \oplus G_0(U_2) \arrow[d] \\
0 & 0
\end{tikzcd} \tag{2.2}$$

where Σ is the sum map.

We note that

$$ker(p_*) \subset ker(\sigma'^* \circ p_*) = ker(\sigma^*) = im(i_{1*} \oplus i_{2*})$$

Thus, if $y = y_1 \oplus y_2$ is in $ker(p_*)$, then there are elements $x_i \in G_0(D_{12})$ with $y_1 = i_{1*}(x_1)$, $y_2 = i_{2*}(x_2)$. Since $p_*(i_{1*}(x_1) \oplus i_{2*}(x_2)) = 0$, we have $i_*(x_1 + x_2) = 0$ hence there are elements $\alpha_i \in G_1(U_i)$ with $\partial_1(\alpha_1) + \partial_2(\alpha_2) = x_1 + x_2$. Replacing x_i with $x_i - \partial_i(\alpha_i)$, we may assume that $x_1 = -x_2$ in $G_0(D_{12})$, i.e., there is an $x \in G_0(D_{12})$ with

$$y_1 = i_{1*}(x), \ y_2 = -i_{2*}(x)$$

which proves the exactness of our sequence (1) at $G_0(D_1)_\beta \oplus G_0(D_2)_\beta$. The surjectivity of ϕ in (1) follows from the diagram (2.2) and the five lemma, noting that the maps Σ and id are surjective.

(2) Using the right exact localization sequence of $\Omega_*^\times$, the same arguement as for the surjectivity in (1) goes through to prove the surjectivity of ϕ.

The proof of the lemma is completed.

$\square$

Lemma 2.1.8. *Let D be a strict normal crossing divisor on a scheme $M \in Sm_k$. Then $\Omega^\times_*(D) \overset{\sim}{\to} G_0(D)_\beta$.*

Proof. We may assume that D is reduced. Let us write $D = D_1 \cup D_2$, where D_2 is an irreducible component of D. We proceed by induction on the number of irreducible components of D as well as on the dimension of D. As in the previous lemma 2.1.7, we write $D_{12} = D_1 \cap D_2$, and use $i_j : D_{12} \to D_j$ and $\phi_j : D_j \to D$ for $j = 1, 2$ to denote the inclusions.

We have the following commutative diagram:

$$\begin{tikzcd}
\Omega^\times_*(D_{12}) \arrow[r, "i^-_*"] \arrow[d, "\sim"] & \Omega^\times_*(D_1) \oplus \Omega^\times_*(D_2) \arrow[r, "\phi"] \arrow[d, "\sim"] & \Omega^\times_*(D) \arrow[r] \arrow[d, two heads] & 0 \\
G_0(D_{12})_\beta \arrow[r, "i^-_*"'] & G_0(D_1)_\beta \oplus G_0(D_2)_\beta \arrow[r, "\phi"'] & G_0(D)_\beta \arrow[r] & 0
\end{tikzcd}$$

where $i^-_* = (i_{1*}, -i_{2*})$ and $\phi = \phi_{1*} + \phi_{2*}$. The first two of the three vertical maps are isomorphisms by induction, while the third one is surjective. Clearly the top row is a complex; in addition, the bottom row is exact by lemma 2.1.7 (1) and the top map ϕ is surjective by lemma 2.1.7 (2).

We fill $K := coker(i_*)$ into the following diagram

$$\begin{tikzcd}
\Omega^\times_*(D_{12}) \arrow[r, "i^-_*"] \arrow[dd, "\sim"] & \Omega^\times_*(D_1) \oplus \Omega^\times_*(D_2) \arrow[rr, "\phi"] \arrow[dd, "\sim"] \arrow[dr, two heads] & & \Omega^\times_*(D) \arrow[r] \arrow[dd, two heads] & 0 \\
& & K \arrow[ur, two heads] \arrow[dr, "\psi"] & & \\
G_0(D_{12})_\beta \arrow[r, "i^-_*"'] & G_0(D_1)_\beta \oplus G_0(D_2)_\beta \arrow[rr] & & G_0(D)_\beta \arrow[r] & 0
\end{tikzcd}$$

with the sequence

$$\Omega^\times_*(D_{12}) \to \Omega^\times_*(D_1) \oplus \Omega^\times_*(D_2) \to K \to 0$$

being exact. Since $\phi \circ i_* = 0$, we have a surjective map $K \to \Omega^\times_*(D)$. By the five lemma $\psi : K \to G_0(D)_\beta$ is an isomorphism, hence the surjection $\Omega^\times_*(D) \to G_0(D)_\beta$ is an isomorphism.

The proof of the lemma is completed. □

Lemma 2.1.9. *Let M be in Sm_k, $Z \subset M$ a reduced closed subscheme. Let $F \subset M$ be a smooth closed subscheme contained in Z. We denote by M_F the blowup of M along F with the canonical projective morphism $p : M_F \to M$. Then the sequence*

$$0 \to ker(p_*) \to G_n(M_F) \overset{p_*}{\to} G_n(M) \to 0$$

is split exact.

Proof. It suffices to show that $p_* \circ p^* = id$ on $G_n(M) = K_n(M)$. We have the projection formula

$$p_*(a \cdot p^*(b)) = p_*(a) \cdot b$$

for all $a \in K_0(M_F)$ and $b \in G_n(M)$. Thus, for any $x \in G_n(M)$, we have

$$p_*(p^*(x)) = p_*([\mathcal{O}_{M_F}] \cdot p^*(x)) = p_*([\mathcal{O}_{M_F}]) \cdot x.$$

But $R^q p_*(\mathcal{O}_{M_F}) = 0$ for $q > 0$ and $p_*(\mathcal{O}_{M_F}) = \mathcal{O}_M$, so $p_*([\mathcal{O}_{M_F}]) = [\mathcal{O}_M]$, and $p_*(p^*(x)) = [\mathcal{O}_M] \cdot x = x$.

The proof of the lemma is completed. □

Lemma 2.1.10. *Borrowing notations as in the preceding lemma 2.1.9, we denote by D the exceptional divisor $p^{-1}(F)$. Let K be the kernel of $p_* : G_0(M_F) \to G_0(M)$ and K' be the kernel of $p_* : G_0(D) \to G_0(Z)$.*

Then the inclusion $i : D \to M_F$ induces an isomorphism $K' \simeq K$.

Proof. Let us look at the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & K' \arrow[r, dashed, "i_*"'] \arrow[d] & K \arrow[d] & & \\
G_1(M_F) \arrow[r] \arrow[d, "p_*"'] & G_1(U) \arrow[r] \arrow[d, "="'] & G_0(D) \arrow[r] \arrow[d] & G_0(M_F) \arrow[r] \arrow[d] & G_0(U) \arrow[r] \arrow[d, "="'] & 0 \\
G_1(M) \arrow[r] \arrow[d] & G_1(U) \arrow[r] & G_0(Z) \arrow[r] \arrow[d] & G_0(M) \arrow[r] \arrow[d] & G_0(U) \arrow[r] & 0 \\
0 & & 0 & 0 & &
\end{tikzcd}$$

where i_* is the natural map induced by $i : D \to M_F$ and the rows are the respective localization sequences.

Surjectivity of i_*: To see this, we pick an element a in K, it goes to 0 in $G_0(M)$, thus goes to 0 in $G_0(U)$ as well by the commutivity property, which implies that there is an element b of $G_0(D)$ whose image in $G_0(M_F)$ is a. Let c be the image of b in $G_0(Z)$, then c goes to 0 in $G_0(M)$, so it comes from an element d in $G_1(U)$. Let e be the image of d in $G_0(D)$, then it is trivial to see that $b - e$ belongs to K' whose image in K is a.

Injectivity of i_*: Let x be such an element that $i_*(x) = 0$. Then it is the image of some element y in $G_1(U)$, which goes to 0 in $G_0(Z)$ by commutativity. Therefore y is the image of some element z in $G_1(M)$. Since p_* is split surjective by lemma 2.1.9, we can lift z to an element $\tilde{z}$ in $G_1(M_F)$, whose image in $G_1(U)$ is y. Therefore, x is the image of $\tilde{z}$ in $G_0(D)$, which is then 0. We conclude that i_* is injective.

Proof of the lemma is completed. □

Remark 2.1.11. Let us consider the following localization commutative diagrams:

$$\begin{tikzcd}
\Omega_*^\times(D) \arrow[r] \arrow[d] & \Omega_*^\times(M_F) \arrow[r] \arrow[d, "\cong"'] & \Omega_*^\times(U) \arrow[r] \arrow[d, "\cong"'] & 0 \\
G_0(D)_\beta \arrow[r] & G_0(M_F)_\beta \arrow[r] & G_0(U)_\beta \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega_*^\times(Z) \arrow[r] \arrow[d] & \Omega_*^\times(M) \arrow[r] \arrow[d, "\cong"'] & \Omega_*^\times(U) \arrow[r] \arrow[d, "\cong"'] & 0 \\
G_0(Z)_\beta \arrow[r] & G_0(M)_\beta \arrow[r] & G_0(U)_\beta \arrow[r] & 0
\end{tikzcd}$$

It is easy to deduce from lemma 2.1.10 that

$$ker(\Omega_*^\times(D) \to \Omega_*^\times(Z)) \to ker(\Omega_*^\times(M_F) \to \Omega_*^\times(M))$$

is surjective. This is because lemma 2.1.10 still holds if we replace Z by F, and D by $E := p^{-1}(F)$, i.e. the map

$$ker(G_0(E)_\beta \to G_0(F)_\beta) \to ker(G_0(M_F)_\beta \to G_0(M)_\beta)$$

is an isomorphism. We can replace the $G_0[\beta, \beta^{-1}]$ by $K_0[\beta, \beta^{-1}]$ since everything is smooth; similarly $K_0[\beta, \beta^{-1}]$ is isomorphic to theory $\Omega_*^\times$ by Corollary 1.6.2. Thus

$$ker(\Omega_*^\times(E) \to \Omega_*^\times(F)) \to ker(\Omega_*^\times(M_F) \to \Omega_*^\times(M))$$

is an isomorphism. Since the map

$$ker(\Omega_*^\times(E) \to \Omega_*^\times(F)) \to ker(\Omega_*^\times(M_F) \to \Omega_*^\times(M))$$

factors through $ker(\Omega_*^\times(D) \to \Omega_*^\times(Z))$, the surjectivity of

$$ker(\Omega_*^\times(D) \to \Omega_*^\times(Z)) \to ker(\Omega_*^\times(M_F) \to \Omega_*^\times(M))$$

follows.

2.2 Main theorem

Let Z be a k-scheme which admits an embedding into some smooth k-scheme M. By the well-known theorem of Hironaka on embedded resolution of singularities, cf. [5], there is a sequence of blowups of M, $p : M^{'} \to M$, along smooth centers lying over Z such that $D := p^{-1}(Z)$ is a strict normal crossing divisor of $M^{'}$.

To be more precise, we have the following diagram of blowups:

$$\begin{tikzcd}
M^{'} = & M_r \arrow[r, "p_r"] & \cdots\cdots \arrow[r] & M_1 \arrow[r, "p_1"] & M_0 \arrow[r, "p_0"] & M \\
D = & D_r \arrow[r] \arrow[u] & \cdots\cdots \arrow[r] & D_1 \arrow[r] \arrow[u] & D_0 \arrow[r] \arrow[u] & Z \arrow[u]
\end{tikzcd}$$

where

- $p_{i+1} : M_{i+1} \to M_i$ is the blowup of M_i along some smooth $F_i \subset D_i$ for $i = 0, \ldots, r-1$
- $D_{i+1} = p_{i+1}^{-1}(D_i)$ for $i = 0, \ldots, r-1$
- $p = p_0 \circ \ldots \circ p_r$

Claim 2.2.1. *In the commutative diagram of short exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & K_i^{''} \arrow[r] & \Omega_*^{\times}(M_i) \arrow[r] & \Omega_*^{\times}(M) \arrow[r] & 0 \\
0 \arrow[r] & K_i^{'} \arrow[r] \arrow[u] & \Omega_*^{\times}(D_i) \arrow[r] \arrow[u] & \Omega_*^{\times}(Z) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

the map $K_i^{'} \to K_i^{''}$ is surjective for all $i = 0, \ldots, r$.

In particular, $K_r^{'} \to K_r^{''}$ is surjective.

Claim 2.2.2. *In the commutative diagram of short exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & L_i \arrow[r] & G_0(M_i)_\beta \arrow[r] & G_0(M)_\beta \arrow[r] & 0 \\
0 \arrow[r] & K_i \arrow[r] \arrow[u] & G_0(D_i)_\beta \arrow[r] \arrow[u] & G_0(Z)_\beta \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

the map $K_i \to L_i$ *is an isomorphism for all* $i = 0, \dots, r$.

In particular, $K_r \to L_r$ *is an isomorphism.*

Proof of claim 2.2.1: We proceed by induction.

For $i = 0$, p_0 is only a single blowup, the claim follows from remark 2.1.11. Let us assume the claim for $i \geqslant 0$, we must show that the claim holds for $i+1$.

Note that $K^{'}_{i+1} \to K^{'}_{i}$ is surjective by lemma 2.1.6 applied to p_{i+1}, and $K^{''}_{i+1} \to K^{''}_{i}$ is surjective since $p_{i+1*} : G_0(M_{i+1}) \to G_0(M_i)$ is (split) surjective and $G_0(M_j)_\beta = \Omega^\times_*(M_j)$ as M_j is smooth. Letting $N' := ker(\Omega^\times_*(D_{i+1}) \to \Omega^\times_*(D_i))$ and $M' := ker(\Omega^\times_*(M_{i+1}) \to \Omega^\times_*(M_i))$, then we have the natural morphism of short exact sequences as follows:

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r] \arrow[d, "f'"'] & K^{'}_{i+1} \arrow[r, "p_{i+1*}"] \arrow[d, "f"'] & K^{'}_{i} \arrow[r] \arrow[d, "f^{''}"'] & 0 \\
0 \arrow[r] & M' \arrow[r] & K^{''}_{i+1} \arrow[r, "p_{i+1*}"'] & K^{''}_{i} \arrow[r] & 0
\end{tikzcd}$$

We see that f' is surjective because p_{i+1} is a single blowup, and that $f^{''}$ is surjective by induction. The claim thus follows by 5-lemma.

Proof of claim 2.2.2: The same argument goes through as in the preceding claim using the isomorphism of lemma 2.1.10 instead of the surjection of remark 2.1.11.

Theorem 2.2.3. $\theta^\times_G(Z) : \Omega^\times_*(Z) \to G_0(Z)_\beta$ *is an isomorphism.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K'_r \arrow[r] \arrow[d] & \Omega^\times_*(D) \arrow[r] \arrow[d, "\cong"'] & \Omega^\times_*(Z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_r \arrow[r] & G_0(D)_\beta \arrow[r] & G_0(Z)_\beta \arrow[r] & 0
\end{tikzcd}$$

where the middle map is an isomorphism by lemma 2.1.8. It follows that $K'_r \to K_r$ is injective.

By claims 2.2.1 and claim 2.2.2, we have the isomorphism $K_r \simeq L_r$, and the epimorphism $K'_r \twoheadrightarrow K^{''}_r$. Moreover, $K^{''}_r \simeq L_r$ because M and M' are both smooth.

We conclude that $K'_r \to K_r$ is surjective in view of the following commutative diagram:

$$\begin{tikzcd} K'_r \arrow[r, two heads] \arrow[d] & K''_r \arrow[d, "\simeq"'] \\ K_r \arrow[r, "\simeq"] & L_r \end{tikzcd}$$

Therefore, $K'_r \simeq K_r$, which implies that $\Omega^{\times}_{*}(Z) \simeq G_0(Z)_\beta$.

This completes the proof that the natural transformation (2.1) is an isomorphism. As we have already remarked, this proves theorem 1.6.3 and corollary 1.6.4 □

Remark 2.2.4. From the proof of the theorem, it is easy to see that the isomorphism $\Omega^{\times}_{*}(Z) \simeq G_0(Z)_\beta$ does not depend on the choice of embeddings $Z \hookrightarrow M$, nor does it depend on the choice of the resolution blowup sequences. This is because what we have proved is actually only the injectivity of the canonical surjective map $\Omega^{\times}_{*}(Z) \twoheadrightarrow G_0(Z)_\beta$.

Chapter 3

Singular Riemann-Roch

3.1 Singular R.R. for l.c.i. morphisms

In remark 1.5.3 we are able to twist the the additive O.B.M. homology theory $CH_* \otimes \mathbb{Q}[\beta, \beta^{-1}]$ into the periodic multiplicative O.B.M. homology theory $CH_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(td)}$ In addition, if we restrict $CH_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(td)}$ to Sm_k, it is isomorphic to the Todd-twisted oriented cohomology theory $CH^* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{td}$. (see lemma 1.5.5).

Corollary 3.1.1. *Suppose k admits resolution of singularities. Then there is a unique natural transformation of O.B.M. homology theories on $qSch_k$*

$$\tau : G_0[\beta, \beta^{-1}] \to CH_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(td)}$$

Proof. By theorem 1.6.3 $G_0[\beta, \beta^{-1}]$ is the universal periodic multiplicative OBM homology theory on $qSch_k$. Thus, for any oriented OBM homology theory A_* on $qSch_k$ with periodic multiplicative formal group law, there exists a unique natural transformation $\tau : G_0[\beta, \beta^{-1}] \to A_*$.

By construction, $CH_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(td)}$ is an OBM theory on $qSch_k$ with mulitplicative periodic formal group law. Thus we have a unique natural transformation of OBM homology theories $\tau : G_0[\beta, \beta^{-1}] \to CH_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(td)}$.

The proof is completed. $\square$

Recall the inverse system of coefficients $(td)^{-1}$ defined in §1.5, For a vector bundle E on X, we thus have the degree 0 endomorphism $\tilde{c}_{(td)^{-1}}(E)$ of $CH_*(X)[\beta, \beta^{-1}]$. We identify $CH_*(X)$ with the degree zero portion of $CH_*(X)[\beta, \beta^{-1}]$ by sending $x \in CH_p(X)$ to $x\beta^{-p}$, and denote by $\tilde{td}(E)$ the restriction of $\tilde{c}_{(td)^{-1}}(E)$ to $CH_*(X)$.

Remark 3.1.2. Since $(td)^{-1}$ is given by the coefficients of $\beta t/(1 - e^{-\beta t})$, it follows that $td(E)$ agrees with the classical Todd class automorphism of $CH_*(X)$, as defined in [3]

Corollary 3.1.3 (Singular Riemann-Roch for l.c.i. morphisms)**.** *Let $f : Y \to X \in qSch_k$ be an l.c.i. morphsim of relative degree d. Then we have the following commutative diagram:*

$$\begin{tikzcd}
G_0(X) \arrow[r, "f^*"] \arrow[d, "\tau_0"'] & G_0(Y) \arrow[d, "\tau_0"] \\
CH(X)_{\mathbb{Q}} \arrow[r, "\tilde{td}(T_f)\circ f^*"] & CH(Y)_{\mathbb{Q}}
\end{tikzcd}$$

where, for a vector bundle $E \to Y$ over Y, $\tilde{td}(E) : CH_*(Y)_{\mathbb{Q}} \to CH_*(Y)_{\mathbb{Q}}$ sending $a \mapsto td(E) \cap a$ by cap-product map $CH^*(Y)_{\mathbb{Q}} \otimes CH_*(Y)_{\mathbb{Q}} \xrightarrow{\cap} CH_*(Y)_{\mathbb{Q}}$ defined in [4]

Proof: By corollary 3.1.1, there is a natural transformation of O.B.M homology theories

$$\tau : G_0[\beta, \beta^{-1}] \to CH_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(td)}.$$

By restricting τ to degree 0, denoted by τ_0, the naturality of τ gives us the following commutative diagram for an l.c.i. morphism $f : Y \to X \in qSch_k$:

$$\begin{tikzcd}
G_0(X) \arrow[r, "f^*"] \arrow[d, "\tau_0"'] & G_0(Y) \arrow[d, "\tau_0"] \\
CH(X)_{\mathbb{Q}} \arrow[r, "f^*_{(td)}"'] & CH(Y)_{\mathbb{Q}}
\end{tikzcd}$$

To finish the proof, it remains to verify that

$$f^*_{(td)} = \tilde{td}(T_f) \circ f^*.$$

By definition of the twisting operation in section 1.5, we have

$$f^*_{(td)} := \tilde{c}_{td}(N_f) \circ f^*$$

Since $N_f = -T_f$ in $K_0(Y)$, and since $\tilde{td}(T_f)$ is the restriction of $\tilde{c}_{(td)^{-1}}(T_f)$ to the degree zero portion, it suffices to show that

$$\tilde{c}_{(td)}(-T_f) = \tilde{c}_{(td)^{-1}}(T_f).$$

But by definition of $(\tau)^{-1}$ and the multiplicative properties of $\tilde{c}_\tau$, we have

$$\tilde{c}_{(\tau)^{-1}}(E) = \tilde{c}_\tau(E)^{-1}$$

for all τ and E. Since $\tilde{c}_\tau(E)$ is multiplicative in E, we thus have

$$\tilde{c}_{(\tau)^{-1}}(E) = \tilde{c}_\tau(E)^{-1} = \tilde{c}_\tau(-E)$$

The proof of corollary is completed.

3.2 Classical Singular R.R.

Corollary 3.2.1 (Classic Singular R.R.). *Let $f : X \to Y$ be a projective morphism in $qSch_k$. Then the following diagram is commutative:*

$$\begin{tikzcd}
G_0(X) \arrow[r, "f_*"] \arrow[d, "\tau_0"'] & G_0(Y) \arrow[d, "\tau_0"] \\
CH(X)_{\mathbb{Q}} \arrow[r, "f_*"'] & CH(Y)_{\mathbb{Q}}
\end{tikzcd}$$

where τ_0 is the restriction to degree 0 of the natural transformation

$$\tau : G_0[\beta, \beta^{-1}] \to CH \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(td)}.$$

Moreover, τ_0 coincides with the local chern class morphism in [2].

Proof: The commutivity of the diagram is clear by restricting the natural transformation τ to degree 0 noting that τ is a transformation of O.B.M. homology theories, and that the twisting construction does not alter the pushforward maps.

We claim that if P is a projective space $\mathbb{P}^n$, the term of degree n in $\tau_0([\mathcal{O}_p])$ is the fundamental class in $CH_n(P)$, $[P]$. For this, we have canonical natural transformations

$$\Omega_* \xrightarrow{\theta_\times} \Omega_*^\times \xrightarrow{\theta_G^\times} G_0[\beta,\beta^{-1}] \xrightarrow{\tau} CH_*[\beta,\beta^{-1}]^{(td)}.$$

Thus the composition

$$\tau \circ \theta_G^\times \circ \theta_\times : \Omega_* \to CH_*[\beta,\beta^{-1}]^{(td)}$$

is the canonical natural transformation $\theta_{CH^{(td)}}$ given by the universality of Ω_*. Similarly, the composition

$$\tau \circ \theta_G^\times : \Omega_* \to G_0[\beta,\beta^{-1}]$$

is similarly the canonical natural transformation $\theta_G : \Omega_* \to G_0[\beta,\beta^{-1}]$.

If A_* is a O.B.M. homology theory on some $\mathcal{V}$, then for a cobordism cycle $[f : Y \to X]$, then the canonical natural transformation $\theta_A : \Omega_* \to A_*$ has

$$\theta_A([f : Y \to X]) = f_*^A(1_Y^A)$$

Here $1_Y^A = p_Y^*(1)$, where $p : Y \to Speck$ is the structure morphism and $1 \in A_0(k)$ is the unit (note that by definition of a cobordism cycle, Y is irreducible and in Sm_k, and f is projective). We use the notation f_*^A to indicate the pushforward for the theory A.

For $A = G_0[\beta,\beta^{-1}]$, this gives $1_Y = [\mathcal{O}_Y]$ and

$$\theta_G([id : P \to P]) = id_*(1_P) = [\mathcal{O}_P].$$

For $A = CH_*[\beta,\beta^{-1}]^{(td)}$ we have

$$1_Y = p_Y^*([Speck]) = c_{td}(N_{p_Y}) = c_{td}(-T_Y).$$

hence

$$\theta_{CH^{(td)}}([id : P \to P]) = c_{td}(-T_P)$$

and thus

$$\tau([\mathcal{O}_P]) = c_{td}(-T_P)$$

In degree zero, this is just the classical total Todd class of T_P, which written in $CH^*(P)$ is:

$$\tau_0([\mathcal{O}_P]) = td(T_P) = td(\mathcal{O}_P(1))^{n+1} = \left[\frac{H}{1-e^{-H}}\right]^{n+1} = (1+\frac{1}{2}H+\ldots)^{n+1}$$

where $H \in CH^1(P)$ is the class of a hyperplane, and $1 \in CH^0(P)$ is the usual fundamental class.

We conclude τ_0 coincides with the localized chern class map of [2] by the following uniqueness theorem of Baum-Fulton-MacPherson.

Theorem 3.2.2 (Baum-Fulton-MacPherson)**.** *There is only one additive natural transformation $\phi : G_0 \to CH.\otimes \mathbb{Q}$ with the property that if P is a projective space, the top dimensional cycle in $\phi(\mathcal{O}_P)$ is $[P]$.*

The proof of the corollary is completed.

Corollary 3.2.3 (Module)**.** *Let X be in Sch_k. Then for any $a \in K_0(X)$ and $b \in G_0(X)$, we have*

$$\tau_0(a\cdot b) = \tilde{ch}(a)(\tau_0(b)).$$

Proof. By linearity, it suffices to prove it for the case where $a = [E]$ and $b = [\mathcal{F}]$ for $E \to X$ a vector bundle and $\mathcal{F}$ a coherent sheaf on X. By the splitting principle, it is further reduced to the case where E is a line bundle L, with projection $p : L \to X$. Let $\mathcal{L}$ denote the associated sheaf of sections of L.

We have the first chern class operator map

$$\tilde{c}_1(L) : G_0(X)_\beta \to G_0(X)_\beta$$

defined as

$$\tilde{c}_1(L)(x) := s^* s_*(x)\beta^{-1}$$

where $s : X \to L$ is the zero section.

We resolve $\mathcal{O}_{s(X)}$, regarded as an $\mathcal{O}_L$-module, as follows

$$0 \to p^*(\mathcal{L}^\vee) \to \mathcal{O}_L \to \mathcal{O}_{s(X)} \to 0.$$

Using the fact that the pullback map p^* is flat we get the exact sequence

$$0 \to p^*(\mathcal{L}^\vee \otimes \mathcal{F}) \to p^*(\mathcal{F}) \to s_*(\mathcal{F}) \to 0.$$

Since s is a closed immersion, the higher direct images of s_* vanish. Thus in $G_0(L)$, we have

$$s_*([\mathcal{F}]) = [s_*\mathcal{F}] = [p^*(\mathcal{F})] - [p^*(\mathcal{L}^\vee \otimes \mathcal{F})]$$

Since p is flat and $s^*p^* = id$, we have

$$s^*([p^*(\mathcal{F})]) = [\mathcal{F}];\ \ s^*([p^*(\mathcal{L}^\vee \otimes \mathcal{F})]) = [\mathcal{L}^\vee \otimes \mathcal{F}]$$

We then have

$$\tilde{c}_1(L)([\mathcal{F}]) = s^*s_*([\mathcal{F}])\beta^{-1} = ([\mathcal{F}] - [\mathcal{L}^\vee \otimes \mathcal{F}])\beta^{-1}.$$

The naturality of the canonical transformation

$$\tau : G_0(X)_\beta \to CH_*(X)[\beta, \beta^{-1}]^{(td)}_{\mathbb{Q}},$$

gives us

$$\tau(\tilde{c}_1(L)([\mathcal{F}]) = \tilde{c}_1^{(td)}(L)(\tau([\mathcal{F}])).$$

Thus,

$$\tau([\mathcal{F}]\beta^{-1} - [\mathcal{L}^\vee][\mathcal{F}]\beta^{-1}) = (\beta^{-1} - \beta^{-1}e^{-\beta\tilde{c}_1(L)})\tau([\mathcal{F}]).$$

We easily deduce that, at degree 0,

$$\tau_0([\mathcal{L}^\vee][\mathcal{F}]) = e^{-\beta\tilde{c}_1(L)}\tau_0([\mathcal{F}]) = ch(L^\vee) \cap \tau_0([\mathcal{F}]).$$

One should notice that the presence of β in $ch(L^\vee)$ is due to the introduction of β in the twisting of CH_*-theory. Under the identification of sending $x \in CH_p(X)$ to $x \cdot \beta^{-p}$, $ch(L^\vee)$ becomes the classical chern character of $L^\vee$, $e^{\tilde{c}_1(L^\vee)}$, which is equal to $e^{-\tilde{c}_1(L)}$. The proof of corollary is thus completed by replacing $L^\vee$ (respectively $\mathcal{L}^\vee$) by L (respectively $\mathcal{L}$). $\square$

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