

f -categories and Tate motives

by

Jörg Wildeshaus *

LAGA

Institut Galilée

Université Paris 13

Avenue Jean-Baptiste Clément

F-93430 Villetaneuse

France

wildesh@math.univ-paris13.fr

October 9, 2008

Abstract

Using the theory of f -categories [B, Appendix], we prove that the triangulated category of Tate motives over a field k is equivalent to the bounded derived category of its heart, provided that k is algebraic over \mathbb{Q} . This answers a question asked by Levine.

Keywords: triangulated categories, t -structures, derived categories, f -categories, Tate motives.

Math. Subj. Class. (2000) numbers: 14F42 (18E10, 18E30, 19E15, 19F27).

*Partially supported by the *Agence Nationale de la Recherche*, project no. ANR-07-BLAN-0142 “Méthodes à la Voevodsky, motifs mixtes et Géométrie d’Arakelov”.

Contents

0	Introduction	2
1	Statement of the main results	3
2	Review of f-categories	7

0 Introduction

Let (\mathcal{C}, t) be a triangulated category with a t -structure. Denote its heart by \mathcal{C}^0 . It appears natural to ask the following question.

“Can the identity on \mathcal{C}^0 be extended to a functor $D^b(\mathcal{C}^0) \rightarrow \mathcal{C}$ from the bounded derived category of \mathcal{C}^0 to \mathcal{C} ?”

To the author’s knowledge, this question was first formulated in [BBD, Sect. 3.1]. It was solved in [loc. cit.] under the additional hypotheses that (i) \mathcal{C} can be embedded into the derived category $D^+(\mathcal{A})$ of complexes over an Abelian category \mathcal{A} , which are bounded from below, (ii) there are enough injectives in \mathcal{A} . This was then generalized in [B, Appendix], by introducing the notion of f -category. Another generalization was developed in [K], where the notion of *tower* over a category is introduced.

The aim of this note is to identify an easy criterion on the pair (\mathcal{C}, t) allowing for a positive answer to the above question. According to our Theorem 1.1, it is sufficient that the triangulated category \mathcal{C} can be embedded into the unbounded derived category over an exact category.

We admit feeling somewhat uneasy about this result. On the one hand, it is an almost immediate consequence of any one of the approaches from [B] and [K]. It is therefore hard to imagine that an expert in category theory might find much originality in the criterion itself. On the other hand, it does not appear to be “well known” in each given context where it has chances to apply. This latter observation is our only source of hope for the reader’s indulgence.

In this note, we use f -categories (which are recalled in Section 2), and thus follow the approach from [B], which we think of as being the more explicit one. The approach from [K], which yields a universal property, and hence a more satisfactory formal aspect, would give the same criterion on the existence of an extension of $\text{id}_{\mathcal{C}^0}$. We ignore whether the extensions we get from the two approaches coincide.

We illustrate the usefulness of Theorem 1.1 by an application for which the original criterion from [BBD] is insufficient. Let k be a number field, or more generally, a field which is algebraic over \mathbb{Q} . Using our criterion, we establish a functor from the bounded derived category of mixed (effective) *Tate motives over k* [L1] to the triangulated category of (effective) Tate motives over k . Given that the induced maps on Ext-groups coincide with those which were studied in [L1], the results from [loc. cit.] then imply that our functor is in fact an equivalence of categories (Theorem 1.6 and Corollary 1.8). This answers a question asked by Levine.

This work was done while I was enjoying a *modulation de service pour les porteurs de projets de recherche*, granted by the *Université Paris 13*. I wish to thank D.-C. Cisinski for useful discussions and comments.

1 Statement of the main results

Fix a pair (\mathcal{C}, t) consisting of a triangulated category \mathcal{C} with a t -structure [BBD, Déf. 1.3.1]. The heart of the t -structure will be denoted by \mathcal{C}^0 . Recall the notion of *exact category* [Q, Def. on p. 100]. The aim of this note is to prove the following result.

Theorem 1.1. *Assume that \mathcal{C} can be embedded as a full triangulated sub-category into $D(\mathcal{A})$, the derived category of (unbounded) complexes over an exact category \mathcal{A} . Fix one such embedding.*

(a) *There exists an exact functor*

$$real : D^b(\mathcal{C}^0) \longrightarrow \mathcal{C}$$

from the derived category of bounded complexes over \mathcal{C}^0 to \mathcal{C} , inducing the identity on \mathcal{C}^0 , viewed as a full sub-category of both its source and target. The functor $real$ is t -exact wrt. the canonical t -structure on $D^b(\mathcal{C}^0)$ and the given t -structure on \mathcal{C} . Its composition with the cohomology functor $H : \mathcal{C} \rightarrow \mathcal{C}^0$ associated to t equals the canonical cohomology functor $D^b(\mathcal{C}^0) \rightarrow \mathcal{C}^0$.

(b) *The functor $real$ depends functorially on the pair (\mathcal{C}, t) satisfying the above hypothesis on embeddability into the derived category of an exact category. More precisely, given a second triangulated category \mathcal{D} with a t -structure, a t -exact functor*

$$\alpha : \mathcal{C} \longrightarrow \mathcal{D}$$

inducing an exact functor $\alpha^0 : \mathcal{C}^0 \rightarrow \mathcal{D}^0$ between the hearts, a second exact category \mathcal{B} , together with an exact functor $\beta : \mathcal{A} \rightarrow \mathcal{B}$, and a commutative

diagram

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & D(\mathcal{A}) \\ \alpha \downarrow & & \downarrow D(\beta) \\ \mathcal{D} & \hookrightarrow & D(\mathcal{B}) \end{array}$$

of full triangulated embeddings, then the associated functors *real* fit into a commutative diagram

$$\begin{array}{ccc} D^b(\mathcal{C}^0) & \xrightarrow{\text{real}} & \mathcal{C} \\ D^b(\alpha^0) \downarrow & & \downarrow \alpha \\ D^b(\mathcal{D}^0) & \xrightarrow{\text{real}} & \mathcal{D} \end{array}$$

(c) Assume that $\text{Hom}_{\mathcal{C}}(M, N[2]) = 0$, for any two objects M, N of \mathcal{C}^0 . Then \mathcal{C}^0 is of cohomological dimension at most one, and

$$\text{real} : D^b(\mathcal{C}^0) \longrightarrow \mathcal{C}$$

is fully faithful.

(d) Assume that $\text{Hom}_{\mathcal{C}}(M, N[2]) = 0$, for any two objects M, N of \mathcal{C}^0 . Then

$$\text{real} : D^b(\mathcal{C}^0) \longrightarrow \mathcal{C}$$

is an equivalence if and only if \mathcal{C}^0 generates \mathcal{C} (as a triangulated category).

The proof of Theorem 1.1 will be given in Section 2.

Remark 1.2. Note that even when \mathcal{A} is Abelian, there is no hypothesis on the compatibility of the embedding $\mathcal{C} \hookrightarrow D(\mathcal{A})$ with the t -structures on its source and target, i.e., the t -structure on \mathcal{C} is not necessarily the one induced by the canonical t -structure on $D(\mathcal{A})$.

Remark 1.3. One of the main results of [K] states that any choice of *epivalent tower* over \mathcal{C} determines an extension of the identity on \mathcal{C}^0 [K, Cor. 2.7]. It seems however that in practice the existence of such a tower is best guaranteed when \mathcal{C} can be embedded into the derived category $D(\mathcal{A})$ of some exact category \mathcal{A} ... We have not tried to see whether in the situation of Theorem 1.1, the approach of [K] yields the same functor *real*.

Remark 1.4. As follows from Theorem 1.1 (b), the functor *real* is not *a priori* independent of the auxiliary data given by the exact category \mathcal{A} and the embedding of \mathcal{C} into $D(\mathcal{A})$: choose a triangulated t -exact equivalence $\kappa : \mathcal{C} \rightarrow \mathcal{C}$ inducing the identity on \mathcal{C}^0 . Then the functor real_i associated to a fixed embedding i of \mathcal{C} into $D(\mathcal{A})$ and the functor $\text{real}_{i \circ \kappa}$ associated to $i \circ \kappa$ satisfy the relation

$$\text{real}_{i \circ \kappa} = \kappa^{-1} \circ \text{real}_i .$$

Theorem 1.1 applies in particular in the setting of Tate motives over a field k which is algebraic over the field \mathbb{Q} of rational numbers. Recall that for any integer m , there is defined a Tate object $\mathbb{Z}(m)$ in the category $DM_{gm}(k)$ of *geometrical motives over k* [V, p. 189–192]. If $m \geq 0$, then $\mathbb{Z}(m)$ belongs to the full sub-category $DM_{gm}^{eff}(k)$ [V, Thm. 4.3.1] of *effective geometrical motives over k* . By imitating the construction of [loc. cit.], replacing the Abelian groups of *finite correspondences* by their tensor product with \mathbb{Q} [A, Sect. 16.2.4 and Sect. 17.1.3], one constructs the \mathbb{Q} -linear analogues $DM_{gm}(k)_{\mathbb{Q}}$ and $DM_{gm}^{eff}(k)_{\mathbb{Q}}$ of the above categories. The images of the Tate objects in $DM_{gm}(k)_{\mathbb{Q}}$ resp. $DM_{gm}^{eff}(k)_{\mathbb{Q}}$ will still be denoted by $\mathbb{Z}(m)$.

Definition 1.5 (cmp. [L1, Def. 3.1]). Define the *triangulated category of Tate motives over k* as the full triangulated sub-category $DMT(k)_{\mathbb{Q}}$ of $DM_{gm}(k)_{\mathbb{Q}}$ generated by the $\mathbb{Z}(m)$, for $m \in \mathbb{Z}$. Define the *triangulated category of effective Tate motives over k* as the full triangulated sub-category $DMT^{eff}(k)_{\mathbb{Q}}$ of $DM_{gm}^{eff}(k)_{\mathbb{Q}}$ generated by the $\mathbb{Z}(m)$, for $m \geq 0$.

According to [L1, Thm. 1.4 i), 4.2 i)], essentially thanks to the validity of the Beilinson–Soulé vanishing conjecture for number fields, there is a canonical non-degenerate t -structure on $DMT(k)_{\mathbb{Q}}$. (The relation of K -theory of k tensored with \mathbb{Q} , to $\text{Hom}_{DM_{gm}(k)_{\mathbb{Q}}}$ is established by work of Bloch [Bl1, Bl2]; see [L2, Section II.3.6.6].) Denote its heart by $MT(k)_{\mathbb{Q}}$. This is the Abelian category of *mixed Tate motives over k* . It contains all Tate objects $\mathbb{Z}(m)$ [L1, Thm. 4.2 ii)]. It is of cohomological dimension one, and $\text{Hom}_{MT(k)_{\mathbb{Q}}}(M, N[2]) = 0$, for any two mixed Tate motives M, N [L1, Cor. 4.3]. Applying [L1, Th. 1.4 i), ii)] for $a = -\infty$ and $b = 0$, one sees that the t -structure on $DMT(k)_{\mathbb{Q}}$ induces a non-degenerate t -structure on $DMT^{eff}(k)_{\mathbb{Q}}$, whose heart $MT^{eff}(k)_{\mathbb{Q}}$ is the Abelian category of *mixed effective Tate motives over k* . It contains all Tate objects $\mathbb{Z}(m)$, for $m \geq 0$. Its inclusion into $MT(k)_{\mathbb{Q}}$ induces isomorphisms of Yoneda Ext-groups. In particular, it is also of cohomological dimension one, and $\text{Hom}_{MT^{eff}(k)_{\mathbb{Q}}}(M, N[2])$ vanishes, for any two mixed effective Tate motives M, N .

Theorem 1.6. *Let k be an algebraic field extension of \mathbb{Q} .*

(a) *There is a canonical t -exact functor*

$$\text{real} : D^b(MT^{eff}(k)_{\mathbb{Q}}) \longrightarrow DMT^{eff}(k)_{\mathbb{Q}},$$

inducing the identity on $MT^{eff}(k)_{\mathbb{Q}}$. Its composition with the cohomology functor $H : DMT^{eff}(k)_{\mathbb{Q}} \rightarrow MT^{eff}(k)_{\mathbb{Q}}$ associated to t equals the canonical cohomology functor on $D^b(MT^{eff}(k)_{\mathbb{Q}})$.

(b) *The functor*

$$\text{real} : D^b(MT^{eff}(k)_{\mathbb{Q}}) \longrightarrow DMT^{eff}(k)_{\mathbb{Q}}$$

is an equivalence of triangulated categories.

Proof of Theorem 1.6, assuming Theorem 1.1. Recall the definition of the category $Shv_{Nis}(SmCor(k))$ of Nisnevich sheaves with transfers [V, Def. 3.1.1]. It is Abelian [V, Thm. 3.1.4], and there is a canonical full triangulated embedding

$$DM_{gm}^{eff}(k) \hookrightarrow D^-(Shv_{Nis}(SmCor(k)))$$

into the derived category of complexes of Nisnevich sheaves bounded from above [V, Thm. 3.2.6, p. 205]. Imitating the construction from [loc. cit.] using rational coefficients, one shows that there is a canonical full triangulated embedding

$$DM_{gm}^{eff}(k)_{\mathbb{Q}} \hookrightarrow D^-(Shv_{Nis}(SmCor(k))_{\mathbb{Q}}),$$

where $Shv_{Nis}(SmCor(k))_{\mathbb{Q}} \subset Shv_{Nis}(SmCor(k))$ denotes the full sub-category of Nisnevich sheaves taking values in \mathbb{Q} -vector spaces. We thus get a canonical embedding into $D(Shv_{Nis}(SmCor(k))_{\mathbb{Q}})$ of any full triangulated category \mathcal{C} of $DM_{gm}^{eff}(k)_{\mathbb{Q}}$. Thus, the hypothesis of Theorem 1.1 is satisfied with $\mathcal{A} = Shv_{Nis}(SmCor(k))_{\mathbb{Q}}$, and a *canonical* choice of embedding for any such sub-category \mathcal{C} , which in addition is equipped with a t -structure. This is the case in particular for $\mathcal{C} = DMT^{eff}(k)_{\mathbb{Q}}$. Our claim thus follows from Theorem 1.1 (a), (d): indeed, by the results recalled before, $\text{Hom}_{DMT^{eff}(k)_{\mathbb{Q}}}(M, N[2]) = 0$, for any two mixed effective Tate motives M, N , and $MT^{eff}(k)_{\mathbb{Q}}$ generates $DMT^{eff}(k)_{\mathbb{Q}}$ (since $MT^{eff}(k)_{\mathbb{Q}}$ contains all $\mathbb{Z}(m)$, $m \geq 0$). **q.e.d.**

Remark 1.7. Note that it would not be possible to perform the above proof in the context developed in [BBD, Sect. 3.1]. Indeed, our triangulated category is contained in $D^-(Shv_{Nis}(SmCor(k))_{\mathbb{Q}})$, while [loc. cit.] supposes its immersion into some $D^+(\mathcal{A})$ [BBD, p. 79].

Corollary 1.8. *Let k be an algebraic field extension of \mathbb{Q} .*

(a) *There is a unique t -exact functor*

$$real : D^b(MT(k)_{\mathbb{Q}}) \longrightarrow DMT(k)_{\mathbb{Q}}$$

compatible with the functor from Theorem 1.6, and sending the object $\mathbb{Z}(-1)$ to $\mathbb{Z}(-1)$. It induces the identity on $MT(k)_{\mathbb{Q}}$. Its composition with the cohomology functor $H : DMT(k)_{\mathbb{Q}} \rightarrow MT(k)_{\mathbb{Q}}$ associated to t equals the canonical cohomology functor on $D^b(MT(k)_{\mathbb{Q}})$.

(b) *The functor*

$$real : D^b(MT(k)_{\mathbb{Q}}) \longrightarrow DMT(k)_{\mathbb{Q}}$$

is an equivalence of triangulated categories.

Proof. By [V, Thm. 4.3.1], the object $\mathbb{Z}(1)$ is quasi-invertible in the category $DMT^{eff}(k)_{\mathbb{Q}}$. Thus, the category $DMT(k)_{\mathbb{Q}}$ is canonically equivalent to the category obtained from $DMT^{eff}(k)_{\mathbb{Q}}$ by inverting $\mathbb{Z}(1)$.

Since $\mathbb{Z}(1)$ is invertible in $D^b(MT(k)_{\mathbb{Q}})$, the functor from Theorem 1.6 (a) extends uniquely to a functor

$$real : D^b(MT(k)_{\mathbb{Q}}) \longrightarrow DMT(k)_{\mathbb{Q}}$$

mapping $\mathbb{Z}(-1)$ to $\mathbb{Z}(-1)$. It is easily seen to have all the properties listed in part (a) of our claim.

In order to prove part (b), note that by [V, Thm. 4.3.1] again, and by Theorem 1.6 (b), the object $\mathbb{Z}(1)$ is quasi-invertible in $D^b(MT^{eff}(k)_{\mathbb{Q}})$. Therefore, the canonical functor from the category obtained from $D^b(MT^{eff}(k)_{\mathbb{Q}})$ by inverting $\mathbb{Z}(1)$ to $D^b(MT(k)_{\mathbb{Q}})$ is an equivalence. Hence so is *real*.

q.e.d.

Remark 1.9. Corollary 1.8 gives an affirmative answer to the question asked in [L2, top of p. 237], and provides a proof of [A, Prop. 20.2.3.1].

Remark 1.10. As the proofs show, parts (a) of Theorem 1.6 and Corollary 1.8 hold more generally for any perfect base field k satisfying the Beilinson–Soulé vanishing conjecture (i.e., such that the data from [L1] define a t -structure on $DMT(k)_{\mathbb{Q}}$).

2 Review of f -categories

In this section, we review definitions and results of Beilinson’s paper [B]. An additional, rather elementary observation (Proposition 2.7) will then allow to give a proof of Theorem 1.1.

Definition 2.1 ([B, Def. A 1]). (a) An f -category (or *filtered triangulated category*) is a triangulated category \mathcal{CF} , together with a quadruple $f = (\mathcal{CF}(\leq 0), \mathcal{CF}(\geq 0), s, \iota)$, whose first and second components are full triangulated sub-categories of \mathcal{CF} closed under isomorphisms in \mathcal{CF} , s is an exact auto-equivalence on \mathcal{CF} , and ι a transformation of functors from the identity on \mathcal{CF} to s , such that, putting

$$\mathcal{CF}(\leq n) := s^n \mathcal{CF}(\leq 0) \quad , \quad \mathcal{CF}(\geq n) := s^n \mathcal{CF}(\geq 0) \quad \forall n \in \mathbb{Z} \quad ,$$

the following conditions are satisfied.

- (1) We have the inclusions

$$\mathcal{CF}(\leq 0) \subset \mathcal{CF}(\leq 1) \quad , \quad \mathcal{CF}(\geq 0) \supset \mathcal{CF}(\geq 1) \quad ,$$

and the equality

$$\bigcup_{n \in \mathbb{Z}} \mathcal{CF}(\leq n) = \mathcal{CF} = \bigcup_{n \in \mathbb{Z}} \mathcal{CF}(\geq n)$$

(on objects).

(2) For any object X in \mathcal{CF} , we have

$$\iota_X = s(\iota_{s^{-1}X}) : X \longrightarrow sX .$$

(3) For any pair of objects $X \in \mathcal{CF} (\geq 1)$ and $Y \in \mathcal{CF} (\leq 0)$, we have

$$\mathrm{Hom}_{\mathcal{CF}}(X, Y) = 0 ,$$

and the maps

$$\iota_* : \mathrm{Hom}_{\mathcal{CF}}(Y, s^{-1}X) \longrightarrow \mathrm{Hom}_{\mathcal{CF}}(Y, X)$$

and

$$\iota^* : \mathrm{Hom}_{\mathcal{CF}}(sY, X) \longrightarrow \mathrm{Hom}_{\mathcal{CF}}(Y, X)$$

are both isomorphisms.

(4) For any object $X \in \mathcal{CF}$, there exists an exact triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

in \mathcal{CF} , such that $A \in \mathcal{CF} (\geq 1)$ and $B \in \mathcal{CF} (\leq 0)$.

(b) An *f*-functor between *f*-categories is an exact functor that respects the quadruple *f*, i.e., that conserves the sub-categories $\mathcal{CF} (\leq 0)$ and $\mathcal{CF} (\geq 0)$, and commutes with *s* and ι .

(c) Let \mathcal{C} be a triangulated category. An *f*-category over \mathcal{C} is a pair (\mathcal{CF}, j) consisting of an *f*-category \mathcal{CF} and an equivalence of triangulated categories

$$j : \mathcal{C} \xrightarrow{\sim} \mathcal{CF} (\leq 0) \cap \mathcal{CF} (\geq 0) .$$

Here, the target $\mathcal{CF} (\leq 0) \cap \mathcal{CF} (\geq 0)$ denotes the full triangulated subcategory of \mathcal{CF} whose objects lie both in $\mathcal{CF} (\leq 0)$ and in $\mathcal{CF} (\geq 0)$.

Example 2.2 ([B, Ex. A 2]). Fix an Abelian category \mathcal{A} , and consider the *filtered derived category* $D^\bullet F(\mathcal{A})$, for $\bullet \in \{\emptyset, b, -, +\}$, over \mathcal{A} [BBD, Sect. 3.1.1]. Recall that by definition, this category is obtained from the category of complexes A^* over \mathcal{A} equipped with a *finite* decreasing filtration F^\bullet by sub-complexes of A^* . One sets

$$D^\bullet F(\mathcal{A}) (\leq 0) := \{(A^*, F^\bullet A^*), \mathrm{Gr}_{F^\bullet}^i A^* = 0 \forall i > 0\}$$

and

$$D^\bullet F(\mathcal{A}) (\geq 0) := \{(A^*, F^\bullet A^*), \mathrm{Gr}_{F^\bullet}^i A^* = 0 \forall i < 0\} .$$

Thus, the category $D^\bullet F(\mathcal{A}) (\leq 0)$ is obtained from the category of complexes over \mathcal{A} equipped with a finite decreasing filtration concentrated in non-positive degrees, and similarly for $D^\bullet F(\mathcal{A}) (\geq 0)$. The auto-equivalence *s* is given by the shift of the filtration:

$$s : (A^*, F^\bullet A^*) \longmapsto (A^*, F^{\bullet-1} A^*) .$$

Thus for example, an object of $D^\bullet F(\mathcal{A})$ lies in the image $D^\bullet F(\mathcal{A}) (\leq 1)$ of $D^\bullet F(\mathcal{A}) (\leq 0)$ under s if and only if its filtration is concentrated in degrees ≤ 1 . The natural transformation $\iota : \text{id} \rightarrow s$ is given by the identity on the underlying complexes. Altogether, the data

$$(D^\bullet F(\mathcal{A}) (\leq 0), D^\bullet F(\mathcal{A}) (\geq 0), s, \iota)$$

define a structure of f -category on $D^\bullet F(\mathcal{A})$. In fact, $D^\bullet F(\mathcal{A})$ is an f -category over $D^\bullet(\mathcal{A})$: the equivalence

$$j : D^\bullet(\mathcal{A}) \xrightarrow{\sim} D^\bullet F(\mathcal{A}) (\leq 0) \cap D^\bullet F(\mathcal{A}) (\geq 0)$$

is induced by $A^* \mapsto (A^*, Tr^\bullet)$, where Tr^\bullet is the trivial filtration concentrated in degree zero.

Remark 2.3. We leave it to the reader to show that the construction described in Example 2.2 generalizes to give an f -category $D^\bullet F(\mathcal{A})$ over $D^\bullet(\mathcal{A})$, for any exact category \mathcal{A} .

Beilinson proved that the forgetful functor from $D^\bullet F(\mathcal{A})$ to $D^\bullet(\mathcal{A})$ generalizes to the context of f -categories.

Proposition 2.4 ([B, Prop. A 3 (iii)]). *Let $(\mathcal{C}F, j)$ be an f -category over \mathcal{C} . Then there is a unique exact functor*

$$\omega : \mathcal{C}F \longrightarrow \mathcal{C}$$

satisfying the following properties (1)–(3).

(1) *The restriction of ω to $\mathcal{C}F (\leq 0)$ is left adjoint to the embedding*

$$\mathcal{C} \xrightarrow{j} \mathcal{C}F (\leq 0) \cap \mathcal{C}F (\geq 0) \hookrightarrow \mathcal{C}F (\leq 0) .$$

(2) *The restriction of ω to $\mathcal{C}F (\geq 0)$ is right adjoint to the embedding*

$$\mathcal{C} \xrightarrow{j} \mathcal{C}F (\leq 0) \cap \mathcal{C}F (\geq 0) \hookrightarrow \mathcal{C}F (\geq 0) .$$

(3) *For any object X of $\mathcal{C}F$, the morphism*

$$\omega(\iota_X) : \omega(X) \longrightarrow \omega(sX)$$

is an isomorphism.

The functor ω also satisfies the following property.

(4) *For any pair of objects $X \in \mathcal{C}F (\leq 0)$ and $Y \in \mathcal{C}F (\geq 0)$, the map*

$$\omega : \text{Hom}_{\mathcal{C}F}(X, Y) \longrightarrow \text{Hom}_{\mathcal{C}}(\omega X, \omega Y)$$

is an isomorphism.

Definition 2.5 ([B, Def. A 4]). Let $(\mathcal{C}F, j)$ be an f -category over \mathcal{C} . Assume given t -structures $(\mathcal{C}^{t \leq 0}, \mathcal{C}^{t \geq 0})$ and $(\mathcal{C}F^{t \leq 0}, \mathcal{C}F^{t \geq 0})$ on \mathcal{C} and on $\mathcal{C}F$. They are said to be *compatible* with each other if $j : \mathcal{C} \hookrightarrow \mathcal{C}F$ is t -exact (in other words, the t -structure on \mathcal{C} is induced by the t -structure on $\mathcal{C}F$ via j), and

$$s(\mathcal{C}F^{t \leq 0}) = \mathcal{C}F^{t \leq -1} .$$

Here is the reason why we are interested in f -categories.

Theorem 2.6 ([B, Prop. A 5, Sect. A 6]). *Let $(\mathcal{C}F, j)$ be an f -category over \mathcal{C} . Assume given a t -structure on \mathcal{C} . Denote its heart by \mathcal{C}^0 .*

(a) *There is a unique t -structure on $\mathcal{C}F$ compatible with the t -structure on \mathcal{C} . Denote its heart by $\mathcal{C}F^0$.*

(b) *There is a canonical equivalence of categories*

$$\eta : \mathcal{C}F^0 \xrightarrow{\sim} C^b(\mathcal{C}^0)$$

between $\mathcal{C}F^0$ and the category of bounded complexes over \mathcal{C}^0 .

(c) *The composition*

$$\widetilde{real} : C^b(\mathcal{C}^0) \xrightarrow{\eta^{-1}} \mathcal{C}F^0 \hookrightarrow \mathcal{C}F \xrightarrow{\omega} \mathcal{C}$$

factors uniquely through an exact functor

$$real : D^b(\mathcal{C}^0) \longrightarrow \mathcal{C} .$$

The functor $real$ induces the identity on \mathcal{C}^0 , and is t -exact. Its composition with the cohomology functor $H : \mathcal{C} \rightarrow \mathcal{C}^0$ associated to t equals the canonical cohomology functor $D^b(\mathcal{C}^0) \rightarrow \mathcal{C}^0$.

Our contribution to the abstract theory of f -structures reads as follows.

Proposition 2.7. *Let $(\mathcal{C}F, j)$ be an f -category over \mathcal{C} , and \mathcal{D} a full triangulated sub-category of \mathcal{C} . Then there is a unique sub- f -category $\mathcal{D}F$ of $\mathcal{C}F$ such that $\mathcal{D}F$, together with the restriction of j to \mathcal{D} , forms an f -category over \mathcal{D} .*

Proof. We need to recall a last construction due to Beilinson. According to [B, Prop. A 3 (i)], there are exact functors

$$\sigma_{\leq n} : \mathcal{C}F \longrightarrow \mathcal{C}F (\leq n)$$

and

$$\sigma_{\geq n} : \mathcal{C}F \longrightarrow \mathcal{C}F (\geq n)$$

which are left resp. right adjoint to the inclusions, for all integers n . The adjunction properties formally imply the relations

$$s \circ \sigma_{\leq n} = \sigma_{\leq n+1} \circ s$$

and

$$s \circ \sigma_{\geq n} = \sigma_{\geq n+1} \circ s$$

for all n . Again by [B, Prop. A 3 (i)], the functors $\sigma_{\leq n}$ and $\sigma_{\geq n}$ respect all sub-categories $\mathcal{CF}(\leq m)$ and $\mathcal{CF}(\geq m)$, and there are canonical isomorphisms

$$\sigma_{\leq n} \circ \sigma_{\geq m} \xrightarrow{\sim} \sigma_{\geq m} \circ \sigma_{\leq n}.$$

The case $m = n$ will be of particular interest. We identify $\sigma_{\leq n} \circ \sigma_{\geq n}$ and $\sigma_{\geq n} \circ \sigma_{\leq n}$, and note that its target equals the category $\mathcal{CF}(\leq n) \cap \mathcal{CF}(\geq n)$. Define the exact functor

$$\mathrm{gr}_f^n : \mathcal{CF} \longrightarrow \mathcal{C}$$

as the composition of $\sigma_{\leq n} \circ \sigma_{\geq n}$, of s^{-n} , and of j^{-1} . Note the relation

$$\mathrm{gr}_f^n = j^{-1} \circ \sigma_{\leq 0} \circ \sigma_{\geq 0} \circ s^{-n}.$$

By [B, Prop. A 3 (ii)], for any integer n , and any object X of \mathcal{CF} , the adjunction morphisms fit into an exact triangle

$$\sigma_{\geq n+1}X \longrightarrow X \longrightarrow \sigma_{\leq n}X \longrightarrow \sigma_{\geq n+1}X[1],$$

which is unique up to unique isomorphism.

In this precise sense, any object of \mathcal{CF} is a successive extension of objects of \mathcal{C} . The analogous statement must be true for the f -category \mathcal{DF} over \mathcal{D} we intend to construct. Conversely, any successive extension of objects of \mathcal{D} in \mathcal{CF} must belong to \mathcal{DF} .

Hence our only choice is to define \mathcal{DF} as the full sub-category of \mathcal{CF} of objects X such that $\mathrm{gr}_f^n X$ belongs to $\mathcal{D} \subset \mathcal{C}$, for all n . All gr_f^n being exact, and \mathcal{D} triangulated, the category \mathcal{DF} is triangulated. The quadruple $(\mathcal{DF}(\leq 0), \mathcal{DF}(\geq 0), s, \iota)$ is induced from $(\mathcal{CF}(\leq 0), \mathcal{CF}(\geq 0), s, \iota)$. More precisely, we define

$$\mathcal{DF}(\leq 0) := \mathcal{DF} \cap \mathcal{CF}(\leq 0)$$

and

$$\mathcal{DF}(\geq 0) := \mathcal{DF} \cap \mathcal{CF}(\geq 0),$$

and s and ι as the restrictions from the corresponding data on \mathcal{CF} . Indeed, s and s^{-1} respect \mathcal{DF} since $\mathrm{gr}_f^n \circ s = \mathrm{gr}_f^{n-1}$ for all integers n .

First, in order to verify conditions 2.1 (a) (1)–(4), observe that (1)–(3) are obvious by construction. Condition (4) is equivalent to stating that the functors $\sigma_{\leq 0}$ and $\sigma_{\geq 1}$ respect \mathcal{DF} . In order to see this, it suffices to note that the composition $\mathrm{gr}_f^n \circ \sigma_{\leq 0}$ equals gr_f^n if $n \leq 0$, and zero otherwise, and similarly for $\mathrm{gr}_f^n \circ \sigma_{\geq 1}$.

Now consider the triangulated category $\mathcal{DF}(\leq 0) \cap \mathcal{DF}(\geq 0)$. Via j^{-1} , it is equivalent to a full triangulated sub-category of \mathcal{C} . Since all gr_f^n are trivial on $\mathcal{DF}(\leq 0) \cap \mathcal{DF}(\geq 0)$ except for $n = 0$, this sub-category equals \mathcal{D} . Therefore, \mathcal{DF} is indeed an f -category over \mathcal{D} . **q.e.d.**

Corollary 2.8. *Let \mathcal{A} be an exact category, and \mathcal{C} a full triangulated sub-category of $D(\mathcal{A})$. Then the embedding of \mathcal{C} into $D(\mathcal{A})$ induces a choice of f -category over \mathcal{C} .*

Proof. Use Proposition 2.7 and Remark 2.3. **q.e.d.**

Proof of Theorem 1.1. We are in the situation of Corollary 2.8, and thus get an f -category \mathcal{CF} over \mathcal{C} . In addition, a t -structure on \mathcal{C} is given. We need to equip \mathcal{CF} with a t -structure, and take the one from Theorem 2.6 (a). Then Theorem 2.6 (c) gives the construction of the functor *real* and states that it has the properties listed in part (a) of our claim.

Part (b) is a special case of the functorial behaviour of *real* under f -functors [B, Lemma A 7.1].

To prove parts (c) and (d), note that our functor

$$real : D^b(\mathcal{C}^0) \longrightarrow \mathcal{C}$$

is fully faithful if and only if for any two objects M, N of \mathcal{C}^0 , and any integer $p \geq 0$, the morphism

$$\mathrm{Ext}_{\mathcal{C}^0}^p(M, N) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M, N[p])$$

($\mathrm{Ext}^p =$ Yoneda Ext-group of p -extensions) induced by *real* is an isomorphism. This is obviously true for $p = 0$. Now recall that the Yoneda Ext-groups form a universal δ -functor [Bu, Prop. 4.1, Prop. 4.3]. Therefore, the above morphism equals the value on M of the p -th member of the natural transformation of functors

$$\tau^p : \mathrm{Ext}_{\mathcal{C}^0}^p(\bullet, N) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\bullet, N[p])$$

corresponding to this universal property. Using the criterion from [Bu, Prop. 4.2], one shows abstractly (see for example [DG, p. 3]) that τ^p is an isomorphism for $p = 1$, and injective for $p = 2$. Our hypothesis on the vanishing of $\mathrm{Hom}_{\mathcal{C}}(\bullet, N[2])$ thus trivially implies that τ^2 is an isomorphism. It also implies that both source and target of τ^p are zero for all $p \geq 2$. **q.e.d.**

References

- [A] Y. André, *Une introduction aux motifs*, Panoramas et Synthèses **17**, Soc. Math. France (2004).
- [BBD] A.A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, in B. Teissier, J.L. Verdier (eds.), *Analyse et topologie sur les espaces singuliers (I)*, Astérisque **100** (1982).
- [B] A.A. Beilinson, *On the derived category of perverse sheaves*, in: Yu.I. Manin (ed.), *K-Theory, Arithmetic and Geometry*, Lect. Notes Math. **1289**, Springer-Verlag (1987), 27–41.

- [Bl1] S. Bloch, *Algebraic cycles and algebraic K-theory*, Adv. in Math. **61** (1986), 267–304.
- [Bl2] S. Bloch, *The moving lemma for higher Chow groups*, J. Alg. Geom. **3** (1994), 537–568.
- [Bu] A. Buchsbaum, *Satellites and exact functors*, Ann. of Math. **71** (1960), 199–209.
- [DG] P. Deligne, A.B. Goncharov, *Groupes fondamentaux motiviques de Tate mixte*, Ann. Scient. ENS **38** (2005), 1–56.
- [K] B. Keller, *Derived categories and universal problems*, Comm. in Alg. **19** (1991), 699–747.
- [L1] M. Levine, *Tate motives and the vanishing conjectures for algebraic K-theory*, in P.G. Goerss, J.F. Jardine (eds.), *Algebraic K-Theory and algebraic topology. Proceedings of the NATO Advanced Study Institute, held at Lake Louise, Alberta, December 12–16, 1991*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **407**, Kluwer (1993), 167–188.
- [L2] M. Levine, *Mixed Motives*, Math. Surveys and Monographs **57**, AMS (1998).
- [Q] D. Quillen, *Higher algebraic K-theory: I*, in H. Bass (ed.), *Algebraic K-Theory I — Higher K-Theories. Proceedings of the Conference held at the Seattle Research Center of the Battelle Memorial Institute, August 28 – September 8, 1972*, Lect. Notes Math. **341**, Springer-Verlag (1973), 85–147.
- [V] V. Voevodsky, *Triangulated categories of motives*, Chapter 5 of V. Voevodsky, A. Suslin, E.M. Friedlander, *Cycles, Transfers, and Motivic Homology Theories*, Ann. of Math. Studies **143**, Princeton Univ. Press (2000).