Notes on Artin–Tate motives

by

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Abstract

We first study the weight structure on the triangulated category of Artin–Tate motives over a perfect base field k, building on the main result of [Bo]. We then study the t-structure on the triangulated category of Artin–Tate motives, when k is algebraic over $\mathbb{Q}$, generalizing the main result of [L1]. We finally study the interaction of the weight structure and the t-structure. When k is a number field, this will give a useful criterion identifying the weight structure via realizations.

Keywords: Artin–Tate motives, Dirichlet–Tate motives, weight structures, t-structures, realizations.

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Contents

0 Introduction

The aim of this article is to exhibit the basic structural properties of the *triangulated category of Artin–Tate motives* over a fixed perfect base field k. The definition of this category will be recalled, and a number of generalizations will be defined in Section 1. Roughly speaking, the properties we shall be interested in, then fall into two classes.

First (Section 2), we apply the main results of [Bo] to Artin–Tate motives. More precisely, we show (Theorem 2.5) that the *weight structure* of [loc. cit.], defined on the category of *geometrical motives* [V1], induces a weight structure on the triangulated category of Artin–Tate motives. We then give a very explicit description of the *heart* of the latter, showing in particular that it is Abelian semi-simple.

Second (Section 3), we generalize the main result from [L1] from Tate motives to Artin–Tate motives, when the base field is algebraic over $\mathbb{Q}$. More precisely, we show (Theorem 3.1) that under this hypothesis, there is a non-degenerate *t-structure* on the triangulated category of Artin–Tate motives. The strategy of proof is identical to the one used by Levine. Using the main result of [W2], we show (Corollary 3.3) that this triangulated category is canonically equivalent to the bounded derived category of its heart (formed with respect to the t-structure), i.e., of the Abelian category of *mixed Artin–Tate motives*.

Our main interest lies then in the simultaneous application of both points of view: that of weight structures and that of t-structures. Still assuming that k is algebraic over $\mathbb{Q}$, we give a characterization (Theorem 3.8) of the weight structure on the triangulated category of Artin–Tate motives in terms of the t-structure. Specializing further to the case of number fields, we get a powerful criterion (Corollary 3.10) allowing to identify the weight structure via the Hodge theoretic or ℓ-adic realization.

We should warn the reader that all our constructions are *a priori* with $\mathbb{Q}$-coefficients. This seems to be necessary for at least two reasons. First, the *triangulated category of Artin motives* is not known to admit a t-structure; by contrast, such a structure becomes obvious after tensoring with $\mathbb{Q}$ (see Section 1). This t-structure is at the very basis of our construction. Second, as pointed out in [L1], the existence of the t-structure on the *triangulated category of Tate motives* necessitates (and is in fact equivalent to) the validity of the Beilinson–Soulé vanishing conjecture; but this vanishing is only known (for algebraic base fields) after tensoring with $\mathbb{Q}$.

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Notation and conventions: Throughout the article, k denotes a fixed perfect base field. The notation of this paper follows that of [V1]. We refer to [loc. cit.] for the definition of the triangulated categories $DM_{gm}^{eff}(k)$ and $DM_{gm}(k)$ of (effective) geometrical motives over k. Let F be a commutative $\mathbb{Q}$-algebra. The notation $DM_{gm}^{eff}(k)_F$ and $DM_{gm}(k)_F$ stands for the F-linear analogues of these triangulated categories defined in [An, Sect. 16.2.4 and Sect. 17.1.3]. Similarly, let us denote by $CHM^{eff}(k)$ and $CHM(k)$ the categories opposite to the categories of (effective) Chow motives, and by $CHM^{eff}(k)_F$ and $CHM(k)_F$ the pseudo-Abelian completion of the category $CHM^{eff}(k) \otimes_{\mathbb{Z}} F$ and $CHM(k) \otimes_{\mathbb{Z}} F$, respectively. Using [V2, Cor. 2] ([V1, Cor. 4.2.6] if k admits resolution of singularities), we canonically identify $CHM^{eff}(k)_F$ and $CHM(k)_F$ with a full additive sub-category of $DM_{gm}^{eff}(k)_F$ and $DM_{gm}(k)_F$, respectively.

1 Definition and first properties

Fix a commutative $\mathbb{Q}$-algebra F, which we suppose to be semi-simple and Noetherian, in other words, a finite direct product of fields of characteristic zero. In this section, we recall the definition of the F-linear triangulated category of Artin–Tate motives (Definition 1.3), and define a number of variants, indexed by certain sub-categories of the category of discrete representations of the absolute Galois group of our perfect base field k (Definition 1.6). We then start the analysis of this category, following the part of [L1] valid without additional assumptions on k.

For any integer m, there is defined a Tate object $\mathbb{Z}(m)$ in $DM_{gm}(k)$, which belongs to $DM_{gm}^{eff}(k)$ if $m \geq 0$ [V1, p. 192]. We shall use the same notation when we consider $\mathbb{Z}(m)$ as an object of $DM_{gm}(k)_F$.

Definition 1.1 (cmp. [L1, Def. 3.1])**.** Define the *triangulated category of Tate motives over* k as the strict full triangulated sub-category $DMT(k)_F$ of $DM_{gm}(k)_F$ generated by the $\mathbb{Z}(m)$, for $m \in \mathbb{Z}$.

Recall that by definition, a strict sub-category is closed under isomorphisms in the ambient category. It is easy to see that $DMT(k)_F$ is tensor triangulated.

Definition 1.2. Define the *triangulated category of Artin motives over* k as the pseudo-Abelian completion of the strict full triangulated sub-category $DMA(k)_F$ of $DM_{gm}^{eff}(k)_F$ generated by the motives $M_{gm}(X)$ of smooth zero-dimensional schemes X over k.

This category is again tensor triangulated.

Definition 1.3. Define the *triangulated category of Artin–Tate motives over* k as the strict full tensor triangulated sub-category $DMAT(k)_F$ of $DM_{gm}(k)_F$ generated by $DMA(k)_F$ and $DMT(k)_F$.

The following observation [V1, Remark 2 on p. 217] is central for what is to follow.

Proposition 1.4. *The triangulated category $DMA(k)_F$ of Artin motives is canonically equivalent to $D^b(MA(k)_F)$, the bounded derived category of the Abelian category $MA(k)_F$ of discrete representations of the absolute Galois group of k in finitely generated F-modules.*

More precisely, if X is smooth and zero-dimensional over k, and $\bar{k}$ a fixed algebraic closure of k, then the absolute Galois group of k, when identified with the group of automorphisms of $\bar{k}$ over k, acts canonically on the set of $\bar{k}$-valued points of X. The object of $MA(k)_F$ corresponding to $M(X)$ under the equivalence of Proposition 1.4 is nothing but the formal F-linear envelope of this set, with the induced action of the Galois group. Note that the category $MA(k)_F$ is semi-simple.

Corollary 1.5. *There is a canonical non-degenerate t-structure on the category $DMA(k)_F$. Its heart is equivalent to $MA(k)_F$.*

By contrast [V1, Remark 1 on p. 217], it is not clear how to construct a non-degenerate t-structure on the triangulated category $DMA(k)$ of *zero motives* (whose F-linearization equals $DMA(k)_F$).

For the rest of this section, let us identify the triangulated categories $DMA(k)_F$ and $D^b(MA(k)_F)$ via the equivalence of Proposition 1.4. Let us also fix a strict full Abelian semi-simple F-linear tensor sub-category $\mathcal{A}$ of $MA(k)_F$, containing the category *triv* of objects of $MA(k)_F$ on which the Galois group acts trivially.

Definition 1.6. Define $D\mathcal{A}T$ as the strict full tensor triangulated sub-category of $DMAT(k)_F$ generated by $\mathcal{A}$, and by $DMT(k)_F$.

Examples 1.7. (a) When $\mathcal{A}$ equals $MA(k)_F$, then $D\mathcal{A}T = DMAT(k)_F$.
(b) When $\mathcal{A}$ equals *triv*, then $D\mathcal{A}T = DMT(k)_F$.

Let us agree to set $\mathbb{Z}(n/2) := 0$ for odd integers n. For any object M of $D\mathcal{A}T$ and any integer n, let us write $M(n/2)$ for the tensor product of M and $\mathbb{Z}(n/2)$.

Following [L1], let us first define $D\mathcal{A}T_{[a,b]}$ as the full triangulated sub-category of $D\mathcal{A}T$ generated by the objects $N(m)$, for $N \in \mathcal{A}$ and $a \le -2m \le b$, for integers $a \le b$ (we allow $a = -\infty$ and $b = \infty$). We denote $D\mathcal{A}T_{[a,a]}$ by $D\mathcal{A}T_a$.

Lemma 1.8. *The category $D\mathcal{A}T_a$ is zero for $a \in \mathbb{Z}$ odd. For $a \in \mathbb{Z}$ even, the exact functor*

$$D\mathcal{A}T_a \longrightarrow DMA(k)_F \; , \; M \longmapsto M(a/2)$$

induces an equivalence between $D\mathcal{A}T_a$ and the bounded derived category of $\mathcal{A}$.

Proof. By construction, the functor is exact, and identifies $D\mathcal{A}T_a$ with the full triangulated sub-category of $DMA(k)_F$ of objects, whose co-homology lies in $\mathcal{A}$. Recall that we identified the categories $DMA(k)_F$ and $D^b(MA(k)_F)$. It remains to see that the obvious exact functor

$$D^b(\mathcal{A}) \longrightarrow D^b(MA(k)_F)$$

is fully faithful. But this an immediate consequence of the fact that the Abelian categories $\mathcal{A}$ and $MA(k)_F$ are semi-simple. **q.e.d.**

In particular, there is a canonical t-structure $(D\mathcal{A}T_a^{\le 0}, D\mathcal{A}T_a^{\ge 0})$ on $D\mathcal{A}T_a$: the category $D\mathcal{A}T_a^{\le 0}$ is the full sub-category of $D\mathcal{A}T_a$ generated by objects $N(-a/2)[r]$, for $N \in \mathcal{A}$ and $r \ge 0$, and $D\mathcal{A}T_a^{\ge 0}$ is the full sub-category generated by objects $N(-a/2)[r]$, for $N \in \mathcal{A}$ and $r \le 0$. The category $\mathcal{A}$ is equivalent to the heart $\mathcal{A}T_a$ of this canonical t-structure via the functor $N \mapsto N(-a/2)$.

Second, we construct auxiliary t-structures.

Lemma 1.9 (cmp. [L1, Lemma 1.2])**.** *Let $a \le n \le b$. Then the pair $(D\mathcal{A}T_{[a,n]}, D\mathcal{A}T_{[n+1,b]})$ defines a t-structure on $D\mathcal{A}T_{[a,b]}$.*

Proof. Imitate the proof of [L1, Lemma 1.2]. The decisive ingredient is the following generalization of the vanishing from [L1, Def. 1.1 i)]:

$$\operatorname{Hom}_{D\mathcal{A}T}\big(N_1(m_1)[r], N_2(m_2)[s]\big) = 0 \; , \; \forall\, m_1 > m_2 \, , \; N_1, N_2 \in \mathcal{A} \, , \; r, s \in \mathbb{Z} \, .$$

It holds because $\mathrm{Hom}_{DAT} = \mathrm{Hom}_{DM_{gm}(k)_F}$, and $\mathrm{Hom}_{DM_{gm}(\bullet)_F}$ satisfies descent for finite extensions L/k of the base field. Choosing an extension L splitting both N_1 and N_2 therefore allows to deduce the desired vanishing from that of

$$\mathrm{Hom}_{DM_{gm}(L)_{\mathbb{Q}}}\big(\mathbb{Z}(m_1)[r], \mathbb{Z}(m_2)[s]\big) .$$

q.e.d.

Remark 1.10. (a) The above proof uses the relation of K-theory of L tensored with $\mathbb{Q}$, with $\mathrm{Hom}_{DM_{gm}(L)_{\mathbb{Q}}}$. This relation is established by work of Bloch [Bl1, Bl2] (see [L2, Section II.3.6.6]), and will be used again in the proofs of Theorem 3.1 and Variant 3.2.
(b) Levine pointed out that the t-structures from Lemma 1.9, for varying n, can be used to show that the category $D\mathcal{A}T$ is pseudo-Abelian. We shall give an alternative proof of this result in Section 2, using Bondarko's theory of weight structures (Corollary 2.6).

Note that since $D\mathcal{A}T_{[a,n]}$ and $D\mathcal{A}T_{[n+1,b]}$ are themselves triangulated, the t-structure from Lemma 1.9 is necessarily degenerate. As in [L1, Sect. 1], denote the truncation functors by

$$W_{\le n} : D\mathcal{A}T_{[a,b]} \longrightarrow D\mathcal{A}T_{[a,n]}$$

and

$$W_{\ge n+1} : D\mathcal{A}T_{[a,b]} \longrightarrow D\mathcal{A}T_{[n+1,b]} ,$$

and note that for fixed n, they are compatible with change of a or b. Write gr_n for the composition of $W_{\le n}$ and $W_{\ge n}$ (in either sense). The target of this functor is the category $D\mathcal{A}T_n$. We are now ready to set up the data necessary for the t-structure we shall actually be interested in.

Definition 1.11 (cmp. [L1, Def. 1.4])**.** Fix $a \le b$ (we allow $a = -\infty$ and $b = \infty$).
(a) Define $D\mathcal{A}T^{\le 0}_{[a,b]}$ as the full sub-category of $D\mathcal{A}T_{[a,b]}$ of objects M such that $\mathrm{gr}_n M \in D\mathcal{A}T^{\le 0}_n$ for all integers n such that $a \le n \le b$.
(b) Define $D\mathcal{A}T^{\ge 0}_{[a,b]}$ as the full sub-category of $D\mathcal{A}T_{[a,b]}$ of objects M such that $\mathrm{gr}_n M \in D\mathcal{A}T^{\ge 0}_n$ for all integers n such that $a \le n \le b$.

As we shall see (Theorem 3.1, Variant 3.2), the pair $(D\mathcal{A}T^{\le 0}_{[a,b]}, D\mathcal{A}T^{\ge 0}_{[a,b]})$ defines a t-structure on $D\mathcal{A}T_{[a,b]}$, provided that the base field k is algebraic over $\mathbb{Q}$. In particular, we then get a canonical t-structure on $D\mathcal{A}T$. The vital point will be the validity of the Beilinson–Soulé vanishing conjecture for all finite field extensions of k.

2 The weight structure

The purpose of this section is to first review Bondarko's definition of weight structures on triangulated categories, and his result on the existence of such a weight structure on the categories $DM_{gm}(k)$ and $DM_{gm}(k)_F$ [Bo]. We will then show (Theorem 2.5) that the latter induces a weight structure on any of the triangulated categories constructed in Section 1.

Definition 2.1 (cmp. [Bo, Def. 1.1.1]). Let $\mathcal{C}$ be a triangulated category. A *weight structure on* $\mathcal{C}$ is a pair $w = (\mathcal{C}_{w\le 0}, \mathcal{C}_{w\ge 0})$ of full sub-categories of $\mathcal{C}$, such that, putting

$$\mathcal{C}_{w\le n} := \mathcal{C}_{w\le 0}[n] \quad , \quad \mathcal{C}_{w\ge n} := \mathcal{C}_{w\ge 0}[n] \quad \forall\, n \in \mathbb{Z} ,$$

the following conditions are satisfied.

(1) The categories $\mathcal{C}_{w\le 0}$ and $\mathcal{C}_{w\ge 0}$ are Karoubi-closed (i.e., closed under retracts formed in $\mathcal{C}$).

(2) (Semi-invariance with respect to shifts.) We have the inclusions

$$\mathcal{C}_{w\le 0} \subset \mathcal{C}_{w\le 1} \quad , \quad \mathcal{C}_{w\ge 0} \supset \mathcal{C}_{w\ge 1}$$

of full sub-categories of $\mathcal{C}$.

(3) (Orthogonality.) For any pair of objects $M \in \mathcal{C}_{w\le 0}$ and $N \in \mathcal{C}_{w\ge 1}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(M, N) = 0 .$$

(4) (Weight filtration.) For any object $M \in \mathcal{C}$, there exists an exact triangle

$$A \longrightarrow M \longrightarrow B \longrightarrow A[1]$$

in $\mathcal{C}$, such that $A \in \mathcal{C}_{w\le 0}$ and $B \in \mathcal{C}_{w\ge 1}$.

It is easy to see that for any integer n and any object $M \in \mathcal{C}$, there is an exact triangle

$$A \longrightarrow M \longrightarrow B \longrightarrow A[1]$$

in $\mathcal{C}$, such that $A \in \mathcal{C}_{w\le n}$ and $B \in \mathcal{C}_{w\ge n+1}$. By a slight generalization of the terminology introduced in condition 2.1 (4), we shall refer to any such exact triangle as a weight filtration of M.

Remark 2.2. Our convention concerning the sign of the weight is opposite to the one from [Bo, Def. 1.1.1], i.e., we exchanged the roles of $\mathcal{C}_{w\le 0}$ and $\mathcal{C}_{w\ge 0}$.

Definition 2.3 ([Bo, Def. 1.2.1]). Let $w = (\mathcal{C}_{w\le 0}, \mathcal{C}_{w\ge 0})$ be a weight structure on $\mathcal{C}$. The *heart of* w is the full additive sub-category $\mathcal{C}_{w=0}$ of $\mathcal{C}$ whose objects lie both in $\mathcal{C}_{w\le 0}$ and in $\mathcal{C}_{w\ge 0}$.

One of the main results of [Bo] is the following.

Theorem 2.4 ([Bo, Sect. 6])**.** *(a) If k is of characteristic zero, then there is a canonical weight structure on the category $DM_{gm}^{eff}(k)$. It is uniquely characterized by the requirement that its heart equal $CHM^{eff}(k)$.*
(b) If k is of characteristic zero, then there is a canonical weight structure on the category $DM_{gm}(k)$, extending the weight structure from (a). It is uniquely characterized by the requirement that its heart equal $CHM(k)$.
(c) Let F be a commutative $\mathbb{Q}$-algebra. Analogues of statements (a) and (b) hold for the F-linearized categories $DM_{gm}^{eff}(k)_F$, $CHM^{eff}(k)_F$, $DM_{gm}(k)_F$, and $CHM(k)_F$, and for a perfect base field k of arbitrary characteristic.

Let us refer to any of these weight structures as *motivic*. For a concise review of the main ingredients of Bondarko's proof, see [W1, Sect. 1].

Now fix a finite direct product F of fields of characteristic zero, and a full Abelian F-linear tensor sub-category $\mathcal{A}$ of $MA(k)_F$, containing the category *triv*. Recall (Definition 1.6) that

$$D\mathcal{A}T \subset DMAT(k)_F \subset DM_{gm}(k)_F$$

denotes the strict full tensor triangulated sub-category generated by $\mathcal{A}$, and by the triangulated category $DMT(k)_F$ of Tate motives. Intersecting with $D\mathcal{A}T$, the motivic weight structure $(DM_{gm}(k)_{F,w\le 0}, DM_{gm}(k)_{F,w\ge 0})$ from Theorem 2.4 (c) yields a pair

$$w := w_{\mathcal{A}} := \big(D\mathcal{A}T_{w\le 0}, D\mathcal{A}T_{w\ge 0}\big)$$

of full sub-categories of $D\mathcal{A}T$.

Theorem 2.5. *(a) The pair w is a weight structure on $D\mathcal{A}T$.*
(b) The heart $D\mathcal{A}T_{w=0}$ equals the intersection of $D\mathcal{A}T$ and $CHM(k)_F$. It generates the triangulated category $D\mathcal{A}T$. It is Abelian semi-simple. Its objects are finite direct sums of objects of the form $N(m)[2m]$, for $N \in \mathcal{A}$ and $m \in \mathbb{Z}$.
(c) The functor from the $\mathbb{Z}$-graded category $\operatorname{Gr}_{\mathbb{Z}} \mathcal{A}$ over $\mathcal{A}$ to $D\mathcal{A}T_{w=0}$

$$\operatorname{Gr}_{\mathbb{Z}} \mathcal{A} = \bigoplus_{m\in\mathbb{Z}} \mathcal{A} \longrightarrow D\mathcal{A}T_{w=0}\ ,\ (N_m)_{m\in\mathbb{Z}} \longmapsto \oplus_{m\in\mathbb{Z}} N_m(m)[2m]$$

is an equivalence of categories.

Proof. Define $\mathcal{K}$ as the full additive sub-category of $D\mathcal{A}T$ of objects, which are finite direct sums of objects of the form $N(m)[2m]$, for $N \in \mathcal{A}$ and $m \in \mathbb{Z}$. Note that $\mathcal{K}$ generates the triangulated category $D\mathcal{A}T$. All objects of $\mathcal{K}$ are Chow motives. In particular, by orthogonality 2.1 (3) for the motivic weight structure (see [V1, Cor. 4.2.6]), $\mathcal{K}$ is *negative*, i.e.,

$$\operatorname{Hom}_{D\mathcal{A}T}(M_1, M_2[i]) = \operatorname{Hom}_{DM_{gm}(k)_F}(M_1, M_2[i]) = 0$$

for any two objects M_1, M_2 of $\mathcal{K}$, and any integer $i > 0$. Therefore, [Bo, Thm. 4.3.2 II 1] can be applied to ensure the existence of a weight structure v on $D\mathcal{A}T$, uniquely characterized by the property of containing $\mathcal{K}$ in its heart. Furthermore [Bo, Thm. 4.3.2 II 2], the heart $D\mathcal{A}T_{v=0}$ of v is equal to the category $\mathcal{K}'$ of retracts of $\mathcal{K}$ in $D\mathcal{A}T$. In particular, it is contained in the heart $CHM(k)_F$ of the motivic weight structure. The existence of weight filtrations 2.1 (4) for the weight structure v then formally implies that

$$D\mathcal{A}T_{v\le 0} \subset DM_{gm}(k)_{F,w\le 0} ,$$

and that

$$D\mathcal{A}T_{v\ge 0} \subset DM_{gm}(k)_{F,w\ge 0} .$$

Now let $M_1 \in D\mathcal{A}T_{w\le 0} = D\mathcal{A}T \cap DM_{gm}(k)_{F,w\le 0}$. Then for any $M_2 \in D\mathcal{A}T_{v\ge 1}$, we have

$$\mathrm{Hom}_{D\mathcal{A}T}(M_1, M_2) = 0 ,$$

thanks to orthogonality 2.1 (3) for the motivic weight structure, and to the fact that $D\mathcal{A}T_{v\ge 1}$ is contained in $DM_{gm}(k)_{F,w\ge 1}$. Axioms 2.1 (1) and (4) easily imply (see also [Bo, Prop. 1.3.3 2]) that $M_1 \in D\mathcal{A}T_{v\le 0}$. Therefore,

$$D\mathcal{A}T_{w\le 0} = D\mathcal{A}T_{v\le 0} .$$

In the same way, one proves that

$$D\mathcal{A}T_{w\ge 0} = D\mathcal{A}T_{v\ge 0} .$$

Altogether, the weight structure v coincides with the data $w = w_{\mathcal{A}}$. This proves part (a) of our claim. We also see that part (b) is formally implied by the following claim. (b') The category $\mathcal{K}$ is Abelian semi-simple. (Since then $\mathcal{K}$ will necessarily be pseudo-Abelian, hence $D\mathcal{A}T_{w=0} = \mathcal{K}'$ coincides with $\mathcal{K}$.)

Now consider two objects N_1, N_2 of $\mathcal{A}$, two integers m_1, m_2, and the group of morphisms

$$\mathrm{Hom}\big(N_1(m_1)[2m_1], N_2(m_2)[2m_2]\big) = \mathrm{Hom}\big(N_1, N_2(m_2-m_1)[2(m_2-m_1)]\big)$$

in $D\mathcal{A}T$. Two essentially different cases occur: if $m_1 \ne m_2$, then the group of morphisms is zero. Indeed, using descent for finite extensions of k as in the proof of Lemma 1.9, we reduce ourselves to the case $N_1 = N_2 = \mathbb{Z}$, where the desired vanishing follows from [V1, Prop. 4.2.9].

If $m_1 = m_2$, then

$$\mathrm{Hom}(N_1(m_1)[2m_1], N_2(m_2)[2m_2]) = \mathrm{Hom}(N_1, N_2)$$

can be calculated in the Abelian category $\mathcal{A}$.

Thus in any of the two cases, the group $\mathrm{Hom}(N_1(m_1)[2m_1], N_2(m_2)[2m_2])$ coincides with

$$\mathrm{Hom}_{\mathrm{Gr}_{\mathbb{Z}}\mathcal{A}}\big((N_1)_{m=m_1}, (N_2)_{m=m_2}\big) .$$

Therefore, the functor defined in part (c) of the claim is fully faithful. Furthermore, it induces an equivalence of categories between $\mathrm{Gr}_{\mathbb{Z}} \mathcal{A}$ and $\mathcal{K}$. The latter is therefore Abelian semi-simple. This shows (b'), hence part (b) of our claim. It also shows part (c). **q.e.d.**

Corollary 2.6. *The category $D\mathcal{A}T$ is pseudo-Abelian.*

Proof. By Theorem 2.5 (b), the heart $D\mathcal{A}T_{w=0}$ is pseudo-Abelian and generates the triangulated category $D\mathcal{A}T$. Our claim thus follows from [Bo, Lemma 5.2.1]. **q.e.d.**

3 The case of an algebraic base field: the t-structure

In this section, we assume k to be algebraic over the field $\mathbb{Q}$ of rational numbers. We first show that the data from Definition 1.11 define a t-structure on the triangulated category $D\mathcal{A}T$ (Theorem 3.1), and more generally, on $D\mathcal{A}T_{[a,b]}$ (Variant 3.2). This provides a generalization of the main result from [L1] (which concerns the case of Tate motives). Our strategy of proof is identical to the one from [loc. cit.]. We then proceed (Theorem 3.8) to give a characterization of the weight structure from Theorem 2.5 in terms of this t-structure. Specializing further to the case of number fields, Theorem 3.8 implies a criterion allowing to identify the weight structure via the Hodge theoretic or ℓ-adic realization (Corollary 3.10).

Theorem 3.1. *The pair $(D\mathcal{A}T^{\le 0}, D\mathcal{A}T^{\ge 0})$ (Definition 1.11) is a t-structure on $D\mathcal{A}T$. It has the following properties.*

(a) The t-structure is non-degenerate.

(b) Its heart $\mathcal{A}T$ is generated (as a full Abelian sub-category of $D\mathcal{A}T$ stable under extensions) by the objects $N(m)$, for $N \in \mathcal{A}$ and $m \in \mathbb{Z}$.

(c) Each object M of $\mathcal{A}T$ has a canonical weight filtration *by sub-objects*

$$0 \subset \ldots \subset W_{n-1}M \subset W_nM \subset \ldots \subset M \; .$$

This filtration is functorial and exact in M. It is uniquely characterized by the properties of being finite (i.e., $W_nM = 0$ for n very small and $W_nM = M$ for n very large), and of admitting sub-quotients

$$\mathrm{gr}_n M := W_nM/W_{n-1}M \; , \; n \in \mathbb{Z}$$

of the form $N_n(-n/2)$, for some $N_n \in \mathcal{A}$.

(d) *The functor*

$$\bigoplus_{m\in\mathbb{Z}} \mathrm{gr}_{2m}(m) : \mathcal{A}T \longrightarrow \mathrm{Gr}_{\mathbb{Z}}\,\mathcal{A}\ ,\ M \longmapsto \big((\mathrm{gr}_{2m}M)(m)\big)_m$$

is a faithful exact tensor functor to the $\mathbb{Z}$*-graded category over* $\mathcal{A}$*. It thus identifies* $\mathcal{A}T$ *with a tensor sub-category of* $\mathrm{Gr}_{\mathbb{Z}}\,\mathcal{A}$*.*

(e) *The natural maps*

$$\mathrm{Ext}^p_{\mathcal{A}T}\big(M_1, M_2\big) \longrightarrow \mathrm{Hom}_{D\mathcal{A}T}\big(M_1, M_2[p]\big)$$

(Ext^p = *Yoneda Ext-group of* p*-extensions) are isomorphisms, for all* p*, and all* $M_1, M_2 \in \mathcal{A}T$*. Both sides are zero for* $p \geq 2$*. In particular, the Abelian category* $\mathcal{A}T$ *is of cohomological dimension one.*

We thus get in particular the existence of two generating Abelian sub-categories, namely $\mathcal{A}T$ and $D\mathcal{A}T_{w=0}$, of the same triangulated category $D\mathcal{A}T$. The first of these is of cohomological dimension one, and the second is semi-simple. In addition (Theorems 3.1 (d) and 2.5 (c)), the first is abstractly tensor equivalent to a tensor sub-category of the second.

Theorem 3.1 is the special case $(a,b) = (-\infty, \infty)$ of the following.

Variant 3.2 (cmp. [L1, Thm. 1.4, Cor. 4.3])**.** *Fix* $a \leq b$*. Then the pair* $(D\mathcal{A}T^{\leq 0}_{[a,b]}, D\mathcal{A}T^{\geq 0}_{[a,b]})$ *is a t-structure on* $D\mathcal{A}T_{[a,b]}$*. It has the following properties.*

(a) *The t-structure is non-degenerate.*

(b) *Its heart* $\mathcal{A}T_{[a,b]}$ *is generated (as a full Abelian sub-category of* $D\mathcal{A}T_{[a,b]}$ *(or of* $DM_{gm}(k)_F$*) stable under extensions) by the objects* $N(-n/2)$*, for* $N \in \mathcal{A}$ *and* $a \leq n \leq b$*.*

(c) *Each object* M *of* $\mathcal{A}T_{[a,b]}$ *has a canonical weight filtration by sub-objects*

$$0 = W_{a-1}M \subset W_aM \subset \ldots \subset W_{b-1}M \subset W_bM = M\ .$$

This filtration is functorial and exact in M*. It is uniquely characterized by the property of admitting sub-quotients*

$$W_nM/W_{n-1}M\ ,\ n \in \mathbb{Z}$$

of the form $N_n(-n/2)$*, for some* $N_n \in \mathcal{A}$*. For all* $n \in \mathbb{Z}$*, we have*

$$W_nM/W_{n-1}M = \mathrm{gr}_nM$$

as objects of the heart $\mathcal{A}T_n$ *of* $D\mathcal{A}T_n$*.*

(d) *The functor*

$$\bigoplus_{m\in\mathbb{Z}, a\leq 2m\leq b} \mathrm{gr}_{2m}(m) : \mathcal{A}T_{[a,b]} \longrightarrow \bigoplus_{m\in\mathbb{Z}, a\leq 2m\leq b} \mathcal{A}$$

is a faithful exact tensor functor.

(e) *The natural maps*

$$\mathrm{Ext}^p_{\mathcal{A}T_{[a,b]}}\big(M_1, M_2\big) \longrightarrow \mathrm{Hom}\big(M_1, M_2[p]\big)$$

(Hom *= morphisms in* $D\mathcal{A}T_{[a,b]}$ *(or in* $DM_{gm}(k)_F$*)) are isomorphisms, for all* p*, and all* $M_1, M_2 \in \mathcal{A}T_{[a,b]}$*. Both sides are zero for* $p \geq 2$*. In particular, the Abelian category* $\mathcal{A}T_{[a,b]}$ *is of cohomological dimension one.*

(f) *For* $a' \leq a$ *and* $b \leq b'$*, the inclusion of* $D\mathcal{A}T_{[a,b]}$ *into* $D\mathcal{A}T_{[a',b']}$ *as a full triangulated sub-category is compatible with the t-structures. That is, the* t*-structure on* $D\mathcal{A}T_{[a,b]}$ *is induced by the* t*-structure on* $D\mathcal{A}T_{[a',b']}$*.*

Proof. The decisive ingredient is the following generalization of the vanishing from [L1, Thm. 1.4]:

$$\mathrm{Hom}_{D\mathcal{A}T_{[a,b]}}\big(N_1(m_1), N_2(m_2)[s]\big) = 0 \ , \ \forall\, m_1 < m_2 \ , \ N_1, N_2 \in \mathcal{A} \ , \ s \leq 0 \ .$$

It holds because $\mathrm{Hom}_{D\mathcal{A}T_{[a,b]}} = \mathrm{Hom}_{DM_{gm}(k)_F}$, and $\mathrm{Hom}_{DM_{gm}(\bullet)_F}$ satisfies descent for finite extensions L/k of the base field. Choosing an extension L splitting both N_1 and N_2 therefore allows to deduce the desired vanishing from the Beilinson–Soulé vanishing conjecture

$$\mathrm{Hom}_{DM_{gm}(L)_{\mathbb{Q}}}\big(\mathbb{Z}(m_1), \mathbb{Z}(m_2)[s]\big) \ ,$$

which by the work of Borel is known for all number fields, hence also for direct limits L of such.

We now faithfully imitate the proof of [L1, Thm. 1.4], to get assertions (a), (b), and (d). We also get the following: the filtration $W_\bullet M$ induced by the grading $\mathrm{gr}_\bullet M$ is functorial. By construction, the sub-quotient $\mathrm{gr}_n M$ lies in $\mathcal{A}T_n$. Its unicity follows from the fact that there are no non-zero morphisms from objects of $\mathcal{A}T$ of weights at most r to objects of weights at least $r+1$. To prove this, use induction on the lenght of weight filtrations, and the vanishing

$$\mathrm{Hom}_{D\mathcal{A}T}\big(N_1(m_1)[r], N_2(m_2)[s]\big) = 0 \ , \ \forall\, m_1 > m_2 \ , \ N_1, N_2 \in \mathcal{A} \ , \ r, s \in \mathbb{Z}$$

(see the proof of Lemma 1.9). We thus get part (c) of our claim.

Part (f) follows from the definition of our t-structure, and from the compatibility of the functors gr_n under the inclusion of $D\mathcal{A}T_{[a,b]}$ into $D\mathcal{A}T_{[a',b']}$.

As for claim (e), we faithfully imitate the proof of [L1, Cor. 4.3]. **q.e.d.**

Corollary 3.3. *The identity on* $\mathcal{A}T$ *extends canonically to an equivalence of triangulated categories*

$$D^b\big(\mathcal{A}T\big) \longrightarrow D\mathcal{A}T$$

between the bounded derived category of $\mathcal{A}T$ *and* $D\mathcal{A}T$*. Its composition with the cohomology functor* $D\mathcal{A}T \to \mathcal{A}T$ *associated to the* t*-structure of Theorem 3.1 equals the canonical cohomology functor on* $D^b(\mathcal{A}T)$*.*

Proof. Recall the definition of the category $Shv_{Nis}(SmCor(k))$ of *Nisnevich sheaves with transfers* [V1, Def. 3.1.1]. It is Abelian [V1, Thm. 3.1.4], and there is a canonical full triangulated embedding

$$DM_{gm}^{eff}(k) \hookrightarrow D^-\big(Shv_{Nis}(SmCor(k))\big)$$

into the derived category of complexes of Nisnevich sheaves bounded from above [V1, Thm. 3.2.6, p. 205]. Imitating the construction from [loc. cit.] using F instead of $\mathbb{Z}$ as ring of coefficients, one shows that there is a canonical full triangulated embedding

$$DM_{gm}^{eff}(k)_F \hookrightarrow D^-\big(Shv_{Nis}(SmCor(k))_F\big) \ ,$$

where $Shv_{Nis}(SmCor(k))_F$ denotes the Abelian category of Nisnevich sheaves with transfers taking values in F-modules. We thus get a canonical embedding into $D\big(Shv_{Nis}(SmCor(k))_F\big)$ of any full triangulated category $\mathcal{C}$ of $DM_{gm}^{eff}(k)_F$, and hence in particular for $\mathcal{C} = D\mathcal{A}T$. Our claim thus follows from [W2, Thm. 1.1 (a), (d)]: indeed, $\mathrm{Hom}_{D\mathcal{A}T}(M_1, M_2[2]) = 0$ for any two objects M_1, M_2 in $\mathcal{A}T$ (Theorem 3.1 (e)), and $\mathcal{A}T$ generates $D\mathcal{A}T$ (Theorem 3.1 (b)). **q.e.d.**

We already mentioned the special cases $\mathcal{A} = triv$ and $\mathcal{A} = MA(k)_F$. A third case appears worthwhile mentioning.

Definition 3.4. (a) Define the category $MD(k)_F$ as the full Abelian F-linear sub-category of $MA(k)_F$ of objects on which the Galois group acts via a commutative (finite) quotient.
(b) Define the *triangulated category of Dirichlet–Tate motives over* k as the strict full tensor triangulated sub-category $DMDT(k)_F$ of $DM_{gm}(k)_F$ generated by $MD(k)_F$ and $DMT(k)_F$.

Similarly, for any algebraic extension K of k, we could define the *triangulated category of Artin-Tate* (resp. *Dirichlet–Tate*, resp...) *motives over* k *trivializable over* K by letting $\mathcal{A}$ equal the full Abelian F-linear sub-category $MA(K/k)_F$ (resp. $MD(K/k)_F$, resp...) of $MA(k)_F$ (resp. $MD(k)_F$, resp...) of objects on which the absolute Galois group of K, when identified with a subgroup of the Galois group of k, acts trivially.

Corollary 3.5. *The conclusions of Theorem 3.1, Variant 3.2 and Corollary 3.3 hold in particular in any of the following three cases.*

(1) $\mathcal{A} = triv$. *In particular, this gives back the main result of [L1]. The heart* $\mathcal{A}T$ *equals the Abelian category* $MT(k)_F$ *of* mixed Tate motives.

(2) $\mathcal{A} = MD(k)_F$. *In this case, the category* $D\mathcal{A}T$ *equals the triangulated category* $DMDT(k)_F$ *of Dirichlet–Tate motives. Its heart* $\mathcal{A}T$ *equals the Abelian category* $MDT(k)_F$ *of* mixed Dirichlet–Tate motives.

(3) $\mathcal{A} = MA(k)_F$. *In this case, the category* $D\mathcal{A}T$ *equals the triangulated category* $DMAT(k)_F$ *of Artin–Tate motives. Its heart* $\mathcal{A}T$ *equals the Abelian category* $MAT(k)_F$ *of* mixed Artin–Tate motives*.*

Remark 3.6. (a) An equivalent construction of the category $MAT(k)_F$, for $F = \mathbb{Q}$, is given in [DG, Sect. 2.17].
(b) Note that by construction, an inclusion $\mathcal{A} \subset \mathcal{B}$ of strict full Abelian semi-simple F-linear tensor sub-categories of $MA(k)_F$ containing $triv$ induces first a strict full tensor triangulated embedding $D\mathcal{A}T \subset D\mathcal{B}T$, and then a strict full exact tensor embedding $\mathcal{A}T \subset \mathcal{B}T$. An object of $D\mathcal{B}T$ belongs to $D\mathcal{A}T$ if and only if its cohomology objects (with respect to the t-structure from Theorem 3.1) lie in $\mathcal{A}T$. The equivalences of Corollary 3.3 for $\mathcal{A}$ and $\mathcal{B}$ fit into a commutative diagram

$$\begin{array}{ccc} D^b(\mathcal{A}T) & \xrightarrow{\cong} & D\mathcal{A}T \\ \downarrow & & \downarrow \\ D^b(\mathcal{B}T) & \xrightarrow{\cong} & D\mathcal{B}T \end{array}$$

In particular, the bounded derived category $D^b(\mathcal{A}T)$ is canonically identified with a full sub-category of the bounded derived category $D^b(\mathcal{B}T)$.

Definition 3.7. Let M be a mixed Artin–Tate motive, with weight filtration

$$0 \subset \ldots \subset W_{r-1}M \subset W_rM \subset \ldots \subset M \, .$$

Let n be an integer.
(a) We say that M is *of weights* $\leq n$ if $W_nM = M$.
(b) We say that M is *of weights* $\geq n$ if $W_{n-1}M = 0$.
(c) We say that M is *pure of weight* n if it is both of weights $\leq n$ and of weights $\geq n$, i.e., if $W_{n-1}M = 0$ and $W_n = M$.

Denote by

$$\tau^{\leq n} \, , \; \tau^{\geq n} : D\mathcal{A}T \longrightarrow D\mathcal{A}T$$

the truncation functors, and by

$$\mathcal{H}^n : D\mathcal{A}T \longrightarrow \mathcal{A}T$$

the cohomology functors associated to the t-structure from Theorem 3.1. Here is the main result of this section.

Theorem 3.8. *Let* $K \in D\mathcal{A}T$*, and* $r \leq s$.
(a) K *lies in the heart* $D\mathcal{A}T_{w=0}$ *of* w *if and only if the object* $\mathcal{H}^nK$ *of* $\mathcal{A}T$ *is pure of weight* n*, for all* $n \in \mathbb{Z}$.
(b) K *lies in* $D\mathcal{A}T_{w \leq r}$ *if and only if* $\mathcal{H}^nK$ *is of weights* $\leq n+r$*, for all* $n \in \mathbb{Z}$.
(c) K *lies in* $D\mathcal{A}T_{w \geq s}$ *if and only if* $\mathcal{H}^nK$ *is of weights* $\geq n+s$*, for all* $n \in \mathbb{Z}$.

Proof. Observe that the triangulated category $D\mathcal{AT}$ is generated by the heart $D\mathcal{AT}_{w=0}$ of w (Theorem 2.5 (b)) as well as by the heart $\mathcal{AT}$ of t (Theorem 3.1 (a)). This will allow to simplify the proof.

The explicit description of objects K of $D\mathcal{AT}_{w=0}$ from Theorem 2.5 (b) shows that the $\mathcal{H}^n K$ are indeed pure of weight n, for all n (see Theorem 3.1 (c)). To show that any K whose cohomology objects $\mathcal{H}^n K$ are pure of weight n, does belong to $D\mathcal{AT}_{w=0}$, we may assume by the above that K is concentrated in one degree (with respect to the t-structure), say $K = M[d]$ for some $M \in \mathcal{AT}$ and $d \in \mathbb{Z}$. By assumption, the mixed Artin-Tate motive M is pure of weight $-d$, and hence (Theorem 3.1 (c)) of the form $N(d/2)$, for some Artin motive N belonging to $\mathcal{A}$. The latter is clearly a Chow motive, and hence so is its tensor product with the Chow motive $\mathbb{Z}(d/2)[d]$. Therefore, K is a Chow motive belonging to $D\mathcal{AT}$. By Theorem 2.5, it is in the heart $D\mathcal{AT}_{w=0}$. This shows part (a).

We leave it to the reader to deduce (b) and (c) from (a). **q.e.d.**

To conclude, let us now consider realizations ([H, Sect. 2.3 and Corrigendum]; see [DG, Sect. 1.5] for a simplification of this approach). We assume from now on that k is a number field, and concentrate on two realizations (the statement from Corollary 3.10 below then formally generalizes to any of the other realizations "with weights" considered in [H]):

(i) the Hodge theoretic realization

$$R_\sigma : DM_{gm}(k)_F \longrightarrow D$$

associated to a fixed embedding σ of the number field k into the field $\mathbb{C}$ of complex numbers. Here, D is the bounded derived category of mixed graded-polarizable $\mathbb{Q}$-Hodge structures [Be, Def. 3.9, Lemma 3.11], tensored with F,

(ii) the ℓ-adic realization

$$R_\ell : DM_{gm}(k)_F \longrightarrow D$$

for a prime ℓ. Here, D is the bounded "derived category" of constructible $\mathbb{Q}_\ell$-sheaves on $\mathbf{Spec}(k)$ [E, Sect. 6], tensored with F.

Choose and fix one of these two, denote it by R, recall that it is a contravariant tensor functor, and use the same letter for its restriction to the sub-category $D\mathcal{AT}$ of $DM_{gm}(k)_F$. The category $D\mathcal{AT}$ is equipped with a t-structure. The same is true for D; write H^n for the cohomology functors. It is easy to see that R is t-exact (since it maps $\mathcal{AT}$ to the heart of D). In particular, it induces an exact contravariant functor R_0 from the heart $\mathcal{AT}$ of $D\mathcal{AT}$ to the heart of D, which we shall denote by $\mathcal{B}$. As for the weight structure on $D\mathcal{AT}$, note that R_0 maps the pure Tate motive $\mathbb{Z}(m)$ to the pure Hodge structure $\mathbb{Q}(-m)$ (when $R = R_\sigma$) and to the pure $\mathbb{Q}_\ell$-sheaf $\mathbb{Q}_\ell(-m)$ (when $R = R_\ell$), respectively [H, Thm. 2.3.3].

Proposition 3.9. *Assume k to be a number field.*
(a) The realization

$$R : D\mathcal{A}T \longrightarrow D$$

is conservative. In other words, an object K of $D\mathcal{A}T$ is zero if and only if its image $R(K)$ under R is.
(b) The induced functor

$$R_0 : \mathcal{A}T \longrightarrow \mathcal{B}$$

is conservative.
(c) The functor R_0 respects and detects weights up to inversion of the sign. More precisely, an object M of $\mathcal{A}T$ is pure of weight n if and only if $R_0(M)$ is pure of weight $-n$.

Note that there is a notion of purity and mixedness for objects of $\mathcal{B}$.

Proof of Proposition 3.9. Let $K \in D\mathcal{A}T$. Given the t-exactness and contravariance of R, we have the formula

$$H^n R(K) = R_0\big(\mathcal{H}^{-n} K\big)$$

for all n. By Theorem 3.1 (a), the t-structure on $D\mathcal{A}T$ is non-degenerate. Hence (b) implies (a).

Recall that by Theorem 3.1 (c), there is a unique finite weight filtration on any object of $\mathcal{A}T$. Also, the functor R_0 is exact. Hence (b) is implied by conservativity of the restriction of R_0 to the sub-category of objects of $\mathcal{A}T$, which are pure of some weight. But this property is clearly implied by (c) (since the zero object of $\mathcal{B}$ is pure of any weight).

Let $M \in \mathcal{A}T$. As before, we may assume that M is pure of some weight, say n. Again by Theorem 3.1 (c), M is of the form $N(-n/2)$, for some Artin-Tate motive N. Thus, $R_0(M) \cong R_0(N)(n/2)$ is pure of weight $-n$. It is zero if and only if $R_0(N)$ is, which is the case if and only if N is. **q.e.d.**

Corollary 3.10. *Assume k to be a number field. Then the realization R respects and detects the weight structure. More precisely, let $K \in D\mathcal{A}T$, and $r \le s$.*
(a) K lies in the heart $D\mathcal{A}T_{w=0}$ of w if and only if the n-th cohomology object $H^n R(K) \in \mathcal{B}$ of $R(K)$ is pure of weight n, for all $n \in \mathbb{Z}$.
(b) K lies in $D\mathcal{A}T_{w\le r}$ if and only if $H^n R(K)$ is of weights $\ge n - r$, for all $n \in \mathbb{Z}$.
(c) K lies in $D\mathcal{A}T_{w\ge s}$ if and only if $H^n R(K)$ is of weights $\le n - s$, for all $n \in \mathbb{Z}$.

Proof. Recall that $H^n R(K) = R_0\big(\mathcal{H}^{-n} K\big)$. The claim thus follows from Theorem 3.8 and Proposition 3.9. **q.e.d.**

Remark 3.11. As the proof shows, the analogues of parts (a) and (b) of Proposition 3.9 continue to hold for any of the realizations (including those "without weights") considered in [H]. This is true in particular for

(iii) the Betti realization, i.e., the composition of the Hodge theoretic realization R_σ with the forgetful functor to the bounded derived category of F-modules of finite type,

(iv) the topological ℓ-adic realization, i.e., the composition of the ℓ-adic realization R_ℓ with the forgetful functor to the bounded derived category of $F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-modules of finite type [E, Thm. 7.2 (i)].

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