BIRATIONAL MOTIVES, I: PURE BIRATIONAL MOTIVES

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Contents

Introduction

In the preprint [19], we toyed with birational ideas in three areas of algebraic geometry: plain varieties, pure motives in the sense of Grothendieck, and triangulated motives in the sense of Voevodsky. These three themes are finally treated separately in revised versions. The first one was the object of [21]; the second one is the object of the present paper; we hope to complete the third one soon.

We work over a field F. Recall that we introduced in [21] two "birational" categories. The first, $\mathbf{place}(F)$, has for objects the function fields over F and for morphisms the F-places. The second one is the Gabriel-Zisman localisation of the category $\mathbf{Sm}(F)$ of smooth F-varieties obtained by inverting birational morphisms: we denoted this category by $S_b^{-1}\mathbf{Sm}(F)$.

We may also invert stable birational morphisms: those which are dominant and induce a purely transcendental extension of function

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fields, and invert the corresponding morphisms in $\mathbf{place}(F)$. We denote the sets of such morphisms by S_r.

In order to simplify the exposition, let us assume that F is of characteristic 0. Then the main results of [21] and its predecessor [20] can be summarised in a diagram

$$\begin{array}{ccccc} \mathbf{place}(F)^{\mathrm{op}} & \longrightarrow & S_b^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F) & \xrightarrow{\sim} & S_b^{-1}\,\mathbf{Sm}(F) \\ \downarrow & & \wr\downarrow & & \wr\downarrow \\ S_r^{-1}\,\mathbf{place}(F)^{\mathrm{op}} & \longrightarrow & S_r^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F) & \xrightarrow{\sim} & S_r^{-1}\,\mathbf{Sm}(F) \end{array}$$

where $\mathbf{Sm}^{\mathrm{proj}}(F)$ is the full subcategory of smooth projective varieties and the symbols $\sim$ denote equivalences of categories: see [20, Prop. 8.5] and [21, Th. 1.7.7 and Cor. 4.4.3].

Moreover, if X is smooth and Y is smooth proper, then $\mathrm{Hom}(X, Y) = Y(F(X))/R$ in $S_b^{-1}\,\mathbf{Sm}(F)$, where R is R-equivalence [21, Th. 5.4.14].

In this paper, we consider the effect of inverting birational morphisms in categories of *effective pure motives*. For simplicity, let us still assume $\operatorname{char} F = 0$, and consider only the category of effective Chow motives $\mathbf{Chow}^{\mathrm{eff}}(F)$, defined by using algebraic cycles modulo rational equivalence. The graph functor then induces a commutative square

$$\begin{array}{ccc} S_b^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F) & \longrightarrow & S_b^{-1}\,\mathbf{Chow}^{\mathrm{eff}}(F) \\ \wr\downarrow & & \downarrow \\ S_r^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F) & \longrightarrow & S_r^{-1}\,\mathbf{Chow}^{\mathrm{eff}}(F). \end{array}$$

One can expect that the right vertical functor is an equivalence of categories, and indeed this is not difficult to prove (Corollary 2.2.4 b)). But we have two other descriptions of this category of "birational motives":

- The functor $\mathbf{Chow}^{\mathrm{eff}}(F) \to S_b^{-1}\,\mathbf{Chow}^{\mathrm{eff}}(F)$ is full, and its kernel is the ideal $\mathcal{L}_{\mathrm{rat}}$ of morphisms which factor through some object of the form $M \otimes \mathbb{L}$, where $\mathbb{L}$ is the *Lefschetz motive* (ibid.).
- If X, Y are smooth projective varieties, then $\mathcal{L}_{\mathrm{rat}}(h(X), h(Y))$ coincides with the group of Chow correspondences represented by algebraic cycles on $X \times Y$ whose irreducible components are not dominant over X (Theorem 2.4.1).

As a consequence, the group of morphisms from $h(X)$ to $h(Y)$ in $S_b^{-1}\,\mathbf{Chow}^{\mathrm{eff}}(F)$ is isomorphic to $CH_0(Y_{F(X)})$. Given the similar description of Hom sets in $S_b^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F)$ recalled above, this places the classical map

$$Y(F(X))/R \to CH_0(Y_{F(X)})$$

in a categorical context.

This paper is organised as follows. In Section 1 we review pure motives. In Section 2 we study pure birational motives, in greater generality than outlined in this introduction. In particular, many results are valid for other adequate equivalence relations than rational equivalence, see §2.3; moreover, some results extend to characteristic p if the coefficients contain $\mathbf{Q}$, by using de Jong's alteration theorem [8], see Theorem 2.4.1.

Section 3 consists of examples. Rationally connected varieties are shown to have trivial birational motives. We also study the Chow-Künneth decomposition in the category of birational motives, special attention being devoted to the case of complete intersections.

Let $\mathbf{Chow}^{\mathrm{o}}(F)$ denote the pseudo-abelian envelope of $S_b^{-1}\,\mathbf{Chow}^{\mathrm{eff}}(F)$. (The superscript o stands for "open".) In Section 4, we examine two questions: the existence of a right adjoint to the projection functor $\mathbf{Chow}^{\mathrm{eff}}(F) \to \mathbf{Chow}^{\mathrm{o}}(F)$ (and similarly for more general adequate equivalences), and whether pseudo-abelian completion is really necessary. It turns out that the answer to the first question is negative (Theorems 4.3.2 and 4.3.3) and the answer to the second question is positive with rational coefficients under a nilpotence conjecture (Conjecture 3.3.1). We can get an unconditional positive answer to the second question if we restrict to a suitable type of motives (Proposition 4.4.1 and Example 4.4.2).

In Section 5, we define a functor $S_r^{-1}\,\mathbf{field}(F)^{\mathrm{op}} \to S_r^{-1}\,\mathbf{Chow}^{\mathrm{eff}}(F,\mathbf{Q})$ in characteristic p, using de Jong's theorem again. Here $\mathbf{field}(F)$ denotes the subcategory of $\mathbf{place}(F)$ with the same objects but morphisms restricted to field extensions (Proposition 5.1.1).

We end this paper by relating the previous constructions to more classical objects. In Section 6, we define a tensor additive category $\mathbf{AbS}(F)$ of *locally abelian schemes*, whose objects are those F-group schemes that are extensions of a lattice (*i.e.* locally isomorphic for the étale topology to a free finitely generated abelian group) by an abelian variety. We then show in Section 7 that the classical construction of the Albanese variety of a smooth projective variety extends to a tensor functor

$$\mathrm{Alb} : \mathbf{Chow}^{\mathrm{o}}(F) \to \mathbf{AbS}(F)$$

which becomes full and essentially surjective after tensoring morphisms with $\mathbf{Q}$ (Proposition 7.2.1). So, one could say that $\mathbf{AbS}(F)$ is the *representable part* of $\mathbf{Chow}^{\mathrm{o}}(F)$. We also show that, after tensoring with $\mathbf{Q}$, Alb has a right adjoint which identifies $\mathbf{AbS}(F)\otimes\mathbf{Q}$ with the thick subcategory of $\mathbf{Chow}^{o}(F)\otimes\mathbf{Q}$ generated by motives of varieties of dimension ≤ 1.

Some results of the preliminary version [19] of this work were used in other papers, namely [22] and [18], and we occasionally refer to these papers to ease the exposition. In order to give a correspondence guide to the reader and also explain that there are no circular arguments, let us describe precisely what results from [19] are used in these papers, and which results replace them here:

- In [18], Lemma 7.2 uses [19, Lemmas 5.3 and 5.4], which correspond to Proposition 2.3.4 and Theorem 2.4.1 of the present paper. The reader will verify that the proofs of Proposition 2.3.4 and Theorem 2.4.1 are the same as those of [19, Lemmas 5.3 and 5.4], mutatis mutandis, and do not use any result from [18].
- In [22], Lemma 7.5.3 uses the same references: the same comment as above applies. Moreover, [19, 9.5] is used on pp. 174–175 of [22]: this result is now Proposition 7.2.4. Again, its proof is identical to the one in the preliminary version and does not use results from [22].

The idea of considering birational Chow correspondences that yield here a category in which $\operatorname{Hom}([X],[Y]) = CH_0(Y_{F(X)})$ for two smooth projective varieties X, Y goes back to S. Bloch's method of "decomposition of the diagonal" in [4, App. to Lecture 1]. There, he attributes the idea of considering the generic point of a smooth projective variety X as a 0-cycle over its function field to Colliot-Thélène: here, it corresponds to the identity endomorphism of $h^{o}(X)\in\mathbf{Chow}^{o}(F)$. We realised the connection with Bloch's ideas after reading H. Esnault's article [10], and this led to another proof of her theorem by the present birational techniques in [18]. M. Rost has considered this category independently [31]: this was pointed out to us by N. Karpenko.

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1. Review of pure motives

In this section, we recall the definition of categories of pure motives in a way which is suited to our needs. A slight variance to the usual exposition is the notion of *adequate pair* which is a little more precise than the notion of adequate equivalence relation (it explicitly takes the coefficients into account).

We adopt the covariant convention, for future comparison with Voevodsky's triangulated categories of motives: here, the functor which sends a smooth projective variety to its motive is covariant. For a dictionary between the covariant and contravariant conventions, the reader may refer to [22, 7.1.2].

1.1. **Adequate pairs.**

We give ourselves:

- a commutative ring of coefficients A;
- an adequate equivalence relation $\sim$ on algebraic cycles with coefficients in A [42].

We refer to $(A,\sim)$ as an *adequate pair*. Classical examples for $\sim$ are rat (rational equivalence), alg (algebraic equivalence), num (numerical equivalence), $\sim_H$ (homological equivalence relative to a fixed Weil cohomology theory H). A less classical example is Voevodsky's smash-nilpotence tnil [47], see [3, Ex. 7.4.3] (a cycle α is smash-nilpotent if $\alpha^{\otimes n} \sim_{\mathrm{rat}} 0$ for some $n > 0$). We then have a notion of domination $(A,\sim) \geq (A,\sim')$ if $\sim$ is finer than $\sim'$ (*i.e.* the groups of cycles modulo $\sim$ surjects onto the one for $\sim'$). It is well-known that $(A,\mathrm{rat}) \geq (A,\sim)$ for any $\sim$ (*cf.* [12, Ex. 1.7.5]), and that $(A,\sim) \geq (A,\mathrm{num}_A)$ if A is a field.

Since the issue of coefficients is sometimes confusing, the following remarks may be helpful. Given a pair $(A,\sim)$ and a commutative A-algebra B, we get a new pair $B\otimes_A (A,\sim)$ by tensoring algebraic cycles with B: for example, $(A,\sim) = A\otimes_{\mathbf{Z}} (\mathbf{Z},\sim)$ for $\sim = \mathrm{rat}, \mathrm{alg}$ or tnil by definition. On the other hand, given a pair $(B,\sim)$ and a ring homomorphism $A \to B$ we get a "restriction of scalars" pair $(A,\sim_{|A})$ by considering cycles with coefficients in A which become ~ 0 after tensoring with B: for example, if H is a Weil cohomology theory with coefficients in K, this applies to any ring homomorphism $A \to K$. Obviously $B\otimes_A (A,\sim_{|A}) \geq (B,\sim)$, but this need not be an equality in general.

In the case of numerical equivalence (a cycle with coefficients in A is numerically equivalent to 0 if the degree of its intersection with any cycle of complementary dimension in good position is 0), we have $B\otimes_A (A,\mathrm{num}_A) \geq (B,\mathrm{num}_B)$, with equality if B is flat over A.

Given a pair $(A,\sim)$, to any smooth projective F-variety X we may associate for each integer $n \geq 0$ its group of cycles of codimension n with coefficients in A modulo $\sim$, that will be denoted by $\mathcal{Z}^n_\sim(X,A)$. If X has dimension d, we also write this group $\mathcal{Z}^\sim_{d-n}(X,A)$.

1.2. **Smooth projective varieties, connected and nonconnected.** In [21] we were only considering (connected) varieties over F. Classically, pure motives are defined using not necessarily connected smooth projective varieties. One could base the treatment on connected smooth varieties, but this would introduce problems with the tensor product, since a product of connected varieties need not be connected in general (*e.g.* if neither of them is geometrically connected). Thus we prefer to use here:

1.2.1. **Definition.** We write $\mathbf{Sm}_{\coprod}(F)$ for the category of smooth separated schemes of finite type over F. For $\% \in \{\mathrm{prop}, \mathrm{qp}, \mathrm{proj}\}$, we write $\mathbf{Sm}^{\%}_{\coprod}(F)$ for the full subcategory of $\mathbf{Sm}_{\coprod}(F)$ consisting of proper, quasi-projective or projective varieties.

Unlike their counterparts considered in [21], these categories enjoy finite products and coproducts.

The following lemma is clear.

1.2.2. **Lemma.** *The categories considered in Definition 1.2.1 are the "finite coproduct envelopes" of those considered in* [21]*, in the sense of* [20, Prop. 6.1].

1.3. **Review of correspondences.** We associate to two smooth projective varieties X, Y the group $\mathcal{Z}^{\dim Y}_{\sim}(X \times Y, A)$ of correspondences from X to Y relative to $(A, \sim)$. The composition of correspondences is defined as follows[1]: if X, Y, Z are smooth projective and $(\alpha, \beta) \in \mathcal{Z}^{\dim Y}_{\sim}(X \times Y, A) \times \mathcal{Z}^{\dim Z}_{\sim}(Y \times Z, A)$, then

$$\beta \circ \alpha = (p_{XZ})_*(p_{XY}^*\alpha \cdot p_{YZ}^*\beta)$$

where p_{XY}, p_{YZ} and p_{XZ} denote the partial projections from $X \times Y \times Z$ onto two-fold factors.

We then get an A-linear tensor (*i.e.* symmetric monoidal) category $\mathbf{Cor}_{\sim}(F, A)$. The graph map defines a *covariant* functor

$$\begin{aligned} \mathbf{Sm}^{\mathrm{proj}}_{\coprod}(F) &\to \mathbf{Cor}_{\sim}(F, A) \\ X &\mapsto [X] \end{aligned} \tag{1.1}$$

so that $[X \coprod Y] = [X] \oplus [Y]$, and $[X \times Y] = [X] \otimes [Y]$ for the tensor structure.

If $f : X \to Y$ is a morphism, let Γ_f denote its graph and $[\Gamma_f]$ denote the class of Γ_f in $\mathcal{Z}^{\dim Y}_{\sim}(X \times Y)$. We write f_* for the correspondence $[\Gamma_f] : [X] \to [Y]$ (the image of f under the functor (1.1)). Note that

[1]We follow here the convention of Voevodsky in [48]. It is also the one used by Fulton [12, §16]. See [22, 7.1.2].

if $f : X \to Y$ and $g : Y \to Z$ are two morphisms, then the cycles $\Gamma_f \times Z$ and $X \times \Gamma_g$ on $X \times Y \times Z$ intersect properly, so that $g_* \circ f_*$ is well-defined as a cycle and not just as an equivalence class of cycles; the equation $g_* \circ f_* = (g \circ f)_*$ is an equality of cycles.

1.4. **Rational maps.** We first define rational maps between not necessarily connected smooth varieties X, Y in the obvious way: it is a morphism from a suitable *dense* open subset of X to Y. Like morphisms, rational maps split as disjoint unions of "connected" rational maps. A rational map f is *dominant* if all its connected components are dominant and if the image of f meets all connected components of Y.

Let $f : X \dashrightarrow Y$ be a rational map between two smooth projective varieties X, Y. To f we associate the correspondence $f_* : [X] \to [Y]$ in $\mathbf{Cor}_\sim(F, A)$, defined as the closure of the graph of f inside $X \times Y$. We also define f^* as the transpose of f_*, as above for solid morphisms. The formula $g_* \circ f_* = (g \circ f)_*$ need not be valid in general, even if $g \circ f$ is defined (but see Proposition 2.3.6 below). Yet we have:

1.4.1. **Lemma.** *Let $X \overset{g}{\dashrightarrow} Y \xrightarrow{p} Z$ be a diagram of smooth projective varieties, where g is a rational map and p is a morphism. Let $f = p \circ g$. Then, we have an equality of cycles*

$$g_* \circ f_* = (g \circ f)_*$$

in $\mathcal{Z}^{\dim Z}(X \times Z)$.

Sketch. As in [12, proof of Prop. 16.1.1 (c) (iii)], we introduce the rational map

$$(g, f) : X \dashrightarrow Y \times Z.$$

Let U be an open subset of X on which g, hence f, is defined. We work in $U \times Y \times Z$. Clearly, the graph $\Gamma_{g,f}$ in this variety is contained in $(U \times \Gamma_p) \cap (\Gamma_f \times Y)$. As explained in §1.3, this inclusion is an equality of reduced closed subschemes. Therefore, equality persists on taking their closures in $X \times Y \times Z$. □

1.5. **Effective pure motives.** We now define as usual the category of effective pure motives $\mathbf{Mot}_\sim^{\text{eff}}(F, A)$ relative to $(A, \sim)$ as the pseudo-abelian envelope of $\mathbf{Cor}_\sim(F, A)$. We denote the composition of (1.1) with the pseudo-abelianisation functor by $h_\sim^{\text{eff}}$. If $\sim =$ rat, we usually abbreviate $h_\sim^{\text{eff}}$ to h^{eff}.

In $\mathbf{Mot}_\sim^{\text{eff}}(F, A)$ we have

- $h_\sim^{\text{eff}}(\operatorname{Spec} F) = \mathbf{1}$ (the unit object for the tensor structure)
- $h_\sim^{\text{eff}}(\mathbf{P}^1) = \mathbf{1} \oplus \mathbb{L}$ where $\mathbb{L}$ is the *Lefschetz motive.*

If $n \geq 0$, we write $M(n)$ for the motive $M \otimes \mathbb{L}^{\otimes n}$ (beware that the "standard" notation is $M(-n)$!)

We then have the formula, for two smooth projective X, Y and integers $p, q \geq 0$

$$\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F,A)(h^{\mathrm{eff}}_{\sim}(X)(p), h^{\mathrm{eff}}_{\sim}(Y)(q)) = \mathcal{Z}^{\dim Y + q - p}_{\sim}(X \times Y). \tag{1.2}$$

In particular, the endofunctor $-\otimes \mathbb{L}$ of $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F,A)$ is fully faithful. If $f : X \to Y$ is a morphism, then the correspondence $[{}^t\Gamma_f] \in \mathcal{Z}^{\dim Y}(Y \times X)$ obtained by the "switch" defines a morphism $f^* : h^{\mathrm{eff}}_{\sim}(Y)(\dim X) \to h^{\mathrm{eff}}_{\sim}(X)(\dim Y)$, *i.e.* from $h^{\mathrm{eff}}_{\sim}(Y)$ to $h^{\mathrm{eff}}_{\sim}(X)(\dim Y - \dim X)$ or from $h^{\mathrm{eff}}_{\sim}(Y)(\dim X - \dim Y)$ to $h^{\mathrm{eff}}_{\sim}(X)$ according to the sign of $\dim X - \dim Y$.

In particular, if f has relative dimension 0 then f^* maps $h^{\mathrm{eff}}_{\sim}(Y)$ to $h^{\mathrm{eff}}_{\sim}(X)$. We recall the well-known

1.5.1. **Lemma.** *Suppose that f is generically finite of degree d. Then $f_* \circ f^* = d1_Y$.*

Proof. It suffices to prove this for the action on cycles, and then the lemma follows by Manin's identity principle. Let $\alpha \in \mathcal{Z}^*_{\sim}(Y, A)$. By the projection formula,

$$f_* f^*(\alpha) = \alpha \cdot f_*(1).$$

But $f_*(1) \in \mathcal{Z}^0_{\sim}(Y, A)$ may be computed after restriction to any open subset U of X and for U small enough it is clear that $f_*(1) = d$. □

1.6. **Pure motives.** The category $\mathbf{Mot}_{\sim}(F,A)$ is now obtained from $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F,A)$ by inverting the endofunctor $-\otimes \mathbb{L}$, *i.e.* adjoining a $\otimes$-quasi-inverse $\mathbb{T}$ of $\mathbb{L}$ (the Tate motive) to $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F,A)$. The resulting category is rigid and the functor $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F,A) \to \mathbf{Mot}_{\sim}(F,A)$ is fully faithful; we refer to [41] for details. We shall write $h_{\sim}(X)$ for the image of $h^{\mathrm{eff}}_{\sim}(X)$ in $\mathbf{Mot}_{\sim}(F,A)$.

2. Pure birational motives

2.1. **A first approach.** The first idea to define a notion of pure birational motives is to localise $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F,A)$ with respect to [the graphs of] stable birational morphisms as in [21], hence getting a functor

$$S_r^{-1}\,\mathbf{Sm}^{\mathrm{proj}}_{\coprod}(F) \to S_r^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F,A).$$

This idea turns out to be the good one in all important cases, but for this we need some preliminary work first.

We start by reviewing the sets of morphisms used in [21, §1.7]:

- S_b^w: compositions of blow-ups with smooth centres;

- S_h: projections of the form $X \times (\mathbf{P}^1)^n \to X$;
- $S_r^w = S_b^w \cup S_h$;
- S_b: birational morphisms;
- S_r stably birational morphisms: $s \in S_r$ if and only if s is dominant and gives a purely transcendental function field extension.

These morphisms, defined for connected varieties in [21], extend trivially to the categories of Definition 1.2.1 as explained in [20, Cor. 6.3]. More precisely, if S is a set of morphisms of $\mathbf{Sm}(F)$, we define $S^{\coprod} \subset \mathbf{Sm}_{\coprod}(F)$ as the set of those morphisms which are dominant and whose connected components are all in S. For simplicity, we shall write S rather than $S^{\coprod}$ in the sequel.

By Lemma 1.2.2 and [20, Th. 6.4], the localisation results of [20] and [21] extend to the category $\mathbf{Sm}_{\coprod}(F)$ and, moreover, the functors

$$S^{-1}\,\mathbf{Sm}(F) \to S^{-1}\,\mathbf{Sm}_{\coprod}(F)$$

identify the right hand side with the "finite coproduct envelope" of the left hand side. Similarly for their likes with decorations $\mathbf{Sm}^{\%}$.

We shall view the above morphisms as correspondences via the graph functor. We introduce two more sets which are convenient here:

2.1.1. **Definition.** We write $\widetilde{S}_b$ and $\widetilde{S}_r$ for the set of dominant rational maps which induce, respectively, an isomorphism of function fields and a purely transcendental extension. We let these rational maps act on pure motives via their graphs, as in §1.4.

Thus we have a diagram of inclusions of morphisms on $\mathbf{Mot}^{\text{eff}}_{\sim}(F,A)$:

$$\begin{array}{ccccc} S_b^w & \subset & S_b^w \cup S_h & = & S_r^w \\ \cap & & \cap & & \cap \\ S_b & \subset & S_b \cup S_h & \subset & S_r \\ \cap & & \cap & & \cap \\ \widetilde{S}_b & \subset & \widetilde{S}_b \cup S_h & \subset & \widetilde{S}_r \end{array} \tag{2.1}$$

Let us immediately notice:

2.1.2. **Proposition.** *Let S be one of the systems of morphisms in* (2.1). *Then the category $S^{-1}\,\mathbf{Mot}^{\text{eff}}_{\sim}(F,A)$ is an A-linear category provided with a tensor structure, compatible with the corresponding structures of $\mathbf{Mot}^{\text{eff}}_{\sim}(F,A)$ via the localisation functor.*

Proof. This follows from Theorem A.3.3, Proposition A.1.2 and the fact that elements of S are stable under disjoint unions and products. □

2.2. A second approach: the Lefschetz ideal.

2.2.1. **Definition.** We denote by $\mathcal{L}_\sim$ the ideal of $\mathbf{Mot}_\sim^{\text{eff}}(F,A)$ consisting of those morphisms which factor through some object of the form $P(1)$: this is the *Lefschetz ideal*. It is a monoidal ideal (*i.e.* , it is closed with respect to composition and tensor products on the left and on the right).

2.2.2. **Remark.** In any additive category $\mathcal{A}$ there is the notion of product of two ideals $\mathcal{I},\mathcal{J}$:

$$\mathcal{I}\circ\mathcal{J} = \langle f\circ g \mid f\in\mathcal{I}, g\in\mathcal{J}\rangle.$$

If $\mathcal{B}$ is some given additive subcategory of $\mathcal{A}$ and $\mathcal{J} = \{f \mid f \text{ factors through some } A\in\mathcal{B}\}$, then $\mathcal{J}$ is idempotent because it is generated by idempotent morphisms, namely the identity maps of the objects of $\mathcal{B}$. In $\mathcal{A}=\mathbf{Mot}_\sim^{\text{eff}}(F,A)$, this applies to $\mathcal{L}_\sim$.

On the other hand, in a tensor additive category $\mathcal{A}$ there is also the tensor product of two ideals $\mathcal{I},\mathcal{J}$: for $A,B\in\mathcal{A}$

$$(\mathcal{I}\otimes\mathcal{J})(A,B) = \langle \mathcal{A}(E\otimes F,B)\circ(\mathcal{I}(C,E)\otimes\mathcal{J}(D,F))\circ\mathcal{A}(A,C\otimes D)\rangle$$

where C,D,E,F run through all objects of $\mathcal{A}$. Coming back to $\mathcal{A}=\mathbf{Mot}_\sim^{\text{eff}}(F,A)$, we have $\mathcal{L}_\sim\otimes\mathcal{L}_\sim = \mathbf{Mot}_\sim^{\text{eff}}(F,A)(2)\neq\mathcal{L}_\sim\circ\mathcal{L}_\sim=\mathcal{L}_\sim$. This is in sharp contrast with the case where $\mathcal{A}$ is rigid [3, (6.15)].

2.2.3. **Proposition.** *a) The localisation functor*

$$\mathbf{Mot}_\sim^{\text{eff}}(F,A)\to (S_b^w)^{-1}\mathbf{Mot}_\sim^{\text{eff}}(F,A)$$

factors through $\mathbf{Mot}_\sim^{\text{eff}}(F,A)/\mathcal{L}_\sim$.
b) The functors

$$\mathbf{Mot}_\sim^{\text{eff}}(F,A)/\mathcal{L}_\sim\to (S_b^w)^{-1}\mathbf{Mot}_\sim^{\text{eff}}(F,A)\to (S_r^w)^{-1}\mathbf{Mot}_\sim^{\text{eff}}(F,A)$$

are both isomorphisms of categories.
c) The functor

$$\mathbf{Mot}_\sim^{\text{eff}}(F,A)/\mathcal{L}_\sim\to S_b^{-1}\mathbf{Mot}_\sim^{\text{eff}}(F,A)$$

is full.
d) For any $s\in\widetilde{S}_r$, s_* *becomes invertible in* $\widetilde{S}_b^{-1}\mathbf{Mot}_\sim^{\text{eff}}(F,A)$.

Proof. a) By Proposition 2.1.2, it is sufficient to show that $\mathbb{L}\mapsto 0$ in $(S_b^w)^{-1}\mathbf{Mot}_\sim^{\text{eff}}(F,A)$. Here as in the proof of b) we shall use the following formula of Manin [30, §9, Cor. p. 463]: if $p:\widetilde{X}\to X$ is a blow-up with smooth centre $Z\subset X$ of codimension n, then

$$h_\sim^{\text{eff}}(\widetilde{X})\simeq h_\sim^{\text{eff}}(X)\oplus\bigoplus_{i=1}^{n-1} h_\sim^{\text{eff}}(Z)\otimes\mathbb{L}^{\otimes i} \tag{2.2}$$

where projecting the right hand side onto $h^{\mathrm{eff}}_\sim(X)$ we get p_*.

In (2.2), take $X = \mathbf{P}^2$ and for $\widetilde{X}$ the blow-up of X at (say) $Z = \{(0:0)\}$. Since p is invertible in $S_b^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_\sim(F,A)$, we get $\mathbb{L}=0$ in this category as requested.

b) It suffices to show that morphisms of S_r^w become invertible in $\mathbf{Mot}^{\mathrm{eff}}_\sim(F,A)/\mathcal{L}_\sim$, which immediately follows from (2.2) and the easier projective line formula.

c) It suffices to show that members of S_b have right inverses in $\mathbf{Mot}^{\mathrm{eff}}_\sim(F,A)$: this follows from Lemma 1.5.1.

d) Let $g : X \dashrightarrow Y$ be an element of $\widetilde{S}_r$. Then X is birational to $Y\times(\mathbf{P}^1)^n$ for some $n\geq 0$, and if $f : X \dashrightarrow Y\times(\mathbf{P}^1)^n$ is the corresponding birational map, its composition with the first projection π is g. By Lemma 1.4.1, it suffices to show that π_* is invertible in $\widetilde{S}_b^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_\sim(F,A)$, which follows from b). □

2.2.4. **Corollary.** *Let* $M = \mathbf{Mot}^{\mathrm{eff}}_\sim(F,A)$.
a) The diagram (2.1) *induces a commutative diagram of categories and functors*
(2.3)

$$\begin{array}{ccccccc}
M/\mathcal{L}_\sim & \xrightarrow{\sim} & (S_b^w)^{-1}M & \xrightarrow{\sim} & (S_b^w\cup S_h)^{-1}M & \xrightarrow{\sim} & (S_r^w)^{-1}M\\
 & & \downarrow{\scriptstyle \text{full}} & & \downarrow{\scriptstyle \text{full}} & & \downarrow\\
 & & S_b^{-1}M & \xrightarrow{\sim} & (S_b\cup S_h)^{-1}M & \longrightarrow & S_r^{-1}M\\
 & & \downarrow & & \downarrow & & \downarrow\\
 & & \widetilde{S}_b^{-1}M & \xrightarrow{\sim} & (\widetilde{S}_b\cup S_h)^{-1}M & \xrightarrow{\sim} & \widetilde{S}_r^{-1}M
\end{array}$$

where the functors with a sign $\sim$ *are isomorphisms of categories and the indicated functors are full.*
**b) If* char $F=0$*, all functors are isomorphisms of categories.*

Proof. a) follows from Proposition 2.2.3; b) follows from Hironaka's resolution of singularities (*cf.* [21, Lemma 1.7.6 b) and c)]). □

2.2.5. **Remark.** Tracking isomorphisms in Diagram (2.3), one sees that without resolution of singularities we get a priori 4 different categories of "pure birational motives". If $p:\widetilde{X}\to X$ is a birational morphism, then at least $h^{\mathrm{eff}}_\sim(X)$ is a direct summand of $h^{\mathrm{eff}}_\sim(\widetilde{X})$ by Lemma 1.5.1. However it is not clear how to prove that the other summand is divisible by $\mathbb{L}$ without using resolution. We shall get by for some special pairs $(A,\sim)$ in characteristic p below, using de Jong's theorem.

2.2.6. **Definition.** The category of *pure birational motives* is

$$\mathbf{Mot}^{\mathrm{o}}_{\sim}(F,A) = \left(\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F,A)/\mathcal{L}_{\sim}\right)^{\natural}.$$

For a smooth projective variety X, we write $h^{\mathrm{o}}_{\sim}(X)$ for the image of $h^{\mathrm{eff}}_{\sim}(X)$ in $\mathbf{Mot}^{\mathrm{o}}_{\sim}(F,A)$. For $\sim=$ rat, we usually write h^{o} rather than $h^{\mathrm{o}}_{\mathrm{rat}}$.
We also set

$$\mathbf{Chow}^{\mathrm{eff}}(F,A) = \mathbf{Mot}^{\mathrm{eff}}_{\mathrm{rat}}(F,A)$$
$$\mathbf{Chow}^{\mathrm{o}}(F,A) = \mathbf{Mot}^{\mathrm{o}}_{\mathrm{rat}}(F,A).$$

When $A = \mathbf{Z}$, we abbreviate this notation to $\mathbf{Chow}^{\mathrm{eff}}(F)$ and $\mathbf{Chow}^{\mathrm{o}}(F)$.

In Section 4, we shall examine to what extent it is really necessary to adjoin idempotents.

2.3. **A third approach: extendible pairs.** To go further, we need to restrict the adequate equivalence relation we are using:

2.3.1. **Definition.** An adequate pair $(A,\sim)$ is *extendible* if

- $\sim$ is defined on cycles over arbitrary quasiprojective F-varieties;
- it is preserved by inverse image under flat morphisms and direct image under proper morphisms;
- if X is smooth projective, Z is a closed subset of X and $U = X - Z$, then the sequence

$$\mathcal{Z}^{\sim}_n(Z,A) \to \mathcal{Z}^{\sim}_n(X,A) \to \mathcal{Z}^{\sim}_n(U,A) \to 0 \tag{2.4}$$

  is exact.

Note that in the latter sequence, surjectivity always holds because this is already true on the level of cycles. So the issue is exactness at $\mathcal{Z}^{\sim}_n(X,A)$.

2.3.2. **Examples.** a) Rational equivalence (with any coefficients) is extendible.
b) Algebraic equivalence (with any coefficients) is extendible, *cf.* [12, Ex. 10.3.4].
c) The status of homological equivalence is very interesting:

(1) Under resolution of singularities and the standard conjecture that homological and numerical equivalences agree, homological equivalence with respect to a "classical" Weil cohomology theory is extendible if $\operatorname{char} F = 0$ (Corti-Hanamura [7, Prop. 6.7]). The proof involves the weight spectral sequences for Borel-Moore Hodge homology, their degeneration at E_2 and the

semi-simplicity of numerical motives (Jannsen [16]). Presumably the same arguments work in characteristic p by using de Jong's alteration theorem [8] instead of Hironaka's resolution of singularities: we thank Yves André for pointing this out.

(2) It seems that the Corti-Hanamura argument implies unconditionally that André's motivated cycles [1] verify the axioms of an extendible pair.

(3) For Betti cohomology with integral coefficients or l-adic cohomology with $\mathbf{Z}_l$ coefficients, homological equivalence is not extendible. (Counterexample: $n = 1$, Z a surface of degree ≥ 4 in $\mathbf{P}^3$.)

(4) Hodge cycles with coefficients $\mathbf{Q}$ verify the axioms of an extendible pair: the proof involves resolving the singularities of Z in (2.4) and using the semi-simplicity of polarisable pure Hodge structures. See also Jannsen [17]. We are indebted to Claire Voisin for explaining these last two points.

(5) Taking Tate cycles for l-adic cohomology, the same argument works if we assume the semi-simplicity of Galois action on the cohomology of smooth projective varieties.

2.3.3. **Lemma.** *If $(A, \sim)$ verifies the first two conditions of Definition 2.3.1, then $(A, \mathrm{rat}) \geq (A, \sim)$ (also over arbitrary quasiprojective varieties).*

Proof. Again, this follows from [12, Ex. 1.7.5]. □

2.3.4. **Proposition.** *Let $(A, \sim)$ be an extendible pair. For two smooth projective varieties X, Y, let $\mathcal{I}_\sim(X, Y)$ be the subgroup of $\mathcal{Z}^{\dim Y}_\sim(X \times Y, A)$ consisting of those classes vanishing in $\mathcal{Z}^{\dim Y}_\sim(U \times Y, A)$ for some open subset U of X. Then $\mathcal{I}_\sim$ is a monoidal ideal in $\mathbf{Cor}_\sim(F, A)$.*

Proof. Note that by Lemma 2.3.3 and the third condition of Definition 2.3.1, the map $\mathcal{I}_{\mathrm{rat}}(X, Y) \to \mathcal{I}_\sim(X, Y)$ is surjective for any X, Y: this reduces us to the case $\sim = \mathrm{rat}$. We further reduce immediately to $A = \mathbf{Z}$.

Let X, Y, Z be 3 smooth projective varieties. If U is an open subset of X, it is clear that the usual formula defines a composition of correspondences

$$CH^{\dim Y}(U \times Y) \times CH^{\dim Z}(Y \times Z) \to CH^{\dim Z}(U \times Z)$$

and that this composition commutes with restriction to smaller and smaller open subsets. Passing to the limit on U, we get a composition

$$CH^{\dim Y}(Y_{F(X)}) \times CH^{\dim Z}(Y \times Z) \to CH^{\dim Z}(Z_{F(X)})$$

or

$$CH_0(Y_{F(X)}) \times CH^{\dim Z}(Y \times Z) \to CH_0(Z_{F(X)}).$$

Here we used the fact that (codimensional) Chow groups commute with filtering inverse limits of schemes, see [4].

We now need to prove that this pairing factors through $CH_0(Y_{F(X)}) \times CH^{\dim Z}(V \times Z)$ for any open subset V of Y. One checks that it is induced by the standard action of correspondences in $CH^{\dim Z}(Y_{F(X)} \times_{F(X)} Z_{F(X)})$ on groups of 0-cycles. Hence it is sufficient to show that the standard action of correspondences factors as indicated, and up to changing the base field we may replace $F(X)$ by F.

We now show that the pairing

$$CH_0(Y) \times CH^{\dim Z}(Y \times Z) \to CH_0(Z)$$

factors as indicated. The proof is a variant of Fulton's proof of the Colliot-Thélène–Coray theorem that CH_0 is a birational invariant of smooth projective varieties [6], [12, Ex. 16.1.11]. Let M be a proper closed subset of Y, and $i : M \to Y$ be the corresponding closed immersion. We have to prove that for any $\alpha \in CH_0(Y)$ and $\beta \in CH_{\dim Y}(M \times Z)$,

$$(i \times 1_Z)_*(\beta)(\alpha) := (p_2)_*((i \times 1_Z)_*\beta \cdot p_1^*\alpha) = 0$$

where p_1 and p_2 are respectively the first and second projections on $Y \times Z$.

We shall actually prove that $(i \times 1_Z)_*\beta \cdot p_1^*\alpha = 0$. For this, we may assume that α is represented by a point $y \in Y_{(0)}$ and β by some integral variety $W \subseteq M \times Z$. Then $(i \times 1_Z)_*\beta \cdot p_1^*\alpha$ has support in $(i \times 1_Z)(W) \cap (\{y\} \times Z) \subset (M \times Z) \cap (\{y\} \times Z)$. If $y \notin M$, this subset is empty and we are done. Otherwise, up to linear equivalence, hence up to $\sim$, we may replace y by a 0-cycle disjoint from M (*cf.* [39]), and we are back to the previous case.

This shows that $\mathcal{I}_\sim$ is an ideal of $\mathbf{Cor}_\sim(F, A)$. The fact that it is a monoidal ideal is essentially obvious. □

2.3.5. **Definition.** We abbreviate the notation $\mathbf{Cor}_\sim(F, A)/\mathcal{I}_\sim$ into $\mathbf{Cor}^o_\sim(F, A)$.

For future reference, let us record here the value of the Hom groups in the most important case, that of rational equivalence (see also Remark 2.3.8 2) below):

$$\mathbf{Cor}^o_{\rm rat}(F, A)([X], [Y]) = CH_0(Y_{F(X)}) \otimes A. \tag{2.5}$$

2.3.6. **Proposition.** *In* $\mathbf{Cor}^{\mathrm{o}}_{\sim}(F, A)$,

a) $(g \circ f)_* = g_* \circ f_*$ *for any composable rational maps* $X \overset{f}{\dashrightarrow} Y \overset{g}{\dashrightarrow} Z$.
b) [12, Ex. 16.1.11] $f^* f_* = 1_X$ *and* $f_* f^* = 1_Y$ *for any birational map* $f : X \dashrightarrow Y$.
c) Morphisms of $\widetilde{S}_r$ *are invertible.*

Proof. a) Let F be the fundamental set of f, G be the fundamental set of g, $U = X - F$, $V = Y - G$. By assumption, $f(U) \cap V \neq \emptyset$, hence $W = f^{-1}(V)$ is a nonempty open subset of U, on which $g \circ f$ is a morphism.

Let us abuse notation and still write f for the morphism f_U, etc. Then, by definition

$$g_* \circ f_* = (p_{XZ})_*((\bar{\Gamma}_f \times Z) \cap (X \times \bar{\Gamma}_g))$$

(note that the two intersected cycles are in good position). This cycle clearly contains the graph $\Gamma_{g \circ f}$ as an open subset, hence also $(g \circ f)_*$ as a closed subset. One sees immediately that the restriction of $g_* \circ f_*$ and $(g \circ f)_*$ to $W \times Z$ are equal.

b) is proven in the same way.

c) Let $g : X \dashrightarrow Y$ be an element of $\widetilde{S}_r$. Then X is birational to $Y \times (\mathbf{P}^1)^n$ for some $n \geq 0$, and if $f : X \dashrightarrow Y \times (\mathbf{P}^1)^n$ is a birational map, its composition with the first projection π is g. By a) and b), it suffices to show that π_* is invertible in $\mathbf{Cor}_{\sim}(F, A)/\mathcal{I}_{\sim}$. For this we may reduce to $n = 1$ and even to $Y = \operatorname{Spec} F$ since $\mathcal{I}_{\sim}$ is a monoidal ideal. Let $s : \operatorname{Spec} F \to \mathbf{P}^1$ be the ∞ section: it suffices to show that $(s \circ \pi)_* = 1_{\mathbf{P}^1}$. But the cycle $(s \circ \pi)_* - 1_{\mathbf{P}^1}$ on $\mathbf{P}^1 \times \mathbf{P}^1$ is linearly equivalent to $\infty \times \mathbf{P}^1$ (this is the idempotent defining the Lefschetz motive), and the latter cycle vanishes when restricted to $\mathbf{A}^1 \times \mathbf{P}^1$. □

We shall also need the following lemma in the proof of Proposition 5.1.1 c).

2.3.7. **Lemma.** *Let* L/K *be an extension of function fields over* F*, with* $K = F(X)$ *and* $L = F(Y)$ *for* X, Y *two smooth projective* F*-varieties. Let* $\varphi : Y \dashrightarrow X$ *be the rational map corresponding to the inclusion* $K \hookrightarrow L$*. Let* Z *be another smooth projective* F*-variety. Then the map*

$$\mathbf{Chow}^{\mathrm{o}}(F, A)(h^{\mathrm{o}}(X), h^{\mathrm{o}}(Z)) \to \mathbf{Chow}^{\mathrm{o}}(F, A)(h^{\mathrm{o}}(Y), h^{\mathrm{o}}(Z))$$

given by composition with $\varphi_* : h^{\mathrm{o}}(Y) \to h^{\mathrm{o}}(X)$ *(see 1.4) coincides via* (2.5) *with the base-change map* $CH_0(Z_K) \otimes A \to CH_0(Z_L) \otimes A$.

Proof. Let $V \subseteq Y$ and $U \subseteq X$ be open subsets such that f is defined on V and $f(V) \subseteq U$. Up to shrinking U, we may assume that f is flat

[EGA IV, 11.1.1]. As in the proof of Proposition 2.3.4, the composition of correspondences induces a pairing

$$CH^{\dim X}(V \times U) \times CH^{\dim Z}(U \times Z) \to CH^{\dim Z}(V \times Z)$$

and the action of $\varphi_* \in CH^{\dim X}(V \times U)$ on $\alpha \in CH^{\dim Z}(U \times Z)$ is given by the flat pull-back of cycles. Therefore, φ_* induces in the limit the flat pull-back of 0-cycles from $CH_0(Z_K)$ to $CH_0(Z_L)$. □

2.3.8. **Remarks.** 1) Propositions 2.3.4 and 2.3.6 a) were independently observed by Markus Rost in the case $\sim=$ rat [31, Prop. 3.1 and Lemma 3.3]. We are indebted to Karpenko for pointing this out and for referring us to Merkurjev's preprint [31].

2) In $\mathbf{Cor}^{\mathrm{o}}_\sim(F, A)$, morphisms are by definition given by the formula

$$\mathbf{Cor}^{\mathrm{o}}_\sim(F, A)([X], [Y]) = \varinjlim_{U \subseteq X} \mathcal{Z}^{\dim Y}_\sim(U \times Y, A).$$

The latter group maps onto $\mathcal{Z}_0^\sim(Y_{F(X)}, A)$. If $\sim=$ rat, this map is an isomorphism (see (2.5)). For other equivalence relations, this is far from being the case: for example, if $\sim=$ alg, F is algebraically closed, X, Y are two curves and (say) $A = \mathbf{Z}$, then

$$\begin{aligned}\mathcal{Z}^1_{\mathrm{alg}}(X \times Y, \mathbf{Z}) = NS(X \times Y) = NS(X) \oplus NS(Y) \oplus \mathrm{Hom}(J_X, J_Y)\\ = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathrm{Hom}(J_X, J_Y)\end{aligned}$$

where NS is the Néron-Severi group and J_X, J_Y are the Jacobians of X and Y. On the other hand,

$$\mathcal{Z}_0^{\mathrm{alg}}(Y_{F(X)}, \mathbf{Z}) = NS(Y_{F(X)}) = \mathbf{Z}.$$

When we remove a point from X, we kill the factor $NS(X) = \mathbf{Z}$. But any two points of X are algebraically equivalent, so removing further points does not modify the group any further. Hence

$$\varinjlim_{U \subseteq X} \mathcal{Z}^{\dim Y}_{\mathrm{alg}}(U \times Y, \mathbf{Z}) = \mathbf{Z} \oplus \mathrm{Hom}(J_X, J_Y).$$

We thank Colliot-Thélène for helping clarify this matter.

2.4. **The main theorem.** We now extend the ideal $\mathcal{I}_\sim$ from $\mathbf{Cor}_\sim(F, A)$ to $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$ in the usual way (*cf.* [3, Lemme 1.3.10]), without changing notation. By Propositions 2.2.3 a) and 2.3.6, we get a composite functor

$$\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)/\mathcal{L}_\sim \to \widetilde{S}_r^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A) \to \mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)/\mathcal{I}_\sim \tag{2.6}$$

for any extendible pair $(A, \sim)$. Since both categories are full images of $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$, this functor is automatically full. We are going to show that it is an equivalence of categories in some important cases.

2.4.1. **Theorem.** *Let* $(A, \sim)$ *be an extendible pair. Suppose that* $\operatorname{char} F = 0$ *or* F *that is perfect*[2] *and* $A \supseteq \mathbf{Q}$. *Then the functor* (2.6) *is an isomorphism of categories.*

Proof. [3] We have to show that $\mathcal{I}_\sim(M,N) \subseteq \mathcal{L}_\sim(M,N)$ for any $M, N \in \mathbf{Mot}^{\text{eff}}_\sim(F,A)$. Clearly we may assume $M = h^{\text{eff}}_\sim(X)$, $N = h^{\text{eff}}_\sim(Y))$ for two smooth projective varieties X, Y.

Let $f \in \mathcal{I}_\sim(h^{\text{eff}}_\sim(X), h^{\text{eff}}_\sim(Y))$. By the third condition in Definition 2.3.1, the cycle class $f \in \mathcal{Z}^\sim_{\dim X}(X \times Y, A)$ is of the form $(i \times 1_Y)_* g$ for some closed immersion $i : Z \to X$, where $g \in \mathcal{Z}^\sim_{\dim X}(Z \times Y, A)$. Let $\widetilde{g}$ be a cycle representing g. Write $\widetilde{g} = \sum_k a_k g_k$, with $a_k \in A$ and g_k irreducible. Then $(i \times 1_Y)_*(g_k) \in \mathcal{I}_\sim(h^{\text{eff}}_\sim(X), h^{\text{eff}}_\sim(Y))$. This reduces us to the case where g is represented by an irreducible cycle $\widetilde{g}$.

Choose Z minimal among the closed subsets of X such that $\widetilde{g}$ is supported on $Z \times Y$. In particular, Z is irreducible.

Consider Z with its reduced structure. We may choose a proper, generically finite morphism $\pi : \widetilde{Z} \to Z$ where $\widetilde{Z}$ is smooth projective (irreducible) and π is

- birational if $\operatorname{char} F = 0$ (Hironaka)
- an alteration if $\operatorname{char} F > 0$ (de Jong [8, Th. 4.1]).

By the minimality of Z, the support of $\widetilde{g}$ has nonempty intersection $\widetilde{g}_1$ with $V \times Y$, where $V = Z - (Z_{sing} \cup T)$ with Z_{sing} the singular locus of Z and T the closed subset over which π_Z is not finite. Let $\pi_V : \pi^{-1}(V) \to V$ be the map induced by π and d be its degree: we have an equality of cycles

$$d\widetilde{g}_1 = (\pi_V)_* \pi_V^* \widetilde{g}_1$$

which implies an equality of cycles ($\widetilde{g}_1$ is dense in $\widetilde{g}$)

$$d\widetilde{g} = \pi_* \pi^* \widetilde{g}.$$

Let $h = d^{-1}[\pi^* \widetilde{g}] \in \mathcal{Z}^\sim_{\dim X}(\widetilde{Z} \times Y, A)$. Then the correspondence $f = ((i \circ \pi) \times 1)_* h \in \mathbf{Mot}^{\text{eff}}_\sim(F)(h^{\text{eff}}_\sim(X), h^{\text{eff}}_\sim(Y))$ factors as

$$h^{\text{eff}}_\sim(X) \xrightarrow{(i\circ\pi)^*} h^{\text{eff}}_\sim(\widetilde{Z})(\dim X - \dim \widetilde{Z}) \xrightarrow{h} h^{\text{eff}}_\sim(Y)$$

(see (1.2)), which concludes the proof. □

2.4.2. **Corollary.** *Under the assumptions of Theorem 2.4.1, all the categories of Diagram* (2.3) *are isomorphic to* $\mathbf{Mot}^{\text{eff}}_\sim(F,A)/\mathcal{I}_\sim$.

[2] It can be shown that this condition is not necessary.

[3] We thank N. Fakhruddin for his help, which removes the recourse to Chow's moving lemma in the earlier version.

Proof. By Proposition 2.2.3 b) and d) we already know that the categories $\mathbf{Mot}_\sim^{\mathrm{eff}}(F,A)/\mathcal{L}_\sim$, $(S_b^w)^{-1}\mathbf{Mot}_\sim^{\mathrm{eff}}(F,A)$ and $(S_r^w)^{-1}\mathbf{Mot}_\sim^{\mathrm{eff}}(F,A)$ are isomorphic and that $(\widetilde{S}_b)^{-1}\mathbf{Mot}_\sim^{\mathrm{eff}}(F,A)$ and $(\widetilde{S}_r)^{-1}\mathbf{Mot}_\sim^{\mathrm{eff}}(F,A)$ are isomorphic. We also know that the functor $\mathbf{Mot}_\sim^{\mathrm{eff}}(F,A)/\mathcal{L}_\sim \to (S_b)^{-1}\mathbf{Mot}_\sim^{\mathrm{eff}}(F,A)$ is full (Proposition 2.2.3 c)): by Theorem 2.4.1, this implies that it is an isomorphism. To conclude the proof, it is sufficient to show that any morphism of $\widetilde{S}_r$, hence of S_r, has a right inverse in $\mathbf{Mot}_\sim^{\mathrm{eff}}(F,A)/\mathcal{L}_\sim$ (see (2.3)). Since $\widetilde{S}_r$ is generated by $\widetilde{S}_b$ and projections of the form $X\times\mathbf{P}^1\to X$ (*cf.* proof of Proposition 2.2.3 d)) and since this is obvious for these projections, we are left to prove it for elements $f: X \dashrightarrow Y$ of $\widetilde{S}_b$. But we have $f_*f^*=1_X$ in $\mathbf{Mot}_\sim^{\mathrm{eff}}(F,A)/\mathcal{I}_\sim$ by Proposition 2.3.6 b), hence in $\mathbf{Mot}_\sim^{\mathrm{eff}}(F,A)/\mathcal{L}_\sim$ by Theorem 2.4.1. □

2.4.3. **Remark.** Recent results of Gabber [13] imply that the integer d appearing in the proof of Theorem 2.4.1 may be chosen prime to l for any prime $l\neq \operatorname{char} F$. This in turn implies that, in the above, one may relax the condition that A contains $\mathbf{Q}$ to the condition that the exponential characteristic of F is invertible in A.

3. Examples

We give some examples and computations of birational motives.

3.1. **Rationally connected varieties.**

3.1.1. **Proposition.** *Let X be a smooth projective F-variety which is rationally chain connected. Then $h^o(X)=\mathbf{1}$ in $\mathbf{Mot}^o_{\mathrm{rat}}(F,\mathbf{Q})$. (See Definition 2.2.6 for the notation h^o.)*

Proof. Let $\overline{F(X)}$ be an algebraic closure of $F(X)$. The hypothesis implies that $X(\overline{F(X)})/R=*$. Since the group of 0-cycles on $X_{\overline{F(X)}}$ is generated by $X(\overline{F(X)})$, this in turn implies that $CH_0(X_{\overline{F(X)}})\xrightarrow{\sim}\mathbf{Z}$, which implies by a transfer argument that $CH_0(X_{F(X)})\otimes\mathbf{Q}\xrightarrow{\sim}\mathbf{Q}$. By Theorem 2.4.1, the left hand side is the endomorphism ring of $h^o(X)$. If we write $h^o(X)\simeq\mathbf{1}\oplus h^o_{>0}(X)$, it follows that $\operatorname{End}(h^o_{>0}(X))=0$, which implies $h^o_{>0}(X)=0$. □

3.1.2. **Remark.** As noted in [18, Ex. 7.3], an Enriques surface gives a counterexample to the converse of Proposition 3.1.1.

3.2. **Quadrics.** Suppose char $F \neq 2$ and let X be a smooth projective quadric over F. By a theorem of Swan and Karpenko, the degree map

$$\deg : CH_0(X) \to \mathbf{Z}$$

is injective, with image $\mathbf{Z}$ if X has a rational point and $2\mathbf{Z}$ otherwise. This implies:

3.2.1. **Proposition.** *Let X, Y be two smooth projective quadrics over F. Then, in $\mathbf{Mot}_{\text{rat}}(F, \mathbf{Z})/\mathcal{I}_{\text{rat}}$, we have*

$$\operatorname{Hom}(h^o(X), h^o(Y)) = \begin{cases} \mathbf{Z} & \text{if } Y_{F(X)} \text{ is isotropic} \\ 2\mathbf{Z} & \text{otherwise} \end{cases}$$

where we have used the degree map $\deg : CH_0(Y_{F(X)}) \to \mathbf{Z}$. *Similarly, in $\mathbf{Mot}_{\text{rat}}(F, \mathbf{Z}/2)/\mathcal{I}_{\text{rat}}$, we have*

$$\operatorname{Hom}(h^o(X), h^o(Y)) = \begin{cases} \mathbf{Z}/2 & \text{if } Y_{F(X)} \text{ is isotropic} \\ 0 & \text{otherwise.} \end{cases}$$

3.3. **The nilpotence conjecture.** It is:

3.3.1. **Conjecture.** *For any two adequate pairs $(A, \sim), (A, \sim')$ with $A \supseteq \mathbf{Q}$ and $\sim \geq \sim'$, and any $M \in \mathbf{Mot}_\sim(F, A)$,* $\operatorname{Ker}(\operatorname{End}(M) \to \operatorname{End}(M_{\sim'}))$ *is nilpotent. (We say that the kernel of $\mathbf{Mot}_\sim(F, A) \to \mathbf{Mot}_{\sim'}(F, A)$ is* locally nilpotent.*)*

Since rat is the coarsest (*resp.* num is the finest) adequate equivalence relation, this conjecture is clearly equivalent to the same statement for $\sim =$ rat and $\sim' =$ num, but it may be convenient to consider it for selected adequate equivalence relations. For example:

3.3.2. **Proposition.** *a) Conjecture 3.3.1 is true for $M \in \mathbf{Mot}_\sim^{\text{eff}}(F, A)$ (and any $\sim' \leq \sim$) provided M is finite-dimensional in the sense of Kimura-O'Sullivan* [26, Def. 3.7]. *In particular, it is true if M is of abelian type, i.e. M is a direct summand of $h(A_K)$ for A an abelian F-variety and K a finite extension of F.*
b) If $\sim =$ hom, $\sim' =$ num, the condition of a) is equivalent to the sign conjecture*: if H is the Weil cohomology theory defining* hom*, the projector of* $\operatorname{End} H(M)$ *projecting $H(M) = H^+(M) \oplus H^-(M)$ onto its summand $H^+(M)$ is algebraic. In particular, it is true if M satisfies the Standard conjecture C (algebraicity of the Künneth projectors).*
c) Conjecture 3.3.1 is true in the following cases:

(i) $\sim =$ rat, $\sim' =$ tnil*;*
(ii) $\sim =$ rat, $\sim' =$ alg.

Proof. a) This is a theorem of Kimura and O'Sullivan, *cf.* [26, Prop. 7.5], [3, Prop. 9.1.14]. The second assertion follows from Kimura's results, *cf.* [22, Ex. 7.6.3 4)]. b) See [3, Th. 9.2.1 c)]. c) (i) follows from the Voevodsky-Kimura lemma that smash-nilpotent correspondences are nilpotent, *cf.* [47, Lemma 2.7], [26, Prop. 2.16], [3, Lemma 7.4.2 ii)]. (ii) follows from (i) and Voevodsky's theorem that alg $\geq$ tnil, [47, Cor. 3.2]. □

Let us recall some conjectures which imply Conjecture 3.3.1:

3.3.3. **Proposition.** *a) Conjecture 3.3.1 is implied by Voevodsky's conjecture that smash-nilpotence equivalence equals numerical equivalence* [47, Conj. 4.2].
b) It is also implied by the sign conjecture plus the Bloch-Beilinson–Murre conjecture [17, 35].

Proof. a) This follows from Proposition 3.3.2 c) (i). b) Recall that the Bloch-Beilinson conjecture is equivalent to Murre's conjecture in [35] by [17, Th. 5.2]. Now the formulation of the former conjecture, [17, Conj. 2.1], implies the existence of an increasing chain of equivalence relations $(\sim_\nu)_{1\leq \nu\leq\infty}$ such that

- $\sim_1=$ hom;
- if α,β are composable Chow correspondences such that $\alpha\sim_\mu 0$ and $\beta\sim_\nu 0$, then $\beta\circ\alpha\sim_{\mu+\nu} 0$;
- for any smooth projective variety X, there exists $\nu=\nu(X)$ such that $A_{\sim_\nu}(X\times X)=A_{\rm rat}(X\times X)$.

There properties, together with the sign conjecture, imply Conjecture 3.3.1 by Proposition 3.3.2 b). □

3.3.4. **Remark.** In fact, one has more precise but slightly weaker implications: (Bloch-Beilinson–Murre conjecture + hom = num conjecture) $\Rightarrow$ (Voevodsky's conjecture) $\Rightarrow$ (Kimura-O'Sullivan conjecture [any Chow motive is finite-dimensional]) $\Rightarrow$ (Conjecture 3.3.1): see the synoptic table in [2, end of Ch. 12].

For the first implication, see [2, Th. 11.5.3.1]. For the second one, see [2, Th. 12.1.6.6]. The third one is in Proposition 3.3.2 a).

3.3.5. **Definition.** Let $M\in\mathcal{M}_\sim(F,A)$. For $n\in\mathbf{Z}$, we write $\nu(M)\geq n$ if $M\otimes\mathbb{L}^{\otimes -n}$ is effective.

3.3.6. **Proposition.** *Suppose $A\supseteq\mathbf{Q}$ and the nilpotence conjecture holds for $\sim\geq\sim'$. Then:*
a) The functor $\mathbf{Mot}_\sim(F,A)\to\mathbf{Mot}_{\sim'}(F,A)$ is conservative, and for

$M \in \mathbf{Mot}_\sim(F,A)$, any set of orthogonal idempotents in the endomorphism ring of $M_{\sim'}$ lifts.
b) If $M \in \mathbf{Mot}_\sim(F,A)$ and $M_{\sim'}$ is effective, then M is effective.
c) If $M \in \mathbf{Mot}_\sim(F,A)$ and $\nu(M_{\sim'}) \geq n$, then $\nu(M) \geq n$.
d) [2, 13.2.1] *The map $K_0(\mathbf{Mot}_\sim(F,A)) \to K_0(\mathbf{Mot}_{\sim'}(F,A))$ is an isomorphism (here, the K_0-groups are those of additive categories).*

Proof. a) is classical (see [17, Lemma 5.4] for the second statement). b) By definition, $M_{\sim'}$ effective means that $M_{\sim'}$ is isomorphic to a direct summand of $h_{\sim'}(X)$ for some smooth projective X. By a), one may lift the corresponding idempotent $e_{\sim'}$ to an idempotent endomorphism e of $h^{\text{eff}}_\sim(X)$, and the isomorphism $M_{\sim'} \simeq (h_{\sim'}(X), e_{\sim'})$ to an isomorphism $M \simeq (h^{\text{eff}}_\sim(X), e)$. c) follows from b) applied to $M \otimes \mathbb{L}^{\otimes -n}$. d) follows from a), since then the functor $\mathbf{Mot}_\sim(F,A) \to \mathbf{Mot}_{\sim'}(F,A)$ is conservative and essentially surjective. □

The importance of Conjecture 3.3.1 will appear again in the next subsection and in Section 4 (see Remark 4.3.4 2) and Proposition 4.4.1).

3.4. **The Chow-Künneth decomposition.** Here we take $A = \mathbf{Q}$. Recall that Murre [35] strengthened the standard conjecture C (algebraicity of the Künneth projectors) to the existence of a *Chow-Künneth decomposition*

$$h(X) \simeq \bigoplus_{i=0}^{2d} h_i(X)$$

in $\mathbf{Mot}_{\text{rat}}(F,\mathbf{Q})$. (This is part of the Bloch-Beilinson–Murre conjecture appearing in Proposition 3.3.3 b)). By Proposition 3.3.6 a), the nilpotence conjecture together with the standard conjecture C imply the existence of Chow-Künneth decompositions.

Here are some cases where the existence of a Chow-Künneth decoomposition is known independently of any conjecture:

(1) Varieties of dimension ≤ 2 (Murre, [34], see also [41]). In fact, Murre constructs for any X a partial decomposition

$$h(X) \simeq h_0(X) \oplus h_1(X) \oplus h_{[2,2d-2]}(X) \oplus h_{2d-1}(X) \oplus h_{2d}(X).$$

(2) Abelian varieties (Shermenev, [45]).
(3) Complete intersections in $\mathbf{P}^N$ (see next subsection).
(4) If X and Y have a Chow-Künneth decomposition, then so does $X \times Y$.

Suppose that the nilpotence conjecture holds for $h(X) \in \mathbf{Mot}_{\text{rat}}(F,\mathbf{Q})$ and that homological and numerical equivalences coincide on $X \times X$. The latter then implies the standard conjecture C for X, hence the existence of a Chow-Künneth decomposition by the remark above. In [22, Th. 14.7.3 (iii)], it is proven:

3.4.1. **Proposition.** *Under this hypothesis, there exists a further decomposition for each $i \in [0, 2d]$:*

$$h_i(X) \simeq \bigoplus h_{i,j}(X)(j)$$

such that $h_{i,j}(X) = 0$ for $j \notin [0, [i/2]]$ and, for each j, $\nu(h_{i,j}^{\mathrm{hom}}(X)) = 0$ (see Definition 3.3.5). Moreover, one has isomorphisms

$$h_{2d-i,d-i+j}(X) \xrightarrow{\sim} h_{i,j}(X) \tag{3.1}$$

for $i \leq d$. In particular, $\nu(h_i(X)) > 0$ for $i > d$.

Let us justify the last assertion: the isomorphisms (3.1) imply that, for $i > d$, $h_{i,j}(X) = 0$ for $j < d - i$.

Since $\mathbf{Mot}_{\mathrm{rat}}^{\mathrm{eff}}(F, \mathbf{Q}) \to \mathbf{Mot}_{\mathrm{rat}}(F, \mathbf{Q})$ is fully faithful, all the above (refined) Chow-Künneth decompositions hold for the effective Chow motives $h^{\mathrm{eff}}(X) \in \mathbf{Mot}_{\mathrm{rat}}^{\mathrm{eff}}(F, \mathbf{Q})$. We deduce:

3.4.2. **Corollary.** *Under the nilpotence conjecture and the conjecture that homological and numerical equivalences coincide, for any smooth projective variety X the image of its Chow-Künneth decomposition in $\mathbf{Mot}_{\mathrm{rat}}^{\mathrm{o}}(F, \mathbf{Q})$ is of the form*

$$h^{\mathrm{o}}(X) \simeq \bigoplus_{i=0}^{d} h_i^{\mathrm{o}}(X).$$

Moreover, with the notation of Proposition 3.4.1, one has $h_i^{\mathrm{o}}(X) \simeq h_{i,0}^{\mathrm{o}}(X)$ for $i \leq d$.

Examples where this conclusion is true unconditionally follow faithfully the examples where the Chow-Künneth decomposition is unconditionally known:

3.4.3. **Proposition.** *The conclusion of Corollary 3.4.2 holds in the following cases:*

(1) *Varieties of dimension ≤ 2.*
(2) *Abelian varieties.*
(3) *Complete intersections in $\mathbf{P}^N$.*
(4) *If X and Y have a Chow-Künneth decomposition and verify this conclusion, then so does $X \times Y$.*

Proof. In cases (1) and (2), the conclusion holds because one has "Lefschetz isomorphisms" $h_{2d-i}(X) \xrightarrow{\sim} h_i(X)(d-i)$ for $i > d$. For curves, it is trivial, for surfaces they are constructed in [34] (see [41, Th. 4.4.

(ii)]: the isomorphism is constructed for $i = 0, 1$ and any X), and for abelian varieties they are constructed in [45]. For (3), see next section. Finally, (4) is clear. □

In the case of a surface, [22] constructs a refined Chow-Künneth decomposition

$$h(X) = h_0(X) \oplus h_1(X) \oplus NS_X(1) \oplus t_2(X) \oplus h_3(X) \oplus h_4(X)$$

where NS_X is the Artin motive corresponding to the Galois representation defined by $NS(\bar{X}) \otimes \mathbf{Q}$, and $t_2(X)$ is the *transcendental part of* $h(X)$. (In the notation of Proposition 3.4.1, $h_{2,0}(X) = t_2(X)$ and $h_{2,1}(X) = NS_X$.) This translates on the birational motive of X as

$$h^o(X) = h_0^o(X) \oplus h_1^o(X) \oplus t_2^o(X).$$

3.5. **Motives of complete intersections.** These computations will be used in Section 4. Here we take $A \supseteq \mathbf{Q}$.

For convenience, we take the notation of [9]: so let $X \subset \mathbf{P}^r$ be a smooth complete intersection of multidegree $\underline{a} = (a_1, \dots, a_d)$, and let $n = r - d = \dim X$. Then the cohomology of X coincides with the cohomology of $\mathbf{P}^r$ except in middle dimension [9], and in particular it is fully algebraic except in middle dimension. This allows us to easily write down a Chow-Künneth decomposition for $h(X)$ in the sense of Murre [35] (see also [11, Cor. 5.3]):

(1) (Murre) For each $i \neq n/2$, let $c^i \in \mathcal{Z}^i(X)$ be an algebraic cycle whose cohomology class generates $H^{2i}(X)$. Then the Chow-Künneth projector π_{2i} is given by $c^i \times c^{n-i}$. We take $\pi_i = 0$ for i odd $\neq n$, and $\pi_n := \Delta_X - \sum_{i \neq n} \pi_i$.
(2) Consider the inclusion $i : X \hookrightarrow \mathbf{P}^r$. This yields morphisms of motives

$$h(\mathbf{P}^r)(-d) \xrightarrow{i^*} h(X) \xrightarrow{i_*} h(\mathbf{P}^r).$$

Given the decomposition $h(\mathbf{P}^r) \simeq \bigoplus_{j=0}^r \mathbb{L}^j$, this yields for each $j \in [0, n]$ morphisms

$$\mathbb{L}^j \xrightarrow{i_j^*} h(X) \xrightarrow{i_*^j} \mathbb{L}^j$$

with composition $a = \prod a_i$. Then $(1/a) i_j^* i_*^j$ defines the $2i$-th Chow-Künneth projector of X (denoted π_{2i} in (1)), except if $2i = n$. Let $\pi_n^{prim} := 1_{h(X)} - \sum_{i=0}^n (1/a) i_j^* i_*^j$: the image $p_n(X)$ of the projector π_n^{prim} is *the primitive part* of $h_n(X)$.

Note that the Chow-Künneth projectors of (1) and (2) are actually equal. Let us record here the corresponding (refined) Chow-Künneth decomposition:

$$h(X) \simeq \mathbf{1} \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^n \oplus p_n(X). \tag{3.2}$$

3.5.1. **Lemma.** *a) Homological and numerical equivalences agree on all (rational) Chow groups of X provided n is odd or (if $\operatorname{char} F = 0$) the Hodge realisation of $p_n(X)$ does not contain any direct summand isomorphic to $\mathbb{L}^{n/2}$.*
b) Suppose a) is satisfied. Then for any adequate pair $(\sim, A)$ with $A \supseteq \mathbf{Q}$ and any $j \in [0, n]$, we have

$$\mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, p_n(X)) = \operatorname{Ker}(A_j^\sim(X, A) \to A_j^{\mathrm{num}}(X, A)).$$

Proof. We have

$$\begin{aligned} A_j^\sim(X, A) &= \mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, h(X)) \\ &= \bigoplus_{i=0}^n \mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, \mathbb{L}^i) \oplus \mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, p_n(X)) \\ &= \mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, \mathbb{L}^j) \oplus \mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, p_n(X)). \end{aligned}$$

For $\sim =$ hom, we have $\mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, p_n(X)) = 0$ by weight reasons for $2j \neq n$ and under the hypothesis of a) for $2j = n$ (note that the Hodge realization of $p_n(X)$ is semi-simple, as a polarisable Hodge structure). Hence the same is true for any $\sim$ finer than hom, in particular $\sim =$ num. This proves a). Moreover, $\mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, \mathbb{L}^j) = A$ for any choice of $\sim$. Hence b). □

This shows that the birational motive of X reduces to $\mathbf{1} \oplus p_n^\sim(X)^{\mathrm{o}}$. In fact, it is possible to be much more precise:

3.5.2. **Proposition.** *Let $\underline{a} = (a_1, \dots, a_d)$ be the multidegree of X.*
a) If $a_1 + \cdots + a_d \leq r$, $h^{\mathrm{o}}_{\mathrm{rat}}(X) = \mathbf{1}$.
b) If $a_1 + \cdots + a_d > r$, $h^{\mathrm{o}}_{\mathrm{num}}(X) \neq \mathbf{1}$ (equivalently, $p_n^{\mathrm{num}}(X)^{\mathrm{o}} \neq 0$) provided $\operatorname{char} F = 0$ or X is generic.

Proof. a) Under the hypothesis, we conclude from Roitman's theorem [40] that $CH_0(X_K) \otimes \mathbf{Q} = \mathbf{Q}$ for any extension K/F. Assertion a) then follows from Theorem 2.4.1, (2.5) and (3.2). For b), it suffices to prove the statement for homological equivalence, since the kernel of $\mathbf{Mot}_{\mathrm{hom}}(F, \mathbf{Q})(h(X), h(X)) \to \mathbf{Mot}_{\mathrm{num}}(F, \mathbf{Q})(h(X), h(X))$ is a nilpotent ideal (see Propositions 3.3.2 b) and 3.3.6 a)).

If $\operatorname{char} F = 0$, we may use Hodge cohomology and Deligne's theorem [9, Th. 2.5 (ii) p. 54]. Namely, with the notation of loc. cit., the condition $p_n^{\hom}(X)^o = 0$ implies $h_0^{0,n}(\underline{a}) = 0$, which is equivalent by loc. cit., Th. 2.5 (ii) to

$$0 \leq \left[\frac{n+d-\sum a_i}{\sup(a_i)}\right]$$

that is, $\sum a_i \leq n+d = r$.

If $\operatorname{char} F > 0$ and X is generic, we may use Katz's theorem [24, p. 382, Th.4.1]. □

3.5.3. **Remarks.** 1) Katz also has a result concerning a generic hyperplane section of a given complete intersection, [24, Th. 4.2].

2) It seems possible to remove the genericity assumption in positive characteristic by lifting the coefficients of the equations defining X to characteristic 0. We have not worked out the details.

4. On adjoints and idempotents

We now want to examine two related questions:

(1) Does the projection functor $\mathbf{Mot}_\sim^{\text{eff}}(F, A) \to \mathbf{Mot}_\sim^{\text{eff}}(F, A)/\mathcal{L}_\sim$ have a right adjoint? This question was raised by Luca Barbieri-Viale and is closely related to a conjecture of Voevodsky [46, Conj. 0.0.11].

(2) Is the category $\mathbf{Mot}_\sim^{\text{eff}}(F, A)/\mathcal{L}_\sim$ pseudo-abelian? i.e., is it necessary to take the pseudo-abelian envelope in Definition 2.2.6?

The answer to both questions is "yes" for $\sim =$ num and $A \supseteq \mathbf{Q}$, as an easy consequence of Jannsen's semi-simplicity theorem for numerical motives [16]. In fact:

4.0.4. **Proposition** ([18, Prop. 7.7])**.** *a) The projection functor*

$$\pi : \mathbf{Mot}_{\text{num}}^{\text{eff}} \to \mathbf{Mot}_{\text{num}}^{o}$$

is essentially surjective.
b) π has a section i which is also a left and right adjoint.
c) The category $\mathbf{Mot}_{\text{num}}^{\text{eff}}$ is the coproduct of $\mathbf{Mot}_{\text{num}}^{\text{eff}} \otimes \mathbb{L}$ and $i(\mathbf{Mot}_{\text{num}}^{o})$, i.e. any object of $\mathbf{Mot}_{\text{num}}^{\text{eff}}$ can be uniquely written as a direct sum of objects of these two subcategories.

In the sequel, we want to examine these questions for a general adequate pair: see Theorems 4.3.2 and 4.3.3 for (1) and Proposition 4.4.1 for (2). This requires some preparation.

4.1. **A lemma on base change.** Let $P:\mathcal{A}\to\mathcal{B}$ be a functor. Recall that one says that "its" right adjoint is *defined at* $B\in\mathcal{B}$ if the functor

$$\mathcal{A}\ni A\mapsto \mathcal{B}(PA,B)$$

is representable. We write $P^\sharp B$ for a representing object (unique up to unique isomorphism).

Let

$$\begin{array}{ccc}\mathcal{A} & \xrightarrow{\varphi} & \mathcal{B}\\ {\scriptstyle P}\downarrow & & {\scriptstyle Q}\downarrow\\ \mathcal{C} & \xrightarrow{\psi} & \mathcal{D}\end{array}$$

be a naturally commutative diagram of pseudo-abelian additive categories, and let $A\in\mathcal{A}$.

Suppose that "the" right adjoint $P^\sharp$ of P is defined at $PA\in\mathcal{C}$ and that the right adjoint $Q^\sharp$ of Q defined at $\psi PA\simeq Q\varphi A$. We then have two corresponding unit maps (adjoint to the identities of PA and $Q\varphi A$)

$$\varepsilon_P: A\to P^\sharp PA$$
$$\varepsilon_Q:\varphi A\to Q^\sharp Q\varphi A.$$

4.1.1. **Lemma.** *Suppose that ε_Q is an isomorphism. Then $\varphi\varepsilon_P$ has a retraction. If moreover φ is full and* $\operatorname{Ker}(\operatorname{End}_{\mathcal{A}}(A)\to\operatorname{End}_{\mathcal{B}}(\varphi A))$ *is a nilideal, then ε_P has a retraction.*

Proof. Let $\eta_P: PP^\sharp PA\to PA$ be the counit map of the adjunction at PA (adjoint to the identity of $P^\sharp PA$), and let $u: Q\varphi A\xrightarrow{\sim}\psi PA$, $v: Q\varphi P^\sharp PA\xrightarrow{\sim}\psi PP^\sharp PA$ be the natural isomorphism from $Q\varphi$ to ψP evaluated respectively at A and $P^\sharp PA$. We then have a composition

$$Q\varphi P^\sharp PA\xrightarrow{v}\psi PP^\sharp PA\xrightarrow{\psi\eta_P}\psi PA$$

which yields by adjunction a "base change morphism"

$$\varphi P^\sharp PA\xrightarrow{b} Q^\sharp\psi PA.$$

Inspection shows that the diagram

$$\begin{array}{ccc}\varphi A & \xrightarrow{\varphi\varepsilon_P} & \varphi P^\sharp PA\\ {\scriptstyle \varepsilon_Q}\downarrow & & {\scriptstyle b}\downarrow\\ Q^\sharp Q\varphi A & \xrightarrow{Q^\sharp u} & Q^\sharp\psi PA\end{array}$$

commutes. The first claim follows, and the second claim follows from the first. $\square$

4.2. **Right adjoints.** We come back to Question (1) posed at the beginning of this section. In [22, 14.8.7] and [18, 7.8 3)], it was announced that one can show the non-existence of the right adjoint for $\sim=$ rat, using the results of [15, Appendix]. The proof turns out not to be exactly along these lines, but is closely related: see Lemma 4.2.1, Theorem 4.3.2 and Theorem 4.3.3.

Let us abbreviate the notation to $\mathbf{Mot}^{\text{eff}} = \mathbf{Mot}^{\text{eff}}_{\sim}(F, A)$, $\mathbf{Mot}^{\text{o}} = \mathbf{Mot}^{\text{o}}_{\sim}(F, A)$. Let $P : \mathbf{Mot}^{\text{eff}} \to \mathbf{Mot}^{\text{o}}$ denote the projection functor, and let $P^\sharp$ denote its (a priori partially defined) right adjoint. Let ${}^\perp\mathcal{L}$ be the full subcategory of $\mathbf{Mot}^{\text{eff}}$ consisting of those M such that $\operatorname{Hom}(N(1), M) = 0$ for all $N \in \mathbf{Mot}^{\text{eff}}$. Recall from [22, Prop. 7.8.1] that

- If $P^\sharp$ is defined at M, then $P^\sharp M \in {}^\perp\mathcal{L}$;
- The full subcategory $\mathbf{Mot}^\sharp$ of $\mathbf{Mot}^{\text{o}}$ where $P^\sharp$ is defined equals $P({}^\perp\mathcal{L})$;
- $P^\sharp$ and the restriction of P to ${}^\perp\mathcal{L}$ define quasi-inverse equivalences of categories between ${}^\perp\mathcal{L}$ and $\mathbf{Mot}^\sharp$.

The right adjoint $P^\sharp$ is defined at birational motives of varieties of dimension ≤ 2 for any adequate pair $(A, \sim)$ such that $A \supseteq \mathbf{Q}$ by [22, Cor. 7.8.6]. (The proof there is given for $(A, \sim) = (\mathbf{Q}, \text{rat})$, but the argument works in general.)

The following lemma gives a sufficient condition for the nonexistence of $P^\sharp PM$ for an effective motive M.

4.2.1. **Lemma.** *Let $(\mathbf{Q}, \sim)$ be an adequate pair, and let $M \in \mathbf{Mot}^{\text{eff}}_{\sim}(F, \mathbf{Q})$. Assume that*

(i) *$M_{\text{num}} \in \mathbf{Mot}^{\text{eff}}_{\text{num}}(F, \mathbf{Q})$ does not contain any direct summand divisible by $\mathbb{L}$;*
(ii) *$\operatorname{Ker}(\operatorname{End}(M) \to \operatorname{End}(M_{\text{num}}))$ is a nilideal;*
(iii) *There exists $r > 0$ such that $\operatorname{Hom}(\mathbb{L}^r, M) \neq 0$.*

Then $P^\sharp PM$ does not exist.

Proof. Suppose that $P^\sharp$ is defined at PM. Consider the unit map

$$\varepsilon_\sim : M \to P^\sharp PM. \tag{4.1}$$

For $\sim=$ num, $P^\sharp_{\text{num}} P_{\text{num}} M_{\text{num}}$ exists by Proposition 4.0.4. Moreover, part c) of this proposition shows that, under Condition (i) of the lemma, ε_{num} is an isomorphism. By Lemma 4.1.1, the image of $\varepsilon_\sim$ modulo numerical equivalence then has a retraction, and so does $\varepsilon_\sim$ itself under Condition (ii). If this is the case, $M \in {}^\perp\mathcal{L}$, and in particular, $\operatorname{Hom}(\mathbb{L}^r, M) = 0$ for all $r > 0$, contradiction. □

4.3. **Counterexamples.** To give examples where the conditions of Lemma 4.2.1 are satisfied, we appeal as in [15] to the nontriviality of the Griffiths group.

We start with an example which a priori only works for a specific adequate equivalence, because the proof is simpler. Unlike in [15], we don't need the full force of Clemens' theorem [5, Th. 0.2], but merely the previous results of Griffiths [14].

4.3.1. **Definition** ("Abel-Jacobi equivalence")**.** Let $k = \mathbf{C}$. For Y smooth projective, $\mathcal{Z}^j_{\mathrm{AJ}}(X, \mathbf{Q})$ is the image of $CH^j(X) \otimes \mathbf{Q}$ in Deligne-Beilinson cohomology via the (Deligne-Beilinson) cycle class map. This defines an adequate equivalence relation.

4.3.2. **Theorem.** *Let* $F = \mathbf{C}$ *and* $\sim = \mathrm{AJ}$. *Then*
a) Condition (ii) of Lemma 4.2.1 is satisfied for any pure motive M*.*
Let X *be a generic hypersurface of degree* a *in* $\mathbf{P}^{n+1}$*.*
b) Condition (i) of Lemma 4.2.1 is satisfied for $M = p_n(X)$ *(see* (3.2)*) provided* X *is not a quadric, a cubic surface or an even-dimensional intersection of two quadrics, and* $a \geq n + 1$*.*
c) If $n = 2m - 1$ *is odd and* $a \geq 2 + 3/(m - 1)$*, then Condition (iii) of Lemma 4.2.1 is satisfied for* $r = m - 1$*.*
d) $P^\sharp$ *is not defined at* $h^o(X)$ *in the following cases:* n *is odd and*

(i) $n = 3$*:* $a \geq 5$*.*
(ii) $n > 3$*:* $a \geq n + 1$*.*

Proof. a) holds because $\mathrm{Ker}(\mathrm{End}_{\mathrm{AJ}}(M) \to \mathrm{End}_{\mathrm{hom}}(M))$ has square 0 and $\mathrm{Ker}(\mathrm{End}_{\mathrm{hom}}(M) \to \mathrm{End}_{\mathrm{num}}(M))$ is nilpotent.

b) By [37, Ex. 5 and Cor. 18], the Hodge realisation $P_n(X)$ of $p_n(X)$ is an absolutely simple pure Hodge structure: this, together with Proposition 3.5.2 b), is amply sufficient to imply Condition (i) of Lemma 4.2.1.

c) By [14, Cor. 13.2 and 14.2], $\mathrm{Ker}(A^{\sim}_{m-1}(X, \mathbf{Q}) \to A^{\mathrm{num}}_{m-1}(X, \mathbf{Q})) \neq 0$. But by Lemma 3.5.1, this group is $\mathrm{Hom}(\mathbb{L}^{m-1}, p_n(X))$.

d) Note that, by the refined Chow-Künneth decomposition (3.2), $P^\sharp$ is defined at $Ph(X)$ if and only if it is defined at $Pp_n(X)$. The conclusion now follows from Lemma 4.2.1 and from collecting the results of a), b) and c). □

To get a conterexample with rational equivalence, we appeal to a result of Nori [36]. We thank Srinivas for pointing out this reference.

4.3.3. **Theorem.** *Let* X *be a generic abelian threefold over* $k = \mathbf{C}$*. If* $\sim \geq \mathrm{alg}$*, then* $P^\sharp$ *is not defined at* $h^o_\sim(X)$*.*

Proof. It is similar to that of Theorem 4.3.2, except that the motive of an abelian variety is more complicated than that of a hypersurface. We only sketch the argument (details will appear elsewhere):

It is enough to show that $P^\sharp$ is not defined at $h_{3,0}^o(X)$, where $h_{3,0}(X)$ is as in Proposition 3.4.1 (here we use that the nilpotence conjecture is true for motives of abelian varieties, see Proposition 3.3.2 a)). We check the conditions of Lemma 4.2.1 for $M = h_{3,0}(X)$. (i) is true by definition. (ii) is true by Proposition 3.3.2 a). For (iii), one can show that computing the decomposition

$$A_1^\sim(X) = \mathbf{Mot}_\sim^{\text{eff}}(\mathbb{L}, h(X)) \simeq \bigoplus_{i=0}^{6} \bigoplus_{j=0}^{[i/2]} \mathbf{Mot}_\sim^{\text{eff}}(\mathbb{L}, h_{i,j}(X)(j))$$

yields a surjection

$$\mathbf{Mot}_\sim^{\text{eff}}(\mathbb{L}, h_{3,0}(X)) \twoheadrightarrow \operatorname{Griff}_1(X)$$

for $\sim \geq$ alg, where $\operatorname{Griff}_1(X) = \operatorname{Ker}(A_1^{\text{alg}}(X) \to A_1^{\text{num}}(X))$ is the Griffiths group of X. By Nori's theorem [36], $\operatorname{Griff}_1(X) \neq 0$, and the proof is complete. □

4.3.4. **Remark.** It is easy to get examples of any dimension ≥ 4 by multiplying the example of Theorem 4.3.3 with $\mathbf{P}^n$.

4.4. **Idempotents.** We now address Question (2) from the beginning of this section.

4.4.1. **Proposition.** *Let $(A, \sim)$ be an adequate pair with $A \supseteq \mathbf{Q}$, and let $\mathcal{M}$ be a full subcategory of $\mathbf{Mot}_\sim^{\text{eff}}(F, A)$ closed under direct summands. If Conjecture 3.3.1 holds for the objects of $\mathcal{M}$, then the category $\mathcal{M}/\mathcal{L}_\sim$ is pseudo-abelian.*

Proof. Let $\mathcal{M}_{\text{num}}$ denote the pseudo-abelian envelope of the image of $\mathcal{M}$ in $\mathbf{Mot}_{\text{num}}^{\text{eff}}(F, A)$. We have a commutative diagram of categories:

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{P} & \mathcal{M}/\mathcal{L}_\sim \\
{\scriptstyle\pi}\downarrow & & {\scriptstyle\bar\pi}\downarrow \\
\mathcal{M}_{\text{num}} & \xrightarrow{P_{\text{num}}} & \mathcal{M}_{\text{num}}/\mathcal{L}_{\text{num}}
\end{array}$$

Under the hypothesis, π is essentially surjective (one can lift idempotents). Hence $\bar\pi$ is essentially surjective as well. Since P is essentially surjective and π, P_{num} are full, $\bar\pi$ is full, and its kernel is locally nilpotent as a quotient of the kernel of π (fullness of P). Thus $\bar\pi$ is full, essentially surjective and conservative.

Since $\mathbf{Mot}^{\mathrm{eff}}_{\mathrm{num}}(F,A)$ is semi-simple, $\mathcal{M}_{\mathrm{num}}$ is also semi-simple, hence so is $\mathcal{M}_{\mathrm{num}}/\mathcal{L}_{\mathrm{num}}$ which is in particular pseudo-abelian.

Let now $M \in \mathcal{M}/\mathcal{L}_\sim$, and let $p = p^2 \in \mathrm{End}(M)$. Write $M_{\mathrm{num}} \simeq M_1 \oplus M_2$, where $M_1 = \mathrm{Im}\, p_{\mathrm{num}}$ and $M_2 = \mathrm{Ker}\, p_{\mathrm{num}}$. By essential surjectivity, we may lift M_1 and M_2 to objects $\widetilde{M}_1, \widetilde{M}_2 \in \mathcal{M}/\mathcal{L}_\sim$.

By fullness, we may lift the isomorphism $M_1 \oplus M_2 \xrightarrow{\sim} M_{\mathrm{num}}$ to a morphism $\widetilde{M}_1 \oplus \widetilde{M}_2 \to M$ in $\mathcal{M}/\mathcal{L}_\sim$, and this lift is an isomorphism by conservativity. This concludes the proof. □

4.4.2. **Example.** Proposition 4.4.1 applies taking for $\mathcal{M}$ the category of motives of abelian type (direct summands of the tensor product of an Artin motive and the motive of an abelian variety), since such motives are finite-dimensional (Kimura [26]).

The situation when A does not contain $\mathbf{Q}$, for example $A = \mathbf{Z}$, is unclear.

5. Birational motives and birational categories

In this section, we relate the categories studied in [21] with the categories of pure birational motives introduced here.

5.1. As in [21], let $\mathbf{place}(F)$ denote the category of finitely generated extensions of F, with F-places as morphisms. From [21, Cor. 4.4.3] and the above, it follows that we have a composite functor:

$$(5.1)\quad S_r^{-1}\,\mathbf{place}_{\mathbf{Sm}^{\mathrm{proj}}}(F)^{\mathrm{op}} \to S_r^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F) \to S_r^{-1}\,\mathbf{Chow}^{\mathrm{eff}}(F) \to \mathbf{Chow}^{\mathrm{o}}(F)$$

where $\mathbf{place}_{\mathbf{Sm}^{\mathrm{proj}}}(F)$ denotes the full subcategory of $\mathbf{place}(F)$ defined by those K/F which have a cofinal set of smooth projective models and S_r is the set of purely transcendental field extensions. If $\mathrm{char}\, F = 0$, the morphisms in the second category can be described by means of R-equivalence [21, Th. 5.4.14], and by Theorem 2.4.1, the morphisms in the last category can be described by means of Chow groups of 0-cycles. One checks easily that the action of the composite functor on Hom sets is just the map which sends R-equivalence classes of rational points to 0-cycles modulo rational equivalence. This puts this map within a functorial setting.

In characteristic zero, we can also describe the image of a place $\lambda : K \rightsquigarrow L$ in $CH_0(X_L)$, where X is a smooth projective model of K: it is just the class of the centre of λ. Hence the image of the functor

$$\mathbf{place}(F)^{\mathrm{op}} \to \mathbf{Chow}^{\mathrm{o}}(F)$$

on morphisms consists of the classes of rational points. This answers a question of Déglise.

Recall that, in characteristic 0, $\mathbf{place}_{\mathbf{Sm}^{\mathrm{proj}}}(F) = \mathbf{place}(F)$ and $S_r^{-1}\mathbf{Sm}^{\mathrm{proj}}(F) = S_r^{-1}\mathbf{Sm}(F)$ [20, Prop. 8.5]. In characteristic p, we would ideally like to get functors

$$S_r^{-1}\mathbf{place}(F)^{op} \to \mathbf{Chow}^{\mathrm{o}}(F)$$
$$S_r^{-1}\mathbf{Sm}(F) \to \mathbf{Chow}^{\mathrm{o}}(F)$$

fitting with (5.1). This looks technically difficult: we shall content ourselves with extending [18, Rk. 7.4] to all finitely generated fields K/F, by using an adjunction result to appear in [23].

5.1.1. Proposition. *a) There is a unique functor (up to unique isomorphism)*

$$h^{\mathrm{o}} : S_r^{-1}\mathbf{field}(F)^{op} \to \mathbf{Chow}^{\mathrm{o}}(F, \mathbf{Q})$$

such that, for any $K \in \mathbf{field}(F)$ and any $Y \in \mathbf{Sm}^{\mathrm{proj}}(F)$, one has

$$\mathbf{Chow}^{\mathrm{o}}(F, \mathbf{Q})(h^{\mathrm{o}}(K), h^{\mathrm{o}}(Y)) \simeq CH_0(Y_K) \otimes \mathbf{Q}. \tag{5.2}$$

This functor transforms purely inseparable extensions into isomorphisms.
b) If $K \subseteq L$, the map $h^{\mathrm{o}}(L) \to h^{\mathrm{o}}(K)$ has a section.
c) We have $h^{\mathrm{o}}(K) = h^{\mathrm{o}}(X)$ if $K = F(X)$ for a smooth projective variety X. Moreover, if $K = F(X)$, $L = F(Y)$ with X, Y smooth projective, and if $f : K \to L$ corresponds to a rational map $\varphi : Y \dashrightarrow X$, then $h^{\mathrm{o}}(f)$ is given by the graph of φ.

Proof. a) Note that the isomorphism (5.2) determines $h^{\mathrm{o}}(K)$ up to unique isomorphism, by Yoneda's lemma. This isomorphism may be rewritten as

$$\mathbf{Chow}^{\mathrm{o}}(F, \mathbf{Q})(h^{\mathrm{o}}(K), h^{\mathrm{o}}(Y)) \simeq \mathbf{Chow}^{\mathrm{o}}(K, \mathbf{Q})(\mathbf{1}_K, h^{\mathrm{o}}(Y_K)).$$

where $\mathbf{1}_K = h^{\mathrm{o}}(\operatorname{Spec} K)$ is the unit object of $\mathbf{Chow}^{\mathrm{o}}(K)$.

By [23], the base-change functor

$$\mathbf{Chow}^{\mathrm{o}}(F, \mathbf{Q}) \to \mathbf{Chow}^{\mathrm{o}}(K, \mathbf{Q})$$

has a left adjoint $l_{K/F}$. Therefore we may define $h^{\mathrm{o}}(K) = l_{K/F}(\mathbf{1}_K)$.

Suppose $F \to K \xrightarrow{f} L$ are successive finitely generated extensions. Since the base-change of $\mathbf{1}_K$ is $\mathbf{1}_L$, the identity map $\mathbf{1}_L \to \mathbf{1}_L$ gives by adjunction a map

$$l_{L/K}\mathbf{1}_L \to \mathbf{1}_K$$

hence a map

$$h^{\mathrm{o}}(f) : h^{\mathrm{o}}(L) = r_{L/F}(\mathbf{1}_L) \to r_{K/F}(\mathbf{1}_K) = h^{\mathrm{o}}(K).$$

We just used the transitivity of adjoints; using it a second time on a 3-layer extension shows that we have indeed defined a functor $\mathbf{field}(F)^{op} \to \mathbf{Chow}^{\mathrm{o}}(F,\mathbf{Q})$.

Suppose that $L = K(t)$. Then $l_{L/K}(\mathbf{1}_L) = h^{\mathrm{o}}(\mathbf{P}^1) = \mathbf{1}_K$, hence $h^{\mathrm{o}}(f)$ is an isomorphism. This shows that our functor induces a functor $h^{\mathrm{o}} : S_r^{-1}\,\mathbf{field}(F)^{op} \to \mathbf{Chow}^{\mathrm{o}}(F,\mathbf{Q})$, as required.

Suppose now that $K \xrightarrow{f} L$ is a finite and purely inseparable extension of finitely generated fields over F. If X is a smooth projective K-variety, the map $CH_0(X) \otimes \mathbf{Z}[1/p] \to CH_0(X_L) \otimes \mathbf{Z}[1/p]$ is well-known to be an isomorphism: this shows that $l_{L/K}(\mathbf{1}_L) = \mathbf{1}_K$, hence that $h^{\mathrm{o}}(f)$ is invertible.

b) The proof is the same as in [18, Rk. 7.4]: write L as a finite purely inseparable extension of a finite separable extension of a purely transcendental extension of K. Then a) reduces us to the case where L/K is finite and separable. We may write $L = \operatorname{Spec} X$ where X is a 0-dimensional smooth projective K-variety, and $l_{L/K}(\mathbf{1}_L) = h^{\mathrm{o}}(X)$. The conclusion now follows from Lemma 1.5.1.

c) If $K = F(X)$ for X smooth projective, then (2.5) and Yoneda's lemma show that $h^{\mathrm{o}}(K) \simeq h^{\mathrm{o}}(X)$. For the claim on morphisms, we are reduced (again by Yoneda's lemma) to determining the map

$$\mathbf{Chow}^{\mathrm{o}}(F,\mathbf{Q})(h^{\mathrm{o}}(K), h^{\mathrm{o}}(Z)) \xrightarrow{h^{\mathrm{o}}(f)^*} \mathbf{Chow}^{\mathrm{o}}(F,\mathbf{Q})(h^{\mathrm{o}}(L), h^{\mathrm{o}}(Z))$$

for a smooth projective F-variety Z. By definition of $h^{\mathrm{o}}(f)$, an adjunction computation shows that this map may be rewritten as the map

$$\begin{aligned} CH_0(Z_K) \otimes \mathbf{Q} = \mathbf{Chow}^{\mathrm{o}}(K,\mathbf{Q})(\mathbf{1}_K, h^{\mathrm{o}}(Z_K)) \\ \to \mathbf{Chow}^{\mathrm{o}}(L,\mathbf{Q})(\mathbf{1}_L, h^{\mathrm{o}}(Z_L)) = CH_0(Z_L) \otimes \mathbf{Q} \end{aligned}$$

given by extension of scalars. The conclusion now follows from Lemma 2.3.7. □

6. Locally abelian schemes

In this section, F is perfect.

6.1. The Albanese scheme of a smooth projective variety.

6.1.1. Definition. a) Let X be a smooth F-scheme (not necessarily of finite type). For each connected component X_i of X, let E_i be its field of constants, that is, the algebraic closure of F into $F(X_i)$. We define

$$\pi_0(X) = \coprod_i \operatorname{Spec} E_i.$$

There is a canonical F-morphism $X \to \pi_0(X)$; $\pi_0(X)$ is called the *scheme of constants* of X.
b) If $\dim X = 0$ (equivalently $X \xrightarrow{\sim} \pi_0(X)$), we write $\mathbf{Z}[X]$ for the 0-dimensional group scheme representing the étale sheaf $f_*\mathbf{Z}$, where $f : X \to \operatorname{Spec} F$ is the structural morphism.

6.1.2. Definition. a) For an F-group scheme G, we denote by G^0 the kernel of the canonical map $G \to \pi_0(G)$ of Definition 6.1.1: this is the *neutral component* of G.
b) An F-group scheme G is called a *lattice* if $G^0 = \{1\}$ and the geometric fibre of $\pi_0(G)(= G)$ is a free finitely generated abelian group.

6.1.3. Definition ([38])**.** a) Recall that a *semi-abelian variety* is an extension of an abelian variety by a torus. We denote by $\mathbf{SAb}(F)$ the category of semi-abelian varieties, and by $\mathbf{Ab}(F)$ the full subcategory of abelian varieties.
b) We denote by $\mathbf{SAbS}(F)$ the full subcategory of the category of commutative F-group schemes consisting of those objects $\mathcal{A}$ such that

- $\pi_0(\mathcal{A})$ is a lattice;
- $\mathcal{A}^0$ is a semi-abelian variety.

Objects of $\mathbf{SAbS}(F)$ will be called *locally semi-abelian F-schemes*.
c) We denote by $\mathbf{AbS}(F)$ the full subcategory of $\mathbf{SAbS}(F)$ consisting of those $\mathcal{A}$ such that $\mathcal{A}^0$ is an abelian variety. Its objects are called *locally abelian F-schemes*.

For any smooth F-variety X, let $\mathcal{A}_{X/F} = \mathcal{A}_X$ be the Albanese scheme of X over F [38]: it is an object of $\mathbf{SAbS}(F)$ and there is a canonical morphism

$$\varphi_X : X \to \mathcal{A}_X \tag{6.1}$$

which is universal for morphisms from X to objects of $\mathbf{SAbS}(F)$. There is an exact sequence of group schemes

$$0 \to \mathcal{A}_X^0 \to \mathcal{A}_X \to \mathbf{Z}[\pi_0(X)] \to 0$$

where $\mathcal{A}_X^0$ is the Albanese variety of X (a semi-abelian variety) and $\pi_0(X)$ has been defined above.

The aim of this section is to endow the pseudo-abelian category $\mathbf{SAbS}(F)$ and its full subcategory $\mathbf{AbS}(F)$ with symmetric monoidal structures, and to relate the latter one to birational motives (see Propositions 6.2.6 and 7.2.1).

Let us recall from [38] a description of $\mathcal{A}_X$. Let $\mathbf{Z}[X]$ be the "free" presheaf on F-schemes defined by $\mathbf{Z}[X](Y) = \mathbf{Z}[X(Y)]$ and $\mathcal{Z}_{X/F} = \mathcal{Z}_X$ the associated sheaf on the big fppf site of $\operatorname{Spec} F$. Then $\mathcal{A}_X$ is the universal representable quotient of $\mathcal{Z}_X$. In other words, there is a homomorphism

$$\mathcal{Z}_X \to \mathcal{A}_X$$

where $\mathcal{A}_X$ is considered as a representable sheaf, which is universal for homomorphisms from $\mathcal{Z}_X$ to sheaves of abelian groups representable by a locally semi-abelian F-scheme.

Let us also denote by P_X the universal torsor under $\mathcal{A}_X^0$ constructed by Serre [43]. There is a map $X \xrightarrow{\tilde{\varphi}_X} P_X$ which is universal for maps from X to torsors under semi-abelian varieties. The torsor P_X and the group scheme $\mathcal{A}_X$ have the same class in $\operatorname{Ext}^1_{(Sch/F)_{\text{ét}}}(\pi_0(\mathcal{A}_X), \mathcal{A}_X^0) = H^1_{\text{ét}}(\pi_0(X), \mathcal{A}_X^0)$ (here we identify $\mathcal{A}_X^0$ with the corresponding representable étale sheaf over the big étale site of $\operatorname{Spec} F$). A beautiful concrete description of this correspondence is given in [38, 1.2]. The map $\widetilde{\varphi}_X$ induces an isomorphism

$$\mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{P_X}.$$

We repeat some properties of $\mathcal{A}_X$ as taken from [38, Prop. 1.6 and Cor. 1.12] and add one.

6.1.4. Proposition. *a) $\mathcal{A}_X$ is covariant in X.*
b) Let K/F be an extension. Then the natural map

$$\mathcal{A}_{X_K/K} \to \mathcal{A}_{X/F} \otimes_F K$$

stemming from the universal property is an isomorphism.
c) If $X = Y \coprod Z$, then the natural map $\mathcal{A}_{Y/F} \oplus \mathcal{A}_{Z/F} \to \mathcal{A}_{X/F}$ is an isomorphism.
d) Let E/F be a finite extension. For any E-scheme S, let $S_{(F)}$ denote the (ordinary) restriction of scalars of S, i.e. we view S as an F-scheme. Then there is a natural isomorphism for X smooth

$$R_{E/F}\mathcal{A}_{X/E} \to \mathcal{A}_{X_{(F)}/F}$$

where $R_{E/F}$ denotes Weil's restriction of scalars.

Proof. The only thing which is not in [38] is d). We shall construct the isomorphism by descent from c), using b).

Let $f : \operatorname{Spec} E \to \operatorname{Spec} F$ be the structural morphism. Recall that, for any abelian sheaf $\mathcal{G}$ on $(Sch/E)_{\text{ét}}$, the trace map defines an isomorphism [32, Ch. V, Lemma 1.12]

$$f_*\mathcal{G} \xrightarrow{\sim} f_!\mathcal{G}$$

where $f_!$ (*resp.* f_*) is the left (*resp.* right) adjoint of the restriction functor f^*. This isomorphism is natural in $\mathcal{G}$.

This being said, the additive version of Yoneda's lemma immediately yields

$$f_! \mathcal{Z}_{X/E} = \mathcal{Z}_{X_{(F)}/F}$$

hence a composition of homomorphisms of sheaves

$$f_* \mathcal{Z}_{X/E} \xrightarrow{\sim} \mathcal{Z}_{X_{(F)}/F} \to Shv(\mathcal{A}_{X_{(F)}/F}) \tag{6.2}$$

where, for clarity, $Shv(\mathcal{A}_{X_{(F)}/F})$ denotes the sheaf associated to the group scheme $\mathcal{A}_{X_{(F)}/F}$. We also have a chain of homomorphisms

$$f_* \mathcal{Z}_{X/E} \to f_* Shv(\mathcal{A}_{X/E}) \xrightarrow{\sim} Shv(R_{E/F}\mathcal{A}_{X/E}) \tag{6.3}$$

where the last isomorphism is formal. If we can prove that (6.2) factors through (6.3) into an isomorphism, we are done by Yoneda.

In order to do this, we may assume via b) that F is algebraically closed, hence that f is completely split. Then the claim follows from c). □

We record here similar properties for the torsor $P_X = P_{X/F}$ (proofs are similar):

6.1.5. Proposition. *a) $X \mapsto P_X$ is a functor.*
b) Let K/F be an extension. Then the natural map $P_{X_K/K} \to P_{X/F} \otimes_F K$ stemming from the universal property is an isomorphism.
c) If $X = Y \coprod Z$, then there is an isomorphism $P_{Y/F} \times P_{Z/F} \xrightarrow{\sim} P_{X/F}$ which is natural in (Y, Z).
d) Let E/F be a finite extension. Then there is a natural isomorphism

$$P_{X_{(F)}/F} \to R_{E/F} P_{X/E}. \square$$

(In c), the map stems from the fact that coproducts correspond to scheme-theoretic products in an appropriate category of torsors.)

6.2. The tensor category of locally semi-abelian schemes. Recall the Yoneda full embedding $Shv : \mathbf{SAbS}(F) \to Ab((Sch/F)_{\text{ét}})$, where the latter is the category of sheaves of abelian groups over the big étale site of Spec F.

6.2.1. Lemma. *a) If a sheaf $\mathcal{F} \in Ab((Sch/F)_{\text{ét}})$ is an extension of a lattice L by a semi-abelian variety A, then it is represented by an object of $\mathbf{SAbS}(F)$.*
b) Let A be a semi-abelian variety and L a lattice. Then the étale sheaf $B = A \otimes L$ is represented by a semi-abelian variety.

Proof. a) If L is constant, then the choice of a basis of L determines a section of the projection $\mathcal{F} \to Shv(L)$, hence an isomorphism $\mathcal{F} \simeq Shv(A) \oplus Shv(L)$. Then $\mathcal{F}$ is represented by $\coprod_{l \in L} A$. In general, L

becomes constant on some finite extension E/F, hence $\mathcal{F}_E$ is representable. By full faithfulness, the descent data of $\mathcal{F}_E$ are morphisms of schemes; then we may apply [44, Cor. V.4.2 a) or b)].

b) Same method as in a). □

6.2.2. Example. If $L = \mathbf{Z}[\operatorname{Spec} E]$, where E is an étale F-algebra, then $A \otimes L = R_{E/F} A_E$.

Let $\mathcal{A}, \mathcal{B} \in \mathbf{SAbS}(F)$. Viewing them as étale sheaves, we may consider their tensor product $\mathcal{A} \otimes_{shv} \mathcal{B}$. This tensor product contains the subsheaf $\mathcal{A}^0 \otimes_{shv} \mathcal{B}^0$, which is clearly not representable. We define

$$\mathcal{A} \otimes_{\mathrm{rep}} \mathcal{B} = \mathcal{A} \otimes_{shv} \mathcal{B} / \mathcal{A}^0 \otimes_{shv} \mathcal{B}^0.$$

6.2.3. Proposition. *a) $\mathcal{A} \otimes_{\mathrm{rep}} \mathcal{B}$ is representable by an object of* $\mathbf{SAbS}(F)$.
b) For $X, Y \in \mathbf{Sm}(F)$, the natural map

$$\mathcal{Z}_X \otimes_{shv} \mathcal{Z}_Y = \mathcal{Z}_{X \times Y} \to \mathcal{A}_{X \times Y}$$

factors into an isomorphism

$$\mathcal{A}_X \otimes_{\mathrm{rep}} \mathcal{A}_Y \xrightarrow{\sim} \mathcal{A}_{X \times Y}.$$

(This corrects [38, Cor. 1.12 (vi)].*)*

Proof. a) We have a short exact sequence

$$0 \to \mathcal{A}^0 \otimes \pi_0(\mathcal{B}) \oplus \mathcal{B}^0 \otimes \pi_0(\mathcal{A}) \to \mathcal{A} \otimes_{\mathrm{rep}} \mathcal{B} \to \pi_0(\mathcal{A}) \otimes \pi_0(\mathcal{B}) \to 0.$$

By Lemma 6.2.1 b), the left hand side is representable by a semi-abelian variety, and the right hand side is clearly a lattice. We conclude by Lemma 6.2.1 a).

b) It is enough to show that this holds over the algebraic closure of F. Using Proposition 6.1.4 c) (and the similar statement for $\mathcal{Z}$), we may assume that X and Y are connected. We shall show more generally that, for any locally semi-abelian scheme $\mathcal{B}$ and any map $X \times Y \to \mathcal{B}$, the induced sheaf-theoretic map

$$\mathcal{Z}_X \otimes_{shv} \mathcal{Z}_Y \to \mathcal{B} \tag{6.4}$$

factors through $\mathcal{A}_X \otimes_{\mathrm{rep}} \mathcal{A}_Y$. By a), this will show that the latter has the universal property of $\mathcal{A}_{X \times Y}$.

For $n \in \mathbf{Z}$, we shall denote by $\mathcal{Z}_X^n$ or $\mathcal{A}_X^n$ the inverse image of n under the augmentation map $\mathcal{Z}_X \to \mathbf{Z}$ or $\mathcal{A}_X \to \mathbf{Z}$ stemming from the structural morphism $X \to \operatorname{Spec} F$. It is a subsheaf of $\mathcal{Z}_X$ or $\mathcal{A}_X$, and $\mathcal{A}_X^n$ is clearly representable (by a variety $\bar{F}$-isomorphic to the semi-abelian variety $\mathcal{A}_X^0$). We shall also identify varieties with representable sheaves: this should create no confusion in view of Yoneda's lemma.

We first show that (6.4) factors through $\mathcal{A}_X \otimes_{shv} \mathcal{A}_Y$. It suffices to show that the composition

$$\mathcal{Z}_X \times Y \to \mathcal{Z}_X \otimes \mathcal{Z}_Y \to \mathcal{B}$$

factors through $\mathcal{A}_X \times Y$, and to conclude by symmetry. But $X \times Y$ is connected, so its image in $\mathcal{B}$ falls in some connected component $\mathcal{B}^t$ of $\mathcal{B}$, which is a torsor under $\mathcal{B}^0$; applying the "Variation en fonction d'un paramètre" statement in [43, p. 10-05], we see that it extends to a morphism $\mathcal{A}_X^1 \times Y \to \mathcal{B}^t$. Including $\mathcal{B}^t$ into $\mathcal{B}$, we get a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_X^1 \times Y & \longrightarrow & \mathcal{B} \\ \uparrow & & \uparrow \\ \mathcal{Z}_X^1 \times Y & \longrightarrow & \mathcal{Z}_X \times Y. \end{array}$$

Let $\mathcal{K} = \operatorname{Ker}(\mathcal{Z}_X \to \mathcal{A}_X) = \operatorname{Ker}(\mathcal{Z}_X^0 \to \mathcal{A}_X^0)$. The diagram shows that the following diagram

$$\begin{array}{ccc} \mathcal{K} \times \mathcal{Z}_X^1 \times Y & \longrightarrow & \mathcal{Z}_X^1 \times Y \\ \downarrow & & \downarrow \\ \mathcal{Z}_X^1 \times Y & \longrightarrow & \mathcal{B} \end{array}$$

commutes, where the top horizontal map is given by the action of $\mathcal{K}$ on $\mathcal{Z}_X^1$ by left translation and the left vertical map is given by $(k, z, y) \mapsto (z, y)$. Since $\mathcal{Z}_X \times Y \to \mathcal{B}$ is a homomorphism in the first variable, this implies the desired factorisation.

We now show that the composition

$$\mathcal{A}_X^0 \otimes_{shv} \mathcal{A}_Y^0 \to \mathcal{A}_X \otimes \mathcal{A}_Y \to \mathcal{B}$$

is 0. It is sufficient to show that the composition of this map with the inclusion $\mathcal{A}_X^0 \times \mathcal{A}_Y^0 \to \mathcal{A}_X^0 \otimes \mathcal{A}_Y^0$ is 0. But $\mathcal{A}_X^0 \times \mathcal{A}_Y^0$ is connected, hence its image falls in some connected component, in fact in $\mathcal{B}^0$. This map verifies the hypothesis of Corollary B.1.2, hence it is 0. □

As a variant, we have:

6.2.4. Proposition. *$P_{X \times Y} \xrightarrow{\sim} R_{\pi_0(X)/F}(P_Y \times_F \pi_0(X)) \times R_{\pi_0(Y)/F}(P_X \times_F \pi_0(Y))$.*

Since we are not going to use this, we leave the easy proof to the reader.

6.2.5. Remark. In Proposition 6.2.3, the end of the proof of b) is much easier in the case where X and Y are smooth projective: instead of using Corollary B.1.2, it suffices to use [33, Th. 2.1] (rigidity theorem).

Proposition 6.2.3 a) endows $\mathbf{SAbS}(F)$ with a symmetric monoidal structure compatible with its additive structure, hence also its full subcategory $\mathbf{AbS}(F)$. From now on we concentrate on this latter category.

6.2.6. Proposition. *The category $\mathbf{AbS}(F)$ is symmetric monoidal (for $\otimes_{\mathrm{rep}}$) and pseudo-abelian. Its Kelly radical $\mathcal{R}$ is monoidal and has square 0. After tensoring with $\mathbf{Q}$, $\mathbf{AbS}(F)/\mathcal{R}$ becomes isomorphic to the semi-simple category product of the category of abelian varieties up to isogenies and the category of G_F-$\mathbf{Q}$-lattices.*

Recall that the *Kelly radical* $\mathcal{R}$ of an additive catgegory $\mathcal{A}$ is defined by

$$\mathcal{R}(A,B) = \{f \in \mathcal{A}(A,B) \mid \forall g \in \mathcal{A}(B,A)\ 1_A - gf \text{ is invertible}\}$$

and that it is a [two-sided] ideal of $\mathcal{A}$ [25].

Proof. For the first claim, we just observe that kernels exist in the category of commutative F-group schemes, and that a direct summand of an abelian variety (*resp.* of a lattice) is an abelian variety (*resp.* a lattice). For the second claim, consider the functor

$$T : \mathbf{AbS}(F) \to \mathbf{Ab}(F) \times \mathbf{Lat}(F)$$
$$\mathcal{A} \mapsto (\mathcal{A}^0, \pi_0(\mathcal{A}))$$

where $\mathbf{Ab}(F)$ and $\mathbf{Lat}(F)$ are respectively the category of abelian varieties and the category of lattices over F (viewed, for example, as full subcategories of the category of étale sheaves over Sm/F). This functor is obviously essentially surjective. After tensoring with $\mathbf{Q}$, it becomes full, because any extension

$$0 \to \mathcal{A}^0 \to \mathcal{A} \to \pi_0(\mathcal{A}) \to 0$$

is rationally split. Now the collection of sets

$$\mathcal{I}(\mathcal{A},\mathcal{B}) = \{f : \mathcal{A} \to \mathcal{B} \mid T(f) = 0\}$$

defines an ideal $\mathcal{I}$ of $\mathbf{AbS}(F)$. If $f \in \mathcal{I}(\mathcal{A},\mathcal{B})$, then f induces a map

$$\bar{f} : \pi_0(\mathcal{A}) \to \mathcal{B}^0$$

and this gives a description of $\mathcal{I}$. From this description, it follows immediately that $\mathcal{I}^2 = 0$. In particular, $\mathcal{I} \subseteq \mathcal{R}$.

If we tensor with $\mathbf{Q}$, then $\mathbf{Ab}(F) \times \mathbf{Lat}(F)$ becomes semi-simple; since $\mathbf{AbS}(F)/\mathcal{I} \otimes \mathbf{Q}$ is semi-simple and $\mathcal{I} \otimes \mathbf{Q}$ is nilpotent, it follows that $\mathcal{I} \otimes \mathbf{Q} = \mathcal{R} \otimes \mathbf{Q}$. In other words, $\mathcal{R}/\mathcal{I}$ is torsion.

Let $f \in \mathcal{R}(\mathcal{A},\mathcal{B})$. There exists $n > 0$ such that $nf(\mathcal{A}^0) = 0$. But $f(\mathcal{A}^0)$ is an abelian subvariety of $\mathcal{B}^0$, hence $f(\mathcal{A}^0) = 0$ and $f \in \mathcal{I}(\mathcal{A},\mathcal{B})$. So $\mathcal{R} = \mathcal{I}$.

If we endow the category $\mathbf{Ab}(F)\times\mathbf{Lat}(F)$ with the tensor structure

$$(A,L)\otimes(B,M)=(A\otimes M\oplus B\otimes L, L\otimes M)$$

then T becomes a monoidal functor, which shows that $\mathcal{R}=\mathcal{I}$ is monoidal. This completes the proof of Proposition 6.2.6. $\square$

6.2.7. Remarks. a) The morphisms in $\mathbf{AbS}(F)$ are best represented in matrix form:

$$\operatorname{Hom}(\mathcal{A},\mathcal{B})=\begin{pmatrix}\operatorname{Hom}(\mathcal{A}_0,\mathcal{B}_0) & \operatorname{Hom}(\pi_0(\mathcal{A}),\mathcal{B}_0)\\ 0 & \operatorname{Hom}(\pi_0(\mathcal{A}),\pi_0(\mathcal{B}))\end{pmatrix}$$

(note that $\operatorname{Hom}(\mathcal{A}_0,\pi_0(\mathcal{B}))=0$). This clarifies the arguments in the proof of Proposition 6.2.6 somewhat.

b) The Hom groups of $\mathbf{Ab}(F)\times\mathbf{Lat}(F)$ are finitely generated $\mathbf{Z}$-modules. It follows from the proof of Proposition 6.2.6 that, for $\mathcal{A},\mathcal{B}\in\mathbf{AbS}(F)$, $T(\operatorname{Hom}(\mathcal{A},\mathcal{B}))$ has finite index in $\operatorname{Hom}(T(\mathcal{A}),T(\mathcal{B}))$. In particular, for any $\mathcal{A}\in\mathbf{AbS}(F)$, $\operatorname{End}(\mathcal{A})$ is an extension of an order in a semi-simple $\mathbf{Q}$-algebra by an ideal of square 0.

c) The functor T has the explicit section

$$(A,L)\mapsto A\oplus L.$$

This section is symmetric monoidal.

7. Chow birational motives and locally abelian schemes

7.1. The Albanese map. For any smooth projective variety X, there is a canonical map

$$CH_0(X)\xrightarrow{\operatorname{Alb}_X^F}\mathcal{A}_X(F). \tag{7.1}$$

Recall the construction of Alb_X: the map φ_X of (6.1) defines for any extension E/F a map $X(E)\to\mathcal{A}_X(E)$, still denoted by φ_X. When E/F is finite, viewing $\mathcal{A}_X$ as an étale sheaf, we have a trace map $Tr_{E/F}:\mathcal{A}_X(E)\to\mathcal{A}_X(F)$. Then Alb_X maps the class of a closed point $x\in X$ with residue field E to $Tr_{E/F}\,\varphi_X(x)$.

The map Alb_X is injective for $\dim X=1$ and surjective if F is algebraically closed. For a curve, this map corresponds to the isomorphism $\operatorname{Pic}_X\simeq\mathcal{A}_X$, where Pic_X is the Picard scheme of X; we then also have $\mathcal{A}_X^0\simeq J_X$, where J_X is the Jacobian variety of X.

The functoriality of $\mathcal{A}$ shows that there is a chain of isomorphisms

$$\Phi_{X,Y}:\operatorname{Hom}(\mathcal{A}_X,\mathcal{A}_Y)\xrightarrow{\sim}\operatorname{Mor}(X,\mathcal{A}_Y)\xrightarrow{\sim}\mathcal{A}_Y(F(X)) \tag{7.2}$$

(the latter by Weil's theorem on extension of morphisms to abelian varieties [33, Th. 3.1]), hence a canonical map

$$CH_0(Y_{F(X)}) \xrightarrow{\mathrm{Alb}_{X,Y}} \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y) \tag{7.3}$$

which generalises (7.1); more precisely, we have

$$\Phi_{X,Y} \circ \mathrm{Alb}_{X,Y} = \mathrm{Alb}_Y^{F(X)}. \tag{7.4}$$

On the other hand, there is an exact sequence

$$\begin{aligned} 0 \to \mathcal{A}_Y(\pi_0(X)) = \mathrm{Hom}(\mathbf{Z}[\pi_0(X)], \mathcal{A}_Y) \to \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y) \\ \to \mathrm{Hom}(\mathcal{A}_X^0, \mathcal{A}_Y) \to \mathrm{Ext}^1(\mathbf{Z}[\pi_0(X)], \mathcal{A}_Y) = H^1(\pi_0(X), \mathcal{A}_Y) \end{aligned}$$

and the map $\mathrm{Hom}(\mathcal{A}_X^0, \mathcal{A}_Y^0) \to \mathrm{Hom}(\mathcal{A}_X^0, \mathcal{A}_Y)$ is an isomorphism. From this we get a zero sequence

$$0 \to CH_0(Y) \to CH_0(Y_{F(X)}) \to \mathrm{Hom}(\mathcal{A}_X^0, \mathcal{A}_Y^0) \to 0. \tag{7.5}$$

7.1.1. Lemma. *Let Y, Z be two smooth projective varieties and $\beta \in CH_0(Z_{F(Y)})$. Then the following diagram commutes:*

$$\begin{array}{ccc} CH_0(Y) & \xrightarrow{\beta_*} & CH_0(Z) \\ {\scriptstyle \mathrm{Alb}_Y^F}\downarrow & & {\scriptstyle \mathrm{Alb}_Z^F}\downarrow \\ \mathcal{A}_Y(F) & \xrightarrow{\mathrm{Alb}_{Y,Z}(\beta)_*} & \mathcal{A}_Z(F). \end{array}$$

Proof. Without loss of generality, we may assume that β is given by an integral subscheme W in $Y \times Z$. Then the composite $f = p_Y i_W$ is a proper surjective generically finite morphism, where p_Y denotes the projection and i_W is the inclusion of W in $Y \times Z$.

Let V be an affine dense open subset of Y such that $f_{|f^{-1}(V)}$ is finite. Any element of $CH_0(Y)$ may be represented by a zero-cycle with support in V (*cf.* [39]), so it is enough to check the commutativity of the diagram on zero-cycles on Y of the form y, where $y \in V_{(0)}$. For such a y, we have $\beta_* y = p_*(f^{-1}(y))$, where $p = p_Z i_W$.

On the other hand, the composition $\mathrm{Alb}_{Y,Z}(\beta)_* \circ (\mathrm{Alb}_Y^F)_{|V}$ may be described as follows: let d be the degree of $f_{|f^{-1}(V)}$, $f^{-1}(V)^{[d]}$ the d-fold symmetric power of $f^{-1}(V)$ and $f^* : V \to f^{-1}(V)^{[d]}$ the map $x \mapsto f^{-1}(x)$. Then

$$\mathrm{Alb}_{Y,Z}(\beta)_* \circ (\mathrm{Alb}_Y^F)_{|V} = \Sigma_d \circ (\varphi_Z)^{[d]} \circ p_*^{[d]} \circ f^*$$

where $\Sigma_d : \mathcal{A}_Z^{[d]} \to \mathcal{A}_Z$ is the summation map. The commutativity of the diagram is now clear. □

7.2. The Albanese functor.

7.2.1. Proposition. *The assignment* $X \mapsto \mathcal{A}_X$ *defines via* (7.3) *a symmetric monoidal additive functor*

$$\mathrm{Alb} : \mathbf{Chow}^o(F) \to \mathbf{AbS}(F)$$

which becomes full and essentially surjective after tensoring with $\mathbf{Q}$.

Proof. Since $\mathbf{AbS}(F)$ is pseudo-abelian, it suffices to construct the functor on $\mathbf{Cor}^o(F)$. Let $\alpha \in CH_0(Y_{F(X)})$ and $\beta \in CH_0(Z_{F(Y)})$. We want to show that $\mathrm{Alb}_{X,Z}(\beta \circ \alpha) = \mathrm{Alb}_{Y,Z}(\beta) \circ \mathrm{Alb}_{X,Y}(\alpha)$. But β induces a map

$$\beta_* : CH_0(Y_{F(X)}) \to CH_0(Z_{F(X)}),$$

and we have the equality $\beta_*\alpha = \beta \circ \alpha$ (*cf.* proof of Proposition 2.3.4). Hence, applying Lemma 7.1.1 in which we replace F by $F(X)$, we get

$$\mathrm{Alb}_Z^{F(X)}(\beta \circ \alpha) = \mathrm{Alb}_Z^{F(X)}(\beta_*\alpha) = \mathrm{Alb}_{Y,Z}(\beta)_*(\mathrm{Alb}_Y^{F(X)}(\alpha)).$$

Applying now (7.4), we get

$$\Phi_{X,Z} \circ \mathrm{Alb}_{X,Z}(\beta \circ \alpha) = \mathrm{Alb}_{Y,Z}(\beta)_*(\Phi_{X,Y} \circ \mathrm{Alb}_{X,Y}(\alpha)).$$

On the other hand, the diagram

$$\begin{array}{ccc} \mathcal{A}_Y(F(X)) & \xrightarrow{\mathrm{Alb}_{Y,Z}(\beta)_*} & \mathcal{A}_Z(F(X)) \\ {\scriptstyle \Phi_{X,Y}}\big\uparrow\wr & & {\scriptstyle \Phi_{X,Z}}\big\uparrow\wr \\ \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y) & \xrightarrow{\mathrm{Alb}_{Y,Z}(\beta)_*} & \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y) \end{array}$$

obviously commutes, which concludes the proof that Alb is a functor.

Compatibility with the monoidal structures follows from Proposition 6.2.3 b). It remains to show the assertions on fullness and essential surjectivity.

Fullness: for any Y, the map $\mathrm{Alb}_Y^F \otimes \mathbf{Q}$ is surjective. This follows from the case where F is algebraically closed (in which case Alb_Y^F itself is surjective) by a transfer argument. Replacing the ground field F by $F(X)$ for some other X, we get that $\mathrm{Alb}_{X,Y} \otimes \mathbf{Q}$ is surjective. This shows that the restriction of $\mathrm{Alb} \otimes \mathbf{Q}$ to $\mathbf{Cor}^o(F) \otimes \mathbf{Q}$ is full; but the pseudo-abelianisation of a full functor is evidently full (a direct summand of a surjective homomorphism of abelian groups is surjective).

Essential surjectivity: we first note that, after tensoring with $\mathbf{Q}$, the extension

$$0 \to \mathcal{A}^0 \to \mathcal{A} \to \pi_0(\mathcal{A}) \to 0$$

becomes split for any $\mathcal{A}\in \mathbf{AbS}(F)$. Indeed the extension class belongs to $\operatorname{Ext}^1_F(\pi_0(\mathcal{A}),\mathcal{A}^0)$; this group sits in an exact sequence (coming from an Ext spectral sequence)

$$0\to H^1(F,\operatorname{Hom}_{\bar F}(\pi_0(\mathcal{A})_{|\bar F},\mathcal{A}^0_{|\bar F}))\to \operatorname{Ext}^1_F(\pi_0(\mathcal{A}),\mathcal{A}^0)\\ \to H^0(F,\operatorname{Ext}^1_{\bar F}(\pi_0(\mathcal{A})_{|\bar F},\mathcal{A}^0_{|\bar F})).$$

Since the restriction $\pi_0(\mathcal{A})_{|\bar F}$ is a constant sheaf of free finitely generated abelian groups, the group $\operatorname{Ext}^1_{\bar F}(\pi_0(\mathcal{A})_{|\bar F},\mathcal{A}^0_{|\bar F})$ is 0, while the left group is torsion as a Galois cohomology group. It is now sufficient to show separately that L and A are in the essential image of $\operatorname{Alb}\otimes\mathbf{Q}$, where L (*resp.* A) is a lattice (*resp.* an abelian variety).

A lattice L corresponds to a continuous integral representation ρ of G_F. But it is well-known that $\rho\otimes\mathbf{Q}$ is of the form $\theta\otimes\mathbf{Q}$, where θ is a direct summand of a permutation representation of G_F. If E is the corresponding étale algebra, we therefore have an isomorphism of L with a direct summand of $(\operatorname{Alb}\otimes\mathbf{Q})(E)$.

Given an abelian variety A, we simply note that

$$A=\operatorname{Alb}(\widetilde{h}(A))$$

where $\widetilde{h}(A)$ is the reduced motive of A: $h(A)=\mathbf{1}\oplus\widetilde{h}(A)$, where the splitting is given by the rational point $0\in A(F)$. □

7.2.2. Remark. Let $\mathcal{R}$ be the Kelly radical of $\mathbf{AbS}(F)$ (*cf.* Proposition 6.2.6). If F is a finitely generated field, the groups $\mathcal{R}(\mathcal{A},\mathcal{B})$ are finitely generated by the Mordell-Weil-Néron theorem. To see this, note that if L is a lattice and A an abelian variety, then

$$\operatorname{Hom}(L,A)\xrightarrow{\sim}\operatorname{Hom}(L_{|\bar F},A_{|\bar F})^{G_F}$$

and that the right term may be rewritten as $B(F)$, where $B=L^*\otimes A$ (compare Lemma 6.2.1). Hence the Hom groups in $\mathbf{AbS}(F)$ are finitely generated as well. In this case, Proposition 7.2.1 implies that, for any $M,N\in\mathbf{Chow}^{\mathrm{o}}(F)$, *the image of the map* $\operatorname{Alb}_{M,N}$ *has finite index in the group* $\operatorname{Hom}(\operatorname{Alb}(M),\operatorname{Alb}(N))$.

7.2.3. Lemma. *Suppose that Y is a curve. Then the map* (7.3) *fits into an exact sequence*

$$0\to CH_0(Y_{F(X)})\xrightarrow{\operatorname{Alb}_{X,Y}}\operatorname{Hom}(\mathcal{A}_X,\mathcal{A}_Y)\\ \to Br(F(X))\to Br(F(X\times Y))$$

where Br denotes the Brauer group. In particular, (7.3)$\otimes\mathbf{Q}$ *is an isomorphism.*

Proof. In view of the construction of (7.3), we may assume that X is a point; then (7.3) reduces to (7.1). Suppose first that F is separably closed. Then (7.1) is bijective (see comments at the beginning of this section). In the general case, let F_s be a separable closure of F, and $G = Gal(F_s/F)$. Since $\mathcal{A}_Y$ is a sheaf for the étale topology, we get a commutative diagram

$$\begin{array}{ccc} CH_0(Y_s)^G & \xrightarrow[\sim]{\mathrm{Alb}_Y^{F_s}} & \mathcal{A}_Y(F_s)^G \\ \uparrow & & \wr\uparrow \\ CH_0(Y) & \xrightarrow{\mathrm{Alb}_Y^F} & \mathcal{A}_Y(F) \end{array}$$

where $Y_s = Y \times_F F_s$ and the top horizontal and right vertical maps are bijective. The lemma then follows from the classical exact sequence

$$0 \to CH_0(Y) \to CH_0(Y_s)^G \to Br(F(X)) \to Br(F(X \times Y)).$$

□

7.2.4. Proposition. *Let* $\mathbf{Chow}^{\mathrm{o}}_{\leq 1}(F)$ *denote the thick subcategory of* $\mathbf{Chow}^{\mathrm{o}}(F)$ *generated by motives of varieties of dimension* ≤ 1*, and let* $\iota : \mathbf{Chow}^{\mathrm{o}}_{\leq 1}(F) \to \mathbf{Chow}^{\mathrm{o}}(F)$ *be the canonical inclusion. Then*
a) After tensoring morphisms by $\mathbf{Q}$*,* $\mathrm{Alb} \circ \iota : \mathbf{Chow}^{\mathrm{o}}_{\leq 1}(F) \to \mathbf{AbS}(F)$ *becomes an equivalence of categories.*
b) Let j *be a quasi-inverse. Then* $\iota \circ j$ *is right adjoint to* Alb*.*

Proof. a) The full faithfulness follows from Lemma 7.2.3. For the essential surjectivity, we may reduce as in the proof of Proposition 7.2.1 to proving that lattices and abelian varieties are in the essential image. For lattices, this is proven in *loc. cit.* . For an abelian variety A, use the fact that A is isogenous to a quotient of the Jacobian of a curve, and Poincaré's complete reducibility theorem.

b) Let $(M, \mathcal{A}) \in \mathbf{Chow}^{\mathrm{o}}_{\leq 1}(F) \times \mathbf{AbS}(F)$. To produce a natural isomorphism $\mathbf{Chow}^{\mathrm{o}}_{\leq 1}(F)(M, \iota j(\mathcal{A})) \simeq \mathbf{AbS}(F)(\mathrm{Alb}(M), \mathcal{A}))$, it is sufficient by a) to handle the case $M = h^{\mathrm{o}}(X), \mathcal{A} = \mathcal{A}_Y$ for some smooth projective varieties X, Y with $\dim Y \leq 1$. Then the isomorphism follows from the adjunctions (7.2) and from Lemma 7.2.3. □

7.2.5. Remarks. a) Of course the functor $\iota \circ j$ is not a tensor functor (since its image is not closed under tensor product).
b) In particular, the inclusion functor ι has the left-adjoint-left inverse $j \circ \mathrm{Alb}$. This is a birational version of Murre's results for effective Chow motives ([34], [35, §2.1], see also [41, §4], and in the triangulated context [48, §3.4]). Beware however that we have taken the opposite

to the usual convention for the variance of Chow motives (our functor $X \mapsto h(X)$ is covariant rather than contravariant), so the direction of arrows has to be reversed with respect to Murre's work.

Appendix A. Complements on localisation of categories

A.1. Localisation of symmetric monoidal categories.

A.1.1. Lemma. *a) Localisation commutes with products of categories.*
b) Let $T_0, T_1 : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors and $f : T_0 \Rightarrow T_1$ a natural transformation. Let S, S' be collections of morphisms in $\mathcal{C}$ and $\mathcal{D}$ such that $T_i(S) \subseteq S'$, so that T_0 and T_1 pass to localisation. Then f remains a natural transformation between the localised functors.

Proof. a) is clear (*cf.* [29, Lemma 2.1.7], and b) is true because f commuted with the members of S, hence it now commutes with their inverses. □

A.1.2. Proposition. *Let $\mathcal{C}$ be a category with a product $\bullet : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and let S be a collection of morphisms in $\mathcal{C}$. Assume that $S \bullet S \subseteq S$. Then*
a) There is a unique product $S^{-1}\mathcal{C} \times S^{-1}\mathcal{C} \to S^{-1}\mathcal{C}$ such that the localisation functor $P_S : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with the two products.
b) If $\bullet$ is monoidal (resp. braided, symmetric, unital), the induced product on $S^{-1}\mathcal{C}$ enjoys the same properties and P_S is monoidal (resp. braided, symmetric, unital).

Proof. a) follows from Lemma A.1.1 a); b) follows from Lemma A.1.1 b). □

A.2. Semi-additive categories. This subsection is a reformulation of [28, Ch. VIII, §2], see also [27, §18 and beginning of §19].

A.2.1. Lemma. *a) For a category $\mathcal{A}$, the following conditions are equivalent:*

(i) *$\mathcal{A}$ has a 0 object (initial and final), binary products and coproducts, and for any $A, B \in \mathcal{A}$, the map*
$$A \coprod B \to A \times B$$
given on A by $(1_A, 0)$ and on B by $(0, 1_B)$ is an isomorphism.
(ii) *$\mathcal{A}$ has finite products, and for any $A, B \in \mathcal{A}$, $\mathcal{A}(A, B)$ has a structure of a commutative monoid, and composition is distributive with respect to these monoid laws.*
(iii) *Same as* (ii)*, replacing product by coproduct.*

We then say that $\mathcal{A}$ is a semi-additive category *and write $A \oplus B$ for the product or coproduct of two objects A, B.*
b) If $\mathcal{A}$ is a semi-additive category, the law $(A, B) \mapsto A \oplus B$ endows $\mathcal{A}$ with a canonical unital symmetric monoidal structure.

Proof. a) By duality, we only need to show (i) $\iff$ (ii). (i) $\Rightarrow$ (ii) follows from [28, Ch. VIII, §2, ex. 4 (a)]: recall that for two morphisms $f, g : A \to B$ in $\mathcal{A}$, Mac Lane defines their sum $f+g$ as the composition

$$\begin{array}{ccccc} A & \xrightarrow{\Delta_A} & A \times A & & \\ & & & \searrow^{f\times g} & \\ \Big\downarrow{\scriptstyle f+g} & & & & B \times B \\ & & & \nearrow_{\sim} & \\ B & \xleftarrow{\nabla_B} & B \coprod B & & \end{array}$$

where Δ_A is the diagonal and ∇_B is the codiagonal.

As for (ii) $\Rightarrow$ (i), it is implicit in the proof of [28, Ch. VIII, §2, Th. 2]. Indeed, Mac Lane defines a biproduct of two objects $A, B \in \mathcal{A}$ as a diagram

$$A \underset{i_1}{\overset{p_1}{\leftrightarrows}} C \underset{i_2}{\overset{p_2}{\leftrightarrows}} B$$

satisfiyng $p_1 i_1 = 1_A$, $p_2 i_2 = 1_B$ and $i_1 p_1 + i_2 p_2 = 1_C$. Let us say that such a diagram is a *biproduct** if the further identities $p_1 i_2 = 0$ and $p_2 i_1 = 0$ hold. Then, Mac Lane proves that a biproduct* is a product and that a product is a biproduct*. Dually, a biproduct* is the same as a coproduct, hence binary products and coproducts are canonically isomorphic, and one checks from his proof that the isomorphism is given by the map of (i).

(Let us clarify that Mac Lane proves that a biproduct is a biproduct* if the addition law on morphisms has the cancellation property; but we don't use this part of his proof.)

b) This is obvious: already finite products or coproducts define a canonical symmetric monoidal structure. □

Define a *semi-additive functor* between two semi-additive categories $\mathcal{A}, \mathcal{B}$ as a functor $F : \mathcal{A} \to \mathcal{B}$ which preserves addition of morphisms. Note that any semi-additive functor preserves $\oplus$, by the characterisation of biproducts via equations (see proof of Lemma A.2.1 a)).

A.3. Localisation of R-linear categories.

A.3.1. Theorem. *Let $\mathcal{A}$ be a semi-additive category and S a family of morphisms of $\mathcal{A}$, stable under direct sum. Then there exists a unique structure of semi-additive category on $S^{-1}\mathcal{A}$ such that the localisation functor $P_S : \mathcal{A} \to S^{-1}\mathcal{A}$ is semi-additive.*

Proof. We use the characterisation (i) of semi-additive categories in Lemma A.2.1: by [29, 1.3.6 and 2.1.8], P_S preserves products and coproducts, and transforms the isomorphisms $A \coprod B \xrightarrow{\sim} A \times B$ into isomorphisms. □

To "catch" additive categories (as opposed to semi-additive categories), we could do as in Mac Lane [27] and postulate the existence of an endomorphism -1_A for each object A. We prefer to do this more generally by dealing with R-linear categories, where R is an arbitrary ring (an R-linear category is simply a semi-additive R-category).

More precisely, let $\mathcal{A}$ be an R-linear category. Then in particular:

- $\mathcal{A}$ is a semi-additive category.
- It enjoys an action of the multiplicative monoid underlying R, *i.e.* there is a homomorphism of monoids $R \to \operatorname{End}(Id_A)$, where $\operatorname{End}(Id_A)$ is the monoid of natural transformations of the identity functor of A.
- For $\lambda \in R$ and $A \in \mathcal{A}$, let λ_A denote the corresponding endomorphism of A. Then we have identities

$$(\lambda + \mu)_A = \lambda_A + \mu_A. \tag{A.1}$$

Conversely, the following lemma is straightforward.

A.3.2. Lemma. *Let $\mathcal{A}$ be a semi-additive category provided with an action of R verifying* (A.1)*. Then $\mathcal{A}$ is an R-linear category.* □

From this lemma, it follows:

A.3.3. Theorem. *Theorem A.3.1 extends to R-linear categories.* □

A.4. Localisation and pseudo-abelian envelope.

A.4.1. Lemma. *Let $\mathcal{A}$ an additive category and S a family of morphisms in $\mathcal{A}$, stable under direct sums. Let $\mathcal{A} \to \mathcal{A}^\natural$ denote the pseudo-abelian envelope of $\mathcal{A}$, and let us still denote by S the image of S in $\mathcal{A}^\natural$. Then the natural functor*

$$(S^{-1}\mathcal{A})^\natural \to (S^{-1}(\mathcal{A}^\natural))^\natural$$

is an equivalence of categories.

Proof. Both categories are universal for additive functors T from $\mathcal{A}$ to a pseudo-abelian category such that $T(S)$ is invertible. □

Appendix B. Rigidity for semi-abelian varieties

B.1. We include the following results for lack of reference.

B.1.1. Proposition. *Let F be an algebraically closed field and G be a semi-abelian F-variety. Then the map*

$$F^* \times \operatorname{Hom}_F(G, \mathbb{G}_m) \to \operatorname{Mor}_F(G, \mathbb{G}_m)$$

sending (λ, φ) to $\lambda\varphi$ is bijective.

Proof. Write G as an extension $0 \to T \to G \xrightarrow{\pi} A \to 0$, where T is a torus and A is an abelian variety. Then $\pi : G \to A$ is a T-torsor. Since F is algebraically closed, T is split, which implies that this T-torsor is locally trivial for the Zariski topology.

Choose an isomorphism $\mathbb{G}_m^r \xrightarrow{\sim} T$. Let U be an affine open subset of A such that we have a T-isomorphism $\varphi_U : U \times T \xrightarrow{\sim} \pi^{-1}(U)$. Write $U = \operatorname{Spec} R$. Since R is a domain,

$$\operatorname{Mor}(U \times T, \mathbb{G}_m) \simeq \operatorname{Hom}_F(F[t, t^{-1}], R[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]) \simeq R^* \times X(T)$$

where $X(T)$ is the group of characters of T. This shows that there is an exact sequence

$$0 \to \operatorname{Mor}_F(U, \mathbb{G}_m) \xrightarrow{\pi^*} \operatorname{Mor}(\pi^{-1}(U), \mathbb{G}_m) \to X(T) \to 0.$$

Interpreting $\operatorname{Mor}_F(X, \mathbb{G}_m)$ as $\Gamma(X, \mathcal{O}_X^*)$, we may translate this as an exact sequence of Zariski sheaves on A (with $X(T)$ a constant sheaf)

$$0 \to \mathcal{O}_A^* \to \pi_* \mathcal{O}_G^* \to X(T) \to 0. \tag{B.1}$$

Taking the cohomology of this exact sequence, and considering the Hom – Ext-exact sequence of [44, Ch. VII, Prop. 2], we get a diagram of exact sequences

(B.2)

$$\begin{array}{ccccccccc}
0 \to & \operatorname{Hom}_F(A, \mathbb{G}_m) & \to & \operatorname{Hom}_F(G, \mathbb{G}_m) & \to & \operatorname{Hom}_F(T, \mathbb{G}_m) & \xrightarrow{\delta_G} & \operatorname{Ext}^1_F(A, \mathbb{G}_m) \\
& \downarrow & & \downarrow & & & & \\
0 \to & \operatorname{Mor}_F(A, \mathbb{G}_m) & \to & \operatorname{Mor}_F(G, \mathbb{G}_m) & \to & \Gamma(A, X(T)) & \xrightarrow{\delta'_G} & H^1_{\text{Zar}}(A, \mathcal{O}_A^*).
\end{array}$$

The groups $\operatorname{Hom}_F(T, \mathbb{G}_m)$ and $\Gamma(A, X(T))$ both coincide with $X(T)$, while we have isomorphisms $\operatorname{Ext}^1_F(A, \mathbb{G}_m) \simeq \operatorname{Pic}^0(A)$ and $H^1_{\text{Zar}}(A, \mathcal{O}_A^*) \simeq \operatorname{Pic}(A)$. The maps

$$\begin{array}{ccc}
X(T) & & \operatorname{Pic}^0(A) \\
{=}\downarrow & & \downarrow{\scriptstyle \text{inclusion}} \\
X(T) & & \operatorname{Pic}(A)
\end{array}$$

complete (B.2) into
(B.3)

$$\begin{array}{ccccccc} 0\to\operatorname{Hom}_F(A,\mathbb{G}_m) & \to & \operatorname{Hom}_F(G,\mathbb{G}_m) & \to & \operatorname{Hom}_F(T,\mathbb{G}_m) & \xrightarrow{\delta_G} & \operatorname{Ext}^1_F(A,\mathbb{G}_m)\\ \downarrow & & \downarrow & & \downarrow & & \downarrow\\ 0\to\operatorname{Mor}_F(A,\mathbb{G}_m) & \to & \operatorname{Mor}_F(G,\mathbb{G}_m) & \to & \Gamma(A,X(T)) & \xrightarrow{\delta'_G} & H^1_{\mathrm{Zar}}(A,\mathcal{O}_A^*). \end{array}$$

Note also that $\operatorname{Hom}_F(A,\mathbb{G}_m)=0$ and $\operatorname{Mor}_F(A,\mathbb{G}_m)=F^*$ (the latter because A is proper and connected). If we can show that (B.3) commutes, then a diagram chase implies that the map

$$\operatorname{Hom}_F(G,\mathbb{G}_m)\to\operatorname{Mor}_F(G,\mathbb{G}_m)/F^*$$

is bijective, which completes the proof of Proposition B.1.1.

Let us therefore prove that the diagram (B.3) is commutative. Commutation of the left square is clear. For the middle square, let $\varphi\in\operatorname{Hom}_F(G,\mathbb{G}_m)$. Its image χ in $\operatorname{Hom}_F(T,\mathbb{G}_m)$ is just restriction to T. Its image φ' in $\operatorname{Mor}_F(G,\mathbb{G}_m)$ is just φ viewed as a morphism of varieties. To compute the image φ'' of φ' in $\Gamma(A,X(T))$, we may restrict φ' to an open subset of the form $\pi^{-1}(U)$, where U is an affine neighbourhood of 0 in A such that there is a T-isomorphism $U\times T\xrightarrow{\sim}\pi^{-1}(U)$. Then the construction of the exact sequence (B.1) shows that φ'' is given by the restriction of φ' to $\{0\}\times T$: this shows that $\varphi''=\chi$, hence that the middle square commutes.

Finally, we show that the square on the right in (B.3) commutes. Let $\chi:T\to\mathbb{G}_m$ be a character. Let $G'=\chi_*G$ be the extension of A by $\mathbb{G}_m$ obtained as the push-out of G by χ, and let $\chi^*\pi_*\mathcal{O}_G$ be the extension of $\mathbf{Z}$ by $\mathcal{O}_A^*$ obtained as the pull-back of $\pi_*\mathcal{O}_G$ by χ. Let $p:G\to G'$ and $\pi':G'\to A$ be the induced maps. Then the canonical morphism of G'-sheaves

$$\mathcal{O}_{G'}^*\to p_*\mathcal{O}_G^*$$

induces a morphism of A-sheaves $\pi'_*\mathcal{O}_{G'}^*\to\pi_*\mathcal{O}_G^*$ fitting into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_A^* & \longrightarrow & \pi_*\mathcal{O}_G^* & \longrightarrow & X(T) & \longrightarrow & 0\\ & & \| \uparrow & & \uparrow & & \chi\uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_A^* & \longrightarrow & \pi'_*\mathcal{O}_{G'}^* & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \end{array}$$

which in turn induces an isomorphism

$$\pi'_*\mathcal{O}G'\xrightarrow{\sim}\chi^*\pi_*\mathcal{O}_G.$$

Since δ_G and δ'_G are connecting morphisms from the Ext exact sequences respectively associated to the extension of algebraic groups G

and the extension of sheaves $\pi_*\mathcal{O}_G^*$, it is then clear that the diagrams

$$\begin{array}{ccc} \mathrm{Hom}(T,\mathbb{G}_m) & \xrightarrow{\delta_G} & \mathrm{Ext}^1(A,\mathbb{G}_m) \\ {\scriptstyle\chi^*}\uparrow & & \|\uparrow \\ \mathrm{Hom}(\mathbb{G}_m,\mathbb{G}_m) & \xrightarrow{\delta_{\chi_* G}} & \mathrm{Ext}^1(A,\mathbb{G}_m) \end{array} \qquad \begin{array}{ccc} \Gamma(A,X(T)) & \xrightarrow{\delta'_G} & H^1(A,\mathcal{O}_A^*) \\ {\scriptstyle\chi_*}\downarrow & & \|\downarrow \\ \Gamma(A,\mathbf{Z}) & \xrightarrow{\delta'_{\chi_* G}} & H^1(A,\mathcal{O}_A^*) \end{array}$$

commute. This reduces us to the case where $T = \mathbb{G}_m$ and χ is the identity character 1.

But then, $\delta(1)$ is the extension class of G while $\delta'(1)$ is the class of G considered as a $\mathbb{G}_m$-bundle over A, and the indentification of $\mathrm{Ext}^1(A,\mathbb{G}_m)$ with $\mathrm{Pic}^0(A)$ says precisely that these classes coincide. □

B.1.2. Corollary. *Let F be a field, G_1, G_2, G_3 be three semi-abelian F-varieties, and let $\varphi : G_1 \times G_2 \to G_3$ be an F-morphism. Assume that $\varphi(g_1, 0) = \varphi(0, g_2) = 0$ identically. Then $\varphi = 0$.*

Proof. Clearly, we may assume F to be algebraically closed. Write G_3 as an extension

$$0 \to T_3 \to G_3 \to A_3 \to 0$$

where T_3 is a torus and A_3 is an abelian variety.

Composing φ with the projection to A_3 and applying [33, Th. 3.4], we find that the image of φ is contained in T_3. We may further reduce to $T_3 = \mathbb{G}_m$.

By Proposition B.1.1, φ is the sum of a constant morphism and a homomorphism. Since $\varphi(0) = 0$, it is a homomorphism. But then

$$\varphi(g_1, g_2) = \varphi(g_1, 0) + \varphi(0, g_2) = 0.$$

□

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