Relative reciprocities on Dedekind domains and the K_2 of a field

Frans Keune

March 6, 2009

The author's proof [3] of Matsumoto's presentation of the K_2 of a field was based on Bass' description [1] of the SK_1 of an ideal in a Dedekind domain by relative reciprocities together with a reciprocity law for the Milnor ring of a field. Here it is shown that with the use of general exact sequences in algebraic K-theory this description of the SK_1 of an ideal in a Dedekind domain is easily derived. Matsumoto's theorem can be derived from this description as in [3]. A proof that does not use the Milnor ring is given here as well.

Introduction

The SK_1 of a Dedekind domain can be described by Mennicke symbols and also by means of reciprocities. More generally this holds for the SK_1 of an ideal of a Dedekind domain. A complete account of this can be found in [1]. The proofs make use of standard forms of matrices in the general linear group. In classical algebraic K-theory one defines functors K_n for $n = 0, 1, 2$ as in [1] (for $n = 0, 1$) and [7] (for $n = 0, 1, 2$). Exact sequences involving these K-groups are derived using these definitions. In higher algebraic K-theory functors K_n are defined for all non-negative integers n. They fit in long exact sequences that extend the exact sequences of classical algebraic K-theory. For the description of the SK_1 of an ideal in a Dedekind domain here the opposite route is followed, with higher algebraic K-theory as a starting point.

In [3] a proof of Matsumoto's theorem was given that used the computation of the SK_1 of the ideal $(t^2 - t)$ of the polynomial ring $F[t]$ over a field F. That proof used a reciprocity law for the Milnor ring of a field. In this article two proofs of Matsumoto's theorem are given: one similar to the proof in [3] using the Milnor ring, and one that is more direct and avoids the use of the Milnor ring.

1 A relative localization exact sequence

The K-groups of an exact category $\mathcal{C}$ are defined by Quillen [8]. Everything in this section is a direct consequence of results in [8]. The K-groups are defined as the homotopy groups of the space $\mathrm{K}(\mathcal{C}) = BQ(\mathcal{C})$, the classifying space of a category $Q\mathcal{C}$ (as defined in [8]):

$$\mathrm{K}_i(\mathcal{C}) = \pi_{i+1}(\mathrm{K}(\mathcal{C})) \qquad (i \geq 0).$$

For a ring R we denote the category of finitely generated projective R-modules by $\mathcal{P}_R$. The K-groups of R are the K-groups of this category:

$$\mathrm{K}_i(R) = \mathrm{K}_i(\mathcal{P}_R) \qquad (i \geq 0).$$

For an exact functor $\mathcal{C} \to \mathcal{C}'$ the homotopy fibre of the map $\mathrm{K}(\mathcal{C}) \to \mathrm{K}(\mathcal{C}')$ will be denoted by $\mathrm{K}(\mathcal{C} \to \mathcal{C}')$. The relative K-groups of an ideal $\mathfrak{a}$ of a ring R are the homotopy groups of the homotopy fibre of $\mathrm{K}(\mathcal{P}_R) \to \mathrm{K}(\mathcal{P}_{R/\mathfrak{a}})$:

$$\mathrm{K}_i(R, \mathfrak{a}) = \pi_{i+1}(\mathrm{K}(\mathcal{P}_R \to \mathcal{P}_{R/\mathfrak{a}})) \qquad (i \geq 0).$$

For R a Noetherian ring the category $\mathcal{M}_R$ of finitely generated R-modules is an exact category. By resolution the inclusion functor $\mathcal{P}_R \to \mathcal{M}_R$ induces for R regular a homotopy equivalence $\mathrm{K}(\mathcal{P}_R) \to \mathrm{K}(\mathcal{M}_R)$.

For the rest of this article we fix the following notations:

R	a Dedekind domain,
Q	the quotient field of R,
$\mathfrak{a}$	a non-zero ideal of R,
S	a set of maximal ideals $\mathfrak{p}$ of R with $\mathfrak{p} \nmid \mathfrak{a}$,
R_S	the ring of all $u \in Q$ with $v_{\mathfrak{p}}(u) \geq 0$ for all maximal ideals $\mathfrak{p}$ of R with $\mathfrak{p} \notin S$,
$\mathfrak{a}_S$	the ideal $\mathfrak{a}R_S$ of R_S; it consists of all $u \in Q$ with $v_{\mathfrak{p}}(u) \geq v_{\mathfrak{p}}(\mathfrak{a})$ for all maximal ideals $\mathfrak{p}$ of R with $\mathfrak{p} \notin S$.

The commutative triangle of ringhomomorphisms

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & R/\mathfrak{a} \\ & R_S \arrow[ur] & \end{tikzcd}$$

induces a commutative triangle

$$\begin{tikzcd} \mathcal{P}_R \arrow[rr] \arrow[dr] & & \mathcal{P}_{R/\mathfrak{a}} \\ & \mathcal{P}_{R_S} \arrow[ur] & \end{tikzcd}$$

of functors between the categories of finitely generated projective modules.

Since R and R_S are regular Noetherian domains the inclusion functors $\mathcal{P}_R \to \mathcal{M}_R$ and $\mathcal{P}_{R_S} \to \mathcal{M}_{R_S}$ induce homotopy equivalences $\mathrm{K}(\mathcal{P}_R) \to \mathrm{K}(\mathcal{M}_R)$ and $\mathrm{K}(\mathcal{P}_{R_S}) \to \mathrm{K}(\mathcal{M}_{R_S})$. The homotopy fibre $\mathrm{K}(\mathcal{M}_R \to \mathcal{M}_{R_S})$ is homotopy equivalent to $\mathrm{K}(\mathcal{A})$, where $\mathcal{A}$ is the Serre subcategory of S-torsion modules in $\mathcal{M}_R$. By dévissage one has

$$\mathrm{K}_i(\mathcal{A}) \xrightarrow{\sim} \bigoplus_{\mathfrak{p}\in S} \mathrm{K}_i(R/\mathfrak{p}) \qquad (i \geq 0).$$

As a result we have a long exact sequence

$$\cdots \to \mathrm{K}_{i+1}(R_S, \mathfrak{a}_S) \to \bigoplus_{\mathfrak{p}\in S} \mathrm{K}_i(R/\mathfrak{p}) \to \mathrm{K}_i(R, \mathfrak{a}) \to \mathrm{K}_i(R_S, \mathfrak{a}_S) \to \cdots \to \mathrm{K}_0(R_S, \mathfrak{a}_S)$$

of homotopy groups of the homotopy fibres.

2 Relative reciprocities I

In this section S is the set of *all* maximal ideals $\mathfrak{p}$ of R satisfying $\mathfrak{p} \nmid \mathfrak{a}$. Then $\mathfrak{a}_S$ is a radical ideal, i.e. an ideal contained in the Jacobson radical of R. Therefore, for the lower relative K-groups we have:

$$\begin{aligned} \mathrm{K}_0(R_S, \mathfrak{a}_S) &= 0, \\ \mathrm{K}_1(R_S, \mathfrak{a}_S) &= 1 + \mathfrak{a}_S. \end{aligned}$$

Since $\mathrm{SK}_1(R_S, \mathfrak{a}_S) = 0$, from the relative localization exact sequence one obtains an exact sequence

$$\mathrm{K}_2(R_S, \mathfrak{a}_S) \to \bigoplus_{\mathfrak{p}\in S} (R/\mathfrak{p})^* \to \mathrm{SK}_1(R, \mathfrak{a}) \to 0.$$

The homomorphisms $(R/\mathfrak{p})^* \to \mathrm{SK}_1(R, \mathfrak{a})$ will be denoted by $\kappa_\mathfrak{p}$. In [4] a presentation in terms of Dennis-Stein symbols has been given of the group $\mathrm{K}_2(\Lambda, \mathfrak{b})$, where $\mathfrak{b}$ is a radical ideal of a commutative ring Λ. This presentation was based on the presentation given by Maazen and Stienstra [5] for the split case, i.e. the case in which the homomorphism $\Lambda \to \Lambda/\mathfrak{b}$ has a section which is a ring homomorphism. Here we only need generators of $\mathrm{K}_2(R_S, \mathfrak{a}_S)$ and since the map from $\mathrm{K}_2(R_S, \mathfrak{a}_S)$ to $\bigoplus_{\mathfrak{p}\in S}(R/\mathfrak{p})^*$ factors through $\mathrm{K}_2(R_S)$, it suffices to know their images in $\mathrm{K}_2(R_S)$. The image of $\mathrm{K}_2(R_S, \mathfrak{a}_S)$ in $\mathrm{K}_2(R_S)$ is generated by the Dennis-Stein symbols $\langle f, a\rangle$, where $f \in R_S$, $a \in \mathfrak{a}_S$ and $1 - fa \in R_S^*$. The map also factors through $\mathrm{K}_2(Q)$ and the image of $\langle f, a\rangle$ in $\mathrm{K}_2(Q)$ is equal to the Steinberg symbol $\{1 - fa, f\}$. Here the description of the K_2 of a ring as a subgroup of the Steinberg group is used. Steinberg symbols, and more generally Dennis-Stein symbols, are special elements of the Steinberg group.

A prime $\mathfrak{p}$ of R induces a group homomorphism

$$\partial_\mathfrak{p} \colon \mathrm{K}_2(Q) \to (R/\mathfrak{p})^*$$

which maps a Steinberg symbol $\{x, \pi\}$ with $v_{\mathfrak{p}}(x) = 0$ and $v_{\mathfrak{p}}(\pi) = 1$ to $\overline{x}$ in the residue field. The relations for Steinberg symbols imply that for $u, v \in Q^*$:

$$\partial_{\mathfrak{p}}(\{u, v\}) = (u, v)_{\mathfrak{p}},$$

where

$$(u, v)_{\mathfrak{p}} = \overline{u^{v_{\mathfrak{p}}(v)} v^{-v_{\mathfrak{p}}(u)}}.$$

Here the bar denotes the class modulo the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ of the valuation ring $R_{\mathfrak{p}}$ and the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is identified with $R/\mathfrak{p}$. It follows that the image of $\mathrm{K}_2(R_S, \mathfrak{a}_S)$ in $\bigoplus_{\mathfrak{p} \in S}(R/\mathfrak{p})^*$ is generated by the elements

$$((1 - fa, f)_{\mathfrak{p}})_{\mathfrak{p} \in S},$$

where $f \in R_S$ and $a \in \mathfrak{a}_S$.

The presentation of $\mathrm{SK}_1(R, \mathfrak{a})$ that results from the above can conveniently be described by means of reciprocities, the group $\mathrm{SK}_1(R, \mathfrak{a})$ being the target group of the universal reciprocity $(\kappa_{\mathfrak{p}})_{\mathfrak{p}}$.

Definition 1. An $\mathfrak{a}$-*reciprocity* on R with values in an Abelian group A is a collection $(\chi_{\mathfrak{p}})_{\mathfrak{p} \in S}$ of group homomorphisms $\chi_{\mathfrak{p}} \colon (R/\mathfrak{p})^* \to A$, one for each $\mathfrak{p} \in S$, such that

$$\sum_{\mathfrak{p} \in S} \chi_{\mathfrak{p}}((1 - fa, f)_{\mathfrak{p}}) = 0,$$

for all $f \in R_S$ and $a \in \mathfrak{a}_S$.

Summerizing we have:

Theorem 1. *Let* $(\chi_{\mathfrak{p}})_{\mathfrak{p}}$ *be an* $\mathfrak{a}$-*reciprocity on* R *with values in* A. *Then there is a unique group homomorphism* $\chi \colon \mathrm{SK}_1(R, \mathfrak{a}) \to A$ *such that* $\chi\kappa_{\mathfrak{p}} = \chi_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in S$. ☐

The group $\mathrm{SK}_1(R, \mathfrak{a})$ is a factor group of the relative general linear group $\mathrm{GL}(R, \mathfrak{a})$ and can be presented in terms of Mennicke symbols, elements which can be represented by 2×2-matrices. See [1], also for the notion of reciprocity that comes from that approach. In sections 4 and 6 other descriptions of reciprocities will be given. The description in section 4 being only for the special case of a square free ideal of a Dedekind domain.

3 A proof of Matsumoto's theorem

Matsumoto's theorem is about the presentation of the K_2 of a field in terms of Steinberg symbols. In [3] a proof of Matsumoto's theorem was given by constructing a relative reciprocity on the affine line. The proof given here is essentially the same.

Let F be a field. In this section we consider the special case:

$R = F[t]$	the polynomial ring in one indeterminate t over F,
$Q = F(t)$	the quotient field of R,
$\mathfrak{a} = (t^2 - t)$	the ideal of R generated by $t^2 - t$,
S	the set of irreducible polynomials π over F with $\pi \neq t, 1-t$ and $\pi(0) = 1$,
R_S	the semilocal ring of all $u \in Q$ with $v_t(u), v_{1-t}(u) \geq 0$,
$\mathfrak{a}_S$	the ideal $\mathfrak{a}R_S$ of R_S; it consists of all $u \in Q$ with $v_t(u), v_{1-t}(u) \geq 1$.

The ring R_S is the ring of rational functions over F that are defined in 0 and 1. By the Chinese remainder theorem we have isomorphisms

$$R/\mathfrak{a} \xrightarrow{\sim} R_S/\mathfrak{a}_S \xrightarrow{\sim} F \times F,$$

induced by $R \subseteq R_S \to F \times F$, $u \mapsto (u(0), u(1))$. We have a commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{K}_2(F[t]_S, t^2 - t) \arrow[r] \arrow[d, equal] & \mathrm{K}_2(F[t]_S, t) \arrow[r] \arrow[d, "\partial", "\sim"'] & \mathrm{K}_2(F) \arrow[r] \arrow[d, "\sim"'] & 0 \\
\mathrm{K}_2(F[t]_S, t^2 - t) \arrow[r, "\partial"] & \bigoplus_{\pi \in S}(F[t]/\pi)^* \arrow[r] & \mathrm{SK}_1(F[t], t^2 - t) \arrow[r] & 0
\end{tikzcd}$$

The top row results from the exact sequence of relative groups for the ideals (t) and $(t^2 - t)$ of $F[t]_S$. Note that $\mathrm{SK}_1(F[t]_S, t^2 - t) = 0$ since $(t^2 - t)$ is a radical ideal and moreover that $\mathrm{K}_2(F[t]/(t^2 - t), (t)/(t^2 - t)) \cong \mathrm{K}_2(F \times F, 0 \times F) \cong \mathrm{K}_2(F)$. The bottom row is the sequence of section 1 that comes from the relative localization exact sequence. The middle vertical map is an isomorphism. This follows from a relative localization exact sequence, the groups $\mathrm{K}_n(F[t], t)$ in that sequence being trivial since F is regular.

The Milnor ring of a field F will be denoted as $\mathrm{K}_*^{\mathrm{M}}(F)$. It is the graded ring generated by elements $l(a)$ in degree 1, one for each $a \in F^*$. The defining relations are

$$\begin{aligned} l(ab) &= l(a) + l(b) && \text{for all } a, b \in F^*, \\ l(a)l(1-a) &= 0 && \text{for all } a \in F \text{ with } a \neq 0, 1. \end{aligned}$$

It follows that $\mathrm{K}_0^{\mathrm{M}}(F) = \mathbb{Z}$ and that $F^* \xrightarrow{\sim} \mathrm{K}_1^{\mathrm{M}}(F)$, $a \mapsto l(a)$. The Steinberg relations in $\mathrm{K}_2(F)$ imply that $l(a)l(b) \mapsto \{a, b\}$ induces a homomorphism $\mathrm{K}_2^{\mathrm{M}}(F) \to \mathrm{K}_2(F)$.

Theorem 2. (Matsumoto) *The homomorphism*

$$\mathrm{K}_2^{\mathrm{M}}(F) \to \mathrm{K}_2(F), \quad l(a)l(b) \mapsto \{a, b\}$$

is an isomorphism.

Proof. The easy part of the theorem is the surjectivity, i.e. the Steinberg symbols generate $\mathrm{K}_2(F)$, see [7] or, for another argument, see section 5.

It suffices to constuct a left inverse. As in [3] this is done by giving a $(t^2 - t)$-reciprocity on $F[t]$. For each $\pi \in S$ define

$$\chi_\pi \colon (F[t]/\pi)^* \to \mathrm{K}_2^{\mathrm{M}}(F), \quad \overline{g} \mapsto N_\pi(l(\overline{g})l(\overline{1 - \tfrac{1}{t}})),$$

where $g \in F[x]$ with $\pi \nmid g$. The transfer N_π is the transfer N_v as defined in [2], the discrete valuation v being the one corresponding to the prime ideal (π). We will show that $(\chi_\pi)_{\pi \in S}$ is a $(t^2 - t)$-reciprocity, i.e.

$$\sum_{\pi \in S} \chi_\pi((1 - fa, f)_\pi) = 0$$

for all $a \in \mathfrak{a}_S$ and all $f \in F[t]^*_S$. This will follow from the reciprocity law for the Milnor ring:

$$\sum_v N_v(\partial_v(x)) = 0 \quad (\text{for all } x \in \mathrm{K}^{\mathrm{M}}_*(F)),$$

where the sum is over all discrete valuations on $F(t)$ and ∂_v is the degree -1 map from the Milnor ring of $F(t)$ to the Milnor ring of the residue field of v as defined in [2]. For a and f as above we take $x = l(1 - fa)l(1 - \frac{1}{t})l(f) \in \mathrm{K}^{\mathrm{M}}_3(F)$ and compute $\partial_v(x)$ for all discrete valuations v on $F(t)$. For $b, c, d \in F(t)^*$ with $v(c) = 0$ it follows directly from the definition of ∂_v in [2] that

$$\partial_v(l(b)l(c)l(d)) = \partial_v(l(b)l(d))l(\overline{c}).$$

This identity will be used in the computations. We distinguish four cases.

(1) $v = v_\pi$ with $\pi \in S$.

$$\partial_v(x) = \partial_v(l(1 - fa)l(f))l(\overline{1 - \tfrac{1}{t}}) = l((1 - fa, f)_\pi)l(\overline{1 - \tfrac{1}{t}}).$$

(2) $v = v_t$.
Since $l(t)l(1 - \frac{1}{t}) = 0$ and $v(1 - fa) = 0$ we have

$$\partial_v(x) = -\partial_v(l(1 - fa)l(ft^{-v(f)})l(1 - \tfrac{1}{t}) = l(\overline{1 - fa})l(*) = l(1)l(*) = 0.$$

(3) $v = v_{1-t}$.
Since $l(1 - t)l(1 - \frac{1}{t}) = 0$ and $v(1 - fa) = 0$ we have

$$\partial_v(x) = -\partial_v(l(1 - fa)l(f(1 - t)^{-v(f)})l(1 - \tfrac{1}{t}) = l(\overline{1 - fa})l(*) = l(1)l(*) = 0.$$

(4) $v = v_\infty$.

$$\partial_v(x) = \partial_v(l(ft^{-\deg(f)})l(1 - \tfrac{1}{t})l(1 - af)) = l(*)l(1)v(1 - af) = 0.$$

It follows that $(\chi_\pi)_\pi$ is a $(t^2 - t)$-reciprocity on $F[t]$:

$$\begin{aligned}\sum_{\pi \in S} \chi_\pi((1 - fa, f)_\pi) &= \sum_{\pi \in S} N_\pi(l(1 - fa, f)_\pi)l(\overline{1 - \tfrac{1}{t}})) \\ &= \sum_{\pi \in S} N_{v_\pi}(\partial_{v_\pi}(x)) = \sum_v N_v(\partial_v(x)) = 0.\end{aligned}$$

The reciprocity induces a homomorphism from $\mathrm{K}_2(F)$ to $\mathrm{K}_2^{\mathrm{M}}(F)$. It remains to show that for $b, c \in F^*$ the image of $\{b, c\} \in \mathrm{K}_2(F)$ in $\mathrm{K}_2^{\mathrm{M}}(F)$ is $l(b)l(c)$. Assume that $c \neq 1$. The element $\{b, c\} \in \mathrm{K}_2(F)$ lifts to $\{b, \gamma\}$ in $K_2(F[t]_S, t)$, where $\gamma = (c-1)t + 1$. Under ∂ the image of $\{b, \gamma\}$ in $\bigoplus_{\pi \in S}(F[t]/\pi)^*$ is b in the component with index γ and is trivial in the others. Hence,

$$\chi(\partial(\{b, \gamma\})) = \chi_\gamma(b) = l(b)l(1-(1-c)) = l(b)l(c). \qquad \square$$

4 Relative reciprocities II

Definition 2. An ideal $\mathfrak{a}$ of a Dedekind domain R is called *square free* if $v_{\mathfrak{p}}(\mathfrak{a}) \leq 1$ for all maximal ideals $\mathfrak{p}$ of R.

In this section for square free ideals $\mathfrak{a}$ a description of $\mathfrak{a}$-reciprocity is given which is more flexible than the one given in the definition. In section 6 this will be extended to the general case. We use the notations of section 2.

Proposition 1. *Let $\mathfrak{a}$ be square free and A an Abelian group. A collection $(\chi_{\mathfrak{p}})_{\mathfrak{p} \in S}$ of group homomorphisms $\chi_{\mathfrak{p}} \colon (R/\mathfrak{p})^* \to A$ is an $\mathfrak{a}$-reciprocity iff*

$$\sum_{\mathfrak{p} \in S} \chi_{\mathfrak{p}}((g, f)_{\mathfrak{p}}) = 0$$

for all $g \in 1 + \mathfrak{a}_S$ and all $f \in Q^*$.

Proof. Let $(\chi_{\mathfrak{p}})_{\mathfrak{p} \in S}$ be an $\mathfrak{a}$-reciprocity on R. We have to show that $\sum_{\mathfrak{p} \in S} \chi_{\mathfrak{p}}((g, f)_{\mathfrak{p}}) = 0$ for $g \in 1 + \mathfrak{a}_S$ and $f \in Q^*$. Let there be r prime ideals that divide $\mathfrak{a}$, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. By the Chinese remainder theorem there exist $\pi_1, \ldots \pi_r \in R$ such that for $i = 1, \ldots, r$ the following is satisfied

$$\begin{cases} v_{\mathfrak{p}_i}(\pi_i) = 1 \\ \pi_i \equiv 1 \pmod{\mathfrak{p}_j} \quad \text{for all } j \neq i. \end{cases}$$

Put $f' = \prod_{i=1}^r \pi_i^{v_{\mathfrak{p}_i}(f)}$. Then $f(f')^{-1} \in R_S^*$. For each $\mathfrak{p} \in S$ we have

$$(g, f)_{\mathfrak{p}} = (g, f(f')^{-1})_{\mathfrak{p}} \cdot \prod_{i=1}^r (g, \pi_i)_{\mathfrak{p}_i}^{v_{\mathfrak{p}_i}(f)}.$$

Since $(\chi_{\mathfrak{p}})_{\mathfrak{p} \in S}$ is an $\mathfrak{a}$-reciprocity, we have $\sum_{\mathfrak{p} \in S} \chi_{\mathfrak{p}}((g, f(f')^{-1})_{\mathfrak{p}}) = 0$ and hence

$$\sum_{\mathfrak{p} \in S} \chi_{\mathfrak{p}}((g, f)_{\mathfrak{p}}) = \sum_{\mathfrak{p} \in S} \sum_{i=1}^r v_{\mathfrak{p}_i}(f) \chi_{\mathfrak{p}_i}((g, \pi_i)_{\mathfrak{p}}) = \sum_{i=1}^r v_{\mathfrak{p}_i}(f) \sum_{\mathfrak{p} \in S} \chi_{\mathfrak{p}_i}((g, \pi_i)_{\mathfrak{p}}),$$

so it suffices to prove that $\sum_{\mathfrak{p} \in S} \chi_{\mathfrak{p}_i}((g, \pi_i)_{\mathfrak{p}}) = 0$ for $i = 1, \ldots, r$. In $\mathrm{K}_2(R_S)$ we have

$$\{g, \pi_i\} = \left\langle \frac{1-g}{\pi_i}, \pi_i \right\rangle,$$

which is in the image of $\mathrm{K}_2(R_S, \mathfrak{a}_S)$ since it is in the kernel of

$$\mathrm{K}_2(R_S) \to \mathrm{K}_2(R_S/\mathfrak{a}_S) \xrightarrow{\sim} \mathrm{K}_2(R/\mathfrak{a}) \xrightarrow{\sim} \bigoplus_{i=1}^{r} \mathrm{K}_2(R/\mathfrak{p}_i).$$

Because $(\chi_\mathfrak{p})_\mathfrak{p}$ is an $\mathfrak{a}$-reciprocity it follows that

$$\sum_{\mathfrak{p}\in S} \chi_{\mathfrak{p}_i}((g, \pi_i)_\mathfrak{p}) = 0. \qquad \square$$

5 Another proof of Matsumoto's theorem

In this section Matsumoto's theorem is proved without the use of the Milnor ring. This proof is more technical, especially when the field is not algebraically closed. We use the notations of section 3. From the previous section it follows that the image of $\partial\colon \mathrm{K}_2(R_S, \mathfrak{a}_S) \to \bigoplus_{\pi\in S}(F[t]/(\pi))^*$ is generated by the elements $((g, f)_\pi)_p i$, where $g \in 1 + \mathfrak{a}_S$ and $f \in Q^* = F(t)^*$. This leads to another presentation of $\mathrm{K}_2(F)$. For convenience we use the following notations:

$\overline{F}$	a fixed algebraic closure of F,
α_π	a fixed zero of π in $\overline{F}$, one for each $\pi \in S$.

Instead of the groups $(F[t]/\pi)^*$ we will use $F(\alpha_\pi)^*$. The image of $(g, f)_\pi$ (for $g, f \in F(t)^*$) in $F(\alpha_\pi)^*$ under the canonical isomorphism will also be denoted by $(g, f)_\pi$. Using this notation, we have an exact sequence

$$\mathrm{K}_2(F[t]_S, t - t^2) \xrightarrow{\partial} \bigoplus_{\pi\in S} F(\alpha_\pi)^* \to \mathrm{K}_2(F) \to 0.$$

Denoting $\bigoplus_{\pi\in S} F(\alpha_\pi)^*$ by A and the image of $\mathrm{K}_2(F[t]_S, t - t^2)$ in A by B gives a short exact sequence

$$0 \to B \to A \to \mathrm{K}_2(F) \to 0,$$

which can be interpreted as a presentation of $\mathrm{K}_2(F)$. Elements (α, π) with $\alpha \in F(\alpha_\pi)^*$ and $\pi \in S$ will represent the generators of the presentation. The map from $A \to \mathrm{K}_2(F)$ is as follows: first lift to an element of $\mathrm{K}_2(F[t]_S, t)$ and then apply the map induced by the ring homomorphism $f \mapsto f(1)$. For example: for $c, a \in F^*$ with $a \neq 1$, the element $(c, (a-1)t + 1)$ lifts to $\{c, (a-1)t + 1\}$, which maps to $\{c, a\} \in \mathrm{K}_2(F)$. The class of (α, π) modulo B will be denoted as $[\alpha, \pi]$. From the previous section it follows that B is generated by all $\sum_{\pi\in S}((f, g)_\pi, \pi)$ with $f \in 1 + \mathfrak{a}_S$ and $g \in F(t)^*$.

The following notations will be used

S^+	the set $S \cup \{t, 1-t\}$ (the elements of S^+ correspond to the maximal ideals of $F[t]$),
S_n	the elements π of S with $\deg(\pi) \leq n$,
$F(t)^*_n$	the subgroup of $F(t)^*$ of all f with $v_\pi(f) = 0$ for all $\pi \in S^+$ with $\deg(\pi) > n$,
A_n	the subgroup $\bigoplus_{\pi \in S_n} F(\alpha_\pi)^*$ of A,
B_n	the subgroup of B generated by all $d(f,g)$ with $f \in (1+\mathfrak{a}_S) \cap F(t)^*_n$ and $g \in F(t)^*_n$.

For $f, g \in F(t)^*$ we define $d(f,g) = \sum_{\pi \in S^+} ((f,g)_\pi, \pi)$. For $f \in 1 + \mathfrak{a}_S$ one has $d(f,g) \in B$. The group B is the subgroup of A generated by the $d(f,g)$ where $f \in 1 + \mathfrak{a}_S$.

The result is a presentation of $\mathrm{K}_2(F)$ with more generators and more relations: the generators are the $[\alpha, \pi]$ with $\alpha \in F(\alpha_\pi)^*$ and the relations are

$$[\alpha, \pi] + [\beta, \pi] = [\alpha\beta, \pi] \qquad \text{(for } \pi \in S \text{ and } \alpha, \beta \in F(\alpha_\pi)^*\text{)},$$

$$\sum_{\pi \in S} [(f,g)_\pi, \pi] = 0 \qquad \text{(for all } f \in 1 + \mathfrak{a}_S \text{ and } g \in F(t)^*\text{)}.$$

The groups A_n form a filtration of A and the groups B_n form a filtration of B with $B_n \subseteq A_n$ for all n.

Notation. For $a, b \in F$ let $\tau_{a,b} \in F[t]$ be the unique polynomial of degree ≤ 1 such that $\tau_{a,b}(0) = a$ and $\tau_{a,b}(1) = b$, i.e. $\tau_{a,b} = (b-a)t + a$. We also use the notation $\mu_a = \tau_{1,a}$. So $\mu_a = (a-1)t + 1$.

Proposition 2. *B_1 is the kernel of the homomorphism $\varphi\colon A_1 \to \mathrm{K}_2^{\mathrm{M}}(F)$ which maps (c, μ_a) to $l(c)l(a)$ for all $c \in F^*$ and $a \in F^*$ with $a \neq 1$.*

Proof. First we will show that B_1 is in the kernel. The Abelian group B_1 is generated by elements of the following types:

(1) $d(\mu_a \mu_b \mu_{ab}^{-1}, c)$ with $a, b, c \in F^*$ and $a, b \neq 1$,

(2) $d(\mu_a \mu_b \mu_{ab}^{-1}, t)$ with $a, b \in F^*$ and $a, b \neq 1$,

(3) $d(\mu_a \mu_b \mu_{ab}^{-1}, 1-t)$ with $a, b \in F^*$ and $a, b \neq 1$,

(4) $d(\mu_a \mu_b \mu_{ab}^{-1}, \mu_c)$ with $a, b, c \in F^*$ and $a, b \neq 1$.

For each type of generator we will show that its image in $\mathrm{K}_2^{\mathrm{M}}(F)$ is trivial.

(1) First the special case $ab = 1$:

$$d(\mu_a \mu_b, c) = (c, \mu_a) + (c, \mu_b) \mapsto l(c)l(a) + l(c)l(b) = l(c)l(1) = 0.$$

For $ab \neq 1$ we have

$$d(\mu_a \mu_b \mu_{ab}^{-1}, c) = (c, \mu_a) + (c, \mu_b) - (c, \mu_{ab}),$$

which maps to $l(c)l(a) + l(c)l(b) - l(c)l(ab) = 0$.

(2) The special case $ab = 1$:

$$d(\mu_a\mu_b, t) = \Big(\frac{1}{1-a}, \mu_a\Big) + \Big(\frac{1}{1-b}, \mu_b\Big),$$

and this maps to $-l(1-a)l(a) - l(1-b)l(b) = 0$. When $ab \neq 1$ the image is $-l(1-a)l(a) - l(1-b)l(b) + l(1-ab)l(ab) = 0$.

(3) The special case $ab = 1$:

$$d(\mu_a\mu_b, 1-t) = \Big(1 - \frac{1}{1-a}, \mu_a\Big) + \Big(1 - \frac{1}{1-b}, \mu_b\Big),$$

which maps to $-l(1-\frac{1}{a})l(a) - l(1-\frac{1}{b})l(b) = l(1-\frac{1}{a})l(\frac{1}{a}) + l(1-\frac{1}{b})l(\frac{1}{b}) = 0$. When $ab \neq 1$ the image is $l(1-\frac{1}{a})l(\frac{1}{a}) + l(1-\frac{1}{b})l(\frac{1}{b}) - l(1-\frac{1}{ab})l(\frac{1}{ab}) = 0$.

(4) Assume $a \neq c$. Then

$$d(\mu_a, \mu_c) = \Big(\frac{c-a}{1-a}, \mu_a\Big) - \Big(\frac{a-c}{1-c}, \mu_c\Big)$$

and this maps to

$$\begin{aligned} l\Big(\frac{c-a}{1-a}\Big)l(a) - l\Big(\frac{a-c}{1-c}\Big)l(c) &= l(c-a)l(a) - l(a-c)l(c) \\ &= l(c-a)l(a) - l(c-a)l(c) + l(-1,c) \\ &= l(c-a)l\Big(\frac{a}{c}\Big) + l(-1)l(c) = l(c)\Big(\frac{a}{c}\Big) + l(-1)l(c) \\ &= l(c)l(-a) + l(c)l(-1) = l(c)l(a). \end{aligned}$$

From this it easily follows that the elements of type $d(\mu_a\mu_b\mu_{ab}^{-1}, \mu_c)$ map to 0 in $\mathrm{K}_2^{\mathrm{M}}(F)$.

Hence φ induces a surjective homomorphism $A_1/B_1 \to \mathrm{K}_2^{\mathrm{M}}(F),\ (c, \mu_a) + B_1 \mapsto l(c)l(a)$. The computations above show that the map

$$F^* \times F^* \longrightarrow A_1/B_1,\ (c, a) \mapsto (c, \mu_a) + B_1$$

is bilinear.

Notation. The element (c, μ_a) will be denoted as $\lceil c, a\rceil$

We still have to prove that $\lceil 1-a, a\rceil \in B_1$ for all $a \in F^*$ with $a \neq 1$. Put $\eta(a) = \lceil 1-a, a\rceil + B_1$ if $a \neq 1$ and $\eta(1) = B_1 = 0$. Then, by the computations above, $\eta(ab) = \eta(a) + \eta(b)$ for all $a, b \in F^*$. For $a \neq 1$ we have

$$\lceil -a, a\rceil + B_1 = \lceil 1-a, a\rceil - \Big\lceil 1 - \frac{1}{a}, a\Big\rceil + B_1 = \eta(a) + \eta\Big(\frac{1}{a}\Big) = 0.$$

It follows that

$$\lceil a, b\rceil + B_1 = \lceil a, -ab\rceil + B_1 = -\lceil b, -ab\rceil + B_1 = -\lceil b, a\rceil + B_1.$$

In particular we have $\eta(a) = -\eta(1-a)$ and hence $\eta(a(1-a)) = 0$. Since $-a^3$ is the quotient of $a(1-a)$ and $\frac{1}{a}(1-\frac{1}{a})$, we have $\eta(-a^3) = 0$, or $3\eta(-a) = 0$. So the elements $\eta(a)$ are of order 1 or 3. We also have

$$\eta(1-a) = -\eta(a) = 2\eta(a) = \eta(a^2) = 2\eta(-a) = -\eta(-a).$$

Since $1 - \frac{1}{a} = \frac{1-a}{-a}$ it follows that $\eta(1-\frac{1}{a}) = 0$ for all $a \neq 1$ and therefore $\eta(a) = 0$ for all $a \in F^*$. $\square$

Remark. For F algebraically closed we have $A = A_1$ and $B = B_1$. So in this case Matsumoto's theorem follows.

In the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & A_1 \arrow[r, "\varphi"] \arrow[d] & \mathrm{K}_2^{\mathrm{M}}(F) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & \mathrm{K}_2(F) \arrow[r] & 0 \\
 & B/B_1 \arrow[r] \arrow[d] & A/A_1 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

the rows and columns are exact. The exactness of the first row is Proposition 2. Matsumoto's theorem is equivalent to $B/B_1 \to A/A_1$ being an isomorphism. We will show that this map is indeed an isomorphism by constructing an inverse of the maps $B_n/B_{n-1} \to A_n/A_{n-1}$ for all $n \geq 2$. Since $A_n/A_{n-1} = \bigoplus_{\pi \in S_n \setminus S_{n-1}} F(\alpha_\pi)^*$ we will construct homomorphisms $F(\alpha_\pi)^* \to B_n/B_{n-1}$ for $\pi \in S$ of degree $n \geq 2$.

Let $\pi \in S$ with $\deg(\pi) = n \geq 2$. We define a map

$$\psi_\pi \colon F(\alpha_\pi)^* \to B_n/B_{n-1}$$

by

$$\psi_\pi(\alpha) = d(\tfrac{\pi}{\mu}, g) + B_{n-1},$$

where $\mu = \mu_{\pi(1)}$ and $g \in F[t]$ is the unique polynomial with $g(\alpha_\pi) = \alpha$ and $\deg(g) < n$. We will show that ψ_π is a homomorphism. First some lemma's.

Lemma 1. *Let $n \geq 1$ and $f \in R_S^* \cap F(t)_n^*$. Then*

(i) *$d(t, f) \in B_n$ if $f(0) = 1$.*

(ii) *$d(1-t, f) \in B_n$ if $f(1) = 1$.*

Proof. (i) Let $\mu = \mu_{f(1)}$. Then $\frac{f}{\mu} \in 1 + \mathfrak{a}_S$. Since $d(t, \frac{f}{\mu}) \in B_n$, it suffices to show that $d(t, \mu) \in B_1$. If $f(1) = 1$ this is clearly the case, so assume $f(1) \neq 1$. Then

$$d(t, \mu) = \Big\lceil \frac{1}{1 - f(1)}, f(1) \Big\rceil \in B_1.$$

(ii) Let $\tau = \tau_{f(0),1}$. Then $d(1-t, \frac{f}{\tau}) \in B_n$. Again it suffices to show that $d(1-t, \tau) \in B_1$ when $f(0) \neq 1$. This follows from

$$d(1 - t, \tau) = \Big\lceil \frac{1}{1 - \frac{1}{f(0)}}, \frac{1}{f(0)} \Big\rceil. \qquad \square$$

Lemma 2. *Let $f, g \in F(t)^*_n \cap R^*_S$. Then $d(f, g) \equiv d(\tau_{f(0),f(1)}, \tau_{g(0),g(1)}) \pmod{B_n}$.*

Proof. Put $\tau_1 = \tau_{f(0),f(1)}$ and $\tau_2 = \tau_{g(0),g_(1)}$. Then

$$d(f, g) \equiv d(\tau_1, g) \equiv d(\tau_1, \tau_2) \pmod{B_n}$$

since $d(\frac{f}{\tau_1}, g), d(\tau_1, \frac{g}{\tau_2}) \in B_n$. $\square$

Lemma 3. *Let $n \geq 1$ and $f, g \in F(t)^*_n$ such that $d(f, g) \in A$. Put $f = t^{k_1}(1-t)^{k_2} f_0$ and $g = t^{l_1}(1 - t)^{l_2} g_0$ with $f_0, g_0 \in R^*_S$. Put $a = f_0(0)$, $b = f_0(1)$, $c = g_0(0)$ and $d = g_0(1)$. Then $d(f, g) \equiv d(\tau_{a,b}, \tau_{c,d}) \pmod{B_n}$.*

Proof. We have

$$\begin{aligned} d(f, g) &= k_1 l_1 d(t, t) + k_2 l_2 d(1 - t, 1 - t) + k_1 l_2 d(t, 1 - t) + k_2 l_1 d(1 - t, t) \\ &\quad + k_1 d(t, g_0) + k_2 d(1 - t, g_0) + l_1 d(f_0, t) + l_2 d(f_0, 1 - t) + d(f_0, g_0) \\ &= k_1 d l_1(t, -1) + k_2 l_2 d(1 - t, -1) + d(t, g_0^{k_1}) + d(1 - t, g_0^{l_1}) + d(t, f_0^{-l_1}) \\ &\quad + d(1 - t, f_0^{-l_2}) + d(f_0, g_0) \\ &= d(t, (-1)^{k_1 l_1} f_0^{-l_1} g_0^{k_1}) + d(1 - t, (-1)^{k_2 l_2} f_0^{-l_2} g_0^{k_2}) + d(f_0, g_0). \end{aligned}$$

The condition $d(f, g) \in A_n$ means that $d_t(f, g) = d_{1-t}(f, g) = 1$, so by Lemma 1 the first two of these three terms are in B_n. The Lemma follows from Lemma 2. $\square$

Proposition 3. *The map $\psi_\pi \colon F(\alpha_\pi)^* \to B_n/B_{n-1}$ is a group homomorphism.*

Proof. Let $\alpha_1, \alpha_2 \in F(\alpha_\pi)$ and for $i = 1, 2$ let $g_i \in F[t]$ be the unique polynomial with $g_i(\alpha_\pi) = \alpha_i$ and $\deg(g_i) < n$. Let g be the unique polynomial of degree $< n$ with $g(\alpha_\pi) = \alpha_1\alpha_2$. Then $g_1 g_2 = g + h\pi$ with $h \in F[t]$ of degree $< p$ or $h = 0$. We have to prove that $d(f, g_1 g_2) \equiv d(f, g) \pmod{B_{n-1}}$, where $f = \frac{\pi}{\mu_{\pi(1)}}$. If $h = 0$ there is nothing to prove, so assume $h \neq 0$. From $g_1 g_2 = g + h\pi$ it follows that $\frac{g}{g_1 g_2} + \frac{h\pi}{g_1 g_2} = 1$. Put $\mu = \mu_{\pi(1)}$. Then

$$d\Big(\frac{\pi}{\mu}, \frac{g}{g_1 g_2}\Big) = -d\Big(\frac{\mu h}{g_1 g_2}, \frac{g}{g_1 g_2}\Big).$$

This is an element of $B_n \cap A_{n-1}$. We have to prove that it is in B_{n-1}.

Put $g_1 g_2 = k$. Then $k = g + h\pi$. Dividing by factors t we can assume that at most one of the polynomials k, g and h has a factor t and likewise for $1-t$. This results in sixteen cases to consider. Using the automorphism $t \mapsto 1-t$ this number of cases can be reduced to eleven, and by interchanging the role of g and μh the number can be further reduced to seven. We will prove that in each of these seven cases $d(\frac{h\mu}{k}, \frac{g}{k}) \in B_{n-1}$. For this we apply Lemma 3: we remove factors t and $1-t$ and reduce it modulo B_{n-1} to an element $d(\tau_1, \tau_2)$ with τ_1 and τ_2 polynomials of degree ≤ 1 not vanishing in 0 or 1. In each case we show that $d(\tau_1, \tau_2) \in B_1$. If τ_i is not a constant, we denote by a_i the zero of τ_i. By $\tilde{\tau}_i$ we denote the normalized polynomial: $\tilde{\tau}_i = \frac{\tau_i}{\tau_i(0)}$. When the τ_i are not constant we have $d(\tau_1, \tau_2) = \lceil \tau_2(a_1), \tilde{\tau}_1(1) \rceil - \lceil \tau_1(a_2), \tilde{\tau}_2(1) \rceil$ (also when $a_1 = a_2$). When τ_2 is a constant c and τ_1 is not, the formula becomes $d(\tau_1, c) = \lceil c, \tilde{\tau}_1(1) \rceil$. When both are constant it becomes 0.

(1) $t, 1-t \nmid k, g, h$.
Then $k(0) = g(0) + h(0)$ and $k(1) = g(1) + h(1)\mu(1)$. So $\frac{g(0)}{k(0)} + \frac{h(0)}{k(0)} = 1$ and $\frac{g(1)}{k(1)} + \frac{h(1)\mu(1)}{k(1)} = 1$. In this case $\tau_1 + \tau_2 = 1$ and therefore $d(\tau_1, \tau_2) = 0$.

(2) $t \mid k$, $t \nmid g, h$; $1-t \nmid k, g, h$.
Put $k = t^m k_0$ with $k_0(0) \neq 0$. Then $\frac{g(0)}{k_0(0)} + \frac{h(0)}{k_0(0)} = 0$ and $\frac{g(1)}{k_0(1)} + \frac{h(1)\mu(1)}{k_0(1)} = 1$. Here $\tau_1 + \tau_2 = t$ and so $d(\tau_1, \tau_2) = d(\tau_1, t - \tau_1) = d(\tau_1, 1 - \frac{t}{\tau_1}) = d(t, 1 - \frac{t}{\tau_1}) \in B_{n-1}$ by Lemma 1(i).

(3) $t \mid g$, $t \nmid k, h$; $1-t \nmid k, g, h$.
Put $g = t^m g_0$ with $g_0(0) \neq 0$. Then $\frac{h(0)}{k(0)} = 1$ and $\frac{g_0(1)}{k(1)} + \frac{h(1)\mu(1)}{k(1)} = 1$. Put $\frac{g_0(0)}{k(0)} = a$ and $\frac{g_0(1)}{k(1)} = b$. Then

$$\begin{cases} \tau_1 = (b-a)t + a, \\ \tau_2 = -bt + 1. \end{cases}$$

It suffices to show that $d(\tau_1, \tau_2) \in B_1$. Modulo B_1 we have

$$\begin{aligned} d(\tau_1, \tau_2) &= \left\lceil \frac{ab - a + b}{b - a}, \frac{b}{a} \right\rceil - \left\lceil \frac{ab - a + b}{b}, 1 - b \right\rceil \\ &\equiv \left\lceil ab - a + b, \frac{b}{a(1-b)} \right\rceil - \left\lceil b - a, \frac{b}{a} \right\rceil \equiv \left\lceil -a(1-b), \frac{b}{a(1-b)} \right\rceil - \left\lceil -a, \frac{b}{a} \right\rceil \\ &\equiv \lceil -a, b \rceil - \lceil -a, b \rceil \equiv 0 \pmod{B_1}. \end{aligned}$$

(4) $t, 1-t \mid k$; $t, 1-t \nmid g, h$.
Put $k = t^{m_1}(1-t)^{m_2} k_(0)$ with $k_0(0), k_0(1) \neq 0$. Then $\frac{g(0)}{k_0(0)} + \frac{h(0)}{k_0(0)} = 0$ and $\frac{g(1)}{k_0(1)} + \frac{h(1)\mu(1)}{k_0(1)} = 0$. Here $\tau_1 + \tau_2 = 0$ and so $d(\tau_1, \tau_2) = 0$.

(5) $t \mid g$; $t \nmid k, h$; $1-t \mid k$; $1-t \nmid g, h$. Put $k = (1-t)^{m_1} k_0$ and $g = t^{m_2} g_0$ with $k_0(1) \neq 0$ and $g_0(0) \neq 0$. Then $(1-t)^{m_1} = \frac{t^{m_2} g_0}{k_0} + \frac{h\mu}{k_0}$. In this case $\tau_1 = (b-a)t + a$ and

$\tau_2 = -(b+1)t+1$, where $a = \frac{g_0(0)}{k_0(0)}$ and $b = \frac{g_0(1)}{k_0(1)}$. Here $\tau_1+\tau_2 = (a+1)(1-t)$ and so $d(\tau_1, \tau_2) = d(\tau_1, (a+1)(1-t) - \tau_1) = d(\tau_1, 1 - \frac{(a+1)(1-t)}{\tau_1}) \in B_{n-1}$ by Lemma 1(ii).

(6) $t, 1-t \mid g$; $t, 1-t \nmid k, h$.
Put $g = t^{m_1}(1-t)^{m_2} g_0$ with $g_0(0), g_0(1) \neq 0$. Then $\frac{t^{m_1}(1-t)^{m_2} g_0}{k} + \frac{h}{k} = 1$. In this case $\tau_2 = 1$ and so $d(\tau_1, \tau_2) = 0$.

(7) $t \mid h$; $t \nmid k, g$; $1-t \mid g$; $1-t \nmid k, h$.
Put $h = t^{m_1} h_0$ and $g = (1-t)^{m_2} g_0$ with $h_0(0) \neq 0$ and $g_0(1) \neq 0$. Then $\frac{(1-t)^{m_2} g_0}{k} + \frac{t^{m_1} h_0}{k} = 1$. Here $\tau_1 = (b-1)t+1$ and $\tau_2 = (1-a)t + a$, where $a = \frac{g_0(0)}{k(0)}$ and $b = g_0(1)k(1)$. Modulo B_1 we have

$$\begin{aligned} d(\tau_1, \tau_2) &= \left\lceil \frac{1-ab}{1-b}, b \right\rceil - \left\lceil \frac{1-ab}{1-a}, \frac{1}{a} \right\rceil \\ &\equiv \lceil 1-ab, b \rceil + \lceil 1-ab, a \rceil \equiv \lceil 1-ab, ab \rceil \equiv 0 \pmod{B_1}. \end{aligned}$$ ☐

Finally we have:

Proposition 4. *For all $n \geq 2$ the inclusion $B_n \subseteq A_n$ induces a group isomorphism $B_n/B_{n-1} \xrightarrow{\sim} A_n/A_{n-1}$.*

Proof. The group A_n is generated by A_{n-1} and elements of the type (α, π) with $\deg(\pi) = n$. Write $\alpha = f(\alpha_\pi)$ where $f \in F[t]$ is of degree less than n. Then $d(\frac{\pi}{\mu_{\pi(1)}}, f) \equiv (\alpha, \pi) \pmod{A_{n-1}}$ and it follows that the map is surjective, or equivalently, the group homomorphism

$$\psi\colon \bigoplus_{\pi \in S_n \setminus S_{n-1}} F(\alpha_\pi)^* \to B_n/B_{n-1}$$

induced by the ψ_π is a right inverse of $B_n/B_{n-1} \to A_n/A_{n-1}$. The image of ψ is generated by elements $d(\frac{\pi}{\mu_{\pi(1)}}, f) + B_{n-1}$ with $\pi \in S$, $\deg(\pi) = n$, $f \in F[t]$ and $\deg(f) < n$. For the surjectivity of ψ it remains to show that for $\pi_1, \pi_2 \in S$, $\deg(\pi_1) = \deg(\pi_2) = n$ and $\mu_1 = \mu_{\pi_1(1)}$ the element $d(\frac{\pi_1}{\mu_1}, \pi_2) + B_{n-1}$ is in the image of ψ. Put $\mu_2 = \mu_{\pi_2(1)}$. Then $\mu_2\pi_1 - \mu_1\pi_2 = (t-t^2)h$ with $h \in F[t]$ of degree $< n$. From

$$\frac{\mu_2\pi_1}{(t-t^2)h} - \frac{\mu_1\pi_2}{(t-t^2)h} = 1$$

the following identity follows

$$d\Big(\frac{\pi_1}{\mu_1}, \pi_2\Big) = d\Big(\frac{\pi_1}{\mu_1}, -\frac{(t-t^2)h}{\mu_1}\Big) + d\Big(\frac{(t-t^2)h}{\mu_1\mu_2}, -\frac{\mu_1\pi_2}{(t-t^2)h}\Big),$$

which shows that $d(\frac{\pi_1}{\mu_1}, \pi_2) + B_{n-1}$ is in the image of ψ. ☐

6 Relative reciprocities III

In this section a description of relative reciprocity is derived which is in essence the same as the one in [1]. The notations of section 2 will be used. There are no restrictions on the ideal $\mathfrak{a}$ of the Dedekind domain. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ the prime ideals of R which divide $\mathfrak{a}$. Put $v_{\mathfrak{p}_i}(\mathfrak{a}) = k_i$ for $i = 1, \ldots, r$.

For a maximal ideal $\mathfrak{p}$ of R we write

$$U_{\mathfrak{p}} = \mathrm{Ker}((R/\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})+1})^* \to (R/\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})})^*.$$

For $\mathfrak{p} \in S$ this means that $U_{\mathfrak{p}} = (R/\mathfrak{p})^*$ and for $\mathfrak{p}_i \mid \mathfrak{a}$ we have

$$U_{\mathfrak{p}_i} = \mathrm{Ker}((R/\mathfrak{p}^{k_i+1})^* \to (R/\mathfrak{p}^{k_i})^*)(\cong R/\mathfrak{p}).$$

For $\mathfrak{p}$ a maximal ideal of R, $g \in 1 + \mathfrak{a}_S$ and $f \in Q^*$ we define

$$d_{\mathfrak{p}}(g, f) = \begin{cases} (g, f)_{\mathfrak{p}} & \text{if } \mathfrak{p} \in S \\ \overline{g}^{v_{\mathfrak{p}}(f)} & \text{if } \mathfrak{p} \mid \mathfrak{a}. \end{cases}$$

We put $d(g, f) = (d_{\mathfrak{p}}(g, f))_{\mathfrak{p}} \in \bigoplus_{\mathfrak{p}} U_{\mathfrak{p}}$, where the direct sum is over all maximal ideals of R. Let B be the subgroup of $\bigoplus_{\mathfrak{p} \in S} (R/\mathfrak{p})^*$ generated by all $d(1 - af, f)$ with $a \in \mathfrak{a}_S$ and $f \in R_S \setminus \{0\}$, and B^+ the subgroup of $\bigoplus_{\mathfrak{p}} U_{\mathfrak{p}}$ generated by all $d(g, f)$ with $g \in 1 + \mathfrak{a}_S$ and $f \in Q^*$.

Remark. Note that $d_{\mathfrak{p}}(g, f)$ can differ from $(g, f)_{\mathfrak{p}}$ for $\mathfrak{p} \mid \mathfrak{a}$. Therefore the meaning of the notation $d(g, f)$ differs from the notation used in the previous section.

Theorem 3. *Let A be an Abelian group and let $(\chi_{\mathfrak{p}})_{\mathfrak{p}}$ be a collection of group homomorphisms $\chi_{\mathfrak{p}} \colon U_{\mathfrak{p}} \to A$. Then the following two conditions are equivalent:*

(i) $\sum_{\mathfrak{p}} \chi_{\mathfrak{p}}(d_{\mathfrak{p}}(g, f)) = 0$ *for all* $g \in 1 + \mathfrak{a}_S$ *and* $f \in Q^*$.

(ii) $(\chi_{\mathfrak{p}})_{\mathfrak{p} \in S}$ *is an* $\mathfrak{a}$*-reciprocity on* R.

Proof. We have a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & B \arrow[r] \arrow[d] & \bigoplus_{\mathfrak{p} \in S} (R/\mathfrak{p})^* \arrow[r] \arrow[d] & \mathrm{K}_1(R, \mathfrak{a}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^+ \arrow[r] \arrow[d] & \bigoplus_{\mathfrak{p}} U_{\mathfrak{p}} \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & B^+/B \arrow[r] \arrow[d] & \bigoplus_{\mathfrak{p} \mid \mathfrak{a}} U_{\mathfrak{p}} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

The theorem is equivalent to $B^+/B \to \bigoplus_{\mathfrak{p}|\mathfrak{a}} U_\mathfrak{p}$ being an isomorphism. We will construct an inverse. Choose $\pi_1, \dots, \pi_r \in R$ as in section 4. For each j we have to construct a homomorphism $\rho_j \colon U_{\mathfrak{p}_j} \to B^+/B$. Let $\overline{g} \in U_{\mathfrak{p}_j}$ where $g \in R$. By the Chinese remainder theorem there is a $g' \in R$ with $g' \equiv g \pmod{\mathfrak{p}_j^{k_j+1}}$ and $g' \equiv 1 \pmod{\mathfrak{a}}$. Put

$$\rho_j(\overline{g}) = d(g', \pi_j) + B.$$

This is well-defined: if also $g'' \in R$ with $g'' \equiv g \pmod{\mathfrak{p}_j^{k_j+1}}$ and $g'' \equiv 1 \pmod{\mathfrak{a}}$, then $g'(g'')^{-1} \in 1 + \mathfrak{a}_S$ and $v_{\mathfrak{p}_j}(g'(g'')^{-1}) = 0$, from which it follows that $d(g', \pi_j) - d(g'', \pi_j) \in B$. Since $d(g, f)$ is linear in the first variable, the map ρ_j is a homomorphism. Under $B^+/B \to \bigoplus_{\mathfrak{p}|\mathfrak{a}} U_\mathfrak{p}$ the element $d(g', \pi_j) + B$ maps to $((\overline{g'})^{v_{\mathfrak{p}_i}(\pi_j)})_{\mathfrak{p}_i}$. For $i \neq j$ the ith component is trivial and for $i = j$ it is $\overline{g}$.

It remains to check that we constructed a left inverse. Since $d(g, f)$ is linear in the second variable and $d(g, f) \in B$ for $f \in R_S^*$, it suffices to check this on elements $d(g, \pi_j) \in B^+$ with $g \in 1 + \mathfrak{a}_S$. Such a g is a quotient of two elements in $1 + \mathfrak{a}$, so we can assume that $g \in 1 + \mathfrak{a}$. For such $d(g, \pi_j) + B$ it remains to show that $d(g, \pi_j) + B = d(g', \pi_j) + B$, where $g' \in R$ such that $g' \equiv 1 \pmod{\mathfrak{a}}$ and $g' \equiv g \pmod{\mathfrak{p}_j^{k_j+1}}$. This follows from $\frac{1 - g(g')^{-1}}{\pi_j} \in R_S$. ☐

Finally we show that the notion of relative reciprocity on a Dedekind domain is equivalent to the one defined in [1].

Theorem 4. *Let A be an Abelian group and let $(\chi_\mathfrak{p})_\mathfrak{p}$ be a collection of group homomorphisms $\chi_\mathfrak{p} \colon U_\mathfrak{p} \to A$. Then the following two conditions are equivalent:*

(i) $\sum_\mathfrak{p} v_\mathfrak{p}(f)\chi_\mathfrak{p}(\overline{g}) = \sum_\mathfrak{p} v_\mathfrak{p}(g)\chi_\mathfrak{p}(\overline{f})$ *for all* $f, g \in R$ *with* $g \equiv 1 \pmod{\mathfrak{a}}$, $f \neq 0$ *and* $(f, g) = R$.

(ii) $(\chi_\mathfrak{p})_{\mathfrak{p} \in S}$ *is an* $\mathfrak{a}$*-reciprocity on* R.

Proof. Let D be the subgroup of B generated by all $d(b, c)$ with $b, c \in R$, $b \equiv 1 \pmod{\mathfrak{a}}$, $c \neq 0$ and $(b, c) = R$. We will proof that $D = B$. Then the theorem follows from Theorem 3.

Let $g \in 1 + \mathfrak{a}_S$ and $f \in Q^*$. We will prove that $d(g, f) \in D$. This will be done by induction on the number n of maximal ideals $\mathfrak{p}$ with $v_\mathfrak{p}(g)v_\mathfrak{p}(f) \neq 0$.

(1) Let $n = 0$. In the group of fractional ideals of R write Rg and Rf as quotients of ideals of R which are relatively prime: $Rg = \frac{\mathfrak{b}_1}{\mathfrak{c}_1}$ and $Rf = \frac{\mathfrak{b}_2}{\mathfrak{c}_2}$. Since $g \in 1 + \mathfrak{a}_S$ the ideal $\mathfrak{c}_1$ is relatively prime to $\mathfrak{a}$. Choose an ideal $\mathfrak{c}_1'$ in the inverse ideal class of $\mathfrak{c}_1$ and relatively prime to $\mathfrak{a}\mathfrak{b}_2\mathfrak{c}_2$. Choose $c_1 \in R$ with $\mathfrak{c}_1\mathfrak{c}_1' = Rc_1$ and put $b_1 = c_1 g$. Choose $c_1' \in R$ such that $c_1 c_1' \equiv 1 \pmod{\mathfrak{a}}$ and c_1' relatively prime to $\mathfrak{b}_2\mathfrak{c}_2$. Then $g = \frac{b_1 c_1'}{c_1 c_1'}$ and $b_1 c_1', c_1 c_1' \equiv 1 \pmod{\mathfrak{a}}$. Write $f = \frac{b_2}{c_2}$ with $b_2 c_2$ relatively prime to $b_1 c_1'$ and to $c_1 c_1'$. Clearly $d(-,-)$ is bimultiplicative, so we have

$$d(g, f) = d(b_1c_1', b_2) + d(c_1c_1', c_2) - d(bc_1', c_2) - d(c_1c_1', b_2) \in D.$$

(2) Let $n = 1$ and let $\mathfrak{p}_1$ be the unique prime with $v_{\mathfrak{p}_1}(g)v_{\mathfrak{p}_1}(f) \neq 0$. Choose $\mathfrak{r}$ in the ideal class of $\mathfrak{p}_1$ relatively prime to all maximal ideals $\mathfrak{q}$ with $v_{\mathfrak{q}}(f) \neq 0$ or $v_{\mathfrak{q}}(\mathfrak{a}) \neq 0$. Choose $h \in Q^*$ such that $Rh = \frac{\mathfrak{r}}{\mathfrak{p}_1}$. Then $h \in R_S^*$. Choose an $h' \in R_S^*$ which is an inverse modulo $\mathfrak{a}_S^2$. Then for $k = hh'$ we have: $k \in 1 + \mathfrak{a}_S^2$, $v_{\mathfrak{q}}(k) \geq 0$ and $v_{\mathfrak{q}}(k)v_{\mathfrak{q}}(f) = 0$ for all maximal ideals $\mathfrak{q} \neq \mathfrak{p}_1$. Put $i = v_{\mathfrak{p}_1}(g)$ and $j = v_{\mathfrak{p}_1}(f)$. Then $d(k^i g, f) \in D$ by (1) and so $d(g, f) \equiv -id(k, f) \pmod{D}$. For all maximal ideals $\mathfrak{q}$ we have $v_{\mathfrak{q}}(k)v_{\mathfrak{q}}(f(1-k)^j) = 0$. By (1) we have $d(k, f(1-k)^j) \in D$. Therefore $d(k, f) \equiv -jd(k, 1-k) \pmod{D}$. Since $d_{\mathfrak{q}}(k, 1-k) = 1$ for all $\mathfrak{q} \in S$ it remains to show that $d_{\mathfrak{q}}(k, 1-k) = 1$ for all $\mathfrak{q} \mid \mathfrak{a}$. From $1 - k \in \mathfrak{a}_S^2$ it follows that for such $\mathfrak{q}$ we have $v_{\mathfrak{q}}(1-k) \geq 2v_{\mathfrak{q}}(\mathfrak{a}) \geq v_{\mathfrak{q}}(\mathfrak{a}) + 1$ and this implies that $d_{\mathfrak{q}}(k, 1-k) = 1$.

(3) Assume that $n > 1$. Let $\mathfrak{p}_1$ be a maximal ideal with $v_{\mathfrak{p}_1}(g)v_{\mathfrak{p}_1}(f) \neq 0$. Choose $h \in Q^*$ with $v_{\mathfrak{p}_1}(h) = -v_{\mathfrak{p}_1}(f)$ and $v_{\mathfrak{q}}(h)v_{\mathfrak{q}}(g) = 0$. Then by induction hypothesis and (2) it follows that $d(g, f) = d(g, fh) - d(g, h) \in D$. ☐

Remark. In the definition of an $\mathfrak{a}$-reciprocity given in [1] there is the extra condition

$$v_{\mathfrak{q}}(1-g)\chi_{\mathfrak{p}}(\overline{g}) = 0 \qquad \text{for } g \in 1 + \mathfrak{a} \text{ and } \mathfrak{q} \mid \mathfrak{a}.$$

This condition is however a consequence of the other condition: choose $g' \in R$ such that

$$\begin{cases} g' \equiv g \pmod{\mathfrak{q}^{v_{\mathfrak{q}}(\mathfrak{a})+1}} \\ g' \equiv 1 \pmod{\mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})+1}} \quad \text{for all } \mathfrak{p} \mid \mathfrak{a} \text{ with } \mathfrak{p} \neq \mathfrak{q}. \end{cases}$$

Assume $\overline{g} \neq \overline{1}$. Then $v_{\mathfrak{q}}(1-g) = v_{\mathfrak{p}}(1-g') = v_{\mathfrak{p}}(\mathfrak{a})$ and we have

$$0 = \sum_{\mathfrak{p}} \chi_{\mathfrak{p}}(d_{\mathfrak{p}}(g', 1-g')) = \sum_{\mathfrak{p}\mid\mathfrak{a}} v_{\mathfrak{p}}(1-g')\chi_{\mathfrak{p}}(\overline{g'}) = v_{\mathfrak{q}}(1-g)\chi_{\mathfrak{q}}(\overline{g}).$$

References

[1] Bass, H., *Algebraic K-Theory.* New York: Benjamin 1968

[2] Bass, H. and J. Tate, The Milnor ring of a global field. *Algebraic K-Theory II*, 349–446. Lecture Notes in Mathematics **342**. Berlin-Heidelberg-New York: Springer 1973

[3] Keune, F., $(t^2 - t)$-reciprocities on the affine line and Matsumoto's theorem, *Inventiones math.* **28**, 185–192 (1975)

[4] Keune, F., The relativization of K_2, *J. Algebra* **54**, 159–177 (1978)

[5] Maazen, H. and J. Stienstra, A Presentation for K_2 of split radical pairs, *Journal of Pure and Applied Algebra* **10**, 271–294 (1977)

[6] Matsumoto, H., Sur les sous-groupes arithmétique des groupes semi-simples déployés, *Ann. Sci. Ec. Norm. Sup.* **4**$^{\text{e}}$ série **2**, 1–62 (1969)

[7] Milnor, J., *An Introduction to Algebraic K-Theory.* Princeton: Princeton University Press 1971

[8] Quillen, D., Higher algebraic K-theory, in *Algebraic* K-*Theory*, Vol. I, Lecture Notes in Mathematics **341**, Springer-Verlag, Berlin 1973

Frans Keune
Institute for Mathematics, Astrophysics and Particle Physics
Radboud University Nijmegen
Postbus 9010
6500 GL Nijmegen, The Netherlands
`keune@math.ru.nl`