HIGHER K-THEORY OF ALGEBRAIC CURVES

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0. Introduction

In this paper we will construct a three-stage filtration on the algebraic K-theory of an algebraic curve Γ. An algebraic curve we mean is a scheme of dimension one over a field κ, which is smooth, proper, and geometrically connected. We need a finite morphism $\pi : \Gamma \longrightarrow \Pi$ onto the projective line Π which should satisfy the condition on the induced invertible sheaf $L = \pi^* \mathcal{O}_\Pi(1)$ that $\dim H^0(\Gamma, L) = 2$ and $\dim H^1(\Gamma, L) = 0$. The base field can be arbitrary, and if a curve is given it is possible to have such a morphism after a finite extension of the base field (2.1.1), so it is a mild condition.

The filtration produces a spectral sequence converging to the K-groups of the curve where each E_1-term is a filtered colimit of K-groups of finite dimensional κ-algebras (3.3). As an application we prove that the K-groups with $\mathbb{Q}$-coefficients of a scheme over a finite field are zero in above level greater than the dimension of the scheme (4.1).

For a scheme X we denote by $\mathsf{sPerf}(X)$ the category of bounded complexes of locally free coherent sheaves on X, for it is usually called the category of strict perfect complexes [9]. Applying Waldhausen's S-construction [10] to it, we define the K-theory of X as a topological space $K(X)$. We will review basic definitions and results on the construction in the first section. Other than a technical extension of the devissage theorem in 1.2, these are to fix notations.

It is well-known that the K-theory of the projective line Π is equivalent to the product of two copies of K-theory of the base field. We can prove this fact in the following way. Let $\mathsf{sPerf}(\Pi)\langle a, b\rangle$ be the subcategory of $\mathsf{sPerf}(\Pi)$ whose objects are complexes having a term in each level isomorphic to a direct sum of copies of $\mathcal{O}_\Pi(i)$ with $a \leq i \leq b$. $\mathsf{sPerf}(\Pi)\langle a, a\rangle$ is equivalent to $\mathsf{sPerf}(\operatorname{Spec}\kappa)$, and the K-theory of $\mathsf{sPerf}(\Pi)\langle a, a+1\rangle$ is equivalent to the product of two copies of $K(\operatorname{Spec}\kappa)$. We see that whenever $b \geq a+1$, the inclusion $\mathsf{sPerf}(\Pi)\langle a, a+1\rangle \longrightarrow \mathsf{sPerf}(\Pi)\langle a, b\rangle$ induces an equivalence in K-theory (1.6).

In the second section we will study a subcategory $\mathcal{P}$ of $\mathsf{sPerf}(\Gamma)$, analogous to $\mathsf{sPerf}(\Pi)\langle -2, 0\rangle$. $\mathcal{P}$ is the full subcategory whose objects have a term in each level of complexes isomorphic to a direct sum of copies of invertible sheaves L^i for $-2 \leq i \leq 0$.

We first show that the K-theory of function field of Γ is equivalent to that of $\mathcal{P}$ taking weak equivalences u which are quasi-isomorphisms at the generic point of Γ.

If we take usual quasi-isomorphisms w as weak equivalences, it defines $K(\mathcal{P}, w)$ which is equivalent to the K-theory of Π. We get a homotopy fiber sequence (2.4), where the base is the K-theory of the function field, the middle term is $K(\mathcal{P}, w)$ and the fiber is K -theory of $\mathcal{P}^u$, the subcategory of complexes with torsion homologies.

In the third section we discuss the K-theory of $\mathcal{P}^u$. We reduce it to the K-theory of finite dimensional algebras over κ. These algebras are not commutative and depends on global geometry of the curve. Therefore it seems difficult to describe

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them even in concrete examples. Hence our method does not give any explicit description of the K-groups of the curve, or elements on them. Nevertheless we can prove some vanishing results. In the last section we get an application on the K-groups with $\mathbb{Q}$-coefficients of schemes over finite fields. Another application to the Tate-Thomason conjecture on étale cohomology will be treated in the other paper [5].

0.1. Convention. A complex means a chain complex whose boundary morphisms decrease level of terms by one. The word degree is reserved for that of sheaves on a curve. We always assume complexes are bounded in both upper and lower levels.

Schemes are assumed to be essentially of finite type over a field, irreducible and separated. The dimension of a scheme means the transcendental degree of the function field over the prime field.

Short exact sequences of abelian groups or chain complexes of these will be presented as $\bullet \rightarrowtail \bullet \twoheadrightarrow \bullet$ in diagrams. Similarly, cofibration sequences of objects or exact functors between Waldhausen categories (1.1) will be. For locally free sheaves $a \rightarrowtail b$ means a is a subbundle of b, that is, the quotient b/a is locally free.

We understand K-theory is a functor from Waldhausen categories with exact functors to topological spaces. $|X.|$ means the geometric realization for a simplicial set $X.$. We use the identical symbol for a category and its nerve, and for a bi-simplicial set and its diagonal simplicial set.

If X_i, $i = 1, 2, \dots$ is an infinite sequence of topological spaces, $\prod^{\oplus}$ means the colimit of finite products of X_i,

$$\prod^{\oplus} X_i = \operatorname*{colim}_j \prod_{i=1}^{j} X_i :$$

hence $\pi_*(\prod^{\oplus} X_i) = \oplus \pi_*(X_i)$.

A κ-algebra means an associative and unital, but not necessary commutative ring over a field κ.

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1. Waldhausen categories and K-theory

1.0. We will define the K-theory $K(X)$ associated to a scheme X. Among many equivalent constructions, we use Waldhausen's S-construction ([10]) associated to the category of bounded chain complexes of locally free sheaves on X ([9]). We review basic definitions and constructions in 1.1. Then we see a variant of devissage theorem in 1.2. $K(X)$ is defined in 1.3. 1.4 treats exact categories. We study in 1.5 K-theory of projective line Π.

1.1. Waldhausen categories

1.1.0. Let $\mathcal{A}$ be a category with a specified zero object. We call a subcategory $co(\mathcal{A})$ defines a structure of a category with cofibrations on $\mathcal{A}$ if it satisfies the following three axioms:

- All the isomorphisms of $\mathcal{A}$ are in $co(\mathcal{A})$.
- The unique morphism $0 \longrightarrow a$ is in $co(\mathcal{A})$ for any object a of $\mathcal{A}$.
- For any morphism $a \longrightarrow b$ in $co(\mathcal{A})$ and an arbitrary morphism $a \longrightarrow c$ in $\mathcal{A}$, the pushout $b \cup_a c$ exists in $\mathcal{A}$ and the canonical morphism $c \longrightarrow b \cup_a c$ is in $co(\mathcal{A})$.

Morphisms in $co(\mathcal{A})$ are called cofibrations and are presented as $\rightarrowtail$ in diagrams. For a cofibration $a \rightarrowtail b$, its quotient $b/a = 0 \cup_a b$ is well-defined up to isomorphism. We present such a situation as $a \rightarrowtail b \twoheadrightarrow b/a$, and call it a cofibration sequence.

A category with cofibrations $(\mathcal{A}, co(\mathcal{A}))$ and a subcategory $w(\mathcal{A})$ of $\mathcal{A}$ defines a Waldhausen category if it satisfies the following two axioms:

- All the isomorphisms in $\mathcal{A}$ are in $w(\mathcal{A})$.
- For any commutative diagram

$$\begin{tikzcd} b \arrow[d] & a \arrow[l, tail] \arrow[r] \arrow[d] & c \arrow[d] \\ b' & a' \arrow[l, tail] \arrow[r] & c' \end{tikzcd}$$

  where all three vertical arrows are in $w(\mathcal{A})$, the induced morphism on the pushouts $b \cup_a c \longrightarrow b' \cup_{a'} c'$ is in $w(\mathcal{A})$.

Morphisms in $w(\mathcal{A})$ are called weak equivalences and are presented as $\xrightarrow{\sim}$ in diagrams.

The Waldhausen category consisting of triples $(\mathcal{A}, w(\mathcal{A}), co(\mathcal{A}))$ is often abbreviated as $(\mathcal{A}, w)$ or $\mathcal{A}$ if some of the components are obvious.

1.1.1. All the Waldhausen categories in this paper satisfy the two more properties: the saturation and extension axioms ([10, 1.2]). The saturation axiom says that if there are two morphisms x and y which are composable and either x and $x \circ y$, or y and $x \circ y$ are in $w\mathcal{A}$, so is the third; and the extension axiom says if there are two cofibration sequences combined with the following commutative diagram

$$\begin{tikzcd} a \arrow[r, tail] \arrow[d, "\wr"'] & b \arrow[r, two heads] \arrow[d] & b/a \arrow[d, "\wr"] \\ a' \arrow[r, tail] & b' \arrow[r, two heads] & b'/a', \end{tikzcd}$$

where vertical outside morphisms are weak equivalences, then the middle arrow is also a weak equivalence.

1.1.2. An exact functor between two Waldhausen categories is a functor which preserves cofibrations, pushouts along cofibrations and weak equivalences.

1.1.3. A Waldhausen category $(\mathcal{A}, w)$ defines a series of Waldhausen categories $S_i(\mathcal{A})$ for $i = 0, 1, \ldots$ as follows. An object of $S_i(\mathcal{A})$ is a sequence of cofibrations $a. : a_1 \rightarrowtail a_2 \rightarrowtail \ldots \rightarrowtail a_i$. A morphism between $a.$ and $b.$ is a series of morphisms $a_j \longrightarrow b_j$ making the following commutative diagram.

$$\begin{array}{ccccccc} a_1 & \rightarrowtail & a_2 & \rightarrowtail & \cdots & \rightarrowtail & a_i \\ \downarrow & & \downarrow & & & & \downarrow \\ b_1 & \rightarrowtail & b_2 & \rightarrowtail & \cdots & \rightarrowtail & b_i \end{array}$$

$S_i(\mathcal{A})$ becomes a Waldhausen category, where a cofibration is a morphism such that $a_k \cup_{a_j} b_j \rightarrowtail b_k$ is a cofibration in $\mathcal{A}$ for any j and k, and a weak equivalence is a morphism such that $a_j \longrightarrow b_j$ is a weak equivalence in $\mathcal{A}$ for any j.

Under choices of quotients a_i/a_j, $S.(\mathcal{A})$ becomes a simplicial Waldhausen category ([10, 1.3]).

The categories $wS_i(\mathcal{A})$ of weak equivalences in $S_i(\mathcal{A})$ form a simplicial category, and we will define the K-theory of $\mathcal{A}$ as a topological space

$$K(\mathcal{A}, w) = K(\mathcal{A}) = \Omega|wS.(\mathcal{A})|,$$

where the last term means the loop space of the geometric realization of the simplicial category.

1.1.4. Any exact functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ between Waldhausen categories induces a continuous mapping $K(F) : K(\mathcal{A}) \longrightarrow K(\mathcal{B})$. An exact functor F is an equivalence in K-theory if $K(F)$ induces a homotopy equivalence between the topological spaces. If there is a natural transformation between exact functors F and F' which induces a weak equivalence $F(a) \xrightarrow{\sim} F'(a)$ for any object a, then it induces an equivalence $K(F) \simeq K(F')$ in K-theory ([10, 1.3.1]).

1.1.5. Let F, F', F'' be three exact functors from $\mathcal{A}$ to $\mathcal{B}$ which are connected by natural transformations $F' \longrightarrow F$ and $F \longrightarrow F''$, satisfying the following two conditions:

- For any object a in $\mathcal{A}$, $F'a \rightarrowtail Fa \twoheadrightarrow F''a$ is a cofibration sequence in $\mathcal{B}$.
- For any cofibration $a' \rightarrowtail a$ in $\mathcal{A}$, the induced morphism

  $F'a \cup_{F'a'} Fa \rightarrowtail F'a$ is a cofibration in $\mathcal{B}$.

Then there is a homotopy between two mappings

$$KF \simeq KF' + KF'' : K(\mathcal{A}) \longrightarrow K(\mathcal{B}),$$

where the sum means the homotopy sum of mappings between loop spaces. This is called the additivity theorem ([10, 1.4]). It will be presented in diagrams as $F' \rightarrowtail F \twoheadrightarrow F''$.

There is another formulation of the additivity theorem. Assume there are exact inclusion functors from $\mathcal{A}\,, \mathcal{B}$ into $\mathcal{C}$. We define the Waldhausen category $E(\mathcal{A}\,, \mathcal{C}\,, \mathcal{B})$ consisting of cofibration sequences in $\mathcal{C}$, $a \rightarrowtail c \twoheadrightarrow b$, where a is an object in $\mathcal{A}$ and b is an object in $\mathcal{B}$. Then we have an homotopy equivalence ([10, 1.3.2])

$$K(E(\mathcal{A}\,, \mathcal{C}\,, \mathcal{B})) \simeq K(\mathcal{A}) \times K(\mathcal{B}).$$

1.1.6. Assume there is a Waldhausen category $\mathcal{A}$ whose weak equivalences are only isomorphisms. Suppose each object a of $\mathcal{A}$ is equipped with a finite decreasing filtration of index from m to n $(m \geq n)$ $\;0 = F^{m+1}a \rightarrowtail \cdots \rightarrowtail F^n a = a\;$ so that any morphism preserves the filtrations, and for any cofibration $\;a \rightarrowtail b\;$, the induced morphisms define cofibrations $\;F^i b \cup_{F^i a} b \rightarrowtail b\;$ which satisfy $F^i(F^i b \cup_{F^i a} b) = F^i b$ for all i.

Let us denote by $\mathcal{A}\langle i, j\rangle$ the subcategory whose objects satisfy $0 = F^{i+1}a$ and $F^j a = a$ for each pair $i \geq j$. Assume there exist a pair of integers m and n such that $\mathcal{A}$ is equal to $\mathcal{A}\langle m, n\rangle$. There is an exact functor

$$\mathcal{A} \longrightarrow E(\mathcal{A}\langle m, m\rangle\,, \mathcal{A}\,, \mathcal{A}\langle m-1, n\rangle),$$

and it gives an equivalence $K(\mathcal{A}) \simeq K(\mathcal{A}\langle m, m\rangle) \times K(\mathcal{A}\langle m-1, n\rangle)$. If we repeat the process, we get an equivalence

$$K(\mathcal{A}) \simeq K(\mathcal{A}\langle m, m\rangle) \times K(\mathcal{A}\langle m-1, m-1\rangle) \times \cdots \times K(\mathcal{A}\langle n, n\rangle).$$

We call $\mathcal{A}\langle i, i\rangle$ the subcategory of pure pieces with index i.

1.2. Devissage theorem

1.2.0. An exact functor $\;F : \mathcal{A} \longrightarrow \mathcal{B}\;$ between Waldhausen categories defines a simplicial Waldhausen category $S.(F)$ as follows: Let $S_n(F)$ be the pullback of categories $\;S_n\mathcal{A} \xrightarrow{S_n(F)} S_n\mathcal{B} \xleftarrow{d_0} S_{n+1}\mathcal{B}\;$, that is, an object of $S_n(F)$ is a pair of filtered objects $\;b_0 \rightarrowtail b_1 \rightarrowtail \ldots \rightarrowtail b_n\;$ and $\;a_1 \rightarrowtail a_2 \rightarrowtail \ldots \rightarrowtail a_n\;$, together with isomorphisms of filtered objects

$$F(a_1) \rightarrowtail \ldots \rightarrowtail F(a_n) \;\approx\; b_1/b_0 \rightarrowtail \ldots \rightarrowtail b_n/b_0,$$

plus choices of quotients. Then we have a homotopy fiber sequence ([10, 1.5.4])

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(S.(F)),$$

where the last term means the loop space of the geometric realization of the bi-simplicial category $wS.(S.(F))$.

1.2.1. Assume $\mathcal{A}$ is a Waldhausen category whose weak equivalences are just isomorphisms. The sets of objects of $S_n(\mathcal{A})$, determine a simplicial set $s_n(\mathcal{A})$ and the geometric realization of it is homotopy equivalent to the geometric realization of the bi-simplicial set $iS.(\mathcal{A})$.

If there is an exact functor between such categories $\;F : \mathcal{A} \longrightarrow \mathcal{B}\;$, we get a bi-simplicial set $s.S.(F)$ as above, and it makes a homotopy equivalence $K(S.(F)) \simeq \Omega|s.S.(F)|$ ([10, Corollary of 1.4.1]).

1.2.2. The devissage theorem is usually stated only the cases when there is an abelian subcategory $\mathcal{A}$ in an abelian category $\mathcal{B}$. The theorem says if $\mathcal{A}$ is closed under extensions and direct sums, and every object in $\mathcal{B}$ has a filtration with filtered quotients are belong to $\mathcal{A}$, then $\mathcal{A}$ and $\mathcal{B}$ are equivalent in K-theory. Staffeldt proved ([8, §4]) the devissage theorem in the context of Waldhausen categories. The proof works fine in the following setup.

Let us assume $\mathcal{B}$ is an additive category and it is a category with cofibrations where the inclusions into direct sums are cofibrations. We regard it as a Waldhausen category where all the weak equivalences are isomorphisms. An exact full subcategory $\iota : \mathcal{A} \subset \mathcal{B}$ has the devissage property if the following conditions hold:

(i) All the cofibrations in $\mathcal{B}$ are monomorphisms.

(ii) For any pair of cofibrations $a \rightarrowtail b$ and $c \rightarrowtail b$ in $\mathcal{B}$, the pullback $a \cap c$ and the pushout of $a \leftarrowtail a \cap c \rightarrowtail c$ exist. We denote the pushout by $a + c$. And all the morphisms in the diagram below are cofibrations:

$$\begin{tikzcd} a \cap c \arrow[r, tail] \arrow[d, tail] & a \arrow[d, tail] & \\ c \arrow[r, tail] & a + c \arrow[r, tail] & b. \end{tikzcd}$$

(iii) For any commutative diagram of cofibrations as

$$\begin{tikzcd} d \arrow[r, tail] \arrow[d, tail] & a \arrow[d, tail] \\ c \arrow[r, tail] & b, \end{tikzcd}$$

the induced morphism $d \rightarrowtail a \cap c$ is a cofibration. And it becomes a cofibration in $S_2(\mathcal{B})$ if and only if $d \cong a \cap c$.

(iv) For any sequence of cofibrations $a_0 \rightarrowtail a_1 \rightarrowtail b$, if there is another sequence as $c \rightarrowtail b$ in $\mathcal{B}$, then $a_0 \cap c \rightarrowtail a_1 \cap c$ is a cofibration.

(v) $\mathcal{A}$ is closed under isomorphisms and direct sums in $\mathcal{B}$. And a morphism of $\mathcal{A}$ is a cofibration in $\mathcal{A}$ if and only if it is a cofibration in $\mathcal{B}$.

(vi) If there is an exact sequence in $\mathcal{B}$ $a \rightarrowtail b \twoheadrightarrow c$, and b is contained in $\mathcal{A}$, so are a and c.

(vii) Each object b of $\mathcal{B}$ has a filtration $0 = b_0 \rightarrowtail b_1 \rightarrowtail \cdots \rightarrowtail b_p = b$ such that every quotient b_i/b_{i-1} is contained in $\mathcal{A}$.

Then the devissage theorem says that ι induces an equivalence in K-theory.

1.2.3. We reproduce the proof of Staffeldt for the sake of completeness, since we need a technical modification. The inclusion functor $\iota : \mathcal{A} \subset \mathcal{B}$ defines a simplicial set $s.(\iota)$ as in 1.5.1. Let $\mathrm{simp}(s.(\iota))$ be its category of simplicies ([10, p.337 and p.355]). An object of it is a pair $([k], x)$, where $[k]$ is the finite ordered set consisting of $k{+}1$ elements, and x is an element in $s_k(\iota)$, where a morphism $([k], x) \longrightarrow ([l], y)$ consists of a order preserving mapping $\alpha : [k] \longrightarrow [l]$ so that $\alpha^* x = y$. There is the last vertex functor $L : \mathrm{simp}(s.(\iota)) \longrightarrow co(\mathcal{B})$ as $([k], b_0 \rightarrowtail \ldots \rightarrowtail b_k) \longmapsto b_k$ ([10, p.359]). We want to show that L induces a homotopy equivalence. By the theorem A of Quillen it suffices to show that for any object $c \in \mathcal{B}$, $|L/c|$ is contractible ([7, p.85]). But L/c is equivalent to the category of simplicies of a simplicial set N_c, which has k-simplices as $b_0 \rightarrowtail \cdots \rightarrowtail b_k \rightarrowtail c$ where $b_j/b_0 \in \mathcal{A}$. If $|N_c|$ is contractible, $|s.(\iota)|$ is contractible for $|co(\mathcal{B})|$ is contractible. $|N_c|$ is contractible by the following steps of arguments.

a: Chose a filtration $0 = c_0 \rightarrowtail \ldots \rightarrowtail c_p = c$ so that $c_i/c_{i-1} \in \mathcal{A}$.

b: Define a simplicial mapping $F_i : N_c \longrightarrow N_c$ for each i $(0 \le i \le p)$ which sends $b_0 \rightarrowtail \cdots \rightarrowtail b_k \rightarrowtail c$ to $b_0 + c_i \rightarrowtail \cdots \rightarrowtail b_k + c_i \rightarrowtail c$. Because there exists a quotient morphism $b_j/b_0 \twoheadrightarrow (b_j + c_i)/(b_0 + c_i)$, F_i takes values in N_c. F_0 is the identity and F_p is constant.

c: A simplicial homotopy between F_i and F_{i+1} can be constructed as follows. For a morphism $\alpha : [k] \longrightarrow [1]$ like $0 = \alpha(0) = \ldots \alpha(t)$, $1 = \alpha(t+1) = \cdots = \alpha(k)$ and a k-simplex $b_0 \rightarrowtail \ldots \rightarrowtail b_k \rightarrowtail c$, its value is

$$b_0 + c_{i-1} \rightarrowtail \cdots \rightarrowtail b_t + c_{i-1} \rightarrowtail b_{t+1} + c_i \rightarrowtail \cdots \rightarrowtail b_k + c_i \rightarrowtail c.$$

There is a quotient morphism $b_j/b_0 \oplus c_i/c_{i-1} \twoheadrightarrow b_j + c_i/b_0 + c_{i-1}$ for any $t+1 \le j \le k$, and it defines a homotopy in N_c.

If the arguments are true in $\iota_n : S_n(\mathcal{A}) \longrightarrow S_n(\mathcal{B})$ for all $n \ge 2$, it implies that $K(\mathcal{A}) \simeq K(\mathcal{B})$ (1.2.0).

To apply the construction in step **a**, let $c = \ c_1 \rightarrowtail \ldots \rightarrowtail c_n$ be an element of $S_n(\mathcal{B})$. Chose a filtration as (vii) of 1.2.2, $c_{(n,1)} \rightarrowtail \ldots \rightarrowtail c_{(n,p)} = c_n$, so that all the quotients $c_{(n,j)}/c_{(n,j-1)}$ are contained in $\mathcal{A}$. Define $c_{i,j} = c_i \cap c_{(n,j)}$.

It defines a sequence of cofibrations in $S_n(\mathcal{B})$, $c_{(\cdot,1)} \rightarrowtail \ldots \rightarrowtail c_{(\cdot,p)} = c$. We get $c_{(i,j+1)}/c_{(i,j)} \cong (c_{(i,j+1)} + c_{(n,j)})/c_{(n,j)} \rightarrowtail c_{(n,j+1)}/c_{(n,j)}$, where the last term is contained in $\mathcal{A}$. Then $c_{(\cdot,j+1)}/c_{(\cdot,j)}$ is in $S_n(\mathcal{A})$.

For $b_0 \rightarrowtail \cdots \rightarrowtail b_k \rightarrowtail c$ in $\mathcal{B}$ where b_h/b_0 is contained in $\mathcal{A}$ for any $1 \le h \le k$. Chose a filtration $c_{(\cdot,1)} \rightarrowtail \cdots \rightarrowtail c_{(\cdot,p)} = c$ such that all the quotients $c_{(\cdot,j)}/c_{(\cdot,j-1)}$ belong to $S_n(\mathcal{A})$. We define $b_h + c_{(\cdot,j)}$ be an object in $S_n(\mathcal{B})$ where $(b_h + c_{(\cdot,j)})_i = (b_{(n,j)} + c_{(n,k)}) \cap c_{(i,p)}$. Then we know that

$$(b_h + c_{(\cdot,j)}/b_0 + c_{(\cdot,j)})_i = (b_{(n,j)} + c_{(n,k)}) \cap c_{(i,p)}/(b_{(n,0)} + c_{(n,k)}) \cap c_{(i,p)},$$

which has a cofibration into $b_{(n,j)}/b_{(n,0)}$, hence contained in $\mathcal{A}$. Then the same arguments work in steps **b** and **c**.

1.3. $K(X)$

1.3.0. For a scheme X, let $\mathsf{sPerf}(X)$ be the category of bounded chain complexes of locally free coherent sheaves on X. It has a structure of Waldhausen category as cofibrations are level-wise split monomorphisms and weak equivalences are quasi-isomorphisms of complexes of sheaves ([9, 3.1]). The pushouts in the category of complexes are obtained in level-wise manner. That is, for a cofibration $a \rightarrowtail b$ and an arbitrary morphism $a \longrightarrow c$, the pushout $b \cup_a c$ is defined as $(b \cup_a c)_i = b_i \cup_{a_i} c_i$ with the obvious differentials. $K(X)$ denotes the associated K-theory of $(\mathsf{sPerf}(X), w)$:

$$K(X) = K(\mathsf{sPerf}(X), w) = \Omega|wS.(\mathsf{sPerf}(X))|.$$

When X is an affine scheme $\operatorname{Spec} A$, we abbreviate $K(\operatorname{Spec} A) = K(A)$.

Homotopy groups of $K(X)$ are isomorphic to the higher algebraic K-groups of X in the sense of Quillen in [7] ([9, 1.11.2 and 1.11.7]).

Any morphism between schemes $f : X \longrightarrow Y$ induces an exact functor f^* and a mapping between K-theory : $f^* = K(f^*) : K(Y) \longrightarrow K(X)$.

1.3.1. For a finite flat morphism $f : X \longrightarrow Y$ the direct image functor f_* is an exact functor on sheaves and it induces a transfer mapping $f_* : K(X) \longrightarrow K(Y)$.

1.3.2. For an algebra A we can define the K-theory $K(A)$ as the K-theory associated to bounded complexes of finitely generated projective A-modules. Here cofibrations are level-wise split monomorphisms and weak equivalences are quasi-isomorphisms. It is equivalent to the K-theory of Quillen associated to the exact category of finitely generated projective (left) A-modules (1.4).

1.3.3. $(\mathsf{sPerf}(X), w)$ is a complicial biWaldhausen category as is defined in [9, 1.2.11], which is closed under canonical homotopy pushouts and pullbacks. The usual mapping cylinder [9, 1.3.4] defines a cylinder functor in the sense of [10, 1.6].

Consider a set E of locally free coherent sheaves on X, and complexes having a term isomorphic to a direct sum of copies of elements of E in each level. It determine a subcategory of $\mathsf{sPerf}(X)$ which has a structure of Waldhausen category, where cofibrations are level-wise split monomorphisms and weak equivalences are quasi-isomorphisms. It becomes a complicial biWaldhausen category closed under canonical homotopy pushouts and pullbacks.

1.3.4. Let $\mathcal{A}$ and $\mathcal{B}$ be Waldhausen categories satisfying the saturation axiom, and assume $\mathcal{A}$ has a cylinder functor satisfying the cylinder axiom. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor which satisfies the following two conditions:

- A morphism f is in $w\mathcal{A}$ if and only if Ff is in $w\mathcal{B}$.
- For any object a in $\mathcal{A}$ and any morphism $x : Fa \longrightarrow b$ in $\mathcal{B}$, there exist a cofibration $i : a \rightarrowtail a'$ in $\mathcal{A}$ and a weak equivalence $x' : Fa' \xrightarrow{\sim} b$ in $\mathcal{B}$ so that $x = x' \circ Fi$.

Then F induces an equivalence in K-theory: $KF : K(\mathcal{A}) \simeq K(\mathcal{B})$. This is called the approximation theorem ([10, 1.6.7]).

When $\mathcal{B}$ has a cylinder functor satisfying the cylinder axiom, we can weaken the second condition in the following way ([9, 1.9.1]):

- For any a object in $\mathcal{A}$ and any cofibration $x : Fa \rightarrowtail b$ in $\mathcal{B}$, there exist a morphism $i : a \longrightarrow a'$ in $\mathcal{A}$ and a weak equivalence $x' : Fa' \xrightarrow{\sim} b$ in $\mathcal{B}$ so that $x = x' \circ Fi$.

The reason is that for a morphism $x : Fa \longrightarrow b$, we can apply the weaker assumption on the morphism of cylinder functor $x : Fa \rightarrowtail Tx \xrightarrow{\sim} b$, which implies x has a factorization as $Fa \xrightarrow{Fi} Fa' \xrightarrow{\sim} Tx$, and if we take the cylinder of Fi, and compose all together, it satisfy the original condition.

1.3.5. Let $(\mathcal{A}\,, w)$ be a complicial biWaldhausen category closed under canonical homotopy pushouts and pullbacks. We can define its homotopy category as a localization of weak equivalences, and denote it by $Ho(\mathcal{A}\,, w)$ [9, 1.9.6]. It is a triangulated category and a complicial exact functor induces a triangulated functor between homotopy categories. If the functor is an equivalence between homotopy categories, then it induces an equivalence between K-theory [9, 1.9.8].

1.3.6. Suppose we have a category with cofibrations $\mathcal{A}$, and there exist two subcategories $w\mathcal{A} \subset v\mathcal{A}$ where each of them defines a structure of Waldhausen category. We assume $\mathcal{A}$ has a functor T which works as a cylinder functor both for $w\mathcal{A}$ and $v\mathcal{A}$, and $(\mathcal{A}\,,v\mathcal{A})$ satisfies the cylinder axiom.

Let $\mathcal{A}^v$ be the subcategory on which a is in $\mathcal{A}^v$ if and only if the canonical morphism $0 \longrightarrow a$ is contained in $v\mathcal{A}$. It becomes a Waldhausen subcategory if we define $w\mathcal{A}^v = w\mathcal{A} \cap \mathcal{A}^v$. Then there exists a homotopy fiber sequence

$$K(\mathcal{A}^v\,,w) \longrightarrow K(\mathcal{A}\,,w) \longrightarrow K(\mathcal{A}\,,v).$$

This is called the localization theorem ([10, 1.6.4]). The following two examples are important.

1.3.7. For any category $\mathcal{A}$ with cofibrations we can impose a structure of Waldhausen category when the category $i(\mathcal{A})$ of all the isomorphisms are weak equivalences. If there is a Waldhausen category $(\mathcal{A}\,,w(\mathcal{A}))$ having a cylinder functor for both of $(\mathcal{A}\,,w(\mathcal{A}))$ and $(\mathcal{A}\,,i(\mathcal{A}))$ which satisfy the cylinder axiom for $(\mathcal{A}\,,w(\mathcal{A}))$, then the localization theorem implies that there exists a homotopy fiber sequence

$$K(\mathcal{A}^w\,,i) \longrightarrow K(\mathcal{A}\,,i) \longrightarrow K(\mathcal{A}\,,w).$$

1.3.8. Consider a scheme X and its open subscheme U. Morphisms being quasi-isomorphisms on U will define a subcategory $u\mathsf{sPerf}(X)$ and another structure of Walthausen category on $\mathsf{sPerf}(X)$. By the localization theorem there is a homotopy fiber sequence

$$K(\mathsf{sPerf}(X)^u\,,w) \longrightarrow K(X\,,w) \longrightarrow K(\mathsf{sPerf}(X)\,,u).$$

Now the Thomason localization theorem ([9, 5.1]) says that if X is regular the exact functor induced by the restriction on U induces an equivalence $K(\mathsf{sPerf}(X)\,,u) \simeq K(U)$. We denote $K(\mathsf{sPerf}(X)^u\,,w)$ by $K(X \text{ on } X-U)$.

When X is a Noetherian scheme, we can define the K-theory of complexes of coherent sheaves as $G(X)$ ([9, 3.4]).

And if X is regular, the inclusion induces an equivalence $K(X) \simeq G(X)$ ([9, 3.21]), which is compatible with the Quillen localization sequence ([7, §7 3.2]),

$$G(X-U) \longrightarrow G(X) \longrightarrow G(U). \tag{1}$$

It implies $K(X \text{ on } X-U) \simeq G(X-U)$, where $G(X-U)$ depends only on the reduced part of $X-U$ by the devissage theorem ([7, §7 3.1]).

1.3.9. When X is an one-dimensional, regular and irreducible scheme, the subcategory $\mathsf{sPerf}(X)^u$ of complexes on X whose homologies are torsion has the K-theory equivalent to a directed colimit of finite products of the K-theory of the residue fields at closed points:

$$K(\mathsf{sPerf}(X)^u) \simeq \prod^{\oplus} K(\kappa(x)) \tag{2}$$

where x run over all the closed points and $\kappa(x)$ is the residue field at x. This is a consequence of the localization theorem above and the devissage theorem ([7, §5 Theorem 5 Corollary]).

1.4. Exact categories. An exact category $\mathcal{E}$ is an additive category equipped with a class of short exact sequences satisfying axioms. The set of axioms in [7, §2] is equivalent to that in [6, §4], which is simple enough to check in examples in this paper. It has a full exact embedding into a subsidiary abelian category. Morphisms of components of short exact sequences are called admissible monomorphisms or epimorphisms.

All the exact categories in this paper are idempotent complete. It has a structure of Waldhausen category where weak equivalences are isomorphisms and cofibrations are admissible monomorphisms. $K(\mathcal{E})$ denotes K-theory of this Waldhausen category, which is equivalent to that defined by Quillen.

The category of bounded complexes $\mathsf{Ch}(\mathcal{E})$ of $\mathcal{E}$ has a structure of Waldhausen category, where cofibrations are level-wise admissible monomorphisms. $\mathsf{Ac}(\mathcal{E})$ denotes the subcategory of acyclic complexes in $\mathsf{Ch}(\mathcal{E})$, where a complex is acyclic if it is made on successive splicing of short exact sequences in $\mathcal{E}$, which is equivalent to be acyclic as a chain complex in the subsidiary abelian category containing $\mathcal{E}$. There is a homotopy equivalence $K(\mathsf{Ch}(\mathcal{E})\,,\mathsf{Ac}(\mathcal{E}))\simeq K(\mathcal{E})$ [9, 1.11.7].

1.5. On the projective line Π, $K(\Pi)$ is equivalent to the product of two copies of K-theory of the base field ([7, §8 3.1]). We can examine this fact in the following way.

1.5.0. Let $\mathcal{O}_\Pi(k)$ be the unique invertible sheaf of degree k on Π. Consider the subcategory of complexes whose term in each level is isomorphic to a sum of copies of $\mathcal{O}_\Pi(k)$ for $m \le k \le n$. $\mathsf{sPerf}(\Pi)\langle m,n\rangle$ denotes it. It is a Waldhausen subcategory with cylinder functor as in 1.3.3.

$K(\mathsf{sPerf}(\Pi)\langle n,n\rangle)$ is equivalent to $K(\kappa)$. $K(\mathsf{sPerf}(\Pi)\langle -1,0\rangle)$ is equivalent to the product of two copies of $K(\kappa)$, for if $\ p:\Pi \longrightarrow \operatorname{Spec} k\ $ is the structure morphism, p^*p_* defines an exact functor and the Cokernel of $\ p^*p_* \longrightarrow 1\ $ is in $K(\mathsf{sPerf}(\Pi)\langle -1,-1\rangle)$. We can apply the additivity theorem (1.1.5).

1.5.1. There is a resolution of diagonal in $\Pi\times\Pi$

$$(p_1)^*\mathcal{O}_\Pi(-1)\otimes(p_2)^*\mathcal{O}_\Pi(-1) \rightarrowtail \mathcal{O}_{\Pi\times\Pi} \twoheadrightarrow \mathcal{O}_\Delta,$$

where p_1 and p_2 are the first and second projection onto Π. It implies that if X is a sheaf on Π satisfies the condition that $H^1(X)=H^1(X\otimes\mathcal{O}_\Pi(-1))=0$, there is a functorial exact sequence

$$0 \longrightarrow H^0(X\otimes\mathcal{O}_\Pi(-1))\otimes\mathcal{O}_\Pi(-1) \longrightarrow H^0(X)\otimes\mathcal{O}_\Pi \longrightarrow X \longrightarrow 0.$$

The inclusion $\ \mathsf{sPerf}(\Pi)\langle -1,0\rangle \longrightarrow \mathsf{sPerf}(\Pi)\ $ induces an equivalence in K-theory ([9, 4.7]).

1.5.2. In particular here is an exact sequence

$$\oplus^k\,\mathcal{O}_\Pi(-1) \rightarrowtail \oplus^{k+1}\,\mathcal{O}_\Pi(0) \twoheadrightarrow \mathcal{O}_\Pi(k) \tag{3}$$

for any positive k.

1.5.3. We can prove the fact that $\mathsf{sPerf}(\Pi)\langle -1,0\rangle \longrightarrow \mathsf{sPerf}(\Pi)\langle -1,1\rangle$ induces an equivalence in K-theory by equivalence of the homotopy categories (1.3.5) in the following way.

Both are complicial biwaldhausen categories with canonical homotopy pullbacks and pushouts. Let x be an object in $\mathsf{sPerf}(\Pi)\langle -1,1\rangle$. First we construct an object a in $\mathsf{sPerf}(\Pi)\langle -1,0\rangle$ and a quasi-isomorphism $a \xrightarrow{\sim} x$.

There is the Euler sequence.

$$\mathcal{O}_\Pi(-1) \rightarrowtail \mathcal{O}_\Pi(0) \oplus \mathcal{O}_\Pi(0) \twoheadrightarrow \mathcal{O}_\Pi(1),$$

which can be obtained from (3) in $k = 1$.

Each term x_i is isomorphic to a direct sum of $x'_i \oplus x''_i$, where x_i is isomorphic to a sum of copies of $\mathcal{O}_\Pi(1)$ and x''_i is isomorphic to a sum of copies of $\mathcal{O}_\Pi(k)$ for $k = -1, 0$. Take a resolution of $\mathcal{O}_\Pi(1)$ by the Euler sequence $z_i \rightarrowtail y_i \twoheadrightarrow x'_i$. Let us define $a_i = x''_i \oplus y_i \oplus z_{i-1}$. It defines a complex a and a morphism $\alpha : a \xrightarrow{\sim} x$.

For any morphism $f : c \longrightarrow x$, there exists $g : c \longrightarrow a$ such that $\alpha \circ g = f$. This lift g is unique up to homotopy so that for any representative of morphism in $\mathit{Ho}(\mathsf{sPerf}(\Pi)\langle -1,1\rangle)$: $c \longrightarrow x \xleftarrow{\sim} d$ between objects c, d in $\mathsf{sPerf}(\Pi)\langle -1,0\rangle$, there is a commutative diagram

$$\begin{tikzcd}
 & a \arrow[d, "\sim"] & \\
c \arrow[ur] \arrow[r] & x & d \arrow[l, "\sim"'] \arrow[ul, "\sim"']
\end{tikzcd}$$

where a is in $\mathsf{sPerf}(\Pi)\langle -1,0\rangle$ and it represents a morphism in $\mathit{Ho}(\mathsf{sPerf}(\Pi)\langle -1,0\rangle)$.

For any equivalent relation

$$\begin{tikzcd}
 & e \arrow[d, "\sim"] & \\
c \arrow[ur] \arrow[r] \arrow[dr] & x & d \arrow[l, "\sim"'] \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\
 & f \arrow[u, "\sim"'] &
\end{tikzcd}$$

there is a similar diagram replacing x by a, which is commutative up to homotopy. Then we take a mapping path space ([9, 1.1.2]) of $f : a \xrightarrow{\sim} x$, we get a commutative diagram representing an equivalence relation in $\mathit{Ho}(\mathsf{sPerf}(\Pi)\langle -1,0\rangle)$. It implies $\mathit{Ho}\,(\mathsf{sPerf}(\Pi)\langle -1,0\rangle) \longrightarrow \mathit{Ho}\,(\mathsf{sPerf}(\Pi)\langle -1,1\rangle)$ is an equivalence of categories.

2. K-theory of the function field of Γ

2.1. Let Γ be a curve over a field κ. In our convention, it is smooth, proper and geometrically connected over κ. We may assume that Γ is of positive genus g, for a curve of genus 0 becomes the projective line after taking a finite extension of the base field. In 2.1.2 we define a morphism $\pi : \Gamma \longrightarrow \Pi$ and a line bundle $L = \pi^*\mathcal{O}_\Pi(1)$. Technical conditions are explained in 2.1.1 and the existence of such is discussed in 2.1.2. We will have a model of derived category of function field of Γ using L.

2.1.1. Assumption. There exists a finite morphism $\pi : \Gamma \longrightarrow \Pi$ from Γ onto the projective line Π such that the induced line bundle $L = \pi^*\mathcal{O}_\Pi(1)$ has degree $g+1$, and $\dim H^0(L) = 2$ and $H^1(L) = 0$.

We fix $\pi : \Gamma \longrightarrow \Pi$ and then L. The whole constructions heceforth are depend on them.

2.1.2. When the base field is algebraically closed, a general effective divisor D of degree $g+1$ on an arbitrary curve Γ of genus g satisfies the condition that $\dim H^0(\mathcal{O}_\Gamma(D)) = 2$ and $H^1(\mathcal{O}_\Gamma(D))) = 0$ ([1, I.2]). A non-constant section defines a finite morphism $\pi : \Gamma \longrightarrow \Pi$ satisfying the assumption.

When the base field is not algebraically closed, we can find π in the extension of the base field to its algebraic closure, but π must be defined in a finite extension.

2.1.3. Definition. Let us define the full subcategory $\mathcal{P} = \mathcal{P}\langle -2, 0\rangle$ of $\mathsf{sPerf}(\Gamma)$ whose objects are complexes having a term in each level isomorphic to a sum of copies of L^k for $k = -2, -1, 0$. We can define a structure of complicial biWaldhausen category (1.3.3) on $\mathcal{P}$ where level-wise split monomorphisms are cofibrations and quasi-isomorphisms are weak equivalences.

2.1.4. Consider the direct image sheaf $\pi_*\mathcal{O}_\Gamma$. It is isomorphic to a sum of invertible sheaves on Π, by the theorem of Grothendieck. Since $H^0(\mathcal{O}_\Gamma) \cong H^0(\pi_*\mathcal{O}_\Gamma)$ is one dimensional, $\pi_*\mathcal{O}_\Gamma$ consists of one copy of $\mathcal{O}_\Pi$ and a sum of invertible sheaves of negative degree. And $H^0(L) \cong H^0(\pi_*L) \cong H^0(\pi_*\mathcal{O}_\Gamma \otimes \mathcal{O}_\Pi(1))$ implies $\pi_*\mathcal{O}_\Gamma$ does not contain any $\mathcal{O}_\Pi(-1)$ as a factor. On the other hand $H^1(\pi_*L) = 0$ implies that it does not contains any factor of degree less than -2. Finally, because $H^1(\pi_*\mathcal{O}_\Gamma)$ has dimension g, $\pi_*\mathcal{O}_\Gamma$ has to be of the form below.

$$\pi_*\mathcal{O}_\Gamma \cong \mathcal{O}_\Pi \oplus (\overset{g}{\oplus}\, \mathcal{O}_\Pi(-2)). \tag{4}$$

The first component $\mathcal{O}_\Pi$ is determined uniquely.

L^n means the n-ple tensor product of L and its dual is L^{-n}. We get

$$H^0(\Gamma, L^n) \cong H^0(\Pi, \pi_*L^n) \cong H^0(\Pi, \pi_*\mathcal{O}_\Gamma \otimes \mathcal{O}_\Pi(n)),$$

which is isomorphic to

$$H^0(\mathcal{O}_\Pi(n) \oplus (\overset{g}{\oplus}\, \mathcal{O}_\Pi(n-2))) \cong H^0(\mathcal{O}_\Pi(n)) \oplus H^0((\overset{g}{\oplus}\, \mathcal{O}_\Pi(n-2)))$$

by (4).

2.1.5. Definition. We say a global section of L^n is *linear* if it belongs to the first part of the above decomposition. Linear sections generate a subspace, which is independent of the decomposition in (4), and the quotient space will be called the *non-linear quotient* of the space of global sections.

We use a canonical identification on $\mathrm{Hom}(L^n, L^m)$ and $\mathrm{Hom}(L^0, L^{m-n})$, so that we can think about the subspace of linear morphisms $L^n \longrightarrow L^m$ and the space of non-linear quotients.

2.1.6. The Euler sequence

$$\mathcal{O}_\Pi(-2) \rightarrowtail \mathcal{O}_\Pi(-1) \oplus \mathcal{O}_\Pi(-1) \twoheadrightarrow \mathcal{O}_\Pi$$

has the image of π^*

(5) $$L^{-2} \rightarrowtail L^{-1} \oplus L^{-1} \twoheadrightarrow L^0.$$

It will be called the Euler sequence on Γ.

2.2. We present a morphism which is a quasi-isomorphism on the generic point of Γ as $a \xrightarrow{\approx} b$. Let u be the subcategory of $\mathsf{sPerf}(\Gamma)$ consisting of morphisms which become quasi-isomorphisms on the generic point of Γ. It induces another notion of weak equivalences for Waldhausen subcategories of $\mathsf{sPerf}(\Gamma)$, which will be present as u. A complex belongs to $\mathsf{sPerf}(\Gamma)^u$ (1.3.8) if and only if it has torsion homologies.

2.2.0. Proposition. The inclusion

$$(\mathcal{P}, u) \longrightarrow (\mathsf{sPerf}(\Gamma), u)$$

induces an equivalence in homotopy categories, and then on K-theory.

2.2.1. Lemma. For any object x in $\mathsf{sPerf}(\Gamma)$, there is a morphism $f : x \xrightarrow{\approx} l$ into an object l where each term is isomorphic to a direct sum of a power of L: $l_i = \oplus L^{n_i}$ with zero differential such that f is a quasi-isomorphism at the generic point.

Proof. Let us first recall that for any coherent sheaf M on Γ, we can take a morphism $f : M \longrightarrow \oplus L^k$ for sufficiently large k, which is an isomorphism on the generic point, because L is ample. Now for each x_n there is a subbundle $\operatorname{Ker} d_n \rightarrowtail x_n$ and an epimorphism $\operatorname{Ker} d_n \longrightarrow H_n(x)$. For sufficiently large k_n, there exists a morphism $f'_n : H_n(x) \longrightarrow \oplus L^{k_n}$ such that f'_n is an isomorphism at the generic point and the composition $\operatorname{Ker} d_n \twoheadrightarrow H_n(x) \xrightarrow{f'_n} \oplus L^{k_n}$ can be extended to a morphism $x_n : x_n \longrightarrow \oplus L^{k_n}$. It defines a chain complex l with zero differential and a mapping of chain complex f.

2.2.2. For a morphism from $p \in \mathcal{P}$ to L^k $(k \geq 2)$, $p \longrightarrow L^k$, there is a universal one

(6) $$q^k_{11} \oplus q^k_{10} \longrightarrow q^k_0 \longrightarrow L^k$$

defined as follows.

We have a homomorphism $\gamma_k : H^0(\Gamma, L^k) \otimes L^0 \longrightarrow L^k$ for any $k \geq 2$. It is an epimorphism. Define $C(L^k)$ to be the kernel of γ_k. It defines an exact sequence

(7) $$C(L^k) \rightarrowtail H^0(L^k) \otimes L^0 \xrightarrow{\gamma_k} L^k.$$

Define $R(L^k) = H^0(\pi_*(L^k) \otimes \mathcal{O}_\Pi(-1)) \otimes L^{-1}$. $\pi_*(L^k)$ satisfies the condition of 1.5.1 and we have an exact sequence of functors

$$R(L^k) \rightarrowtail H^0(L^k) \otimes L^0 \twoheadrightarrow \pi^*\pi_* L^k.$$

for $H^0(\Pi, \pi_*(L^k)) = H^0(\Gamma, L^k)$.

Compare the above sequences, we get

$$
\begin{tikzcd}
 & & L^k \otimes (\oplus^g L^{-2}) \arrow[d, tail] \\
R(L^k) \arrow[r, tail] \arrow[d, tail] & H^0(L^k) \otimes L^0 \arrow[r, two heads] \arrow[d, equal] & \pi^*\pi_* L^k \arrow[d, two heads] \\
C(L^k) \arrow[r, tail] \arrow[d, two heads] & H^0(L^k) \otimes L^0 \arrow[r, two heads, "\gamma_k"] & L^k \\
C(L^k)/R(L^k) & & .
\end{tikzcd} \tag{8}
$$

Hence $C(L^k)/R(L^k) \cong \mathrm{Ker}(\pi^*\pi_* L^k \longrightarrow L^k) \cong L^k \otimes (\oplus^g L^{-2})$, and we get an exact sequence of functors

$$R(L^k) \rightarrowtail C(L^k) \xrightarrow{\mu} \!\!\!\!\twoheadrightarrow L^k \otimes (\oplus^g L^{-2}) = \oplus^g L^{k-2}.$$

2.2.3. We have to investigate morphisms from L^{-j} for $j = 0, 1, 2$ into the diagram (8).

It is not difficult to see $\mathrm{Hom}(L^0, C(L^k)) = 0$ for $\mathrm{Hom}(L^0, H^0(L^k) \otimes L^0) \cong \mathrm{Hom}(L^0, L^k)$.

If there is a morphism $f : L^{-1} = \pi^*\mathcal{O}_\Pi(-1) \longrightarrow L^k$, consider its adjunction: $f^\# : \mathcal{O}_\Pi(-1) \longrightarrow \pi_* L^k$. It has a lift onto $H^0(L^k) \otimes \mathcal{O}_\Pi(0)$ which is unique up to a morphisms into $H^0(\pi_*(L^k) \otimes \mathcal{O}_\Pi(-1)) \otimes \mathcal{O}_\Pi(-1)$. It implies there are exact sequences

$$\mathrm{Hom}(L^{-1}, R(L^k)) \rightarrowtail \mathrm{Hom}(L^{-1}, H^0(L^k) \otimes L^0) \twoheadrightarrow \mathrm{Hom}(L^{-1}, L^k),$$

and $\mathrm{Hom}(L^{-1}, R(L^k)) \cong \mathrm{Hom}(L^{-1}, C(L^k))$.

There exists a short exact sequence

$$\mathrm{Hom}(L^{-2}, C(L^k)) \rightarrowtail \mathrm{Hom}(L^{-2}, H^0(L^k) \otimes L^0) \twoheadrightarrow \mathrm{Hom}(L^{-2}, L^k)$$

and a short exact sequence from (8),

$$\mathrm{Hom}(L^{-2}, R(L^k)) \rightarrowtail \mathrm{Hom}(L^{-2}, C(L^k)) \twoheadrightarrow \mathrm{Hom}(L^{-2}, \oplus^g L^{k-2})$$

for $\mathrm{Ext}^1(L^{-2}, R(L^k)) = 0$.

2.2.4. Define $q_0^k = H^0(L^k) \otimes L^0$, $q_{10}^k = R(L^k) = \oplus L^{-1}$ and

$$q_{11}^k = \mathrm{Hom}(L^{-2}, \oplus^g L^{k-2}) \otimes L^{-2} = H^0(L^k) \otimes (\oplus^g L^{-2}).$$

These defines a complex $q_{11}^k \oplus q_{10}^k \longrightarrow q_0^k \xrightarrow{\gamma_k} L^k$ where $q_{11}^k \longrightarrow q_0^k$ is any lift of $H^0(L^k) \otimes (\oplus^g L^{-2}) \xrightarrow{\oplus\gamma_k} \oplus^g L^{k-2}$ to $C(L^k)$ composed with the inclusion into q_0^k. It define a complex

$$q_{10}^k \oplus q_{11}^k \longrightarrow q_0^k. \tag{9}$$

It is a part of the an exact sequence:

$$C(L^{k-2}) \longrightarrow q_{10}^k \oplus q_{11}^k \longrightarrow q_0^k \xrightarrow{\gamma} \!\!\!\!\rightarrow L^k \longrightarrow 0. \tag{10}$$

For any $j = -2, -1, 0$ the exact sequence (10) induces an exact sequence

$$\mathrm{Hom}(L^j,\, q_{10}^k \oplus q_{11}^k) \rightarrowtail \mathrm{Hom}(L^j,\, q_0^k) \twoheadrightarrow \mathrm{Hom}(L^j,\, L^k).$$

Therefore for any morphisms of complex $f : p \longrightarrow L^k$ from an object p in $\mathcal{P}$, there exists a morphism of complex $F : p \longrightarrow q_\cdot^k$ such that $\gamma \circ F = f$. This lift is unique up to unique homotopy.

2.2.5. Lemma. There exists an exact sequence $\oplus L^{-2} \rightarrowtail \oplus L^0 \twoheadrightarrow \oplus L^2$ and a morphism of complexes such that ξ is an isomorphism at the generic point and it makes the following commutative diagram:

$$\begin{tikzcd}
\oplus L^{-2} \arrow[r, tail] \arrow[d, "\xi"'] & \oplus L^0 \arrow[r, two heads] \arrow[d] & \oplus L^2 \arrow[d] \\
C(L^k) \arrow[r, tail] & \oplus L^0 \arrow[r, two heads, "\gamma"'] & L^k.
\end{tikzcd}$$

Proof. Chose nonzero linear morphisms $\beta_1 : L^{-2} \longrightarrow L^{-1}$, $\beta_2 : L^{-2} \longrightarrow L^0$, $\beta_k : L^2 \longrightarrow L^k$, and a basis of non-linear quotients $\alpha_i : L^{-2} \longrightarrow L^0$ for $i = 1, \ldots, g$.

For $R(L^k) = \oplus^m L^{-1} \longrightarrow \oplus L^0$ where $m = k + g(k-1)$, we get

$$\begin{tikzcd}
\oplus^m L^{-2} \arrow[r] \arrow[d, "\oplus\beta_1"'] & \oplus^{2m} L^0 \arrow[r] \arrow[d] & \oplus^m L^2 \arrow[d] \\
R(L^k) \arrow[r] & \oplus L^0 \arrow[r] & L^k,
\end{tikzcd} \tag{11}$$

where the first raw is induced by pullback of the Euler sequence by degree two morphism on the projective line.

There exists an exact sequence for each $i = 1, \ldots, g$

$$L^{-2} \xrightarrow{(\alpha_i, \beta_2)} L^0 \oplus L^0 \xrightarrow{{}^t(-\beta_2, \alpha_i)} L^2 \tag{12}$$

such that

$$\begin{tikzcd}
L^{-2} \arrow[r] \arrow[d] \arrow[dd, bend right=60, "(\ldots,0,\beta_2\circ\beta_k,0,\ldots)"'] & L^0 \oplus L^0 \arrow[r] \arrow[d] & L^2 \arrow[d, "\beta_k"] \\
C(L^k) \arrow[r, tail] \arrow[d] & \oplus L^0 \arrow[r, two heads, "\gamma"'] & L^k \\
\oplus^g L^{k-2} & &
\end{tikzcd}$$

where the leftmost morphism goes into i-th component. (It will be easier to see it if the morphism is into the upper right corner of (8).)

Summing up all these subcomplexes (12) for all i and (11), we get the desired one.

2.2.6. A complex in (2.2.5) with level and degree shifts gives the following diagram:

$$\begin{tikzcd}
 \oplus L^{-4} \arrow[r, tail] \arrow[d, "\xi"'] & \oplus L^{-2} \arrow[r, two heads] \arrow[d] & \oplus L^{0} \arrow[d, dotted] & \\
 C(L^{k-2}) \arrow[r] & q^k_{10} \oplus q^k_{11} \arrow[r] & q^k_0 \arrow[r, two heads, "\gamma"] & L^k,
\end{tikzcd}$$

where dotted arrow is determined uniquely. Define a complex $r^k_\cdot$ as the total complex of the second and third column of the diagram above. It makes a commutative diagram

$$\begin{tikzcd}
 r^k_\cdot \arrow[dr, "\approx"] & \\
 q^k_\cdot \arrow[u] \arrow[r, "\gamma"'] & L^k.
\end{tikzcd}$$

2.2.7. Now we can prove 2.2.0. It suffices to show that the associated derived categories are equivalent by the inclusion. Then by 1.3.5, it induces an equivalence in K-theory. Proof is similar to that in 1.5.3. It is straightforward to see that $\mathit{Ho}(\mathcal{P}, u) \longrightarrow \mathit{Ho}(\mathsf{sPerf}(\Gamma), u)$ is essentially surjective.

For any representative $a \longrightarrow x \underset{\approx}{\longleftarrow} b$ of a morphism between a and b in $\mathit{Ho}(\mathsf{sPerf}(\Gamma), u)$, there exists a commutative diagram

$$\begin{tikzcd}
 & \oplus r^\cdot & \\
 & \oplus q^\cdot \arrow[u] \arrow[d] & \\
 a \arrow[uur] \arrow[r] \arrow[dr] & l & b \arrow[uul, "\approx"'] \arrow[l, "\approx"] \arrow[dl, "\approx"] \\
 & x \arrow[u, "\approx"'] &
\end{tikzcd}$$

so that the morphism has a representative in $\mathit{Ho}(\mathcal{P}, u)$.

For any equivalent relation

$$\begin{tikzcd}
 & \cdot \arrow[d, "\approx"] & \\
 a \arrow[ur] \arrow[r] \arrow[dr] & x & b \arrow[ul, "\approx"'] \arrow[l, "\approx"'] \arrow[dl, "\approx"] \\
 & \cdot \arrow[u, "\approx"'] &
\end{tikzcd}$$

there is a similar diagram replacing x by l and then by q. The result is commutative up to homotopy, and we can replace q by r as above. Then we take a mapping path space, we get a commutative diagram representing an equivalence relation in $\mathit{Ho}(\mathcal{P}, u)$.

2.3. $k(\Gamma)$ denotes the function field of Γ. The localization theorem in 1.3.8 implies $K(k(\Gamma)) \simeq K(\mathsf{sPerf}(\Gamma), u)$, so that 1.3.9 and 2.2.0 implies there is a homotopy fiber sequence

$$\prod\nolimits^{\oplus} K(\kappa(\gamma)) \longrightarrow K(\Gamma) \longrightarrow K(\mathcal{P}, u), \tag{13}$$

where $\kappa(\gamma)$ is the residue field at closed point γ in Γ.

There is another homotopy fiber sequence

$$K(\mathcal{P}^u, w) \longrightarrow K(\mathcal{P}, w) \longrightarrow K(\mathcal{P}, u)$$

by the localization theorem, where w denotes quasi-isomorphisms of complexes of sheaves.

2.3.0. Proposition. π^* induces an exact functor

$$\pi^* : (\mathsf{sPerf}(\Pi)\langle -2, 0\rangle, w) \longrightarrow (\mathcal{P}, w).$$

It induces an equivalence in K-theory.

2.3.1. Consider $\mathsf{sPerf}(\Pi)\langle -2, 0\rangle$ and $\mathcal{P}$ with another notion of weak equivalences (1.3.7): Define only the isomorphisms of complexes as weak equivalences. Let $(\mathcal{P}, i)$ be the associated Waldhausen category.

The localization theorem tells there is a homotopy fiber sequence

$$K(\mathcal{P}^w, i) \longrightarrow K(\mathcal{P}, i) \longrightarrow K(\mathcal{P}, w).$$

There is a corresponding homotopy fiber sequence on $\mathsf{sPerf}(\Pi)\langle -2, 0\rangle$, and these are connected by exact functors associated to π^*:

$$\begin{tikzcd}
K(\mathsf{sPerf}(\Pi)\langle -2, 0\rangle^w, i) \arrow[r] \arrow[d, "\pi^*"'] & K(\mathsf{sPerf}(\Pi)\langle -2, 0\rangle, i) \arrow[r] \arrow[d, "\pi^*"'] & K(\mathsf{sPerf}(\Pi)\langle -2, 0\rangle, w) \arrow[d, "\pi^*"'] \\
K(\mathcal{P}^w, i) \arrow[r] & K(\mathcal{P}, i) \arrow[r] & K(\mathcal{P}, w).
\end{tikzcd}$$

To prove the right vertical π^* is a homotopy equivalence, it suffices to show that π^* on the left and middle induce equivalences.

2.3.2. To see the middle equivalence in the diagram above, let $\mathcal{P}_{[a,b]}$ be the subcategory of $\mathcal{P}$ whose objects are complexes supported on level from a to b. The filtration by level is defined on $\mathcal{P}_{[a,b]}$ and the additivity theorem (1.1.5) says that there is an equivalence

$$K(\mathcal{P}_{[a,b]}, i) \simeq \prod_{b-a} K(\mathcal{P}_{[0,0]}, i)$$
$$\cdots \to x_j \to \cdots \mapsto (\ \ldots, \ x_j \ , \ \ldots \ ),$$

where we identify $(\mathcal{P}_{[j,j]}, i)$ as $(\mathcal{P}_{[0,0]}, i)$.

On the projective line, there are corresponding subcategories $\mathsf{sPerf}(\Pi)\langle -2, 0\rangle_{[a,b]}$ of complexes supported on level from a to b in $\mathsf{sPerf}(\Pi)\langle -2, 0\rangle$. The K-theory of $\mathsf{sPerf}(\Pi)\langle -2, 0\rangle_{[a,b]}$ has the corresponding decomposition.

2.3.3. Each term of a complex in $\mathcal{P}$ has the increasing filtration by degree:

$$0 = F^1 x \subset F^0 x \subset F^{-1} x \subset F^{-2} x = x$$

where $F^j x / F^{j+1} x \cong \oplus L^j$. This filtration is preserved by all the morphisms of sheaves, for there is only zero mapping from L^j to L^k if $j > k$. We can apply 1.1.6 to $\mathcal{P}_{[0,0]}$ so that there is a homotopy equivalence

$$K(\mathcal{P}\langle -2, 0\rangle_{[0,0]}) \simeq K(\mathcal{P}\langle 0, 0\rangle_{[0,0]}) \times K(\mathcal{P}\langle -1, -1\rangle_{[0,0]}) \times K(\mathcal{P}\langle -2, -2\rangle_{[0,0]}).$$

On $K(\mathsf{sPerf}(\Pi)\langle -2, 0\rangle_{[0,0]})$ there is a corresponding decomposition, and $K(\pi^*)$ respects the decompositions.

Because π^* induces isomorphisms

$$\mathrm{Aut}_\Pi(\overset{n}{\oplus}\mathcal{O}_\Pi(k)) \cong \mathrm{Aut}_\Gamma(\overset{n}{\oplus}\pi^*\mathcal{O}_\Pi(k)),$$

π^* induces an exact equivalence of Waldhausen categories between subcategories of pure pieces for each degree (1.1.6). Hence it induces an equivalence in K-theory between $\mathsf{sPerf}(\Pi)\langle -2,0\rangle_{[0,0]}$ and $\mathcal{P}_{[0,0]}$.

For π^* is compatible with the product decomposition in 2.3.2, it induces an equivalence between $K(\mathsf{sPerf}(\Pi)\langle -2,0\rangle_{[a,b]}, i)$ and $K(\mathcal{P}_{[a,b]}, i)$ for any a and b. Then on directed colimits it becomes a homotopy equivalence

$$K(\pi^*) : K(\mathsf{sPerf}(\Pi)\langle -2,0\rangle, i) \xrightarrow{\sim} K(\mathcal{P}, i).$$

2.3.4. Lemma. Suppose there is an element of $\mathcal{P}$, which is a short exact sequence of sheaves $x_1 \overset{\phi}{\rightarrowtail} x_0 \twoheadrightarrow x_{-1}$. Then there exists sheave y_0 and y_1 on Π and a monomorphism $\phi_y : y_1 \rightarrowtail y_0$ such that each x_i is isomorphic to $\pi^* y_i$ by $\psi_i : x_i \longrightarrow \pi^* y_i$ for $i = 0, 1$ and the following diagram is commutative:

$$\begin{tikzcd} x_1 \arrow[r, tail, "\phi"] \arrow[d, "\psi_1"'] & x_0 \arrow[d, "\psi_0"] \\ \pi^* y_1 \arrow[r, tail, "\pi^*\phi_y"'] & \pi^* y_0. \end{tikzcd}$$

Proof. We will define a subcomplex which is isomorphic to an image of π^*, and its quotient is also isomorphic to such.

Let us recall x_i has the filtration $F^j x_i$ (2.3.3). The image of $F^{-1}x_1$ should be contained in $F^{-1}x_0 \cong (\oplus L^0) \oplus (\oplus L^{-1})$. The restriction of $\phi|$ on $F^{-1}x_1$ is linear. The image is isomorphic to a sum of copies of L^{-1} and L^0, and its cokernel is a sum of line bundles.

$$\begin{tikzcd} F^{-1}x_1 \arrow[r, tail, "\phi|"] \arrow[d, tail] & F^{-1}x_0 \arrow[r, two heads] \arrow[d, tail] & \mathrm{Coker}(\phi|) \arrow[d] \\ x_1 \arrow[r, tail, "\phi"] \arrow[d, two heads] & x_0 \arrow[r, two heads] \arrow[d, two heads] & x_{-1} \\ x_1/F^{-1}x_1 \arrow[r] & x_0/F^{-1}x_0 & . \end{tikzcd}$$

If $\mathrm{Coker}(\phi|)$ contains a line bundle of positive degree, it could not have non-zero morphism into x_{-1} or $x_1/F^{-1}x_1 \cong \oplus L^{-2}$. Therefore $\phi|$ is an isomorphism onto a split factor of x_0. It defines an acyclic subcomplex of $x_1 \longrightarrow x_0$ and the quotient complex is a case where $x_1 = \oplus L^{-2}$.

Consider a case where $x_1 = \oplus L^{-2}$. For the composition of ϕ and the projection onto $x_0/F^0 x_0$ is linear, it has a decomposition $x_1 \overset{\psi}{\twoheadrightarrow} x_1' \hookrightarrow x_0/F^0 x_0$. It can be converted to the following exact sequences:

$$\begin{tikzcd}
\operatorname{Ker}\psi \arrow[r, tail] \arrow[d, tail] & F^0x_0 \arrow[r, two heads] \arrow[d, tail] & F^0x_0/\operatorname{Ker}\psi \arrow[d, hook] \\
x_1 \arrow[r, tail, "\phi"] \arrow[d, two heads, "\psi"'] & x_0 \arrow[r, two heads] \arrow[d, two heads] & x_{-1} \\
x_1' \arrow[r, hook] & x_0/F^0x_0 & .
\end{tikzcd}$$

$\operatorname{Ker}\psi$ is a sum of line bundles with degree less than or equal to -2. If it is not zero, the determinant of $F^0x_0/\operatorname{Ker}\psi$ is positive, but it contradicts that it is a subsheaf of an object of $\mathcal{P}$. Therefore $\operatorname{Ker}\psi = 0$. $F^0x_0 \longrightarrow F^0x_{-1}$ determines an acyclic subcomplex whose quotients is the case in which $x_1 = \oplus L^{-2}$ and $F^0x_0 = 0$. In this case ϕ is linear.

2.3.5. For a sheaf M let $D(M)[n]$ be an acyclic complex $M \xrightarrow{1_M} M$ in level n and $n-1$. $[n]$ means shift of level by n. Let $E[n]$ be the Euler sequence (5), shifted in level so that it has non-zero terms in level n , $n-1$, $n-2$. Let $\mathcal{P}^w_{[a,b]}$ be the subcategory of acyclic complexes supported in level from a to b. Similarly $\mathsf{sPerf}(\Pi)\langle -2,0\rangle^w_{[a,b]}$ is defined.

Any object of $\mathcal{P}^w_{[a,a-1]}$ is isomorphic to a sum of $D(M)[a]$ and this category is equivalent to $\mathcal{P}\langle -2,0\rangle_{[a,a]}$. Comparing the situation on Π, we get a homotopy equivalence $K(\pi^*) : K(\mathsf{sPerf}(\Pi)\langle -2,0\rangle^w_{[a,a-1]}) \simeq K(\mathcal{P}^w_{[a,a-1]})$.

Let x be an object of $\mathcal{P}^w_{[a,a-2]}$. We can apply the lemma 2.3.4 to $x_a \longrightarrow x_{a-1}$ for its cokernel is x_{a-2}. As a result each object x has a filtration of complexes whose filtered quotients are $D(\oplus L^0)[a]$, $D(\oplus L^{-1})[a]$, $D(\oplus L^0)[a-1]$, $E[a]$, $D(\oplus L^{-2})[a]$, $D(\oplus L^{-1})[a-1]$ and $D(\oplus L^{-2})[a-1]$ in this order. Because there are only zero mappings from right to left in this order, the above filtration must be preserved by all the morphisms.

Since π^* is an exact equivalence on each subcategory of pure pieces, π^* induces an equivalence in K-theory $K(\pi^*) : K(\mathsf{sPerf}(\Pi)\langle -2,0\rangle^w_{[a,a-2]}) \simeq K(\mathcal{P}^w_{[a,a-2]})$.

Now for a fixed b we make an inductive hypotheses on a that π^* induces an equivalence

$$K(\pi^*) : K(\mathsf{sPerf}(\Pi)\langle -2,0\rangle^w_{[a-1,b]}) \simeq K(\mathcal{P}^w_{[a-1,b]}).$$

$\mathcal{P}^{w,+}_{[a,b]}$ denotes the subcategory whose objects satisfy $F^0x_a = 0$, and $\mathcal{P}^{w,++}_{[a,b]}$ denotes the subcategory whose objects satisfy $F^0x_a = F^0x_{a-1} = 0$.

There are exact equivalences of Waldhausen categories,

$$E(\mathcal{P}\langle 0,0\rangle^w_{[a,a-1]}, \mathcal{P}^w_{[a,b]}, \mathcal{P}^{w,+}_{[a,b]}) \longrightarrow \mathcal{P}^w_{[a,b]}$$

and

$$E(\mathcal{P}\langle 0,0\rangle^w_{[a-1,a-2]}, \mathcal{P}^{w,+}_{[a,b]}, \mathcal{P}^{w,++}_{[a,b]}) \longrightarrow \mathcal{P}^{w,+}_{[a,b]}.$$

We can define the corresponding categories in $\mathsf{sPerf}(\Pi)\langle -2,0\rangle^w_{[a,b]}$, and π^* respects all of the subcategories and categories of cofibration sequences, and it is an exact equivalence on $\mathcal{P}\langle 0,0\rangle^w_{[i,i-1]}$ for $i = a, a-1$. By the additivity theorem and the inductive hypotheses, $K(\mathcal{P}^{w,+}_{[a-1,b]}) \simeq K(\mathsf{sPerf}(\Pi)\langle -2,0\rangle^{w,+}_{[a-1,b]})$, it suffices to show that π^* is a homotopy equivalence between $K(\mathsf{sPerf}(\Pi)\langle -2,0\rangle^{w,++}_{[a,b]})$ and $K(\mathcal{P}^{w,++}_{[a,b]})$.

Any object in $\mathcal{P}^{w,++}_{[a,b]}$ has a filtration $F^{-1}x \rightarrowtail F^{-2}x \rightarrowtail F^{-3}x \rightarrowtail x$ on which $F^{-1}x \cong \oplus D(L^{-1})[a]$, $F^{-2}x/F^{-1}x \cong \oplus E[a]$, $F^{-3}x/F^{-2}x \cong \oplus D(L^{-2})[a]$, and $x/F^{-3}x \in \mathcal{P}^{w,+}_{[a-1,b]}$. The argument as above implies

$$K(\pi^*) : K(\mathsf{sPerf}(\Pi)\langle -2,0\rangle^{w,++}_{[a,b]}) \simeq K(\mathcal{P}^{w,++}_{[a,b]}),$$

hence $K(\mathsf{sPerf}(\Pi)\langle -2,0\rangle^{w}_{[a,b]}) \simeq K(\mathcal{P}^{w}_{[a,b]})$. We can take colimits on a and b, which implies π^* in the left of 2.3.3 is a homotopy equivalence. This is the end of proof of 2.3.2.

2.3.6. We get the following homotopy fiber sequence

$$\begin{tikzcd}
K(\mathcal{P}^u) \arrow[r] \arrow[d] & K(\kappa) \times K(\kappa) \arrow[r] \arrow[d] & K(\mathcal{P}\,, u) \arrow[d, "\simeq"] \\
\prod^{\oplus} K(\kappa(\gamma)) \arrow[r] & K(\Gamma) \arrow[r] & K(\mathsf{sPerf}(\Gamma)\,,\, u).
\end{tikzcd}$$

3. Complexes with torsion homologies

3.1. We have to investigate the K-theory of $\mathcal{P}^u$. First we need to extend 2.3.4 to get a structure on objects in $\mathcal{P}^u$. K-theory of $\mathcal{P}^u$ will be first reduced to that of length one complexes, and then to the K-theory of finite dimensional algebras over κ.

3.1.0. Recall for a subsheaf x in a locally free sheaf y, there exists a unique subbundle $x^{\sim}$ in y containing x as a subsheaf whose inclusion is an isomorphism at the generic point:

$$x \overset{\approx}{\hookrightarrow} x^{\sim} \rightarrowtail y\ .$$

We call $x^{\sim}$ the saturation of x in y. If y is contained in another locally free sheaf z as a subbundle, the saturation of x in y and that in z coincide.

3.1.1. Lemma. Suppose there is an element of $\mathcal{P}^u$,

$$0 \longrightarrow x_1 \xrightarrow{d_1} x_0 \xrightarrow{d_0} x_{-1} \longrightarrow \cdots,$$

Then

$$\operatorname{Ker} d_0 \rightarrowtail x_0 \twoheadrightarrow \operatorname{Im} d_0$$

belongs to $\mathcal{P}$.

Proof. As in 2.3.3, let us recall x_i has the filtration $F^j x_i$ The image of $F^{-1}x_1$ should be contained in $F^{-1}x_0 \cong (\oplus L^0) \oplus (\oplus L^{-1})$. The restriction of d_1 on $F^{-1}x_1$ is linear. The saturation $(F^{-1}x_1)^{\sim}$ (3.1.0) is isomorphic to a sum of copies of L^{-1}

and L^0, and its cokernel is a sum of line bundles.

$$\begin{tikzcd}
F^{-1}x_1 \arrow[r, hook] \arrow[d, tail] & (F^{-1}x_1)^{\sim} \arrow[r, tail, "\alpha"] \arrow[d, tail] & F^{-1}x_0 \arrow[r, two heads] \arrow[d, tail] & \operatorname{Coker}\alpha \arrow[d] \\
x_1 \arrow[r, hook] \arrow[d, two heads] & \operatorname{Ker} d_0 \arrow[r, tail] & x_0 \arrow[r, two heads] \arrow[d, two heads] & \operatorname{Im} d_0 \\
x_1/F^{-1}x_1 \arrow[rr] & & x_0/F^{-1}x_0 & .
\end{tikzcd}$$

If $\operatorname{Coker}\alpha$ contains a line bundle of positive degree as a split factor, this factor could not have non-zero morphism into $\operatorname{Im} d_0$, which is a subsheaf of x_{-1}, or into $x_1/F^{-1}x_1$ which is isomorphic to a sum of L^{-2}. Therefore α is an isomorphism onto a split factor of $F^{-1}x_0$. $F^{-1}x_1 \longrightarrow (F^{-1}x_1)^{\sim}$ is a subcomplex of $x_1 \longrightarrow x_0$ and each term is an object of $\mathcal{P}$. The quotient is a case where $x_1 = \oplus L^{-2}$.

In this case the composition of d_1 and the projection onto x_0/F^0x_0 is linear, and it has a decomposition $x_1 \overset{\gamma}{\twoheadrightarrow} x_1' \hookrightarrow x_0/F^0x_0$, where $\operatorname{Ker}\gamma$ is a sum of line bundles with degree less than -2. It can be converted to the following exact sequences:

$$\begin{tikzcd}
\operatorname{Ker}\gamma \arrow[r, hook] \arrow[d, tail] & (\operatorname{Ker}\gamma)^{\sim} \arrow[r, tail, "\beta"] \arrow[d, tail] & F^0x_0 \arrow[r, two heads] \arrow[d, tail] & \operatorname{Coker}\beta \arrow[d, hook] \\
x_1 \arrow[r, hook] \arrow[d, two heads, "\gamma"'] & \operatorname{Ker} d_0 \arrow[r, tail] \arrow[d, two heads] & x_0 \arrow[r, two heads] \arrow[d, two heads] & \operatorname{Im} d_0 \\
x_1' \arrow[r, hook] & x_1'' \arrow[r, hook] & x_0/F^0x_0 &
\end{tikzcd}$$

If β is not an inclusion of a split factor, $\det(\operatorname{Ker}\gamma)^{\sim}$ is negative, and $\det(\operatorname{Coker}\beta)$ must be positive. But $\operatorname{Coker}\beta$ can not be a subsheaf of $\operatorname{Im} d_0$. It implies that β is a split factor.

$x_1'' = \operatorname{Ker} d_0/(\operatorname{Ker}\gamma)^{\sim}$ is a sum of copies of L^{-1} and L^{-2} and it implies $\operatorname{Ker} d_0$ is an object of $\mathcal{P}$ where the middle column of the above diagram is a split short exact sequence.

It follows that there exists a cofibration $y \rightarrowtail x$ from a length one subcomplex $y : y_1 \longrightarrow y_0$ such that $H_0(y) \cong H_0(x)$, $H_0(x/y) = 0$ and $F^{-1}(x/y) = 0$, $F^0(x/y) = 0$.

3.1.2. Definition. We define a subcategory $\mathcal{R}$ of $\mathcal{P}^u$ whose objects are length one complex $x : x_1 \longrightarrow x_0$ satisifying $F^0x_1 = 0$ and $F^{-1}x_0 = x_0$. It is an exact category where level-wise split monomorphisms are admissible monomorphisms. We can consider $\mathsf{Ch}(\mathcal{R})$ the category of bounded complexes of $\mathcal{R}$. An object in $\mathsf{Ch}(\mathcal{R})$ will be presented as a bi-complex

$$\begin{tikzcd}
\cdots \arrow[r] & x_a^+ \arrow[r, "d_a^+"] \arrow[d] & x_{a-1}^+ \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & x_a^- \arrow[r, "d_a^-"] & x_{a-1}^- \arrow[r] & \cdots ,
\end{tikzcd}$$

where each $x : x_\cdot^+ \longrightarrow x_\cdot^-$ belongs to $\mathcal{R}$. We can define an exact functor which regards an object in $\mathsf{Ch}(\mathcal{R})$ as a bi-complex of sheaves on Γ and takes the total complex $\tau : \mathsf{Ch}(\mathcal{R}) \longrightarrow \mathcal{P}^u$.

3.1.3. There are two structure of Waldhausen category on $\mathsf{Ch}(\mathcal{R})$. One is $(\mathsf{Ch}(\mathcal{R}),\ \mathsf{Ac}(\mathcal{R}))$ where $\mathsf{Ac}(\mathcal{R})$ is the subcategory of acyclic complexes of $\mathsf{Ch}(\mathcal{R})$, and the other is $(\mathsf{Ch}(\mathcal{R}),\ \mathcal{W})$ where $\mathcal{W}$ is the subcategory of complexes of which total complexes are acyclic. Both categories are equipped with cylinder functor satisfying cylinder axiom, taking the usual mapping cylinder in $\pm$-component-wise. There is an equivalance: $K(\mathcal{R}) \simeq K(\mathsf{Ch}(\mathcal{R}),\ \mathsf{Ac}(\mathcal{R}))$ (1.4).

The localization theorem can be applied to have a homotopy fiber sequence

$$K(\mathcal{W}\,, \mathsf{Ac}(\mathcal{R})) \longrightarrow K(\mathsf{Ch}(\mathcal{R})\,, \mathsf{Ac}(\mathcal{R})) \longrightarrow K(\mathsf{Ch}(\mathcal{R})\,, \mathcal{W}).$$

3.1.4. Complexes in $\mathcal{P}^u$ having non-trivial homologies only in level 0 will make a subcategory $\mathcal{P}^u[0]$. It is a Waldhausen category if we define a level-wise split monomorphism $x \longrightarrow y$ to be a cofibration in $\mathcal{P}^u[0]$ if the quotient belongs to $\mathcal{P}^u[0]$. This is equivalent to that $H_0(x) \longrightarrow H_0(y)$ is a monomorphism.

There is an exact inclusion functor $\mathcal{R} \longrightarrow \mathcal{P}^u[0]$. It induces an equivalence. For an object x in $\mathcal{P}^w[0]$ the pullback along $F^{-1}\,\mathrm{Im}\,d_1 \rightarrowtail x_0$ defines an acyclic subcomplex $G^{-1}x \rightarrowtail x$.

$x/G^{-1}x$ will be a complex as in 3.1.1. By the last remark in 3.1.1, there is an acyclic quotient complex $x \twoheadrightarrow x/G^0x$. We get $G^{-1}x \rightarrowtail G^0x \rightarrowtail x$ where $G^0x/G^{-1}x$ is an object of $\mathcal{R}$. G^0, G^{-1} define exact functors on $\mathcal{P}^u[0]$, and then apply the additivity theorem, we get the equivalence.

3.1.5. Proposition. $K(\tau) : K(\mathsf{Ch}(\mathcal{R})\,, \mathcal{W}) \longrightarrow K(\mathcal{P}^u\,, w)$ is an equivalence.

Proof. We like to apply the cell filtration theorem of Waldhausen [10, 1.7]. In our case the natural cell filtration is descending but the formulation in [10] is ascending. We have to take the following modification to read the proof line by line. An object is called k-co-spherical if it has only nontrivial homologies in level $-k$. We define a morphism $f : x \longrightarrow y$ is k-co-connected if it induces an isomorphism in homology in level more than $-k$ and monomorphism in level $-k$.

The hypotheses in the top of p.361 in [10] will be follow from the statement bellow. For any k-co-connected $f : A \longrightarrow B$, there are a cofibration $j : A \rightarrowtail C$ and a $k+1$-co-connected morphism $f' : C \rightarrowtail B$ such that $f = f' \circ j$. To see that, observe there are k-co-spherical A' and B' and morphisms

$$\begin{array}{ccc} A' & \longrightarrow & A \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B' & \longrightarrow & B \end{array}$$

where horizontal morphisms induce isomorphisms in H_{-k}. Define C by the following exact sequence

$$\begin{tikzcd} C(f) \arrow[r, tail] \arrow[d] & \cdot \arrow[r, two heads] \arrow[d] & C(f')[1] \arrow[d, equal] \\ A[1] \arrow[r, tail] & C[1] \arrow[r, two heads] & C(f')[1]. \end{tikzcd}$$

where $C(f)$ is the mapping cone, and $C(f')[1]$ is the shift of the mapping cone.

3.1.6. Proposition Let $\mathcal{P}^u\langle -1, 0\rangle$ be the subcategory of $\mathcal{P}$ (1.5) consisting of complexes whose terms are isomorphic to direct sums of L^{-1} and L^0 with torsion homologies. It is equivalent to those on the projective line Π, and its K-theory is equivalent to that of torsion sheaves on Π (1.3.9).

Taking the $-$-part of $\mathcal{W}$ (3.1.2) will define an exact functor

$$\rho : (\mathcal{W}\,,\, \mathsf{Ac}(\mathcal{R})) \longrightarrow (\mathcal{P}^u\langle -1, 0\rangle, w),$$

and it is an equivalence in K-theory.

Proof. For an object $x : x_\cdot^+ \longrightarrow x_\cdot^-$ in $\mathcal{W}$, $\operatorname{Im} d_i^+ \longrightarrow \operatorname{Im} d_i^-$ and $\operatorname{Ker} d_i^+ \longrightarrow \operatorname{Ker} d_i^-$ are contained in $\mathcal{R}$ for any i, and $\operatorname{Im} d_i^{\pm} \longrightarrow \operatorname{Ker} d_i^{\pm}$ is an object of $\mathcal{W}$. Moreover x is in $\mathsf{Ac}(\mathcal{R})$ if and only if $\operatorname{Im} d_i \cong \operatorname{Ker} d_i$ for all i. Therefore ρ satisfy the first assumption of the approximation theorem (1.3.4). Note $\mathcal{W}$ has a cylinder functor by $\pm$-level-wise construction of the mapping cylinder.

To see the second assumption, let $x : x_\cdot^+ \longrightarrow x_\cdot^-$ be an object in $\mathcal{W}$ and a morphism $x_\cdot^- \longrightarrow a$ in $\mathcal{P}^u\langle -1, 0\rangle$. Take the Euler resolution of L^0 by L^{-1} and L^{-2}, we get a quasi-isomorphism of complexes $a^\sim \longrightarrow a$ and a commutative diagram (1.5.3)

$$\begin{tikzcd} x^+ \arrow[r] \arrow[d] & a^\sim \arrow[d] \arrow[dr] & \\ x^- \arrow[r] & a & a' \arrow[l] \end{tikzcd}$$

where $a^\sim \longrightarrow a'$ defines an object in $\mathsf{Ch}(\mathcal{R})$ and $x : a' \longrightarrow a$ is the quotient by an acyclic complex consisting of L^{-1}. Then $a^\sim \longrightarrow a'$ belongs to $\mathcal{W}$. Finally there exist a morphism $x^- \longrightarrow a'$ which implies the assumption.

3.1.7. In summary we get a homotopy fiber sequence

$$\prod_{\pi\in\Pi}^{\oplus} K(\kappa(\pi)) \longrightarrow K(\mathcal{R}, i) \longrightarrow K(\mathcal{P}^u, w)$$

by identifying terms in 3.1.3

3.2. We will apply the devissage theorem (1.2) to $\mathcal{R}$.

3.2.1. Let $x : x^+ \longrightarrow x^-$ be an object of $\mathcal{R}$. For $y \rightarrowtail x$ and $z \rightarrowtail x$ we define $y \cap z$ and $y + z$ in level-wise manner as sub-sheaves. It is easy to see that these are object in $\mathcal{R}$. The inclusion of sub-sheaf $y \cap z \longrightarrow x$ is a cofibration, for it is a cofibration if and only if $F^0(y \cap z)^\pm \longrightarrow F^0 x^\pm$ and $(y \cap z)^\pm / F^0(y \cap z)^\pm \longrightarrow x^\pm / F^0 x^\pm$ are monomorphisms.

The rank of $x : x_1 \to x_0$ in $\mathcal{R}$ is the rank of x_0 as a locally free sheaf. A cofibration is called non-trivial if it is not an isomorphism or zero morphism.

3.2.2. Definition. We call an object x in $\mathcal{R}$ *simple* if it does not have any non-trivial cofibration into it. A direct sum of simple objects is called *semi-simple*. $\mathcal{R}^{ss}$ denotes the exact subcategory consisting of semi-simple objects.

3.2.3. Proposition. $\mathcal{R}^{ss} \subset \mathcal{R}$ induces an equivalence in K-theory.

Proof. We have to verify the conditions in 1.2.2. The first to fourth conditions on $\mathcal{R}$ are verified in 3.2.1.

For a semi-simple object $x = \oplus_S \, x_s$, if there is a cofibration $y \rightarrowtail x$, then y is also semi-simple, and y is a split factor. To see that, consider a subset T of S so that $y \cap (\oplus_T \, x_t) = 0$. If T is maximal on these subsets, there should be $y + (\oplus_T \, x_t) = x$, for if some x_s is not contained in $y + (\oplus_T \, x_t)$, $x_s \cap y$ is zero. y is then isomorphic to $\oplus_{S-T} \, x_s$. Therefore $\mathcal{R}^{ss}$ satisfy (v) and (vi).

To see the last condition, take an object x of $\mathcal{R}$. We may assume it is not simple. Then there exists a non-trivial cofibration $y \rightarrowtail x$. Let us take a maximal one. If x/y is not simple, there exists a non-trivial $w \rightarrowtail x/y$. But the existence of the pullback v of the diagram

$$\begin{tikzcd} y \arrow[r, tail] \arrow[d, equal] & v \arrow[r, two heads] \arrow[d, tail] & w \arrow[d, tail] \\ y \arrow[r, tail] & x \arrow[r, two heads] & x/y \end{tikzcd}$$

contradicts the assumption that y is maximal. By induction of rank we have a filtration on y and x/y is in $\mathcal{R}^{ss}$.

3.2.4. Simple and semi-simple objects in $\mathcal{R}$ should not be confused with those in abelian categories. As there could be a non-zero morphisms between different simple objects, and the endomorphism ring of a semi-simple object may not be semi-simple as κ-algebra.

Endomorphisms of a semi-simple object defines a finite dimensional algebra over κ, and K-theory of semi-simple objects is a filtered colimit of K-theory of such algebras.

3.3. Conclusion.

3.3.1. Combining 3.1.6 and 3.2.6, we get the following homotopy fiber sequence

$$\prod^{\oplus} K(\kappa(\pi)) \longrightarrow \operatorname{colim} K(A) \longrightarrow K(\mathcal{P}^u)$$

where π run the closed points of the projective line and A are finite dimensional κ-algebras.

Together with 1.3.9 and 2.3.9, we finally get the following theorem:

Theorem. *Let Γ be a smooth proper curve over a field κ having a finite morphism $\pi : \Gamma \to \Pi$ satisfying the condition in 2.1.1. Let $k(\Gamma)$ be the function field of Γ. Then there are long exact sequences*

$$\cdots \longrightarrow \oplus K_*(\kappa(\gamma)) \longrightarrow K_*(\Gamma) \longrightarrow K_*(k(\Gamma)) \longrightarrow \cdots,$$

$$\cdots \longrightarrow \oplus^2 K_*(\kappa) \longrightarrow K_*(k(\Gamma)) \longrightarrow K_{*-1}(\mathcal{P}^u) \longrightarrow \cdots,$$

$$\cdots \longrightarrow \operatorname{colim} K_*(A) \longrightarrow K_*(\mathcal{P}^u) \longrightarrow \oplus K_{*-1}(\kappa(\pi)) \longrightarrow \cdots,$$

where A are finite dimensional κ-algebras, $\kappa(\pi)$ are residue fields of closed points on Π and $\kappa(\gamma)$ are residue fields of closed points on Γ.

4. Application

4.0. We will have an application on K-theory with $\mathbb{Q}$-coefficients of schemes over finite fields.

4.0.1. $K(X) \otimes \mathbb{Q}$ is a space having homotopy groups as

$$\pi_*(K(X) \otimes \mathbb{Q}) \cong (\pi_* K(X)) \otimes \mathbb{Q}.$$

This operation $\otimes\mathbb{Q}$ preserves homotopy fiber sequences of H-spaces and finite products.

Note that for a finite field κ, $K_*(\kappa) \otimes \mathbb{Q}$ is zero for all $* > 0$.

4.0.2. For a finite dimensional algebra R over a field κ, its radical J is a nilpotent ideal ([3, §6 theorem 3]). The quotient R/J is a semi-simple algebra. It implies R/J is isomorphic to a direct product of the matrix algebras $Mat(D_i, n_i)$ of rank n_i over division algebras D_i whose center κ_i are finite over κ ([3, §5 theorem 2]):

$$R/J \cong \prod Mat(D_i, n_i).$$

When κ is a field of positive characteristic, there is an isomorphism of the K-theory with $\mathbb{Q}$-coefficients

$$K_*(R) \otimes \mathbb{Q} \cong K_*(R/J) \otimes \mathbb{Q}$$

by [11], and

$$K_*(\prod Mat(D_i, n_i)) \otimes \mathbb{Q} \cong \oplus K_*(D_i) \otimes \mathbb{Q} \cong \oplus K_*(\kappa_i) \otimes \mathbb{Q},$$

where the first isomorphism is induced by Morita equivalence and the second is due to the fact that $K_*(\kappa_i)$ and $K_*(D_i)$ are isomorphic up to degree of D_i [4].

4.0.3. Proposition. Assume there is an integer N such that for any finite extension κ' of the base field κ, $K_*(\kappa') \otimes \mathbb{Q} = 0$ for $* > N$. Then for any curve Γ over k, $K_*(\Gamma) \otimes \mathbb{Q} = K_*(k(\Gamma)) \otimes \mathbb{Q} = 0$ for $* > N + 2$.

Proof. When there is a finite extension of the base field κ', $K_*(\Gamma) \otimes \mathbb{Q}$ is a submodule of $K_*(\Gamma \times \operatorname{Spec} \kappa') \otimes \mathbb{Q}$, for the transfer mapping (1.3.1) f_* satisfies that $f_* \circ f^*$ is multiplication by the degree of κ' over κ.

We may take a finite extension of the base field so that Γ satisfy the condition 2.1.1. We can get the homotopy fiber sequence (3.3). But 4.0.2 implies $\operatorname{colim} \pi_*(K(A)) \otimes \mathbb{Q} = 0$ for $* > N$. Then we proved $K_*(\Gamma) \otimes \mathbb{Q} = 0$ for $* > N + 2$.

4.1. To study general schemes we recall G-theory in 1.3.8

Theorem *For an irreducible scheme X, essentially of finite type over a finite field or the algebraic closure of finite field, $G_*(X) \otimes \mathbb{Q} = 0$ for $* > 2 \dim X$. When X is regular, it implies $K_*(X) \otimes \mathbb{Q} = 0$.*

4.1.1. For a regular scheme X there is an equivalence $G(X) \simeq K(X)$. And if X_{red} is the reduced subscheme of X, there is an equivalence $G(X_{red}) \simeq G(X)$ ([7, §7 3.1]). If a scheme X is an inverse limit of X_i where transition morphisms are affine and flat, $G(X) \simeq \operatorname{colim} G(X_i)$ ([7, §7 2.2]).

Proof of the theorem. We may assume the base field is an algebraically closed field $\bar{\kappa}$ for $G_*(X) \otimes \mathbb{Q}$ injects into $G_*(X \times \operatorname{Spec} \bar{\kappa}) \otimes \mathbb{Q}$.

We prove the theorem inductively on the dimension of X. Cases $\dim X = 0$ are just K-theory of finite fields. Assume it is true for all irreducible regular schemes of finite type with dim is less than d. Then for any scheme of finite type X there is a sequence of closed immersions $X_0 \subset X_1 \subset \cdots X_n = X$ such that $X_i - X_{i-1}$ is a union of regular irreducible schemes of finite type with dim $= i$. Then $G_*(X) \otimes \mathbb{Q} = 0$ if and only if $G_*(X - X_{n-1}) \otimes \mathbb{Q} = 0$ by the localization theorem (1) in 1.3.8 and the inductive hypothesis.

There exists a projective smooth curve C over a field of dimension $\dim X - 1$ such that the function field $k(X)$ of X is isomorphic to the function field of C. Then the theorem holds for $k(C)$ by the lemma, and then for $k(X)$. Filtered colimits of the localization theorem implies

$$\cdots \longrightarrow \operatorname{colim} G_*(Z) \otimes \mathbb{Q} \longrightarrow G_*(X) \otimes \mathbb{Q} \longrightarrow G_*(k(X)) \otimes \mathbb{Q} \longrightarrow \cdots$$

where the first and the last term vanish if $* > 2 \dim X$.

4.1.2. The best possible result is $G_*(X) \otimes \mathbb{Q} = 0$ for $* > \dim X$. It will be proved in [5].

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