THE TATE-THOMASON CONJECTURE

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0. Introduction

0.0. For a projective smooth variety X over a field k, the i-th Chow group $CH^i(X)$ is generated by cycles of codimension i on X modulo rational equivalence. If a prime l is invertible in k we can define the cycle class morphism from the i-th Chow group into the $2i$-th ℓ-adic étale cohomology group with i-th Tate twist coefficient:

$$\rho^i(X)_{\mathbb{Q}} : CH^i(X) \otimes \mathbb{Q} \to H^{2i}(X, \mathbb{Q}_l(i)).$$

It can extend to a $\mathbb{Q}_l$-linear morphism $\rho^i(X)_{\mathbb{Q}_l} : CH^i(X) \otimes \mathbb{Q}_l \to H^{2i}(X, \mathbb{Q}_l(i))$.

First let us consider the cases when k is a finite field $\mathbb{F}_q$. For a variety X_0 over a finite field $\mathbb{F}_q$, X_∞ denotes the scalar extension of X_0 to the algebraic closure $\mathbb{F}_{q^\infty}$. The main result of this paper is the following.

0.1. Theorem: *For any i the image of the cycle class morphism $\rho^i(X_\infty)_{\mathbb{Q}_l}$ in the cohomology group $H^{2i}(X_\infty, \mathbb{Q}_l(i))$ is exactly the union of fixed parts of open subgroups of the Galois group* $\mathrm{Gal}(\mathbb{F}_{q^\infty}/\mathbb{F}_q)$.

0.1.1. This is the conjecture of Tate [34] in the case of varieties over finite fields. We interpret the cycle class morphism to the localization mapping on the algebraic K-spectrum with respect to the topological K-homology. Namely the left-hand side of the cycle class morphism is a piece of the zeroth homotopy group of algebraic K-spectrum $\mathbf{K}X$ associated to X, tensored with $\mathbb{Q}$,

$$\pi_0(\mathbf{K}X \otimes \mathbb{Q}) \cong \oplus CH^i(X) \otimes \mathbb{Q};$$

and the right-hand side is a piece of the zeroth homotopy group of $\mathbf{K}X$ localized with respect to the topological K-spectrum $\mathcal{K}$ under l-adic completion $(L_{\mathcal{K}}\mathbf{K}X)^\wedge$, tensored with $\mathbb{Q}$,

$$\pi_0((L_{\mathcal{K}}\mathbf{K}X)^\wedge \otimes \mathbb{Q}) \cong \oplus H^{2i}(X; \mathbb{Q}_l(i)).$$

For any spectrum W the composition of $\mathcal{K}$-localization and l-completion mappings

$$\eta(W) : W \to L_{\mathcal{K}}W \to (L_{\mathcal{K}}W)^\wedge$$

is defined, and if $W = \mathbf{K}X$, $\pi_0(\eta(W) \otimes \mathbb{Q})$ corresponds to $\oplus\rho^i(X)$.

In the first non-trivial case where X is a surface and $i = 1$, Artin and Tate showed that triviality of the divisible part of Brauer group of X would imply the theorem on X ([35, §4]). And in [38], Thomason generalized the above argument, as an application of his étale descent theorem (1.3.2) for $L_{\mathcal{K}}\mathbf{K}X$, in the way that if $\pi_{-1}(L_{\mathcal{K}}\mathbf{K}X)$ does not contain any divisible subgroup then the theorem is true for all i simultaneously. It is this form of the theorem we will give a proof here, and we like to call it the *Tate-Thomason conjecture*. The precise statement is presented in 2.1.

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0.2. The proof is basically a combination of land-marking achievements, and the result of [20]. To express the way to assemble them we need a series of set-ups.

We will study $\pi_*(L_{\mathcal{K}}\mathbf{K}X)$ by means of the Adams spectral sequence associated to $\mathcal{K}$ (1.4). As is investigated by Bousfield, the E_2-terms are controlled by $\mathcal{K}$-homology of $\mathbf{K}X$, while its l-completions are controlled by the étale cohomology through the descent theorem of Thomason.

We use $\mathcal{K}$-homology with integral coefficients. The main result of [20] is the existence of long exact sequences (3.0), and it will give us information on the integral groups. We use the theorem of Deligne on Riemann-Weil hypotheses (1.5.3), and the theorem of Voevodsky on the galois cohomology, and its extension to Azumaya algebra by Kahn and Levine.

The theorem of Voevodsky on the galois cohomology mentioned above is the following. Let $H^*(F, \mathbb{Q}/\mathbb{Z}_{(l)})$ be the etale cohomology group of F.

0.2.1. Assumption. Let X_∞ be a smooth variety over an algebraically closed field, and $F = k(X_\infty)$ be the function field of it.

Then the spectral sequence induced from the slice filtration [24]

$$E_2 = H^*(F, \mathbb{Q}/\mathbb{Z}_{(l)}) \Rightarrow K_*(F, \mathbb{Q}/\mathbb{Z}_{(l)})$$

degenerates at E_2, and $H^*(F, \mathbb{Q}/\mathbb{Z}_{(l)})$ are divisible.

This is a consequence of the Milnor-Bloch-Kato conjecture for $F = k(X_\infty)$, saying that the Milnor K-groups are isomorphic to etale cohomology groups;

$$K_*^{Milnor}(F) \otimes \mathbb{Q}/\mathbb{Z}_{(l)} \cong H^i(F, \mathbb{Q}/\mathbb{Z}_{(l)}),$$

because the left hand side is always divisible, with zero differential on the spectral sequence. It is proved when $l = 2$ by Voevodsky [42] for all field of characteristic not 2, and is said to be proved for odd l in general.

In this paper we mainly discuss cases when l is an odd prime number. The argument works properly when $l = 2$ under technical modifications, which we will discuss in 4.4. But if we work only in $l = 2$ cases, we do not have results in $p = 2$ cases.

0.3. In Section 1 we recollect results on the algebraic K-theory spectrum $\mathbf{K}X$ and homotopy theoretic constructions like $L_{\mathcal{K}}$. The Tate-Thomason conjecture is stated in Section 2. The proof of the conjecture is shown in Section 4, while some technical lemmas are prepared in Section 3.

Any kind of generalization of the conjecture to open varieties was tangled with semi-simplicity of the action of the Frobenious operator, which was not known. This Gordian knot is unleashed by virtue of topological K-theory (3.1.3). We can use reduction to open varieties as in 4.3.

0.4. In Section 5, we treats other consequences over finite fields; semi-simplicity of the Frobenious action and the structure theorem of $\mathcal{K}$-homology of K-theory spectra, together with Beilinson-Friedlander-Parshin conjecture.

We can also prove that the coincidence of homological equivalence and numerical equivalence of algebraic cycles on projective smooth varieties over finite fields. We can lift this fact to fields of characteristic zero.

1. Review

1.0. Convention.

1.0.1. $\mathbb{F}_{q^n}$ is the finite field with q^n elements, $\mathbb{F}_{q^\infty}$ is the algebraic closure of $\mathbb{F}_q$. l is a prime number different from the characteristic of $\mathbb{F}_q$. We fix an *odd* l, except in 4.4, though we have results in the cases where $l = 2$ (5.1). $\mathbb{Z}_{(l)}$ is the localization at l of the ring of integers. $\mathbb{Z}_l$ is its completion, and $\mathbb{Q}_l$ is the field of quotients. $\mathbb{Q}/\mathbb{Z}_{(l)}$ is the l-primary torsion irreducible divisible group.

We will fix in 1.4.1 an integer r which becomes a topological generator of the group of units in $\mathbb{Z}_l$ modulo roots of unity. Let l^a be the maximal number of l-power roots of unity in the base field $\mathbb{F}_q$. We may assume a is positive, so that $q \equiv 1(l)$, which means that q is congruent to 1 modulo l.

1.0.2. All the groups are assumed to be commutative and l-local. For a group A, it has a decomposition $A^t \oplus A^f$ into the subgroups of torsion elements and torsion free elements.

We present a monomorphism between groups as $\rightarrowtail$, and an epimorphism as $\twoheadrightarrow$ in diagrams.

A relation $M \sim N$ between groups means that there exists a subgroup M' of M and a morphism $M' \longrightarrow N$ such that the kernel and cokernel of the morphism are of bounded order and M/M' is of bounded order.

A group is said to be of co-finite type if it is a sum of a finite group and finitely many copies of $\mathbb{Q}/\mathbb{Z}_{(l)}$.

When M is a discrete torsion group, $M^\vee = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}_{(l)})$ is a pro-finite group, and when M is a pro-finite group, $M^\vee$ is the group of continuous homomorphisms from M into $\mathbb{Q}/\mathbb{Z}_{(l)}$ which becomes a discrete torsion group. In both cases it is called the dual of M. It defines an anti-equivalence between the category of finitely generated $\mathbb{Z}_l$-modules and that of groups of co-finite type.

For a torsion group A, there exists the maximal divisible subgroup A^{div}. A torsion group is called *reduced* if the maximal divisible subgroup is trivial. For any torsion group A there exists a decomposition

$$A \cong A^{div} \oplus A^{red}$$

where A^{red} is reduced. If A is not torsion, A^{div} denotes the the maximal divisible torsion subgroup.

For a reduced group A, an element has height greater than n if it is contained in the subgroup $\cap_{v \le n} l^v A$. The subgroup $H(A) = \cap_v l^v A$ is called the subgroup of infinite height. If A is countably generated, $H(A) = 0$ if and only if A is a sum of cyclic groups ([12, §18]). For a countably generated torsion group A, $A/H(A)$ is isomorphic to a sum of cyclic groups.

We can define a transfinite filtration on a reduced group. Define $H^{\alpha+1}(A) = H(H^\alpha(A))$ and $H^\beta(A) = \cap_{\alpha<\beta} H^\alpha(A)$ if β is a limit ordinal. If A is countably generated, each filtered quotient is isomorphic to a sum of finite groups, and the filtration terminates at some countable ordinal γ, where $H^\gamma(A) = 0$. It is called the Ulm sequence associated with A and it characterizes countably generated reduced groups ([12, §37]).

1.0.3. A space means a based simplicial set [15]. The base point will be written as $\star$. A weak equivalence is a mapping which becomes a homotopy equivalence after geometric realization. For two spaces X and Y, the wedge product $X \wedge Y$ is quotient of $X \times Y$ by $X \times \star \cup \star \times Y$. The category of spaces make a simplicial model category having the mapping space $\mathrm{Map}(X, Y)$ ([15, II]). Note that every space is cofibrant, and fibrant objects are Kan complexes.

There is a notion of R-completion ([6]) of a space X associated to a ring R. If R is either $\mathbb{Z}_{(l)}$ or $\mathbb{Q}$, it is called R-localization and the result is written as $X \otimes R$. If R is $\mathbb{Z}/l\mathbb{Z}$, it is called l-completion and the result is written as $X^\wedge$. The result of R-completion is assumed to be fibrant. A space is called l-local if the canonical mapping $X \longrightarrow X \otimes \mathbb{Z}_{(l)}$ is an weak equivalence.

For a simplicial space $X : [n] \to X_n$, the realization is given by the application of the diagonal functor ([7, B.1]), and resulting simplicial set is written as $X_\cdot$. It preserves term-wise weak equivalences, and term-wise fibrations if all the spaces in terms are connected ([7, B.5]).

1.0.4. A spectrum means that in algebraic topology. A category of spectra will give a model category for doing the stable homotopy theory, so that the homotopy category of spectra should be the classical one. Following references [36], we use the Bousfield-Friedlander category [7] of simplicial spectra. Namely a spectrum means a sequence of spaces $X_0, X_1, \ldots$ with the structure mappings $S^1 \wedge X_i \to X_{i+1}$. An omega spectrum is a spectrum such that for every i X_i is fibrant and the adjoint of the structure mapping is a weak equivalence $X_i \simeq \Omega X_{i+1}$.

Spectra are always assumed to be l-local. Thus we will work on the category on l-local spectra [33]. Note that fibrant objects in this category are omega spectra with l-local homotopy groups, so that statements in [36] literally true in our setting. Stable homotopy equivalences are presented as $\simeq$.

For an omega spectrum X its $\mathbb{Q}$-localization $X \otimes \mathbb{Q}$ and l-completion $X^\wedge$ is obtained in level-wise manner: $(X \otimes \mathbb{Q})_n = X_n \otimes \mathbb{Q}$ and $(X^\wedge)_n = (X_n)^\wedge$.

For spectra X and Y, its wedge product is written as $X \wedge Y$ ([25, 0.3]), which preserves homotopy cofiber sequences.

1.0.5. T/l^ν will be the homotopy cofiber of self mapping by the multiplication of l^ν:

$$T \xrightarrow{l^\nu} T \longrightarrow T/l^\nu.$$

The homotopy limit of T/l^ν along ν is equivalent to l-adic completion $T^\wedge$. The Moore spectrum associated to $\mathbb{Q}/\mathbb{Z}_{(l)}$ is presented as $\mathcal{M}$:

$$\mathcal{M} \simeq \operatorname*{hocolim}_{\nu} \Sigma^\infty S^0/l^\nu.$$

Here $\Sigma^\infty S^0$ means the sphere spectrum. There is a homotopy cofiber sequence

$$T \to T \otimes \mathbb{Q} \to T \wedge \mathcal{M}.$$

1.0.6. The (l-localized) periodic topological complex K-spectrum with period two is written as $\mathcal{K}$. For it is localized at l, $\mathcal{K}$ decomposes as

$$\mathcal{K} \simeq \bigvee_{i=0}^{l-2} \Sigma^{2i} E(1),$$

where $E(1)$ will be defined in 1.4.1.

For a spectrum T the (stable) topological Adams operator ψ^k is defined on $\mathcal{K}_* S$ for all k prime to l.

1.0.7. All the schemes are assumed to be separated and essentially of finite type over a finite field $\mathbb{F}_q$, or the algebraic closure of finite fields. We may assume schemes are reduced (1.1.2). The dimension of an irreducible scheme means the transcendence degree of the function field, and the dimension of a scheme means the maximum of dimension on the irreducible components. A variety means a scheme of finite type over the base field. We denote by P^1 the projective line, which is a one-dimensional variety.

1.0.8. For a scheme X over a finite field $\mathbb{F}_q$, we can take the scalar extension $X_n = X \otimes \mathbb{F}_{q^{l^n}}$, and X_∞ means the scalar extension to the algebraic closure $\mathbb{F}_{q^\infty}$. $\mathrm{Gal}(\mathbb{F}_{q^\infty}/\mathbb{F}_q)$ acts on $X_\infty = X \otimes \mathbb{F}_{q^\infty}$ through the second factor. The Frobenious operator $x \mapsto x^q$ is a topological generator of $\mathrm{Gal}(\mathbb{F}_{q^\infty}/\mathbb{F}_q)$, and it defines the arithmetic Frobenious operator ϕ_X on X_∞.

1.0.9. We denote by $H^i(X, \mathbb{Z}_l(j))$ the continuous l-adic étale cohomology group with Tate twist coefficient. $H^i(X, \mathbb{Z}_l(j)) \otimes \mathbb{Q} = H^i(X, \mathbb{Q}_l(j))$ defines the cohomology with $\mathbb{Q}_l$-coefficient.

1.0.10. If there is an operator σ on a group M, We denote by M^σ the invariant subgroup $\{x \in M : \sigma x = x\}$ and by M_σ the co-invariant quotient group $M/(\sigma - 1)M$. There is an exact sequence

$$0 \longrightarrow M^\sigma \longrightarrow M \xrightarrow{\sigma-1} M \longrightarrow M_\sigma \longrightarrow 0.$$

1.1. K-theory spectra.

1.1.0. Let $\mathcal{C}$ be a Waldhausen category, which is a category with cofibrations and weak equivalences in the sense of [41, 1.1]. It defines a multi-simplicial category with weak equivalences $S. \cdots S.\mathcal{C}$ and the l-localized spectrum $\mathbf{K}\mathcal{C}$ is defined as the geometric realization of nerves of multi-simplicial categories ([41, 1.3])

$$(\mathbf{K}\mathcal{C})_n = (w \overbrace{S. \cdots S.}^{n} \mathcal{C}) \otimes \mathbb{Z}_{(l)}$$

for $n > 0$.

For a scheme X we can define a structure of Waldhausen category on the category of perfect complexes on X. We denote by $\mathbf{K}X$ the associated spectrum ([40, 3.1]). It defines a contravariant functor for all the morphisms of schemes and the (stable) homotopy groups of $\mathbf{K}X$ are just the usual algebraic K-groups $K_i(X)$ defined by Quillen [30], under localization:

$$\pi_i(\mathbf{K}X) = K_i(X) \otimes \mathbb{Z}_{(l)}.$$

K-groups with mod l^ν-coefficients are defined as $K_i(X\,; \mathbb{Z}/l^\nu) = \pi_i(\mathbf{K}X/l^\nu)$.

For an affine scheme $\mathrm{Spec}\, A$, $\mathbf{K}\,\mathrm{Spec}\, A$ will be written simply as $\mathbf{K}A$. For a non-commutative ring A, we can define its K-theory $\mathbf{K}A$ using bounded complexes of finitely generated projective modules.

1.1.1. Let X be a regular variety and Z be a closed subscheme in X. The subcategory of perfect complexes acyclic on $X - Z$ defines the relative K-theory $\mathbf{K}(X \text{ on } Z)$. It satisfy the localization homotopy cofiber sequence ([40, 3.2.1, 7.4])

$$\mathbf{K}(X \text{ on } Z) \to \mathbf{K}(X) \to \mathbf{K}(X - Z).$$

And if there is a closed subscheme W in Z, there is a relative version of homotopy cofiber sequence

$$\mathbf{K}(X \text{ on } W) \to \mathbf{K}(X \text{ on } Z) \to \mathbf{K}(X - W \text{ on } Z - W).$$

If Z is a regular closed subscheme, the restriction gives an equivalence ([40, 3.2.1 and 5.1.5])

$$\mathbf{K}(X \text{ on } Z) \simeq \mathbf{K}(Z).$$

1.1.2. Because we always take l-localization, $\mathbf{K}X_{red} \simeq \mathbf{K}X$ where X_{red} is the reduced subscheme of X ([40, 9.7]).

1.1.3. Let $f : X' \to X$ be a finite flat morphism. There is a transfer mapping f_* in K-theory. Then f_*f^* is a multiple of $f_*(\mathcal{O}_{X'})$ as an element in $K_0(X)$ ([40, 3.17]). If $f_*(\mathcal{O}_{X'})$ is $d[\mathcal{O}_X]$ in $K_0(X)$, f_*f^* is multiplication by d. This condition will be satisfied when X' is the base change $X \otimes_k k'$ where k' is a finite separable extension of degree d over k. $\pi_*\mathbf{K}X \otimes \mathbb{Q}$ is a submodule of $\pi_*\mathbf{K}X' \otimes \mathbb{Q}$, so that if the second group is zero, so is the first group. It is called the transfer argument.

1.1.4. We ensure that $\pi_*\mathbf{K}X$ are countably generated. For on an affine scheme $\operatorname{Spec} A$, every locally free sheaf is a direct factor of a free sheaf, so that the category of bounded complexes of locally free sheaves is equivalent to a category with countable many objects and morphisms. Because K-theory of this category is equivalent to the original one ([40, 3.8]), we know that $\pi_*\mathbf{K}\operatorname{Spec} A$ are countably generated. General schemes are covered by finitely many affine schemes, and Mayer-Vietoris property implies the claim.

For any spectrum W with countably generated homotopy groups, $\pi_*(W \wedge \mathbf{K}^B X)$ is countably generated. Similarly for relative cases.

1.1.5. For a discrete valuation ring R containing the residue field k and the quotient field K, the localization theorem implies a homotopy cofiber sequence

$$\mathbf{K}k \xrightarrow{i} \mathbf{K}R \xrightarrow{j^*} \mathbf{K}K.$$

Because the exact functor i can be written as homotopy difference of the same exact functor ([14]), for any spectrum W, there is an short exact sequence

$$0 \to W_*\mathbf{K}R \xrightarrow{j^*} W_*\mathbf{K}F \xrightarrow{\partial} W_{*-1}\mathbf{K}k \to 0.$$

It implies that there is the Gersten-Quillen spectral sequence for regular variety

$$E_1^{p,q} = \underset{x \in X^p}{\oplus} W_{-p-q}\mathbf{K}(k(x)) \Rightarrow W_{-p-q}\mathbf{K}(X)$$

where X^p is the set of points of codimension p and $k(x)$ is the residue field.

1.2. Homology localization and completion

1.2.1. The Bousfield localization functor associated with the periodic topological K-theory $\mathcal{K}$ will be presented just as L. There is a natural transformation $\eta(W) : W \to LW$. Since there is an equivalence ([3, 4.3]) for any spectrum W

$$LW \simeq W \wedge L(\Sigma^\infty S^0),$$

L preserves homotopy cofiber sequences and direct sums.

1.2.2. It is convenient to consider $E(1)$ instead of $\mathcal{K}$. Because they define the same localization functor $L_{\mathcal{K}} \simeq L_{E(1)}$ ([3, §4]), and it is consistent with the reference [4, §4].

$E(1)$ is the essential part of $\mathcal{K}$,

$$\mathcal{K} \simeq \bigvee_{i=0}^{l-2} \Sigma^{2i} E(1),$$

having Adams operations such that for any spectrum W, there is an isomorphism $\mathcal{K}_* W \cong \mathcal{K}_* \otimes_{E(1)_*} E(1)_* W$ preserving all Adams operations. Note that $\mathcal{K}_* \cong \mathbb{Z}_{(l)}[\beta, \beta^{-1}]$ where degree of β is 2, and $E(1)_* \cong \mathbb{Z}_{(l)}[\nu, \nu^{-1}]$ where degree of ν is $2(l-1)$, and $\mathcal{K}_* \cong E(1)_*[\beta]/(\beta^{l-1} = \nu\ )$.

1.2.3. For a spectrum W the homotopy groups of the l-completion $W^\wedge$ satisfy the following exact sequences ([6, VI.5]):

$$0 \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}_{(l)}, \pi_* W) \to \pi_* W^\wedge \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}_{(l)}, \pi_{*-1} W) \to 0.$$

An abelian group A is reduced if and only if $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}_{(l)}, A) = 0$. Any subgroup of a reduced group is reduced, and any quotient group of a divisible group is divisible.

1.3. $\mathcal{K}$-localized K-theory and étale topology.

1.3.0. For a separably closed field k^{sep}, we know that ([40, 11.5])

$$L\mathbf{K}k^{sep}/l^\nu \simeq \mathcal{K}/l^\nu,$$

so that

$$\pi_{2i} L\mathbf{K}k^{sep}/l^\nu \cong \mathbb{Z}/l^\nu\mathbb{Z}(i)$$
$$\pi_{2i+1} L\mathbf{K}k^{sep}/l^\nu \cong 0$$

as sheaves of abelian groups on k_{et}. ($\mathbb{Z}/l^\nu\mathbb{Z}(i)$ is the Tate twist in 1.0.9.)

1.3.1. $\mathcal{K}$-localization $L\mathbf{K}X$ will be called LK-theory of X. Since L is a functor on the category of spectra, and it preserves homotopy cofiber sequences and stable homotopy equivalences, LK-theory automatically satisfies properties of K-theory as were explained in 1.0.

K-theory and LK-theory are related in the following way ([36, Appendix A]): they are equivalent in $\otimes\mathbb{Q}$,

$$L\mathbf{K}X \otimes \mathbb{Q} \simeq \mathbf{K}X \otimes \mathbb{Q};$$

and in mod-l^ν, LK is equivalent to the Bott periodic K-theory in the sense of [36]

$$L\mathbf{K}X/l^\nu \simeq \mathbf{K}/l^\nu(X)[\beta^{-1}].$$

1.3.2. K-theory defines a presheaf taking values in spectra on the small étale site X_{et} for any scheme X. LK-theory also defines a presheaf. The most significant difference between two presheaves is the étale descent property. Namely associated to any presheaf of spectra W we can define the hypercohomology spectrum $\mathbb{H}^\cdot(X_{et}, W)$. And the celebrated étale descent theorem of Thomason ([36]) says the augmentation is a weak equivalence

$$L\mathbf{K}X \simeq \mathbb{H}^\cdot(X_{et}, L\mathbf{K}).$$

We only need it for regular X, but it is true for schemes without this property ([40, 11.11]). Note that in [40, 11.11], X has to satisfy a mild finiteness condition. It is satisfied in our conventions for any scheme.

If we take mod-l^ν coefficients, the stalks of the presheaves of homotopy groups are identified to the locally constant sheaf $\mathbb{Z}/l^\nu$ with Tate twist coefficients, and it induces the descent spectral sequence

$$E_2^{s,t} = H^s(X_{et}, \mathbb{Z}/l^\nu(\frac{t}{2})) \Rightarrow \pi_{s-t} L\mathbf{K}X/l^\nu.$$

For a closed subscheme Z in X, there is a relative version of the spectral sequence

$$E_2^{s,t} = H_Z^s(X_{et}, \mathbb{Z}/l^\nu(\frac{t}{2})) \Rightarrow \pi_{s-t} L\mathbf{K}(X \text{ on } Z)/l^\nu.$$

1.3.3. We can take directed colimits of the spectral sequence on ν,

$$E_2^{s,t} = H^s(X_{et}, \mathbb{Q}/\mathbb{Z}_{(l)}(\frac{t}{2})) \Rightarrow \pi_{s-t} L\mathbf{K}X \wedge \mathcal{M}.$$

Similarly for relative cohomologies.

1.3.4. If there is an inverse system of schemes with limit $X = \lim X_i$, where each transition morphism is affine, there is an isomorphism for any s and j

$$\operatorname{colim} H^s(X_i, \mathbb{Q}/\mathbb{Z}_{(l)}(j)) \cong H^s(X, \mathbb{Q}/\mathbb{Z}_{(l)}(j))$$

([19, VII.5.7]), and a weak equivalence

$$\operatorname{colim} \mathbf{K}X_i \simeq \mathbf{K}X$$

([40, 3.20]).

1.4. $E(1)$-homology

1.4.0. The topological $\mathcal{K}$-homology group of $\mathbf{K}X$ is defined as

$$\mathcal{K}_*(\mathbf{K}X) = \pi_*(\mathcal{K} \wedge \mathbf{K}X).$$

Now we need to consider $\mathcal{K}_*(\mathbf{K}X)$ not only as a $\mathbb{Z}/2$-graded abelian group, but with homology operations, so that it is considered to be a $\mathcal{K}_*\mathcal{K}$-comodule. For a spectrum W, there is the Adams spectral sequence associated to F, which converges to the homotopy groups of LW:

$$\operatorname{Ext}^{s,t}_{\mathcal{K}_*\mathcal{K}}(\mathcal{K}_*, \mathcal{K}_*(W)) \Rightarrow \pi_{t-s}(LW),$$

where Ext is that in a subcategory of $\mathcal{K}_*\mathcal{K}$-comodules.

Bousfield [4] studied this spectral sequence in the following way. For a spectrum W, $\mathcal{K}_*(W)$ has the canonical action of the topological (stable) Adams operations ψ^k for all $k \in (\mathbb{Z}_{(l)})^\times$. They are l-adically dense and the category of $\mathcal{K}_*\mathcal{K}$-comodules is equivalent to a subcategory of modules with Adams operations satisfying the conditions below.

1.4.1. We use $E(1)$ because as 1.0.6,

$$\mathcal{K}_*W \cong \oplus E(1)_*\Sigma^i W,$$

and the Adams operations ψ^k with $k \equiv 1(l)$ preserves grading of $E(1)$. Let r be an integer which becomes a topological generator in $(\mathbb{Z}_l)^\times/(\mathbb{F}_l)^\times$. We fix one of such and write just $\psi = \psi^r$.

There is a twist operator T on the category of modules with Adams operations, where TM has the same structure of abelian group as M but ψ^k acts as $k\psi^k$.

$E(1)_*E(1)$ is zero if $* \not\equiv 0$. and $E(1)_0E(1)^\wedge$ is isomorphic to the completed group ring $\Lambda = \lim_n \mathbb{Z}_l[[\mathbb{Z}/l^n]]$. This ring is usually called the Iwasawa algebra Λ, and it is isomorphic to the completion of the polynomial ring of one generator $\mathbb{Z}_l[[t]]$, where ψ corresponds to $t+1$. We get

$$E(1)_0E(1) \otimes \mathbb{Q}/\mathbb{Z}_{(l)} \cong \Lambda^\vee \cong \mathbb{Z}_l[[t]]^\vee.$$

There is a canonical isomorphism $T^j\Lambda \cong \Lambda$, which corresponds $f(t)$ by $f^{(j)}(t) = f(r^j t + r^j - 1)$. The decomposition in 1.0.6 implies

$$E(1)_{2i}\mathcal{K} \otimes \mathbb{Q}/\mathbb{Z}_{(l)} \cong T^i\Lambda^\vee,$$

and $E(1)_{2i+1}\mathcal{K} \otimes \mathbb{Q}/\mathbb{Z}_{(l)} = 0$.

1.4.2. Definition. We will define a subcategory $\mathcal{B}$ of $\mathbb{Z}_{(l)}$-modules with an automorphism ψ, where morphisms should commutate with ψ. A module M with ψ belongs to $\mathcal{B}$ if it satisfies the following conditions:

(1) for each l-torsion element $x \in M$, there exists $u \geq 0$ such that $(\psi)^{l^u}x = x$, or equivalently $(\psi - 1)^{l^v}x = 0$, and
(2) the operator $\psi \otimes 1$ on $M \otimes \mathbb{Q}$ is diagonalizable with eigenvalues all of the form $r^{(l-1)j}$ for $j \in \mathbb{Z}$.

Bousfield denotes the category in [4, §5] by $\mathcal{B}(l)^r$, but we omit l and r which are fixed.

1.4.3. The twist operator T defines an auto-functor T^{l-1} on $\mathcal{B}$. $\mathcal{B}_*$ is the category of $\mathbb{Z}/2(l-1)$-graded objects of $\mathcal{B}$ where the grading is cyclic in such a way that $M_{i+2(l-1)} \cong T^{l-1}M_i$.

T will define a functor where TM_i has the same structure of abelian group as M_{i-2} but ψ acts as $r\psi$.

For a group M, we can regard it as an object in $\mathcal{B}$ where ψ acts as the identity on M. It is also an object in $\mathcal{B}_*$.

For a spectrum W $E(1)_*W$ with ψ^r is an object of $\mathcal{B}_*$ and it can be extended to a categorical equivalence between $\mathcal{B}_*$ and $\mathcal{K}_*\mathcal{K}$-comodules [4].

Torsion object of $\mathcal{B}$ and those of Λ-modules are equivalence [4, §3].

1.4.4. Let M be a divisible torsion Λ-module which is co-finite type. The characteristic polynomial F of ψ defines $M \cong \oplus\mathbb{Z}_l[[t]]/(F_j^{n_j})^\vee$ where $F = \prod F_j^{n_j}$ is a decomposition into irreducible factors ([29, 5.3.8]). Let $\langle F\rangle$ denote it. It is a torsion object of $\mathcal{B}$.

1.4.5. The E_2-terms $\mathcal{E}_2^{s,t}$ of the Adams spectral sequence associated to a spectrum W is isomorphic to $\mathrm{Ext}^{s,t}_{\mathcal{B}_*}(E(1)_*, E(1)_*(W))$ ([4, §10]). These terms are trivial for $s > 2$ and the only possible non-trivial differential is $d_2 : \mathcal{E}_2^{0,t} \to \mathcal{E}_2^{2,t+1}$. It converges strongly to $\pi_{t-s}(LW)$, where $F^s/F^{s+1}(\pi_{t-s}(LW)) \cong \mathcal{E}_3^{s,t} \cong \mathcal{E}_\infty^{s,t}$ ([4, 8.3]).

1.4.6. Bousfield defined a spectral sequence ([4, 7.6]) which converges to the E_2-term of the Adams spectral sequence. Namely for any $L, M \in \mathcal{B}$, there is a spectral sequence

$$E_1^{0,0} = E_1^{1,0} = \mathrm{Hom}_{\mathbb{Z}_{(l)}}(L, M), \quad E_1^{0,1} = E_1^{1,1} = \mathrm{Ext}_{\mathbb{Z}_{(l)}}(L, M)$$

and

$$E_1^{2,0} = \mathrm{Hom}_{\mathbb{Z}_{(l)}}(L \otimes \mathbb{Q}, M \otimes \mathbb{Q})$$

and other terms are 0. Differentials are the followings:

$$d_1 : \mathrm{Ext}^s_{\mathbb{Z}_{(l)}}(L, M) \to \mathrm{Ext}^s_{\mathbb{Z}_{(l)}}(L, M)$$

is given by

$$d_1 f = f \circ \psi_L - \psi_M \circ f$$

and

$$d_1 : E_1^{1,0} = \mathrm{Hom}_{\mathbb{Z}_{(l)}}(L, M) \to E_1^{2,0} = \mathrm{Hom}_{\mathcal{B}_*}(L \otimes \mathbb{Q}, M \otimes \mathbb{Q})$$

carries $f : L \to M$ to the morphism $d_1 f : L \otimes \mathbb{Q} \to M \otimes \mathbb{Q}$ given by the homogeneous components of $f \otimes 1$ with respect to the eigenspace decompositions of $L \otimes \mathbb{Q}$ and $M \otimes \mathbb{Q}$.

The only possible non-trivial d_2 is $d_2 : E_2^{0,1} \to E_2^{2,0}$ and thus $E_3 \cong E_\infty$, and

$$0 \to E_\infty^{s,0} \to \mathrm{Ext}^s_{\mathcal{B}}(L, M) \to E_\infty^{s-1,0} \to 0.$$

1.4.7. When $\mathrm{Hom}_{\mathcal{B}}(L \otimes \mathbb{Q}, M \otimes \mathbb{Q}) = 0$, the spectral sequence reduces to a simple exact sequence ([4, 7.8])

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathcal{B}}(L, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_{(l)}}(L, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_{(l)}}(L, M) \\ \longrightarrow \mathrm{Ext}^1_{\mathcal{B}}(L, M) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}_{(l)}}(L, M) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}_{(l)}}(L, M) \\ \longrightarrow \mathrm{Ext}^2_{\mathcal{B}}(L, M) \longrightarrow 0. \end{aligned}$$

1.4.8. When $L = \mathbb{Z}_{(l)}, \psi = 1$ and M is torsion, $\mathrm{Hom}_{\mathcal{B}}(L, M)$ is isomorphic to the invariant group M^ψ and $\mathrm{Ext}^1_{\mathcal{B}}(L, M)$ is isomorphic to the co-invariant group M_ψ.

If there is a short exact sequence of torsion modules $0 \to M \to N \to L \to 0$, it induces a six-term exact sequence

$$0 \to M^\psi \to N^\psi \to L^\psi \to M_\psi \to N_\psi \to L_\psi \to 0.$$

1.4.9. If there is a torsion module M whose invariant part M^ψ is zero, then M itself is zero because ψ is locally unipotent. If there is a morphism $f : M \to N$ between torsion modules which induces an isomorphism on the invariant part $M^\psi \cong N^\psi$ and $M_\psi = 0$, then f is an isomorphism. This is because $\mathrm{Ker}\, f$ is trivial as above and the six-term exact sequence above implies $\mathrm{Coker}\, f$ is also trivial.

1.5. Etale Cohomology.

1.5.1. When X is a variety over a finite field, etale cohomology groups of X $H^i(X, \mathbb{Q}/\mathbb{Z}_{(l)}(j))$ are of co-finite type for all i, j. ([9]). And $\pi_i LKX \wedge \mathcal{M}$ are of co-finite type for there is a strongly convergent spectral sequence (1.3.3). Similarly for X_∞.

1.5.2. The arithmetic Frobenious operator ϕ_X acts on the geometric cohomology groups $H^{2i}(X_\infty; \mathbb{Q}_l(j))$.

There is a long exact sequence comes from the Leray spectral sequence:

$$\cdots \to H^i(X, \mathbb{Q}/\mathbb{Z}_{(l)}(j)) \longrightarrow H^i(X_\infty, \mathbb{Q}/\mathbb{Z}_{(l)}(j)) \xrightarrow{1-(\phi_X)^*} H^i(X_\infty, \mathbb{Q}/\mathbb{Z}_{(l)}(j)) \to \cdots$$

It implies that there is a homotopy cofiber sequence

$$L\mathbf{K}(X) \wedge \mathcal{M} \longrightarrow L\mathbf{K}(X_\infty) \wedge \mathcal{M} \xrightarrow{1-(\phi_X)^*} L\mathbf{K}(X_\infty) \wedge \mathcal{M}.$$

1.5.3. When X is a smooth projective variety over a finite field, the eigenvalues of ϕ_X on $H^i(X_\infty, \mathbb{Q}/\mathbb{Z}_{(l)}(j))$ can be a power of q only if $2j = i$. This is an special case of the theorem of Deligne on the Weil conjectures ([8]). Therefore $H^i(X, \mathbb{Q}/\mathbb{Z}_{(l)}(j))$ is of bounded order if either $2j \neq i-1\,, i$.

It implies that $\pi_i L\mathbf{K}X \wedge \mathcal{M}$ is of bounded order for $i \neq 0, -1$ because the descent spectral sequence degenerates up to finite exponents ([37]).

1.5.4. Let Z be a closed subvariety of a smooth variety X over a finite field. It follows directly by induction on dimension of Z, that $H^i_Z(X; \mathbb{Q}/\mathbb{Z}_{(l)}(j))$ is of bounded order if $2j - i \leq -2$.

2. Conjecture

2.0. The image of the cycle class morphism ρ from the i-th Chow group in the introduction is contained in the invariant subspaces of $(\phi_X)^n$ for some n.

Now the purpose of this paper is to prove the following conjecture of Tate on the action of ϕ_X.

Tate Conjecture: Let X be a projective and smooth variety over a finite field. For each i, the union of invariant spaces of $(\phi_X)^n$ in the geometric cohomology group $H^{2i}(X_\infty; \mathbb{Q}_l(i))$ is isomorphic to the image of the cycle class morphism.

2.1. Let X_0 be a projective smooth variety over a finite field $\mathbb{F}_q$. Define X_n as $X_0 \otimes \mathbb{F}_{q^{l^n}}$, where $n = 1, 2, \ldots, \infty$. It is consistent of our convention on $n = \infty$, for $\mathbf{K}(X \otimes \mathbb{F}_{q^\infty}) \simeq \mathbf{K}(X \otimes \mathbb{F}_{q^{l^\infty}})$.

In [38] Thomason proved that the following statement in K-localized K-theory implies the Tate conjecture on X for all i:

Thomason Conjecture:

$$\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}_{(l)}, \pi_{-1}(L\mathbf{K}X_n)) \cong 0$$

for all $n < \infty$.

2.1.1. What Thomason observed in [38] is roughly the following: For each X_n ($n = 0, 1, \ldots, \infty$) there is an exact sequence

$$\begin{array}{ccccc} K_0(X_n) \otimes \mathbb{Q}_l & \longrightarrow & \pi_0 L\mathbf{K}(X_n)^\wedge \otimes \mathbb{Q} & \twoheadrightarrow & \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}_{(l)}, \pi_{-1}(L\mathbf{K}X_n)) \\ \nu \downarrow \cong & & \mu \downarrow \cong & & \\ \oplus CH^i(X_n) \otimes \mathbb{Q}_l & \xrightarrow[\rho_n]{} & \oplus H^{2i}(X_n, \mathbb{Q}_l(i)) & & \end{array}$$

where ρ_n is the cycle class morphism on X_n, ν is induced by γ-filtration and μ is an isomorphism by the descent theorem. Comparing the case $n = \infty$ and the directed colimit on n of the exact sequences

$$\begin{array}{ccccc} \operatorname{colim} K_0(X_n) \otimes \mathbb{Q}_l & \longrightarrow & \operatorname{colim} \pi_0 L\mathbf{K}(X_n)^\wedge \otimes \mathbb{Q} & \twoheadrightarrow & \operatorname{colim} \operatorname{Hom}(\mathbb{Q}/\mathbb{Z}_{(l)}, \pi_{-1}(L\mathbf{K}X_n)) \\ \downarrow \cong & & \downarrow \xi & & \\ K_0(X_\infty) \otimes \mathbb{Q}_l & \xrightarrow[\rho_\infty]{} & \pi_0 L\mathbf{K}(X_\infty)^\wedge \otimes \mathbb{Q} & & \end{array}$$

where image of ξ is the union of invariant subspaces of $(\phi_X)^n$ because of the descent exact sequence (1.3.2). Then if $\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}_{(l)}, \pi_{-1}(L\mathbf{K}X_n)) \cong 0$, the images of ρ_∞ and ξ are the same, which implies the Tate conjecture. It therefore suffices to show it for cofinal sequence of n. We may assume $\mathbb{F}_q$ contains l-th roots of unity.

3. Preparations

3.0. Let C_0 be a smooth proper curve over a field k_0. Let g be the genus of C_0. Then there exists a finite extension k of k_0 such that $C = C_0 \times \operatorname{Spec} k$ has a finite morphism onto the projective line P^1 having degree $g+1$, and $H^1(C, f^*\mathcal{O}_{P^1}(1)) = 0$ [20, 2.1.2]. The main result of [20] is that for any homology theory W, there are long exact sequences around K-theory of the function field $k(C)$:

$$\cdots \longrightarrow \oplus^2 W_*\mathbf{K}(k) \longrightarrow W_*\mathbf{K}(k(C)) \longrightarrow W_{*-1}\mathbf{K}(\mathcal{P}^u) \longrightarrow \cdots,$$

$$\cdots \longrightarrow \operatorname{colim} W_*\mathbf{K}(A) \longrightarrow W_*\mathbf{K}(\mathcal{P}^u) \longrightarrow \oplus W_{*-1}\mathbf{K}(k(p)) \longrightarrow \cdots,$$

where A are finite dimensional k-algebras, $k(p)$ and $k(c)$ are field of finite extension over k. $\mathcal{P}^u$ is a Waldhausen subcategory of complexes on C of which details are not important here.

3.1. We have to calculate $E(1)_*(\mathbf{K}X)$ with the Adams operation ψ. First we calculate $E(1)_*(\mathbf{K}X) \wedge \mathcal{M}$. We assume X is regular.

3.1.0. Let us start with the case of the algebraically closed field $\mathbb{F}_{q^\infty}$. There is an equivalence $\mathbf{K}\mathbb{F}_{q^\infty} \wedge \mathcal{M} \simeq \mathcal{K} \wedge \mathcal{M}$ ([31]) and, as in 1.3.0,

$$E(1)_*\mathbf{K}\mathbb{F}_{q^\infty} \wedge \mathcal{M} \cong E(1)_*\mathcal{K} \wedge \mathcal{M} \cong \oplus T^*\Lambda^\vee.$$

3.1.1. For a scheme X over $\mathbb{F}_q$, $\mathbf{K}X_\infty$ is a module spectrum of $\mathbf{K}\mathbb{F}_{q^\infty}$. Hence $\mathbf{K}X_\infty \wedge \mathcal{M}$ is an $E(1)$-module spectrum (1.2.2). $E(1)_*\mathbf{K}X_\infty \wedge \mathcal{M}$ is treated with a functor $\mathcal{U}$ defined by Bousfield ([4, 6.1]. It is also treated in [10, §6] from a different point of view). $\mathcal{U}$ is a functor from the category of $\mathbb{Z}_{(l)}$-modules to $\mathcal{B}$, which has an extension on graded modules:

$$\mathcal{U} : E(1)_*\text{-Mod} \to \mathcal{B}_*.$$

$\mathcal{U}$ is right adjoint to the forgetful functor ([4, 4.3]). For a torsion group A, $\mathcal{U}(A) \cong \oplus^\infty A$ with $\psi(a_1, a_2, a_3, \dots) = \psi(a_1 + a_2, a_2 + a_3, \dots)$. It is easy to see that $\mathcal{U}(A)^\psi = A$ and $\mathcal{U}(A)_\psi = 0$. Note that there is a canonical isomorphism $T^i\mathcal{U}(A) \cong \mathcal{U}(A)$. When D is a divisible torsion group, $\mathcal{U}(D)$ is an injective object in $\mathcal{B}$ ([4, 7.1]). $\mathcal{U}(\mathbb{Q}/\mathbb{Z}_{(l)}) \cong \Lambda^\vee$ as 1.4.1. It is proved that for an $E(1)$-module spectrum Y, $E(1)_*Y \cong \mathcal{U}(\pi_*Y)$.

3.1.2. The descent spectral sequence converges strongly on $\pi_*\mathbf{K}X_\infty \wedge \mathcal{M}$ with E_2-term $H^{\cdot}(X_\infty;\mathbb{Q}/\mathbb{Z}_{(l)})$, which degenerates up to finite exponents. $\mathcal{U}((\phi_X)^*)$ can be identified with ψ^q (1.4.1), which acts as ψ^{l^a} modulo l ([10, §6]).

$$E(1)_*\mathbf{K}X_\infty \wedge \mathcal{M} \sim H^*(X_\infty;\mathbb{Q}/\mathbb{Z}_{(l)}) \otimes \Lambda.$$

3.1.3. It induces a short exact sequence [10, §6.9]

$$E(1)_*\mathbf{K}(X) \wedge \mathcal{M} \rightarrowtail E(1)_*\mathbf{K}(X_\infty) \wedge \mathcal{M} \xrightarrow{1-\psi^q} E(1)_*\mathbf{K}(X_\infty) \wedge \mathcal{M}.$$

Let F_i be the characteristic polynomial of the Frobenious operator ϕ_X on $H^i(X_\infty;\mathbb{Q}/\mathbb{Z}_{(l)})$ and $F_i'(x)$be $F_i(x^{l^a})$. We get

$$E(1)_*(\mathbf{K}X \wedge \mathcal{M}) \sim \oplus T^k\langle F_i'\rangle.$$

3.1.4. Lemma.

(1) Let M be a divisible object in $\mathcal{B}$ such that $M_\psi = 0$ and M^ψ is divisible. Then M is isomorphic to a direct sum of $\Lambda^\vee$: $M \cong \oplus\Lambda^\vee$.
(2) Let M be a divisible object in $\mathcal{B}$ such that M^ψ is reduced. Then $M_\psi = 0$.
(3) Let M be an object in $\mathcal{B}$ whose underlying group is reduced. Then M^ψ and M_ψ are reduced.

Proof: (1) We can extend $M^\psi \longrightarrow U(M^\psi)$ to a morphism $M \longrightarrow U(M^\psi)$ for $U(M^\psi)$ is injective. It is a monomorphism for it is isomorphic on the ψ-invariant part, and then it is an isomorphism as 1.4.9.

(2) If M is of co-finite type, $M \sim \langle F\rangle$ and $F|_{\psi=1} \neq 0$. Then $M_\psi = 0$.

For any element m in M, it is contained in a finite sub-ψ-group for ψ-action is locally unipotent. Let N be its divisible hull of the group in M. N is of co-finite type ([12, 25.1]), and it is a sub-ψ-module. Then M is a filtered colimit of co-finite sub-ψ-modules having trivial co-invariants.

(3) It is trivial for invariant part. If M is a sum of cyclic group, it is a sequential colimit with split transition morphisms, so that the co-invariant part is again a sum of cyclic groups. For general reduced groups, induction on Ulm sequence will prove the statement.

3.1.5. For a field of functions on a variety X we made an assumption 0.2.1 on etale cohomology of $k(X_\infty)$ that there is a spectral sequence [24]

$$E_2 = H^*(k(X_\infty),\mathbb{Q}/\mathbb{Z}_{(l)}) \Rightarrow \pi_*\mathbf{K}(k(X_\infty)) \wedge \mathcal{M}$$

degenerates at E_2, and $H^*(k(X_\infty),\mathbb{Q}/\mathbb{Z}_{(l)})$ are divisible. It implies $\pi_*\mathbf{K}(k(X_\infty)) \wedge \mathcal{M}$ is divisible for any $*$.

Kahn and Levine generalized the above results to Azumaya algebras in the following way [22]. Let A be a division algebra over a field k. Instead of $\mathbf{K}(-)$, they defined a presheaf $\mathbf{K}^A(-)$ associated to A-modules. $\mathbf{K}^A(k) = \mathbf{K}(A)$. There is a spectral sequence

$$E_2 = H^*(k(X_\infty),\mathbb{Q}/\mathbb{Z}_{(l)}) \Rightarrow \pi_*\mathbf{K}(A) \wedge \mathcal{M}.$$

There is a morphism from the spectral sequence for $k(X_\infty)$ to that for A, where on E_2-terms it is an epimorphism. Then the spectral sequence for A degenerates at E_2. It implies the following statement.

3.1.6. Proposition. $E(1)_*\mathbf{K}A \wedge \mathcal{M} \cong \mathcal{U}(\pi_*\mathbf{K}(A) \wedge \mathcal{M})$ is divisible for any division algebra over $k(X_\infty)$.

3.1.7. For a division algebra A over a function field of variety over a finite field $\mathbb{F}_q$, there is a similar short exact sequence as 3.1.3:

$$E(1)_*\mathbf{K}(A) \wedge \mathcal{M} \rightarrowtail E(1)_*\mathbf{K}(A_\infty) \wedge \mathcal{M} \xrightarrow{1-\psi^q} E(1)_*\mathbf{K}(A_\infty) \wedge \mathcal{M}.$$

Since L-localized $\mathbf{K}^A$-theory satisfies the descent property, so is $E(1) \wedge \mathbf{K}^A$.

$\mathcal{U}((\psi)^*)$ can be identified with ψ^q because it is determined by its action on the stalks $\mathbf{K}^A((k)^{sep}) = \mathbf{K}((k)^{sep})$, where k^{sep} means the separable closure.

3.1.8. $E(1)_*\mathbf{K}(A) \wedge \mathcal{M}$ is divisible. To see that, look at the long exact sequence as 1.4.8 from3.1.7. It follows that $(E(1)_*\mathbf{K}(A) \wedge \mathcal{M})_\psi$ is divisible for it is the quotient image of a divisible group $\pi_*\mathbf{K}(A_\infty) \wedge \mathcal{M}$. Then 3.1.4 (3) implies the claim.

3.2. Integral theory We call an object in $\mathcal{B}$ divisible, reduced or torsion free if the underlying group is. For an object M in $\mathcal{B}$, M^{div} is the subobject whose underlying group is the maximal divisible torsion subgroup.

There is a long exact sequence (0.1.5)

$$\cdots \to E(1)_{*+1}\mathbf{K}X \wedge \mathcal{M} \to E(1)_*\mathbf{K}X \to E(1)_*\mathbf{K}X \otimes \mathbb{Q} \to \cdots.$$

3.2.0. We know that

$$E(1)_*\mathbf{K}X \otimes \mathbb{Q} \cong (\pi_*\mathbf{K}X \otimes \mathbb{Q}) \otimes E(1)_*$$

where $\pi_*\mathbf{K}X \otimes \mathbb{Q}$ are not zero only if $0 \le i \le 2\dim X$ [20, 4.1].

We know $E(1)_i\mathbf{K}X \otimes \mathbb{Q}$ is isomorphic to a sum of $\mathbb{Q}[\psi]/(\psi - r^{(l-1)j})$ where $0 \le i + 2(l-1)j \le 2\dim X$.

For negative i we denote by A_j the torsion free subobject of $E(1)_i\mathbf{K}X$. We get $(A_i)^\psi = 0$ and $(A_i)_\psi$ is bounded torsion module, whose order is bounded by a number depends only on $\dim X$: $\prod(r^{(l-1)j} - 1)$ for $0 \le i + 2(l-1)j \le 2\dim X$.

3.2.1. Example. $E(1)_0\mathbf{K}\mathbb{F}_{q^{l^n}} \otimes \mathbb{Q} = \mathbb{Q}[\psi]/(\psi - 1)$ and $E(1)_{2i}\mathbf{K}\mathbb{F}_{q^{l^n}} \otimes \mathbb{Q} = 0$ for $i \not\equiv 0$ ($\equiv$ means modulo $l-1$) and $E(1)_{2i+1}\mathbf{K}\mathbb{F}_{q^{l^n}} \otimes \mathbb{Q} = 0$ for all i.

3.1.3 imply that

$$E(1)_{2i-1}\mathbf{K}\mathbb{F}_{q^{l^n}} = E(1)_{2i-2}\mathbf{K}\mathbb{F}_{q^{l^n}} \wedge \mathcal{M} \sim T^i(\langle 1 - \psi^{l^{a+n}} \rangle)$$

if $i \not\equiv 0$, and $E(1)_{2i}\mathbf{K}\mathbb{F}_{q^{l^n}} = 0$ if $i \not\equiv 0$.

$$E(1)_{-1}\mathbf{K}\mathbb{F}_{q^{l^n}} \sim \langle 1 + \psi + \psi^2 + \cdots \psi^{l^{a+n}-1} \rangle$$

and

$$E(1)_0\mathbf{K}\mathbb{F}_{q^{l^n}} = \mathbb{Z}_{(l)} = E(1)_0S^0.$$

3.2.2. The Adams spectral sequence on $\mathbf{K}X$ in 1.4

$$\mathcal{E}_2^{s,t}(X) \Rightarrow \pi_{t-s}(L\mathbf{K}X)$$

associates the filtration on $\pi_i(L\mathbf{K}X) = F^0(X) \supset F^1(X) \supset F^2(X)$, where the filtered quotients are described as $F^0/F^1 \cong \mathcal{E}_3^{0,i}$, $F^1/F^2 \cong \mathcal{E}_3^{1,i+1}$, $F^2 \cong \mathcal{E}_3^{2,i+2}$, which satisfy

$$0 \to \mathcal{E}_3^{0,i} \to \mathcal{E}_2^{0,i} \to \mathcal{E}_2^{2,i+1},$$

$$\mathcal{E}_2^{0,i+1} \to \mathcal{E}_2^{2,i+2} \to \mathcal{E}_3^{2,i+2} \to 0,$$

and

$$\mathcal{E}_2^{1,i} = \mathcal{E}_3^{1,i}.$$

3.2.3. For a spectrum W whose homotopy groups are torsion, $\mathcal{E}_2^{0,i}(W) = (E(1)_i W)^{\psi}$, $\mathcal{E}_2^{1,i} = (E(1)_i W)_{\psi}$ and $\mathcal{E}_2^{2,i}(W) = 0$ (1.4.6). When $W = L\mathbf{K}X \wedge \mathcal{M}$, there are short exact sequences

$$(E(1)_{i+1}\mathbf{K}(X) \wedge \mathcal{M})_{\psi} \rightarrowtail \pi_i L\mathbf{K}X \wedge \mathcal{M} \twoheadrightarrow (E(1)_i \mathbf{K}X \wedge \mathcal{M})^{\psi}$$

3.2.4. Lemma. For a smooth variety X over a finite field, $(E(1)_*\mathbf{K}X \wedge \mathcal{M})^{div}$ does not contain $\Lambda^{\vee}$. It is also true for the function field $k(X)$ of a smooth variety X over a finite field.

As a consequence, $(E(1)_*\mathbf{K}X \wedge \mathcal{M})^{\psi}$ is reduced if and only if $((E(1)_*\mathbf{K}X \wedge \mathcal{M})^{div})_{\psi} = 0$.

Proof: If it is in case, $(T^i E(1)_*\mathbf{K}X \wedge \mathcal{M})^{\psi}$ contains $\mathbb{Q}/\mathbb{Z}_{(l)}$ for arbitrary i. But it implies $\pi_{*+2(l-1)i} L\mathbf{K}X \wedge \mathcal{M}$ contains a non-trivial divisible subgroup (3.2.3), which contradicts 1.5.3.

For the function field $k(X)$ of a smooth variety X over a finite field, we will use an induction on dimension of fields. For a field of dimension 0, it is true by 3.2.1. If $E(1)_*\mathbf{K}k(X) \wedge \mathcal{M}$ contains $\Lambda^{\vee}$, consider the E_1-terms of Gersten-Quillen spectral sequence, $d_1 : E(1)_*\mathbf{K}k(X) \wedge \mathcal{M} \to \oplus E(1)_*\mathbf{K}k(x) \wedge \mathcal{M}$. By the inductive hypotheses, $\Lambda^{\vee}$ must be contained in $\operatorname{Ker} d_1$, which contradicts a fact that the edge homomorphism $E(1)_*\mathbf{K}X \wedge \mathcal{M} \twoheadrightarrow \operatorname{Ker} d_1 \cong H^0(X_{Zar}, E(1)_*\mathbf{K} \wedge \mathcal{M})$ is an epimorphism.

If $((E(1)_*\mathbf{K}X \wedge \mathcal{M})^{div})_{\psi}$ is not zero, it is non-zero divisible (3.1.4), which implies the last consequence.

4. Proof of the theorem.

4.0. Let X be a smooth projective variety over a finite field. To prove the conjecture (2.1) that $\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}_{(l)}, \pi_{-1}(L\mathbf{K}X)) \cong 0$, we use the filtration on $\pi_{-1}(L\mathbf{K}X)$ as $\pi_{-1}(L\mathbf{K}X) = F^0(X) \supset F^1(X) \supset F^2(X)$ (3.2.2). If we can show that on the filtered quotients $F^{\cdot}/F^{\cdot+1}$ is reduced, it implies that $\pi_{-1}(L\mathbf{K}X))$ is reduced, for $\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}_{(l)}, -)$ is left exact. The filtered quotients are described as $F^0/F^1 \cong \mathcal{E}_3^{0,-1}$, $F^1/F^2 \cong \mathcal{E}_3^{1,0}$, $F^2 \cong \mathcal{E}_3^{2,1}$.

4.0.0. When X is a variety over a finite field, $\pi_i L\mathbf{K}X^{\wedge} \cong \pi_{i+1}(L\mathbf{K}X \wedge \mathcal{M})^{\wedge}$ is of bounded order for $i \leq -2$ (1.5.3).

Then 3.2.3 implies that $(E(1)_{i+1}\mathbf{K}X \wedge \mathcal{M})_{\psi}$ and $(E(1)_i\mathbf{K}X \wedge \mathcal{M})^{\psi}$ are of bounded order for $i \leq -2$, and $(E(1)_i\mathbf{K}X)_{\psi}$ and $(E(1)_{i-1}\mathbf{K}X)^{\psi}$ are of bounded order for $i \leq -2$ (3.2.0). If Z is a closed subvariety in a smooth variety X, $(E(1)_i\mathbf{K}(X \text{ on } Z))^{\psi}$ and $(E(1)_{i-1}\mathbf{K}(X \text{ on } Z))_{\psi}$ are bounded if $i \leq -2$.

4.0.1. If X is a projective smooth variety over a finite field, $\pi_i L\mathbf{K}X^{\wedge} \cong \pi_{i+1}(L\mathbf{K}X \wedge \mathcal{M})^{\wedge}$ is of bounded order for $i \neq 0, -1$ (1.5.3).

4.1. Consider first $\mathcal{E}_3^{1,0} = \mathcal{E}_2^{1,0} = (E(1)_0\mathbf{K}X)_{\psi}$.

4.1.1. If $(E(1)_0\mathbf{K}X)_\psi$ contains a divisible subgroup, $(E(1)_0\mathbf{K}X)^\psi$ contains a divisible group by 3.3.0 (2). It follows that $\mathcal{E}_3^{0,0}$ contains a divisible group for $\mathcal{E}_2^{2,1}$ is always torsion free. But it would imply $\pi_1 LBX^\wedge$ is not bounded by 3.2.3, which contradicts 4.0.1.

4.2. There is an exact sequence

$$\mathcal{E}_2^{0,0} \to \mathcal{E}_2^{2,1} \to \mathcal{E}_3^{2,1} \to 0,$$

and $\mathcal{E}_3^{2,1}$ could contain a non-trivial torsion divisible subgroup only if the torsion free divisible group $\mathcal{E}_2^{2,1}$ is non-trivial. But if it is the case, 1.4.7 implies $\operatorname{Hom}(\mathbb{Z}_{(l)} \otimes \mathbb{Q}, E(1)_1\mathbf{K}X \otimes \mathbb{Q}) / \operatorname{Hom}(\mathbb{Z}_{(l)}, E(1)_1\mathbf{K}X)$ is non-trivial, and Ext-l-completion of $E(1)_1\mathbf{K}X$ would have a non-trivial torsion free subgroup ([6, VI.2.1]), which can not be true.

4.3. To see that $\mathcal{E}_3^{0,-1}$ is reduced, it suffices to show that $\mathcal{E}_2^{0,-1}$ is, for there is an exact sequence

$$0 \to \mathcal{E}_3^{0,-1} \to \mathcal{E}_2^{0,-1} \to \mathcal{E}_2^{2,0}.$$

For any projective smooth variety X over a finite field, $\mathcal{E}_2^{0,-1} = (E(1)_{-1}\mathbf{K}X)^\psi$ is reduced if and only if $((E(1)_{-1}\mathbf{K}X)^{div})_\psi$ is zero (3.2.4).

4.3.1. Lemma. Let L be a field of dimension d over a finite field. Assume for all field K of dimension less than d, $((E(1)_i\mathbf{K}(K))^{div})_\psi = 0$ for any $i \le -1$. Then $((E(1)_{-1}\mathbf{K}(L))^{div})_\psi = 0$ for any $i \le -1$.

Proof: There is a smooth proper curve C over a field K of dimension $d-1$ such that the function field $k(C)$ is isomorphic to L. We may assume C satisfies the condition in 3.0 for the claim is invariant under finite extension of base field, so that there is an exact sequence (3.0),

$$(1) \qquad \cdots \longrightarrow \oplus^2 E(1)_*\mathbf{K}(K) \xrightarrow{a} E(1)_*\mathbf{K}(L) \xrightarrow{b} E(1)_{*-1}\mathbf{K}(\mathcal{P}^u) \longrightarrow \cdots .$$

It implies $E(1)_*\mathbf{K}(\mathcal{P}^u)$ does not contains $\Lambda^\vee$ by 3.2.4.

Using the inductictive hypotheses and 3.1.8, we way assume that for a field K of dimension less than d, and any $* \le -1$,

$$E(1)_*\mathbf{K}(K) = (E(1)_*\mathbf{K}(K))^{div} \oplus F_*^2,$$

where F_*^1 is torsion free and $(F_*^1)^\psi = 0$, $(F_*^1)_\psi \sim 0$, and $(E(1)_*\mathbf{K}(K))_\psi^{div} = 0$. It implies by the transfer argument and 3.1.8 that the similar decomposition exists for any division algebra A over K:

$$E(1)_*\mathbf{K}(A) = (E(1)_*\mathbf{K}(A)))^{div} \oplus F_*^1.$$

Then for arbitrary filtered colimit of finite dimensional K-algebras, we get that

$$\operatorname{colim} E(1)_*\mathbf{K}(A) = (\operatorname{colim} E(1)_*\mathbf{K}(A)))^{div} \oplus F_*^3.$$

such that $((\operatorname{colim} E(1)_*\mathbf{K}(A)))^{div})_\psi = 0$.

Recall the other exact sequence in (3.0):

$$(2) \qquad \cdots \longrightarrow \operatorname{colim} E(1)_*\mathbf{K}(A) \xrightarrow{c} E(1)_*\mathbf{K}(\mathcal{P}^u) \xrightarrow{d} \oplus E(1)_{*-1}\mathbf{K}(k(p)) \longrightarrow \cdots .$$

It induced

$$\operatorname{Im} c_* \rightarrowtail E(1)_*\mathbf{K}(\mathcal{P}^u) \twoheadrightarrow M_*^1$$

where $(\operatorname{Im} c_*)_\psi \sim 0$, and M^1_* is a submodule of $\oplus E(1)_{*-1}\mathbf{K}(k(p))$, so that $((E(1)_*\mathbf{K}(\mathcal{P}^u)^{div})^\psi$ is reduced, and then $(E(1)_*\mathbf{K}(\mathcal{P}^u)_\psi \sim 0$. It imples $(E(1)_{*-1}\mathbf{K}(\mathcal{P}^u)_\psi \sim 0$. for $* \leq -1$.

Go back to (1), There are short exact sequences of divisible groups

$$\operatorname{Im} a_* \rightarrowtail E(1)_*\mathbf{K}(L) \twoheadrightarrow M^2_*$$

where M^2_* is a submodule of $E(1)_{*-1}\mathbf{K}(\mathcal{P}^u)$. Similarly $(E(1)_*\mathbf{K}(L))^{\div}_\psi \sim 0$ and $(E(1)_*\mathbf{K}(L))^\psi$ is reduced. Then $(E(1)_*\mathbf{K}(L))^{\div}_\psi = 0$.

4.3.2. Let X be a smooth variety of dimension d. If $d = 0$, 3.2.1 implies the assumption of 4.3.1 is true. Therefore We knew that $((E(1)_*\mathbf{K}(k(X)))^{div})_\psi = 0$ for $* \leq -1$. We will prove inductively on dimension that for any smooth variety X $((E(1)_*\mathbf{K}(X))^{div})_\psi = 0$.

Now assume there is a smooth variety X such that $((E(1)_*\mathbf{K}(X))^{div})_\psi \neq 0$ for some $*$. There should exist a non-empty subvariety Z such that $(E(1)_*\mathbf{K}(X \text{ on } Z)^{div}_\psi \neq 0$ for the long exact sequence associated to the homotopy cofiber sequence 1.1.1. Then there would be a sequence of closed subvarieties in X,

$$Z = Z^0 \supset Z^1 \supset \cdots \supset Z^d,$$

such that $Z^i - Z^{i-1}$ are regular and of dimension $d - i$, or empty. Then there is a homotopy cofiber sequence

$$\mathbf{K}(X \text{ on } Z^{i-1}) \to \mathbf{K}(X \text{ on } Z^i) \to \mathbf{K}(X - Z^{i-1} \text{ on } Z^i - Z^{i-1})$$

where $\mathbf{K}(X - Z^{i-1} \text{ on } Z^i - Z^{i-1}) \simeq \mathbf{K}(Z^i - Z^{i-1})$ and $\mathbf{K}(X \text{ on } Z^d) \simeq \mathbf{K}(Z^d)$ by 1.1.1. It induces a long exact sequence of $E(1)_*\mathbf{K}(X \text{ on } Z^\cdot)$. The dual of these groups are finitely generated Λ- modules. The inductive hypotheses that for any regular variety W of dimension less than d, $((E(1)_*\mathbf{K}(W))^{div})_\psi = 0$ make a contradiction.

4.4. When $l = 2$, Bousfield developed a similar spectral sequence related the Adams spectral sequence. Necessary points to modify are the followings: Instead of $E(1)$ we need combinations of K-theories associated to real vector bundles and complex vector bundles, so that we must replace the category of $\mathbb{Z}_l$-modules by $\mathcal{CRT}$ in [5, 2.1] and $\mathcal{B}$ by $\mathcal{ACRT}$ ([5, 5.5]). There is a similar functor $\mathcal{U}$ ([5, 6.6]), and r must be an integer congruent to ± 3 modulo 8 ([5, 6.4]).

5. Consequences in positive characteristics

5.0. We can induce other properties of the action of the Frobenious on the etale cohomology groups associated to varieties. For any variety X over a finite field, it is conjectured that the fixed space of ϕ_X in $H^*(X_\infty; \mathbb{Q}_l)$ is semi-simple ([35, §2]). We can prove it in the following way.

$E(1)_*\mathbf{K}(X_\infty)^\wedge \otimes \mathbb{Q}$ is determined by Dwyer-Mitchell ([11]). It is a free module over $E(1)^\wedge \otimes \mathbb{Q}$ with $\phi_X = \psi^{l^a}$ as in 3.2.0. Then ϕ_X-invariant space is the same as ψ^{l^a}-invariant space.

$\pi_* E(1)\mathbf{K}X_\infty$ is isomorphic to the sequential colimit of $\pi_* E(1)\mathbf{K}X_n$, where invariant spaces of ψ is contained in the torsion free part. Therefore $E(1)_*\mathbf{K}(X) \otimes \mathbb{Q}_l$ has the ψ-invariant spaces as a direct factor, which contains the invariant spaces of $H^*(X_\infty; \mathbb{Q}_l)$. It follows that the eigenspace of ψ or ϕ_X with eigenvalue one is a direct factor.

5.1. Theorem: *On the geometric cohomology group $H^i(\bar{X}; \mathbb{Q}_l(*))$ of a smooth projective variety X over a finite field, the action of ϕ_X is semi-simple. And the numerical equivalence in the Chow groups agrees with the homological equivalence.*

Proof. It is known that the Tate conjecture 2.1 and semi-simplicity of eigenspace belongs to 1, proved in 5.0, imply all of the above statement ([35], in the case of finite base fields. See [27, 1.14] also). This form of the Tate conjecture is known to be independent of l. So the final result is valid even for $l = 2$ cases ([27, 1.14]).

5.2. Theorem: *Let X be a smooth variety over a finite field. The Chern character mapping induces isomorphisms*

$$K_j(X) \otimes \mathbb{Q}_l \cong \oplus H^{2i-j}(\bar{X}; \mathbb{Q}_l(i))^{\phi_X}$$

for all j .

It is called Beilinson-Friedlander-Parshin conjecture. T. Geisser [13] showed that this is consequence of the theorem 5.1. When $j = 0$ and X is projective, it says that rational equivalences, algebraic equivalences and numerical equivalences are the same on the Chow groups tensored with $\mathbb{Q}$.

6. Other consequences

6.0. Let K be a field which is finitely generated over the rational numbers $\mathbb{Q}$, or a finite field $\mathbb{F}_p$. We define a fixed separable closure as $\bar{K}$ and the absolute Galois group as G_K.

Let $\bar{X}$ be an irreducible, projective and smooth scheme of dimension d over $\bar{K}$. For any $0 \leq i \leq d$, we can consider a rational vector space $A^i(\bar{X})$ in $H^{2i}(\bar{X}_{et}; \mathbb{Q}_l(i))$ which is generated by classes of cycles of codimension i [23, p.11]. $A^i(\bar{X})$ is the image of $CH^i(\bar{X})_{\mathbb{Q}}$ or $K_0(\bar{X})$ by $\rho(\bar{X})_{\mathbb{Q}}$ as in the introduction. The elements of it will be presented as $\rho(M)$ $(M \in CH^i(\bar{X}))$. There are bilinear mapping corresponding to the intersection pairing of cycles

$$A^i(\bar{X}) \times A^{d-i}(\bar{X}) \longrightarrow A^d(\bar{X}) \xrightarrow[\sim]{tr} \mathbb{Q}.$$

$\rho(M) \cdot \rho(N)$ is the result of this pairing.

6.1. Theorem: *The pairing above is non-degenerate for any $\bar{X}$ and i. In other words, homological equivalence agrees with numerical equivalence for all varieties over algebraically closed fields.*

6.1.1. Let $\mathfrak{X}$ be a projective smooth model of X over a finitely generated integral scheme $\operatorname{Spec} A$. It means that $\mathfrak{X}$ is a scheme over an affine scheme $\operatorname{Spec} A$, where A is finitely generated, integral, regular, and its field of quotients K, such that $\mathfrak{X} \times \operatorname{Spec} \bar{K}$ is isomorphic to X [16, IV.8.8.2]. We may assume l is invertible in A. From now on, for any A-algebra C, we define $\mathfrak{X} \times_{\operatorname{Spec} A} \operatorname{Spec} C$ as $\mathfrak{X}_C$. In our convention $\mathfrak{X}_{\bar{K}} = \bar{X}$.

Let p be a closed point of $\operatorname{Spec} A$. The residue field of p is a finite field k, G_k is the absolute Galois group $\operatorname{Gal}(\bar{k}/k)$. Consider the open immersion $j : \mathfrak{X}_K \to \mathfrak{X}_A$ and the closed immersion $i : \mathfrak{X}_k \to \mathfrak{X}_A$. By the proper smooth base changing theorem [26, VI.4.2], there are isomorphisms

$$H^{2i}(\mathfrak{X}_{\bar{K}}, \mathbb{Q}_l(i)) \xleftarrow[\sim]{\bar{j}^*} H^{2i}(\mathfrak{X}_{\bar{A}}, \mathbb{Q}_l(i)) \xrightarrow[\sim]{\bar{i}^*} H^{2i}(\mathfrak{X}_{\bar{k}}, \mathbb{Q}_l(i)),$$

where $\bar{A}$ is the integral closure of A in $\bar{K}$ and G_K acts on the left hand side through $G_K \to \pi_1(\operatorname{Spec} A)$, and G_k acts on the right hand side through $G_k \to \pi_1(\operatorname{Spec} A)$, which is determined up to rational points $\mathfrak{X}(\bar{k})$. The assumption implies $(\bar{i}^*)(\bar{j}^*)^{-1}w$ is not zero and contained in $H^{2i}(\mathfrak{X}_{\bar{k}}, \mathbb{Q}_l(i))^{G_k}$.

6.1.2. Let $\mathfrak{X} \longrightarrow \operatorname{Spec}(A)$ be a proper smooth morphism as in 6.1.1 where the fiber over a closed point $\operatorname{Spec} k$ is Y.

There exists a vector bundle (A^n-torsor) over $\mathfrak{X}$ which is an affine scheme $\operatorname{Spec}(B)$ ([21, 1.5]) such that the structure morphism $F : \operatorname{Spec}(B) \longrightarrow \mathfrak{X}$ and the zero section $s : \mathfrak{X} \longrightarrow \operatorname{Spec}(B)$ induce equivalences in K-theory. The fiber over a closed point $\operatorname{Spec} k$ is a vector bundle $\operatorname{Spec}(B/I) \longrightarrow Y$, so that the zero section $s_0 : Y \longrightarrow \operatorname{Spec}(B/I)$ induces an equivalence in K-theory.

For any element x in $K_0(Y) \cong K_0(B/I)$ there exists an etale morphism $p : \operatorname{Spec} B' \longrightarrow \operatorname{Spec} B$ which is the identity on $\operatorname{Spec}(B/I)$ and an element N in $K_0(B')$ such that $(i')^*N = x$ ([32, XI. Theorem 1]).

$$\begin{array}{ccc} & & \operatorname{Spec}(B') \\ & \overset{i'}{\nearrow} & \downarrow{\scriptstyle p} \\ \operatorname{Spec}(B/I) & \xrightarrow{i} & \operatorname{Spec}(B) \\ \downarrow & & \downarrow{\scriptstyle F} \\ Y & \xrightarrow[i]{} & \mathfrak{X} \end{array}$$

Taking the pullback along the zero section, we get

$$\begin{array}{ccccc} Y \cup Y' & \xrightarrow{i' \cup i''} & \mathfrak{W} & \longleftarrow & W \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ Y & \xrightarrow[i]{} & \mathfrak{X} & \xleftarrow{j} & X, \end{array}$$

where f is the normalization of an finite extension at the generic point of $\mathfrak{X}$, and $\mathfrak{W}$ is regular scheme having an open subscheme contains Y. Then there is an element N in $K_0(\mathfrak{W})$ such that $(i')^*N = x$.

Proof of 6.1. To prove it, take a non-zero element $M \in K_0(X)$ such that $\rho(M) \neq 0$. We have to find an element N in K_0 such that $\rho(M) \cdot \rho(N) \neq 0$.

We may take a model $\mathfrak{X}$ as in 6.1.1 and an element M' in $K_0(\mathfrak{X})$ such that $j^*M' = M$ and i^*M' has non-zero image in cohomology. Then there is an element $x \in K_0(Y)$ such that $x \cdot i^*M \neq 0$ in $K_0(Y)$ by 5.1.

Apply 6.1.2 to x, there exists a pullback diagram

$$\begin{array}{ccc} \cup V & \xrightarrow{\cup l} & \mathfrak{Z} \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y \cup Y' & \xrightarrow{i' \cup i''} & \mathfrak{W} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[i]{} & \mathfrak{X}, \end{array}$$

where g and $f \circ g$ are the normailizations of Galois extensions at the generic point of $\mathfrak{X}$, V is an irreducible component of the pullback of Y, and there is N' in $K_0(\mathfrak{W})$ such that $(i')^*N' = x$.

Let an element L of $K_0(\mathfrak{Z})$ be $\sum \sigma^* g^* N'$ where σ are elements of the Galois group of $f \circ g$.

$tr_Y i^*(M' \cdot (f \circ g)_* L) = C(tr_V((i \circ f' \circ g')^* M' \cdot l^* L)$ where C is a non-zero number, and tr_V means $tr_Y \circ (f' \circ g'|_V)_*$. On the other hand, $0 \neq tr_Y(i^* M' \cdot (g')_*(g')^* x) = C'(tr_V((i \circ f' \circ g')^* M' \cdot l^* L))$. Therefore $N = j^*(f \circ g)_* L$ will be an element we want.

6.2. Finally we get the Grothendieck's standard conjectures ([18]). They consist of two parts; of Lefschetz type and of Hodge type. Over fields of zero characteristic the conjecture of Hodge type is known to be true by the Hodge index theorem. Theorem 6.1 implies the conjecture of Lefschetz type over arbitrary characteristics.

Y.André proved ([2]) that theorem 6.1 together with the global invariant cycle theorem of Deligne implies another conjecture of Grothendieck on parallel transport of cycle classes ([17, footnote 13]), and it implies the conjecture of Hodge for abelian varieties.

6.3. Theorem: *The Hodge conjecture for abelian varieties is true.*

J. S. Milne showed that the above conjecture only for CM abelian varieties implies the conjecture of Hodge type for all varieties of positive characteristics ([28]) if we assume 5.1 ([27, 2.47, 2.7]).

6.4. Theorem: *All the standard conjectures of Grothendieck are true.*

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