Hermitian K-theory of totally real 2-regular number fields

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Abstract

We completely determine the 2-primary torsion subgroups of the hermitian K-groups of rings of 2-integers in totally real 2-regular number fields. The result is almost periodic with period 8. We also identify the homotopy fibers of the forgetful and hyperbolic maps relating hermitian and algebraic K-theory. The result is then exactly periodic of period 8. In both the orthogonal and symplectic cases, we prove the 2-primary hermitian Quillen-Lichtenbaum conjecture.

1 Introduction and statement of results

Let F be a number field with r real embeddings and ring of integers $\mathcal{O}_F$. It follows from a theorem of Tate [28, Theorem 6.2] that the 2-primary part of the finite abelian group $K_2(\mathcal{O}_F)$ has order at least 2^r. We call F 2-*regular* when this order is exactly 2^r. See Proposition 2.1 below for alternative characterizations. In the totally real case, which is our concern here, the simplest examples are the rational numbers $\mathbb{Q}$ and the following fields recorded in [26, §4].

1. Let $b \geq 2$. The maximal real subfield $F = \mathbb{Q}(\zeta_{2^b} + \bar{\zeta}_{2^b})$ of $\mathbb{Q}(\zeta_{2^b})$ is a totally real 2-regular number field with $r = 2^{b-2}$.
2. Let m be an odd prime power such that 2 is a primitive root modulo m. Then $F = \mathbb{Q}(\zeta_m + \bar{\zeta}_m)$ is a totally real 2-regular number field when Euler's ϕ-function $\phi(m) \leq 66$ (except for $m = 29$), and also for Sophie Germain primes (m and $(m-1)/2$ both prime) with $m \not\equiv 7 \pmod 8$ (the first few instances are $m = 5, 11, 59, 83, 107$ and 179). The number r of real embeddings is $\phi(m)/2$.
3. Let $F = \mathbb{Q}(\sqrt{d})$ be a quadratic number field with $d > 0$ square free. Then F is 2-regular if and only if $d = 2$, $d = p$ or $d = 2p$ with $p \equiv \pm 3 \pmod 8$ prime [7]. Here, $r = 2$.

The main purpose of this paper is to generalize to totally real 2-regular number fields the results on the hermitian K-theory of $\mathbb{Z}[\frac{1}{2}]$ obtained in [4]. Thus, for the theorems of this Introduction we assume that F is totally real 2-regular, with ring of 2-integers $R_F = \mathcal{O}_F[\frac{1}{2}]$. In the following we consider the fields of real and complex numbers with their usual topology, and let $\varepsilon = \pm 1$ according to whether orthogonal ($\varepsilon = +1$) or symplectic ($\varepsilon = -1$) actions are involved. We also implicitly consider only 2-adic completions of abelian groups and spectra. Our proof of the first theorem is based on the techniques employed in the case of the rational numbers [4][1] and the analogous algebraic K-theoretic result established in [15], [24] and [26] (see Appendix A for an overview).

[1]We take this opportunity to mention some notational changes compared to [4]. In order to avoid a potential confusion between hermitian K-theory and surgery theory, we shall write "${}_\varepsilon\mathcal{KQ}$" and "${}_\varepsilon KQ_n$" for "${}_\varepsilon\mathcal{L}$" and "${}_\varepsilon L_n$" respectively, cf. [4].

Theorem 1.1: *There is a homotopy cartesian square of hermitian K-theory connective spectra:*

$$\begin{array}{ccc} {}_\varepsilon\mathcal{KQ}(R_F) & \to & \bigvee^r {}_\varepsilon\mathcal{KQ}(\mathbb{R}) \\ \downarrow & & \downarrow \\ {}_\varepsilon\mathcal{KQ}(\mathbb{F}_q) & \to & \bigvee^r {}_\varepsilon\mathcal{KQ}(\mathbb{C}) \end{array}$$

Here, $\mathbb{F}_q$ is some residue field of R_F of prime power order q such that the two elements corresponding to the Adams operations Ψ^q and Ψ^{-1} in the ring of operations of the periodic complex topological K-theory spectrum generate the Galois group of $F(\mu_{2^\infty}(\mathbb{C}))/F$ obtained by adjoining all 2-primary roots of unity $\mu_{2^\infty}(\mathbb{C}) \subset \mathbb{C}$ to F [24, §1]. The Cebotarev density theorem guarantees the existence of infinitely many such residue fields of R_F. By Dirichlet's theorem on arithmetic progressions we may assume that q is a prime number. Note that the homotopy type of ${}_\varepsilon\mathcal{KQ}(R_F)$ is independent of this choice of q. According to [24], if $a_F := (|\mu_{2^\infty}(F(\sqrt{-1}))|)_2$ is the 2-adic valuation, then q is $\equiv \pm 1 \pmod{2^a}$ but not $\pmod{2^{a+1}}$. In the examples above: when $F = \mathbb{Q}(\zeta_{2^b} + \bar{\zeta}_{2^b})$, then $a_F = b$; when $F = \mathbb{Q}(\zeta_m + \bar{\zeta}_m)$ or $\mathbb{Q}(\sqrt{d})$ with $d > 2$, $a_F = 2$; and finally, when $F = \mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\zeta_8 + \bar{\zeta}_8)$, we have $a_F = 3$.

Theorem 1.1 allows us to compute explicitly the hermitian K-groups ${}_\varepsilon KQ_n(R_F) := \pi_n({}_\varepsilon\mathcal{KQ}(R_F))$. In the next theorem we compare these ${}_\varepsilon KQ$-groups with the algebraic K-groups $K_n(R_F)$ computed in [15] and [26].

Theorem 1.2: *Up to finite groups of odd order, the groups ${}_\varepsilon KQ_n(R_F)$ are given in Table 1. (If m is even, let $w_m = 2^{a_F+\nu_2(m)}$; also, δ_{n0} denotes the Kronecker symbol.)*

Table 1:

$n \geq 0$	${}_{-1}KQ_n(R_F)$	${}_1KQ_n(R_F)$	$K_n(R_F)$
$8k$	$\delta_{n0}\mathbb{Z}$	$\delta_{n0}\mathbb{Z} \oplus \mathbb{Z}^r \oplus \mathbb{Z}/2$	$\delta_{n0}\mathbb{Z}$
$8k+1$	0	$(\mathbb{Z}/2)^{r+2}$	$\mathbb{Z}^r \oplus \mathbb{Z}/2$
$8k+2$	$\mathbb{Z}^r$	$(\mathbb{Z}/2)^{r+1}$	$(\mathbb{Z}/2)^r$
$8k+3$	$(\mathbb{Z}/2)^{r-1} \oplus \mathbb{Z}/2w_{4k+2}$	$\mathbb{Z}/w_{4k+2}$	$(\mathbb{Z}/2)^{r-1} \oplus \mathbb{Z}/2w_{4k+2}$
$8k+4$	$(\mathbb{Z}/2)^r$	$\mathbb{Z}^r$	0
$8k+5$	$\mathbb{Z}/2$	0	$\mathbb{Z}^r$
$8k+6$	$\mathbb{Z}^r$	0	0
$8k+7$	$\mathbb{Z}/w_{4k+4}$	$\mathbb{Z}/w_{4k+4}$	$\mathbb{Z}/w_{4k+4}$

The proof of Theorem 1.2 uses a splitting result for ${}_{\varepsilon}\mathcal{KQ}(R_F)$ shown in §4 and an explicit computation carried out in §5.

Corollary 1.3: *For a totally real number field F the following are equivalent.*

(i) *F is 2-regular.*

(ii) *Theorem 1.1 holds for ${}_{1}\mathcal{KQ}(R_F)$.*

(iii) *The counterpart of Theorem 1.1 holds for $\mathcal{K}(R_F)$ (as in the square (2) in Section 2 below).*

The forgetful and hyperbolic functors induce the two homotopy fiber sequences

$$ {}_{\varepsilon}\mathcal{V}(R_F) \to {}_{\varepsilon}\mathcal{KQ}(R_F) \to \mathcal{K}(R_F), $$

and

$$ {}_{\varepsilon}\mathcal{U}(R_F) \to \mathcal{K}(R_F) \to {}_{\varepsilon}\mathcal{KQ}(R_F). $$

The fundamental theorem in hermitian K-theory [18] says that there is a natural weak homotopy equivalence

$$ {}_{\varepsilon}\mathcal{V}(R_F) \simeq \Omega_{-\varepsilon}\mathcal{U}(R_F). $$

Theorem 1.4: *Up to finite groups of odd order the groups*

$$ {}_{\varepsilon}V_n(R_F) := \pi_n({}_{\varepsilon}\mathcal{V}(R_F)) \cong \pi_n(\Omega_{-\varepsilon}\mathcal{U}(R_F)) =: {}_{-\varepsilon}U_{n+1}(R_F) $$

are given in Table 2.

Table 2:

$n \geq 0$	${}_{-1}V_n(R_F)$	${}_{1}V_n(R_F)$
$8k$	$\mathbb{Z}^r \oplus \mathbb{Z}/2$	$\mathbb{Z}^{2r}$
$8k+1$	0	$(\mathbb{Z}/2)^{2r}$
$8k+2$	$\mathbb{Z}^r$	$(\mathbb{Z}/2)^{2r}$
$8k+3$	0	0
$8k+4$	$\mathbb{Z}^r$	$\mathbb{Z}^{2r}$
$8k+5$	$\mathbb{Z}/2$	0
$8k+6$	$\mathbb{Z}^r \oplus \mathbb{Z}/2$	0
$8k+7$	$\mathbb{Z}/2$	0

More precisely, the spectrum ${}_1\mathcal{V}(R_F)$ has the homotopy type of

$$\bigvee^{2r} \mathcal{K}(\mathbb{R}) \simeq {}_1\mathcal{V}(R_{\mathbb{Q}}) \vee \bigvee^{2(r-1)} \mathcal{K}(\mathbb{R}),$$

while ${}_{-1}\mathcal{V}(R_F)$ has the homotopy type of

$$\Omega^2\mathcal{K}(\mathbb{R}) \vee \Omega^4\mathcal{K}(\mathbb{R}) \vee \bigvee^{r-1} \mathcal{K}(\mathbb{C}) \simeq {}_{-1}\mathcal{V}(R_{\mathbb{Q}}) \vee \bigvee^{r-1} \mathcal{K}(\mathbb{C}).$$

For $\varepsilon = \pm 1$, cup-product with a generator of $K_8(\mathbb{R})$ induces a periodicity homotopy equivalence

$${}_\varepsilon\mathcal{V}(R_F) \simeq \Omega^8{}_\varepsilon\mathcal{V}(R_F).$$

It would be interesting to have more information about the class of number fields for which the periodicity result in this theorem holds.

Recall from [4, §7] that the standard ${}_\varepsilon\mathbb{Z}/2$-action on ${}_\varepsilon\mathcal{K}(R_F)$ defined via conjugation of the involute transpose of a matrix by

$${}_\varepsilon J_n = \begin{bmatrix} 0 & \varepsilon I_n \\ I_n & 0 \end{bmatrix}$$

induces an isomorphism between the fixed point spectrum $\mathcal{K}(R_F)^{{}_\varepsilon\mathbb{Z}/2}$ and ${}_\varepsilon\mathcal{KQ}(R_F)$. Our next result solves in the affirmative for totally real 2-regular number fields the 2-primary homotopy limit problem, also known as the Quillen-Lichtenbaum conjecture, for ${}_\varepsilon\mathbb{Z}/2$-homotopy fixed point K-theory spectra. For algebraic K-theory, the Quillen-Lichtenbaum conjecture holds for every real number field [25]. Analogous results inspired a conjecture formulated by B. Williams in [29, 3.4.2]. Recall the homotopy fixed point spectrum

$$\mathcal{K}(R_F)^{h({}_\varepsilon\mathbb{Z}/2)} := \mathrm{map}_{{}_\varepsilon\mathbb{Z}/2}(\Sigma^\infty E(\mathbb{Z}/2)_+, \mathcal{K}(R_F)).$$

Here $\mathrm{map}_{{}_\varepsilon\mathbb{Z}/2}$ denotes the function spectrum of ${}_\varepsilon\mathbb{Z}/2$-equivariant maps and $E(\mathbb{Z}/2)$ is a free contractible $\mathbb{Z}/2$-space, such as the CW-complex S^∞ with antipodal action.

Theorem 1.5: *There is a natural weak equivalence of 2-completed spectra*

$${}_\varepsilon\mathcal{KQ}(R_F) \simeq \mathcal{K}(R_F)^{h({}_\varepsilon\mathbb{Z}/2)}.$$

2 Preliminaries

We begin with a list of alternative characterizations of the class of number fields of interest. To fix terminology, recall that the r real embeddings of a number field F define the *signature* map $F^\times/(F^\times)^2 \to (\mathbb{Z}/2)^r$ where a unit is mapped to the signs of its images under the real embeddings. One says that $R_F = \mathcal{O}_F[\frac{1}{2}]$, the ring of 2-integers in F, has *units of independent signs* if the signature map remains surjective when restricted to the square classes $R_F^\times/(R_F^\times)^2$ of $R_F^\times$. A *dyadic prime* in F is a prime ideal in the ring of integers $\mathcal{O}_F$ lying over the rational prime ideal (2). The *narrow Picard group* $\mathrm{Pic}_+(R_F)$ consists of fractional R_F-ideals modulo totally positive principal ideals, defined as in [13, V§1]. Finally, the *Witt ring* $W(A)$ of a commutative unital ring A with involution is defined in terms of Witt classes of symmetric nondegenerate bilinear forms [22, I (7.1)]; in K-theoretic terms, as an abelian group it coincides with the cokernel ${}_1W_0(A)$ of the hyperbolic map $K_0(A) \to {}_1KQ_0(A)$ if 2 is invertible in A [18].

Proposition 2.1: *Let F be a number field with r real embeddings and c pairs of complex embeddings. Then the following are equivalent.*

1. *F is 2-regular; that is, the 2-Sylow subgroup of the finite abelian group $K_2(\mathcal{O}_F)$ has order 2^r.*

2. *The real embeddings of F induce isomorphisms on 2-Sylow subgroups*

$$K_2(\mathcal{O}_F)\{2\} \xrightarrow{\cong} K_2(R_F)\{2\} \xrightarrow{\cong} K_2(\prod^r \mathbb{R}) \cong (\mathbb{Z}/2)^r.$$

3. *The finite abelian group $K_2(\mathcal{O}_{F(\sqrt{-1})})$ has odd order.*

4. *There is a unique dyadic prime in F and the narrow Picard group $Pic_+(R_F)$ has odd order.*

5. *There is a unique dyadic prime in F, the Picard group $Pic(R_F)$ has odd order, and R_F has units with independent signs.*

6. *The nilradical of the Witt ring $W(R_F)$ is a finite abelian group of order 2^{c+1}. In particular, if F is a totally real number field, so that $c = 0$, then the nilradical of $W(R_F)$ has order 2.*

7. *The Witt ring $W(R_F)$ is a finitely generated abelian group of rank r with torsion subgroup of order 2^{c+1}. In particular, if F is a totally real number field, then $W(R_F)$ is isomorphic to $\mathbb{Z}^r \oplus \mathbb{Z}/2$.*

If F is a totally real number field and satisfies any of the equivalent conditions above, then the free part of $W(R_F)$ is generated by elements $\langle 1\rangle, \langle u_1\rangle, \dots, \langle u_{r-1}\rangle$ where $u_i \in R_F^\times$ is negative at the i th embedding and positive elsewhere.

Proof. Obviously, (2) implies (1). The converse follows from Tate's 2-rank formula for K_2 [28, Theorem 6.2], which forces r to be a lower bound for the 2-rank of $K_2(\mathcal{O}_F)$. See [26, Proposition 2.2] for the equivalence between (2) and (4). By [9, (4.1),(4.6)], (2), (3) and (5) are equivalent. The equivalences between (4), (6), and (7) are immediate from [11, Corollary 3.6, Theorem 4.7]. Finally, the claim concerning the generators of the free part of $W(R_F)$ follows as in [22, IV (4.3)]. □

Remark 2.2. 1. Further equivalent conditions, in terms of étale cohomology, appear in [26, Proposition 2.2].

2. Berger [3] calls a totally real number field F satisfying (5) above 2^+-*regular*, and has shown in [3] that each totally real 2-regular number field has infinitely many totally real quadratic field extensions satisfying the equivalent conditions in Proposition 2.1. In particular, there exist totally real 2-regular number fields of arbitrarily high degree.

3. In the other direction, by [9, (4.1)] every subfield of a totally real 2-regular number field also enjoys this property.

4. In [22, pg. 95], it is shown that for totally real F there is also an equivalence between amended conditions (4)–(7) above, in which R_F is replaced by $\mathcal{O}_F$ and 2^{c+1} by 2^c. However, in view of the surjection $\mathrm{Pic}_+(\mathcal{O}_F) \twoheadrightarrow \mathrm{Pic}_+(R_F)$, these amended conditions define a strict subclass of those considered here. After Gauss, one knows that for a real quadratic number field $F = \mathbb{Q}(\sqrt{d})$ the narrow Picard group $\mathrm{Pic}_+(\mathcal{O}_F)$ is an odd torsion group if and only if F has prime-power discriminant; that is, $d = 2$ or $d = p$ with $p \equiv 1 \pmod 4$ a prime number. Thus $\mathbb{Q}(\sqrt{p})$ with $p \equiv 3 \pmod 4$ a prime number and $\mathbb{Q}(\sqrt{2p})$ with $p \equiv \pm 3 \pmod 8$ a prime number fail to lie in the subclass, cf. [10].

Suppose henceforth that 2 is a unit in a domain A, which lies in some number field F. The *coWitt group* $W'(A)$ (or ${}_1W'_0(A)$) is defined as the kernel of the forgetful rank map ${}_1KQ_0(A) \to K_0(A)$. Let $k_0(A)$ denote the 0 th Tate cohomology group

$\widehat{H}^0(\mathbb{Z}/2, K_0(A))$ of $\mathbb{Z}/2$ acting on $K_0(A)$, and $k_0'(A)$ denote the 1st Tate cohomology group $\widehat{H}^1(\mathbb{Z}/2, K_0(A))$.

There is a well defined induced *rank map*

$$\rho\colon W(A) \to k_0(A),$$

whose image always has a $\mathbb{Z}/2$ summand. From the 12-term exact sequence established in [18, pg. 278] there is an exact sequence

$$k_0'(A) \longrightarrow W'(A) \xrightarrow{\varphi} W(A) \xrightarrow{\rho} k_0(A),$$

where φ is simply the composition $W'(A) \hookrightarrow {}_1KQ_0(A) \twoheadrightarrow W(A)$.

The next result is almost immediate, but worth recording. Part (c) uses the computations $k_0'(A) = 0$ and $k_0(A) \cong \mathbb{Z}/2$ when $\mathrm{Pic}(A)$ is an odd torsion group. Also, (d) refers to the fact that, when A is a field, its *fundamental ideal* is the unique ideal I of $W(A)$ with $W(A)/I \cong \mathbb{F}_2$ [22, III (3.3)].

Lemma 2.3: *(a) In general, φ maps the coWitt group of A onto the kernel of the rank map ρ.*

(b) When $k_0'(A) = 0$, then φ is an isomorphism between $W'(A)$ and $\mathrm{Ker}\rho$.

(c) For rings A such that Pic(A) is an odd torsion group, there is a natural short exact sequence

$$0 \to W'(A) \xrightarrow{\varphi} W(A) \xrightarrow{\rho} \mathbb{Z}/2 \to 0.$$

(d) When A is a field, φ is an isomorphism between the coWitt group of A and the fundamental ideal of the Witt ring.

We can now give a further characterization of the class of 2-regular totally real number fields, in terms of the coWitt group.

Lemma 2.4: *A totally real number field F with r real embeddings is 2-regular if and only if both*

(i) the Picard group Pic(R_F) has odd order, and

(ii) the coWitt group $W'(R_F)$ is isomorphic to $\mathbb{Z}^r \oplus \mathbb{Z}/2$.

Proof. In both directions (one way uses Proposition 2.1(5)), we have from Lemma 2.3(c) the map of short exact sequences (where the middle vertical map is surjective):

$$\begin{array}{ccccccc} 0 \to & W'(R_F) & \xrightarrow{\varphi} & W(R_F) & \xrightarrow{\rho} & \mathbb{Z}/2 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow \mathrm{id} & \\ 0 \to & W'(\mathbb{R}) & \xrightarrow{\varphi} & W(\mathbb{R}) & \xrightarrow{\rho} & \mathbb{Z}/2 & \to 0 \end{array}$$

Since $W(\mathbb{R}) \cong \mathbb{Z}$ [22, III (2.7)], in the upper sequence ρ must be trivial on torsion elements. By combining with Proposition 2.1(7), we obtain the result. $\square$

When F has r real embeddings, for $A = R_F, F$, the map $W(A) \to W(\mathbb{R})^r \cong \mathbb{Z}^r$ is the *total signature* σ [22, III (2.9)].

To define further invariants, we recall from [18] generalizations of some of the definitions above. For $\varepsilon = \pm 1$, and $n \geq 1$, we set

$$\begin{aligned} {}_{\epsilon}W_n(A) &= \mathrm{Coker}[K_n(A) \to {}_{\epsilon}KQ_n(A)], \\ {}_{\epsilon}W'_n(A) &= \mathrm{Ker}[{}_{\epsilon}KQ_n(A) \to K_n(A)], \\ k_n(A) &= \widehat{H}^0(\mathbb{Z}/2, K_n(A)), \\ k'_n(A) &= \widehat{H}^1(\mathbb{Z}/2, K_n(A)). \end{aligned}$$

The next invariant we shall employ is the *discriminant map*

$${}_{\epsilon}W'_0(A) \longrightarrow k'_1(A).$$

To recall the definition, suppose that M and N are quadratic modules with isomorphic underlying A-modules. The elements of ${}_{\epsilon}W'_0(A)$ are of the form $M - N$ with M and N isomorphic modules. An isomorphism $\alpha\colon M \to N$ induces an automorphism $\alpha^*\alpha$ of M that is antisymmetric for the $\mathbb{Z}/2$-action. Its class in $k'_1(A)$, which is independent of α, defines the desired invariant. For a generalization of the above we refer to [18].

Note that if $SK_1(A) = 0$ (*e.g.* A a ring of S-integers in a number field [2] or A a field), then $k'_1(A)$ is isomorphic to the group of square classes $A^\times/(A^\times)^2$ of units in A. Moreover, taking $\varepsilon = 1$, $W'(A)$ surjects onto $A^\times/(A^\times)^2$ since the discriminant maps $\langle u\rangle - \langle 1\rangle$ to $u \in A^\times/(A^\times)^2$. In the case of $A = \mathbb{R}$, from [22, III (2.7)] this is the surjection $\mathbb{Z} \to \mathbb{Z}/2$, while for $A = \mathbb{F}_q$ by [22, III (5.2), (5.9)] it is the isomorphism $\mathbb{Z}/2 \to \mathbb{Z}/2$.

What follows is a key ingredient in the proof of Theorem 1.1.

Proposition 2.5: *Suppose that F is a totally real 2-regular number field with r real embeddings. Then the residue field map $R_F \to \mathbb{F}_q$ and the real embeddings of F induce an isomorphism*

$$W'(R_F) \xrightarrow{\cong} \bigoplus^r W'(\mathbb{R}) \oplus W'(\mathbb{F}_q) \cong \mathbb{Z}^r \oplus \mathbb{Z}/2.$$

Proof. We consider the exact sequence

$${}_1W_1(A) \longrightarrow k_1(A) \longrightarrow {}_{-1}W_2(A) \longrightarrow {}_1W_0'(A) \longrightarrow k_1'(A)$$

from [18]. Provided that $SK_1(A)$ is the trivial group, the rank map ${}_1W_1(A) \to k_1(A)$ is surjective and then ${}_{-1}W_2(A)$ identifies with the kernel of the induced discriminant map $W'(A) \to k_1'(A) \cong A^\times/(A^\times)^2$. Recalling that SK_1 is trivial for R_F and F [2], we thus obtain a map of short exact sequences

$$\begin{array}{ccccccccc} 0 \to & {}_{-1}W_2(R_F) & \to & W'(R_F) & \to & R_F^\times/(R_F^\times)^2 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \bigoplus^r {}_{-1}W_2(\mathbb{R}) \oplus {}_{-1}W_2(\mathbb{F}_q) & \to & \bigoplus^r W'(\mathbb{R}) \oplus W'(\mathbb{F}_q) & \to & \bigoplus^r \mathbb{R}^\times/(\mathbb{R}^\times)^2 \oplus \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2 & \to 0 \end{array}$$

which by Lemma 2.4 and the Dirichlet S-unit theorem for R_F takes the form:

$$\begin{array}{ccccccccc} 0 \to & \mathbb{Z}^r & \to & \mathbb{Z}^r \oplus \mathbb{Z}/2 & \to & (\mathbb{Z}/2)^{r+1} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \bigoplus^r \mathbb{Z} \oplus 0 & \to & \bigoplus^r \mathbb{Z} \oplus \mathbb{Z}/2 & \to & \bigoplus^r \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \to 0 \end{array} \qquad (1)$$

(The groups ${}_{-1}W_2(\mathbb{F}_q)$ and $W'(\mathbb{F}_q)$ are determined in [22, III (5.9), IV (1.5)].)

From the homotopy cartesian square

$$\begin{array}{ccc} \mathcal{K}(R_F) & \to & \bigvee^r \mathcal{K}(\mathbb{R}) \\ \downarrow & & \downarrow \\ \mathcal{K}(\mathbb{F}_q) & \to & \bigvee^r \mathcal{K}(\mathbb{C}) \end{array} \qquad (2)$$

established in [15] and [24], we deduce the short exact sequence

$$0 \to \bigoplus^r K_2(\mathbb{C}) \to K_1(R_F) \to \bigoplus^r K_1(\mathbb{R}) \oplus K_1(\mathbb{F}_q) \to 0. \qquad (3)$$

Therefore, the final vertical map in (1) is an isomorphism. Thus, to show that the middle vertical map is an isomorphism, it suffices to show that either of the homomorphisms $W'(R_F) \to \bigoplus^r W'(\mathbb{R})$ or ${}_{-1}W_2(R_F) \to \bigoplus^r {}_{-1}W_2(\mathbb{R})$ is surjective.

For the former homomorphism, we use the map of short exact sequences afforded by Lemma 2.3(c):

$$\begin{array}{ccccccc} 0 \to & W'(R_F) & \longrightarrow & W(R_F) & \longrightarrow & \mathbb{Z}/2 & \to 0 \\ & \downarrow \sigma' & & \downarrow \sigma & & \downarrow \Delta & \\ 0 \to & \bigoplus^r W'(\mathbb{R}) & \longrightarrow & \bigoplus^r W(\mathbb{R}) & \longrightarrow & \bigoplus^r \mathbb{Z}/2 & \to 0 \end{array}$$

Since the diagonal map Δ is injective, the snake lemma gives the exact sequence

$$0 \longrightarrow \mathrm{Coker}\sigma' \longrightarrow \mathrm{Coker}\sigma \longrightarrow (\mathbb{Z}/2)^{r-1} \longrightarrow 0.$$

According to [11, Corollary 4.8] (applicable because by Proposition 2.1(4) $\mathrm{Pic}_+(R_F)$ has odd order), $\sigma : W(R_F) \to \bigoplus^r W(\mathbb{R})$ has the same cokernel as $\sigma : W(F) \to \bigoplus^r W(\mathbb{R})$. Now, by [22, pp. 64-65] (applicable because by Proposition 2.1(5) R_F has units with independent signs), $\mathrm{Coker}\sigma$ is $(\mathbb{Z}/2)^{r-1}$. Hence, $\mathrm{Coker}\sigma' = 0$, as desired.

A second approach, showing that ${}_{-1}W_2(R_F) \to \bigoplus^r {}_{-1}W_2(\mathbb{R})$ is surjective, uses the "classical" treatment of [22] instead of [11], by invoking the final quadratic form invariant for A. If A is a field, then $W'(A)$ is the fundamental ideal $I(A)$; hence ${}_{-1}W_2(A)$ is isomorphic to $I^2(A)$ by [22, III (5.2)]. The composite map

$$K_2(A) \longrightarrow {}_{-1}KQ_2(A) \longrightarrow k_2(A)$$

is induced by sending a module E to the direct sum of E and its dual; this makes its image symmetric with respect to the involution on the K-theory spectrum. Thus by the definition of $k_2(A)$, the composite map is trivial and induces a map

$${}_{-1}W_2(A) \longrightarrow k_2(A)$$

employed in the definition of the 12-term exact sequence in [18]. According to [14], this map from $I^2(A)$ to $k_2(A) = K_2(A)/2$ gives an equivalent definition of the classical *Hasse-Witt invariant* if A is a field.

Moreover, the Hasse-Witt invariants for R_F and F induce a commutative diagram with exact rows:

$$\begin{array}{ccccc} {}_{-1}W_2(R_F) & \to & k_2(R_F) & \to & 0 \\ \downarrow & & \downarrow & & \\ {}_{-1}W_2(F) & \to & k_2(F) & \to & 0 \end{array}$$

Now, from (3), the real embeddings of F induce a surjective map from $R_F^\times$ to $\oplus^r K_1(\mathbb{R})$. Hence, ${}_{-1}W_2(R_F)$ surjects onto $k_2(R_F) \cong (\mathbb{Z}/2)^r$ because the tensor product $(\langle u\rangle - \langle 1\rangle) \otimes (\langle v\rangle - \langle 1\rangle)$ maps to the symbol $\{u, v\}$ in $k_2(R_F)$. By reference to [22, IV §4], where the arguments can be extended to any ring D of S-integers in a number field, the kernels of the Hasse-Witt surjections for R_F and F are isomorphic to $I^3(F) \cong \bigoplus^r I^3(\mathbb{R}) \cong 8\mathbb{Z}^r$ via the signature map, and hence there is a map of short exact sequences:

$$\begin{array}{ccccccc} 0 \to & 8\mathbb{Z}^r & \longrightarrow & {}_{-1}W_2(R_F) & \longrightarrow & (\mathbb{Z}/2)^r & \to 0 \\ & \downarrow\cong & & \downarrow & & \downarrow\cong & \\ 0 \to & 8\mathbb{Z}^r & \longrightarrow & \oplus^r {}_{-1}W_2(\mathbb{R}) & \longrightarrow & (\mathbb{Z}/2)^r & \to 0 \end{array}$$

The 5-lemma now completes the proof. □

We finish this section by relating the numbers w_m in the formulation of Theorem 1.2 to t_n, the 2-adic valuation $(q^{(n+1)/2} - 1)_2$ of $q^{(n+1)/2} - 1$ for n odd.

Lemma 2.6: *Suppose that $q \equiv 1 \pmod 4$, and write $(-)_2$ for the 2-adic valuation. Then $(q^m - 1)_2 = (q - 1)_2(m)_2$.*

Proof. With $q = 4r + 1$ and $t = (m)_2$, note that $q^t - 1$ is divisible by $4rt$ but not by $8rt$, due to the binomial identity. Set $s = 8t(r)_2$ and $u = m/t$. Since $(\mathbb{Z}/2^s)^\times$ has even order and u is odd, $q^m - 1 = (q^t)^u - 1$ is divisible by $s/2$ but not by s. □

Lemma 2.7: *Let q be an odd number. If m is odd, then $(q^m - 1)_2 = (q - 1)_2$. If $m = 2m'$ is even, then $(q^m - 1)_2 = (q^2 - 1)_2(m')_2$.*

Proof. If m is odd, writing $q^m - 1 = (q-1)(q^{m-1} + \cdots + 1)$ shows that $(q^m - 1)_2 = (q-1)_2$ since $(q^{m-1} + \cdots + 1)$ is odd. If $m = 2m'$ is even , write $q^m - 1$ as $(q^2)^{m'} - 1$ where $q^2 \equiv 1 \pmod 4$, and apply the previous lemma. □

If $n \equiv 3, 7 \pmod 8$, then $t_n = (q^{(n+1)2} - 1)_2 = (q^2 - 1)_2[(n+1)/4]_2$ and for $q \equiv 1 \pmod 4$ we have $t_n = (q-1)_2[(n+1)/2]_2$. More precisely, $t_{8k+3} = 2(q-1)_2 = w_{4k+2}$ and $t_{8k+7} = (q-1)_2 \cdot 4 \cdot (k+1)_2 = w_{4k+4}$. In case $q \equiv 3 \pmod 4$, $t_n = (q+1)_2[(n+1)/2]_2$ and the same conclusion holds with $q+1$ instead of $q-1$.

3 Proof of Theorem 1.1

In this section, we take all spectra to be 2-completed, and all abelian group computations are modulo odd torsion.

Denote by ${}_{\varepsilon}\overline{\mathcal{KQ}}(R_F)$ the homotopy cartesian product of ${}_{\varepsilon}\mathcal{KQ}(\mathbb{F}_q)$ and $\vee^r{}_{\varepsilon}\mathcal{KQ}(\mathbb{R})$ over $\vee^r{}_{\varepsilon}\mathcal{KQ}(\mathbb{C})$, affording the homotopy cartesian square of hermitian K-theory connective spectra:

$$\begin{array}{ccc} {}_{\varepsilon}\overline{\mathcal{KQ}}(R_F) & \rightarrow & \bigvee^{r}{}_{\varepsilon}\mathcal{KQ}(\mathbb{R}) \\ \downarrow & & \downarrow \\ {}_{\varepsilon}\mathcal{KQ}(\mathbb{F}_q) & \rightarrow & \bigvee^{r}{}_{\varepsilon}\mathcal{KQ}(\mathbb{C}) \end{array}$$

(${}_{\varepsilon}\overline{\mathcal{KQ}}(R_F)$ is thereby connective because of the epimorphism ${}_{\varepsilon}KQ_0(\mathbb{R}) \twoheadrightarrow {}_{\varepsilon}KQ_0(\mathbb{C})$.) Thus, Theorem 1.1 becomes the assertion that there is a weak equivalence of connective spectra

$$ {}_{\varepsilon}\mathcal{KQ}(R_F) \rightarrow {}_{\varepsilon}\overline{\mathcal{KQ}}(R_F). \tag{4}$$

We write ${}_{\varepsilon}\overline{KQ}_n(R_F)$ for the integral homotopy groups of the target spectrum above, and note that by [4, (3.6)] all groups ${}_{\varepsilon}KQ_n(R_F)$ must be finitely generated.

We prove the theorem by means of the following argument. The low-dimensional computations in Theorem 3.1 below show that (4) induces an isomorphism (modulo odd torsion) on integral homotopy groups ${}_{\varepsilon}\pi_n$ for $n = -1, 0, 1$, and thereby [4], [27, pg. 32], on 2-completed homotopy groups ${}_{\varepsilon}\pi_n$ for $n = 0, 1$. When this is combined with the fact that the corresponding algebraic K-theory square in (2) is homotopy cartesian (which is shown in both [15] and [24], see also Appendix A), and the induction methods for hermitian K-groups in [4, §3], the weak equivalence in (4) follows by arguments given for the rational numbers in [4, §5]. In particular, the latter makes use of Brauer liftings in the context of hermitian K-theory and Borel's computation of the ranks of the finitely generated abelian groups ${}_{\varepsilon}KQ_n(R_F)$ [5]. We refer the reader to [4, §3,§5] for the methodology, especially the use of the machinery of Fredholm operators in an Hilbert space, an important analytic tool in the proof.

The next result extends the low-dimensional computations for the rational integers in [4, §4] to every totally real 2-regular number field.

Theorem 3.1: *Modulo odd torsion, the map of homotopy groups*

$$ {}_\varepsilon\pi_n\colon {}_\varepsilon KQ_n(R_F) \to {}_\varepsilon\overline{KQ}_n(R_F) $$

is an isomorphism for $\varepsilon = \pm 1$ *and* $n = -1, 0, 1$.

Proof. According to the homotopy cartesian square, there is a naturally induced long exact sequence

$$ \cdots \to \bigoplus^r {}_\varepsilon KQ_{n+1}(\mathbb{C}) \to {}_\varepsilon\overline{KQ}_n(R_F) \to {}_\varepsilon KQ_n(\mathbb{F}_q) \oplus \bigoplus^r {}_\varepsilon KQ_n(\mathbb{R}) \to \cdots . \tag{5} $$

We shall deal in turn with each of the six cases $\varepsilon = \pm 1$, $n = -1, 0, 1$, in Lemmas 3.2-3.7. □

Lemma 3.2: *The map* ${}_{-1}\pi_1\colon {}_{-1}KQ_1(R_F) \to {}_{-1}\overline{KQ}_1(R_F)$ *is an isomorphism.*

Proof. Recall that ${}_{-1}KQ_1(A) = 0$ when $A = R_F, \mathbb{F}_q, \mathbb{R}$ by for example [16, Théorème 2.13] (using [2]). Thus (5) shows that it suffices to note that ${}_{-1}KQ_2(\mathbb{C}) = \pi_1(\mathrm{Sp}) = 0$ for the infinite symplectic group Sp. □

Lemma 3.3: *The map* ${}_{-1}\pi_0\colon {}_{-1}KQ_0(R_F) \to {}_{-1}\overline{KQ}_0(R_F)$ *is an isomorphism.*

Proof. By [22, I (3.5)] there are isomorphisms ${}_{-1}KQ_0(A) \cong \mathbb{Z}$ when $A = R_F, \mathbb{F}_q, \mathbb{R}$, and $\mathbb{C}$ detected by the (even) rank of the free symplectic A-inner product space. Now since every ring map preserves the rank, we obtain a cartesian square:

$$ \begin{array}{ccc} {}_{-1}KQ_0(R_F) & \to & \bigoplus^r {}_{-1}KQ_0(\mathbb{R}) \\ \downarrow & & \downarrow \\ {}_{-1}KQ_0(\mathbb{F}_q) & \to & \bigoplus^r {}_{-1}KQ_0(\mathbb{C}) \end{array} \tag{6} $$

Combining (5), (6), and the fact that ${}_{-1}KQ_1(\mathbb{C})$ is the trivial group, it follows that ${}_{-1}\pi_0$ is an isomorphism. □

Lemma 3.4: *The map* ${}_1\pi_1\colon {}_1KQ_1(R_F) \to {}_1\overline{KQ}_1(R_F)$ *is an isomorphism.*

Proof. We first note that the determinant and the spinor norm of R_F induce an isomorphism

$$ {}_1KQ_1(R_F) \xrightarrow{\cong} R_F^\times/(R_F^\times)^2 \oplus \mathbb{Z}/2. \tag{7} $$

To see this, consider the exact sequence of units, discriminant modules, and Picard groups in [1, (2.1)]

$$0 \to \mu_2(F) \to R_F^\times \xrightarrow{(\ )^2} R_F^\times \to \mathrm{Discr}(R_F) \to \mathrm{Pic}(R_F) \xrightarrow{2\cdot} \mathrm{Pic}(R_F). \tag{8}$$

Combining (8) with the 2-regular assumption on F and the vanishing of $SK_1(R_F)$, the isomorphism in (7) follows from [1, (4.7.6)]. The group ${}_1KQ_2(\mathbb{R}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ surjects onto ${}_1KQ_2(\mathbb{C}) \cong \mathbb{Z}/2$. Hence by (5) there is an exact sequence

$$0 \to {}_1\overline{KQ}_1(R_F) \to {}_1KQ_1(\mathbb{F}_q) \oplus \bigoplus^r {}_1KQ_1(\mathbb{R}) \to \bigoplus^r {}_1KQ_1(\mathbb{C}) \to 0. \tag{9}$$

Thus, since ${}_1KQ_1(\mathbb{F}_q) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, ${}_1KQ_1(\mathbb{R}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and ${}_1KQ_1(\mathbb{C}) \cong \mathbb{Z}/2$, using (7) and (9), we deduce that ${}_1\overline{KQ}_1(R_F)$ and ${}_1KQ_1(R_F)$ are abstractly isomorphic to direct sums of $r+2$ copies of $\mathbb{Z}/2$. Therefore, to finish the proof of the lemma, it suffices to show that ${}_1KQ_1(R_F) \to {}_1\overline{KQ}_1(R_F)$ is surjective. Let $SKQ_1(A)$ denote the kernel of the determinant map ${}_1KQ_1(A) \to \mathbb{Z}/2$ for $A = R_F, \mathbb{F}_q, \mathbb{R}$ and $\mathbb{C}$. With this definition, the exact sequence (9) implies that there is a commutative diagram with exact rows:

$$\begin{array}{ccccc} K_1(R_F) & \to & K_1(\mathbb{F}_q) \oplus \bigoplus^r K_1(\mathbb{R}) & \to & 0 \\ \downarrow & & \downarrow & & \\ S\overline{KQ}_1(R_F) & \xrightarrow{\cong} & SKQ_1(\mathbb{F}_q) \oplus \bigoplus^r SKQ_1(\mathbb{R}) & & \end{array}$$

Because the right vertical map is surjective, so too is $K_1(R_F) \to S\overline{KQ}_1(R_F)$, which factors through $SKQ_1(R_F)$. Therefore, the left vertical map in the commutative diagram of split short exact sequences

$$\begin{array}{ccccccc} 0 \to & SKQ_1(R_F) & \to & {}_1KQ_1(R_F) & \xrightarrow{\det} & \mathbb{Z}/2 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow^{\gamma} & \\ 0 \to & S\overline{KQ}_1(R_F) & \to & {}_1\overline{KQ}_1(R_F) & \longrightarrow & G & \to 0 \end{array}$$

is also surjective. On the other hand, combining (9) with the computations above gives a short exact sequence

$$0 \to G \to \mathbb{Z}/2 \oplus (\mathbb{Z}/2)^r \to (\mathbb{Z}/2)^r \to 0.$$

This shows that G has order 2. Now since the forgetful map ${}_1\overline{KQ}_1(R_F) \to \overline{K}_1(R_F) \cong K_1(R_F)$, cf. [15], [24], induces an injection of G into the units of R_F it follows that the composite map $\mathbb{Z}/2 \xrightarrow{\gamma} G \to R_F^\times$ is injective. Hence γ is an isomorphism. Applying the snake lemma we deduce the required surjection.

□

Lemma 3.5: *The map* ${}_1\pi_0\colon {}_1KQ_0(R_F) \to {}_1\overline{KQ}_0(R_F)$ *is an isomorphism.*

Proof. Since $\mathrm{Pic}(A)$ is an odd torsion group we obtain for $A = R_F, \mathbb{F}_q, \mathbb{R}$, and $\mathbb{C}$ the short exact sequence (as always modulo odd torsion)

$$0 \to W'(A) \to {}_1KQ_0(A) \to K_0(A) \to 0. \tag{10}$$

For the last three of these four cases we obtain the vertical map of short exact sequences:

$$\begin{array}{ccccccccc} 0 \to & W'(\mathbb{F}_q) \oplus \bigoplus^r W'(\mathbb{R}) & \to & {}_1KQ_0(\mathbb{F}_q) \oplus \bigoplus^r {}_1KQ_0(\mathbb{R}) & \to & K_0(\mathbb{F}_q) \oplus \bigoplus^r K_0(\mathbb{R}) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \bigoplus^r W'(\mathbb{C}) & \to & \bigoplus^r {}_1KQ_0(\mathbb{C}) & \to & \bigoplus^r K_0(\mathbb{C}) & \to 0 \end{array}$$

By [15] and [24], $K_0(R_F)$ is the kernel of the rightmost vertical map. Further, the bottom right horizontal epimorphism maps between two copies of $\mathbb{Z}^r$, making the coWitt group $W'(\mathbb{C})$ trivial. As already noted, ${}_1KQ_1(\mathbb{R})$ maps onto ${}_1KQ_1(\mathbb{C})$, making ${}_1\overline{KQ}_0(R_F)$ the kernel of the center vertical map. Therefore, the desired isomorphism between ${}_1KQ_0(R_F)$ and ${}_1\overline{KQ}_0(R_F)$ follows by combining the short exact sequence (10) for $A = R_F$ and Proposition 2.5. □

For the cases $n = -1$ with $\varepsilon = \pm 1$, following the pattern in [4, pg. 799] we show that the map $R_F \to \mathbb{F}_q$ induces an isomorphism

$${}_\varepsilon KQ_{-1}(R_F) \xrightarrow{\cong} {}_\varepsilon KQ_{-1}(\mathbb{F}_q).$$

As in [4], we exploit the two exact sequences (for $A = R_F$ and $\mathbb{F}_q$)

$$K_1(A) \longrightarrow {}_{-\varepsilon}KQ_1(A) \longrightarrow {}_{-\varepsilon}U_0(A) \longrightarrow K_0(A) \longrightarrow {}_{-\varepsilon}KQ_0(A),$$

and (via the fundamental theorem proved in [18])

$${}_\varepsilon KQ_0(A) \xrightarrow{\varphi} K_0(A) \longrightarrow {}_{-\varepsilon}U_0(A) \longrightarrow {}_\varepsilon KQ_{-1}(A) \longrightarrow K_{-1}(A) = 0.$$

Lemma 3.6: *The map* ${}_1\pi_{-1}\colon {}_1KQ_{-1}(R_F) \to {}_1\overline{KQ}_{-1}(R_F)$ *is an isomorphism between trivial groups modulo odd torsion.*

Proof. For $\varepsilon = 1$, ${}_{-\varepsilon}KQ_1(A) = 0$ and the map $K_0(A) \to {}_{-\varepsilon}KQ_0(A)$ is injective. Hence ${}_{-\varepsilon}U_0(A) = 0$ and it follows from the second exact sequence above that ${}_{\varepsilon}KQ_{-1}(A)$ is the trivial group. □

Lemma 3.7: *The map* ${}_{-1}\pi_{-1}\colon {}_{-1}KQ_{-1}(R_F) \to {}_{-1}\overline{KQ}_{-1}(R_F)$ *is an isomorphism between trivial groups modulo odd torsion.*

Proof. For $\varepsilon = -1$, we can compare the first exact sequence above for $A = R_F$ and $\mathbb{F}_q$, using the previous computations. We get the following diagram (modulo odd torsion):

$$\begin{array}{ccccccccc}
\mathbb{Z}^r \oplus \mathbb{Z}/2 & \xrightarrow{\alpha} & (\mathbb{Z}/2)^{r+2} & \longrightarrow & {}_1U_0(R_F) & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^r \oplus \mathbb{Z}/2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{F}_q^\times & \xrightarrow{\beta} & (\mathbb{Z}/2)^2 & \longrightarrow & {}_1U_0(\mathbb{F}_q) & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}/2
\end{array}$$

The cokernel of β, induced by the hyperbolic map, is $\mathbb{Z}/2$. A generator is just the 1×1 matrix (-1). According to the computations made for the proof of Lemma 3.4, the same can be said for $\mathrm{Coker}\alpha$. Since the last horizontal maps are injective, we obtain the isomorphisms ${}_1U_0(R_F) \cong {}_1U_0(\mathbb{F}_q) \cong \mathbb{Z}/2$. At this stage, we can use the second exact sequence

$$ {}_{-1}KQ_0(A) \xrightarrow{\varphi} K_0(A) \longrightarrow {}_1U_0(A) \longrightarrow {}_{-1}KQ_{-1}(A) \longrightarrow K_{-1}(A) = 0$$

mentioned above for $A = R_F$ and $\mathbb{F}_q$. As before, by [22, I (3.5)] there are isomorphisms ${}_{-1}KQ_0(A) \cong \mathbb{Z}$ detected by the (even) rank of the free symplectic A-inner product space. Modulo odd torsion, $K_0(A)$ is then infinite cyclic. We claim that the subgroup $2\mathbb{Z}$ of $K_0(A)$ generated by the trivial hyperbolic module of rank 2 comprises the image of φ. This follows by comparing with the fraction field F of A (where again ${}_{-1}KQ_0(F) \cong \mathbb{Z}$), using the commutative diagram:

$$\begin{array}{ccc}
{}_{-1}KQ_0(A) & \xrightarrow{\varphi} & K_0(A) \\
\downarrow & & \downarrow \\
{}_{-1}KQ_0(F) & \longrightarrow & K_0(F)
\end{array}$$

The right vertical map is an isomorphism modulo odd torsion, and the lower horizontal map identifies with the multiplication by 2 map on the integers. Hence $K_0(A) \to {}_1U_0(A)$ is surjective and ${}_{-1}KQ_{-1}(A)$ is the trivial group. Moreover, modulo odd torsion, there is an isomorphism ${}_1\overline{KQ}_{-1}(R_F) \cong {}_{-1}KQ_{-1}(\mathbb{F}_q)$, and the latter is trivial according to the computations above. □

For completeness, we note that the computations in Theorem 3.1 have consequences for the odd torsion subgroups of Witt groups, coWitt groups, and hermitian K-groups of R_F as shown in [4, 6.1] for the rational numbers. The next corollaries follow by comparing with the splittings of the hermitian K-theory spectrum and the K-theory spectrum according to their canonical involutions [18, pg. 253].

Proposition 3.8: *The Witt and coWitt groups of R_F have trivial odd torsion subgroups.*

Proposition 3.9: *The odd torsion subgroup of ${}_\varepsilon KQ_n(R_F)$ is the invariant part of the odd torsion subgroup of $K_n(R_F)$ induced by the involution $M \mapsto {}^tM^{-1}$ on $\mathrm{GL}(R_F)$.*

4 Splitting results

The purpose of this section is to prove some generic splitting results employed in the proofs of Theorems 1.2 and 1.4.

To start with, fix some residue field $\mathbb{F}_q$ of R_F as in Theorem 1.1, and define the connective spectrum ${}_\varepsilon\mathcal{KQ}(\overline{R}_F)$ (this is just a convenient notation) by the homotopy cartesian square:

$$\begin{array}{ccc} {}_\varepsilon\mathcal{KQ}(\overline{R}_F) & \rightarrow & {}_\varepsilon\mathcal{KQ}(\mathbb{R}) \\ \downarrow & & \downarrow \\ {}_\varepsilon\mathcal{KQ}(\mathbb{F}_q) & \rightarrow & {}_\varepsilon\mathcal{KQ}(\mathbb{C}) \end{array}$$

Likewise, define ${}_\varepsilon\mathcal{V}(\overline{R}_F)$ by the homotopy cartesian square:

$$\begin{array}{ccc} {}_\varepsilon\mathcal{V}(\overline{R}_F) & \rightarrow & {}_\varepsilon\mathcal{V}(\mathbb{R}) \\ \downarrow & & \downarrow \\ {}_\varepsilon\mathcal{V}(\mathbb{F}_q) & \rightarrow & {}_\varepsilon\mathcal{V}(\mathbb{C}) \end{array}$$

The first homotopy cartesian square can be recast as a homotopy fiber sequence

$${}_\varepsilon\mathcal{KQ}(\overline{R}_F) \;\rightarrow\; {}_\varepsilon\mathcal{KQ}(\mathbb{R}) \vee {}_\varepsilon\mathcal{KQ}(\mathbb{F}_q) \;\rightarrow\; {}_\varepsilon\mathcal{KQ}(\mathbb{C}). \tag{11}$$

In the next step we form the naturally induced diagram with horizontal homotopy fiber sequences:

$$\begin{array}{ccccc}
{}_{\varepsilon}\mathcal{KQ}(\overline{R}_F) & \rightarrow & {}_{\varepsilon}\mathcal{KQ}(\mathbb{R}) \vee {}_{\varepsilon}\mathcal{KQ}(\mathbb{F}_q) & \rightarrow & {}_{\varepsilon}\mathcal{KQ}(\mathbb{C}) \\
\downarrow & & \downarrow{\scriptstyle \nabla\vee \mathrm{id}} & & \downarrow{\scriptstyle \nabla} \\
{}_{\varepsilon}\mathcal{KQ}(R_F) & \rightarrow & \bigvee^r {}_{\varepsilon}\mathcal{KQ}(\mathbb{R}) \vee {}_{\varepsilon}\mathcal{KQ}(\mathbb{F}_q) & \rightarrow & \bigvee^r {}_{\varepsilon}\mathcal{KQ}(\mathbb{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\bigvee^{r-1} {}_{\varepsilon}\mathcal{F} & \rightarrow & \bigvee^{r-1} {}_{\varepsilon}\mathcal{KQ}(\mathbb{R}) & \rightarrow & \bigvee^{r-1} {}_{\varepsilon}\mathcal{KQ}(\mathbb{C})
\end{array} \tag{12}$$

The two right-hand columns are clearly split. Moreover, since all rows are homotopy fiber sequences the same holds for the left-hand column, cf. [8, Lemma 2.1]. Using the compatibility of the two evident splittings we obtain a splitting of ${}_{\varepsilon}\mathcal{KQ}(\overline{R}_F) \rightarrow {}_{\varepsilon}\mathcal{KQ}(R_F)$.

In ${}_1\mathcal{KQ}$-theory, we make use of the following fact.

Lemma 4.1: *There is a homotopy split fibration of spectra*

$$\mathcal{K}(\mathbb{R}) \longrightarrow {}_1\mathcal{KQ}(\mathbb{R}) \longrightarrow {}_1\mathcal{KQ}(\mathbb{C}) = \mathcal{K}(\mathbb{R}).$$

This is immediate from the identification of ${}_1\mathcal{KQ}(\mathbb{R})$ with $\mathcal{K}(\mathbb{R}) \vee \mathcal{K}(\mathbb{R})$, where the map ${}_1\mathcal{KQ}(\mathbb{R}) \rightarrow {}_1\mathcal{KQ}(\mathbb{C})$ is induced by the Whitney sum of real vector bundles, *cf.* the elaboration of Sylvester's classical theorem on real vector spaces in [17, Exercise 9.22].

When applied to (12), this yields the following.

Lemma 4.2:

$${}_1\mathcal{KQ}(R_F) \simeq {}_1\mathcal{KQ}(\overline{R}_F) \vee \bigvee^{r-1} \mathcal{K}(\mathbb{R}).$$

Next, the fiber ${}_{-1}\mathcal{F}$ of ${}_{-1}\mathcal{KQ}(\mathbb{R}) = \mathcal{K}(\mathbb{C}) \rightarrow {}_{-1}\mathcal{KQ}(\mathbb{C}) = \mathcal{K}\mathrm{Sp}(\mathbb{C})$ identifies with the fiber $\Omega^6\mathcal{K}(\mathbb{R})$ of $\Omega^4\mathcal{K}(\mathbb{C}) \rightarrow \Omega^4\mathcal{K}(\mathbb{R})$.

Lemma 4.3:

$${}_{-1}\mathcal{KQ}(R_F) \simeq {}_{-1}\mathcal{KQ}(\overline{R}_F) \vee \bigvee^{r-1} \Omega^6\mathcal{K}(\mathbb{R}).$$

In order to obtain the diagram in ${}_{\varepsilon}\mathcal{V}$-theory corresponding to (12), we start with the diagram

$$\begin{array}{ccccc} {}_\varepsilon\mathcal{V}(R_F) & \to & {}_\varepsilon\mathcal{V}(\mathbb{R}) \vee {}_\varepsilon\mathcal{V}(\mathbb{F}_q) & \to & {}_\varepsilon\mathcal{V}(\mathbb{C}) \\ \downarrow & & \downarrow{\scriptstyle \nabla\vee\mathrm{id}} & & \downarrow{\scriptstyle\nabla} \\ {}_\varepsilon\mathcal{KQ}(R_F) & \to & \bigvee^r {}_\varepsilon\mathcal{KQ}(\mathbb{R}) \vee {}_\varepsilon\mathcal{KQ}(\mathbb{F}_q) & \to & \bigvee^r {}_\varepsilon\mathcal{KQ}(\mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}(R_F) & \to & \bigvee^{r-1} \mathcal{K}(\mathbb{R}) \vee \mathcal{K}(\mathbb{F}_q) & \to & \bigvee^{r-1} \mathcal{K}(\mathbb{C}) \end{array} \tag{13}$$

in which by definition all three columns are fiber sequences; by Theorem 1.1 and its counterpart in algebraic K-theory (2), the lower two rows are fiber sequences, and hence the top row is as well [8, Lemma 2.1]. Thereby we have:

$$\begin{array}{ccccc} {}_\varepsilon\mathcal{V}(\overline{R}_F) & \to & {}_\varepsilon\mathcal{V}(\mathbb{R}) \vee {}_\varepsilon\mathcal{V}(\mathbb{F}_q) & \to & {}_\varepsilon\mathcal{V}(\mathbb{C}) \\ \downarrow & & \downarrow{\scriptstyle \nabla\vee\mathrm{id}} & & \downarrow{\scriptstyle\nabla} \\ {}_\varepsilon\mathcal{V}(R_F) & \to & \bigvee^r {}_\varepsilon\mathcal{V}(\mathbb{R}) \vee {}_\varepsilon\mathcal{V}(\mathbb{F}_q) & \to & \bigvee^r {}_\varepsilon\mathcal{V}(\mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \\ \bigvee^{r-1} {}_\varepsilon\mathcal{G} & \to & \bigvee^{r-1} {}_\varepsilon\mathcal{V}(\mathbb{R}) & \to & \bigvee^{r-1} {}_\varepsilon\mathcal{V}(\mathbb{C}) \end{array} \tag{14}$$

Lemma 4.4: *The fiber ${}_\varepsilon\mathcal{G}$ of ${}_\varepsilon\mathcal{V}(\mathbb{R}) \to {}_\varepsilon\mathcal{V}(\mathbb{C})$ is*

$$\begin{cases} \mathcal{K}(\mathbb{R}) \vee \mathcal{K}(\mathbb{R}) & \varepsilon = 1, \\ \mathcal{K}(\mathbb{C}) & \varepsilon = -1. \end{cases}$$

In both cases we use the description of ${}_\varepsilon\mathcal{V}(\mathbb{R}) \to {}_\varepsilon\mathcal{V}(\mathbb{C})$ as a $K_*(\mathbb{R})$-module map given in Appendix B.

In ${}_1\mathcal{V}$-theory, this map is easily seen to be nullhomotopic, as follows. Since it is a $K_*(\mathbb{R})$-module map, it is induced by the cup-product with an element in $K_{-1}(\mathbb{R})$. But $K_{-1}(\mathbb{R})$ is the trivial group. This observation allows us to identify ${}_1\mathcal{G}$ with the wedge sum $\mathcal{K}(\mathbb{R}) \vee \mathcal{K}(\mathbb{R})$.

For ${}_{-1}\mathcal{V}$, note that ${}_{-1}\mathcal{V}(\mathbb{R})$ is the fiber of ${}_{-1}\mathcal{KQ}(\mathbb{R}) = \mathcal{K}(\mathbb{C}) \to \mathcal{K}(\mathbb{R})$ and so identifies with $O/U = \Omega^2\mathcal{K}(\mathbb{R})$. Meanwhile, as the fiber of ${}_{-1}\mathcal{KQ}(\mathbb{C}) \to \mathcal{K}(\mathbb{C})$, ${}_{-1}\mathcal{V}(\mathbb{C})$ identifies with the fiber of $B\mathrm{Sp} \to BU$, or equally with that of $\Omega^4 BO \to \Omega^4 BU$, namely $\Omega^4(U/O) = \Omega^3\mathcal{K}(\mathbb{R})$. The $K_*(\mathbb{R})$-module map ${}_{-1}\mathcal{V}(\mathbb{R}) \to {}_{-1}\mathcal{V}(\mathbb{C})$ is either trivial or induced by the cup-product with the nontrivial element in $K_1(\mathbb{R})$. The simplest way to resolve this ambiguity is to apply the fundamental theorem of hermitian K-theory [18] and consider the induced map ${}_1\mathcal{U}(\mathbb{R}) \to {}_1\mathcal{U}(\mathbb{C})$ in ${}_1\mathcal{U}$-theory. More precisely, we form the diagram:

$$\begin{array}{ccccccccc} K_1(\mathbb{R}) & \to & {}_1KQ_1(\mathbb{R}) & \to & {}_1U_0(\mathbb{R}) & \to & K_0(\mathbb{R}) & \to & {}_1KQ_0(\mathbb{R}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_1(\mathbb{C}) & \to & {}_1KQ_1(\mathbb{C}) & \to & {}_1U_0(\mathbb{C}) & \to & K_0(\mathbb{C}) & \to & {}_1KQ_0(\mathbb{C}) \end{array}$$

Since by inspection the composite

$$\begin{aligned} {}_1KQ_1(\mathbb{R}) = K_1(\mathbb{R}) \oplus K_1(\mathbb{R}) & \longrightarrow {}_1KQ_1(\mathbb{C}) = K_1(\mathbb{R}) \\ & \longrightarrow \mathrm{Coker}[K_1(\mathbb{C}) \to {}_1KQ_1(\mathbb{C})] \end{aligned}$$

is nontrivial, it follows that ${}_1U_0(\mathbb{R}) \to {}_1U_0(\mathbb{C})$ is also nontrivial. Thus we obtain a nontrivial fibration

$$\Omega^3\mathcal{K}(\mathbb{R}) \longrightarrow \Omega^4\mathcal{K}(\mathbb{R}) \longrightarrow {}_{-1}\mathcal{G},$$

which corresponds to $U/\mathrm{Sp} \longrightarrow B\mathrm{Sp} \longrightarrow BU$, leaving ${}_{-1}\mathcal{G}$ as $\mathcal{K}(\mathbb{C})$.

Lemma 4.5:

$${}_1\mathcal{V}(R_F) \simeq {}_1\mathcal{V}(\overline{R}_F) \vee \bigvee^{r-1} \mathcal{K}(\mathbb{R}) \vee \bigvee^{r-1} \mathcal{K}(\mathbb{R}).$$

Lemma 4.6:

$${}_{-1}\mathcal{V}(R_F) \simeq {}_{-1}\mathcal{V}(\overline{R}_F) \vee \bigvee^{r-1} \mathcal{K}(\mathbb{C}).$$

It remains to compute explicitly the groups ${}_\varepsilon KQ_n$ and ${}_\varepsilon V_n$ of $\overline{R}_F$. We note that the same type of splitting result holds for the algebraic K-groups of R_F.

5 Proof of Theorem 1.2

The splitting results Lemma 4.2 and Lemma 4.3 in §4 show that in order to prove Theorem 1.2, it suffices to compute the groups ${}_\varepsilon KQ_n(\overline{R}_F)$, and then sum with $r-1$ copies of the well-known K-groups of $\mathbb{R}$. We formulate the computation in terms of the numbers t_m introduced at the end of §2.

Theorem 5.1: *Up to finite groups of odd order, the groups ${}_\varepsilon KQ_n(\overline{R}_F)$ are given in Table 3. (Recall that δ_{n0} denotes the Kronecker symbol.)*

Table 3:

$n \geq 0$	${}_{-1}KQ_n(\overline{R}_F)$	${}_1KQ_n(\overline{R}_F)$
$8k$	$\delta_{n0}\mathbb{Z}$	$\delta_{n0}\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$
$8k{+}1$	0	$(\mathbb{Z}/2)^3$
$8k{+}2$	$\mathbb{Z}$	$(\mathbb{Z}/2)^2$
$8k{+}3$	$\mathbb{Z}/2t_{8k+3}$	$\mathbb{Z}/t_{8k+3}$
$8k{+}4$	$\mathbb{Z}/2$	$\mathbb{Z}$
$8k{+}5$	$\mathbb{Z}/2$	0
$8k{+}6$	$\mathbb{Z}$	0
$8k{+}7$	$\mathbb{Z}/t_{8k+7}$	$\mathbb{Z}/t_{8k+7}$

Proof. Throughout the proof, we exploit Friedlander's computation of ${}_\varepsilon KQ_n(\mathbb{F}_q)$ given in [12, Theorem 1.7].

$\varepsilon = 1$. Applying Lemma 4.1 to the homotopy fiber sequence (11) gives for each n a split short exact sequence

$$0 \to {}_1KQ_n(\overline{R}_F) \longrightarrow K_n(\mathbb{R}) \oplus K_n(\mathbb{R}) \oplus {}_1KQ_n(\mathbb{F}_q) \longrightarrow K_n(\mathbb{R}) \to 0.$$

From this splitting we deduce an isomorphism

$${}_1KQ_n(\overline{R}_F) \cong K_n(\mathbb{R}) \oplus {}_1KQ_n(\mathbb{F}_q).$$

$\varepsilon = -1$. For the computation of ${}_{-1}\mathcal{KQ}(\overline{R}_F)$ we need to make several case distinctions arising from the homotopy cartesian square:

$$\begin{array}{ccc} {}_{-1}\mathcal{KQ}(\overline{R}_F) & \longrightarrow & {}_{-1}\mathcal{KQ}(\mathbb{R}) = \mathcal{K}(\mathbb{C}) \approx \Omega^4\mathcal{K}(\mathbb{C}) \\ \downarrow & & \downarrow \\ {}_{-1}\mathcal{KQ}(\mathbb{F}_q) & \longrightarrow & {}_{-1}\mathcal{KQ}(\mathbb{C}) = \mathcal{K}(\mathbb{H}) \approx \Omega^4\mathcal{K}(\mathbb{R}) \end{array}$$

Note that the vertical homotopy fiber is $\Omega^6\mathcal{K}(\mathbb{R})$, while the horizontal homotopy fiber is $\Omega^5\mathcal{K}(\mathbb{R})$. In particular, there is the "vertical" exact sequence

$$\cdots \to {}_{-1}KQ_{n+1}(\mathbb{F}_q) \to K_{n+6}(\mathbb{R}) \to {}_{-1}KQ_n(\overline{R}_F) \to {}_{-1}KQ_n(\mathbb{F}_q) \to K_{n+5}(\mathbb{R}) \to \cdots, \tag{15}$$

and the "horizontal" exact sequence

$$\cdots \to K_{n+5}(\mathbb{C}) \to K_{n+5}(\mathbb{R}) \to {}_{-1}KQ_n(\overline{R}_F) \to K_{n+4}(\mathbb{C}) \to K_{n+4}(\mathbb{R}) \to \cdots . \quad (16)$$

If $n \equiv 0, 1 \pmod 8$ is nonzero, then (15) implies ${}_{-1}KQ_n(\overline{R}_F) = 0$. Likewise, when $n \equiv 2 \pmod 8$, (15) shows that ${}_{-1}KQ_n(\overline{R}_F) \cong K_8(\mathbb{R}) \cong \mathbb{Z}$.

For $n \equiv 4 \pmod 8$, we use the segment

$$K_{n+5}(\mathbb{C}) \;\longrightarrow\; K_{n+5}(\mathbb{R}) \;\longrightarrow\; {}_{-1}KQ_n(\overline{R}_F) \;\longrightarrow\; K_{n+4}(\mathbb{C})$$

of (16). By analyzing (15) (or applying Borel's computations on the K-theory of S-integers in a number field or the fundamental theorem in hermitian K-theory), it follows that ${}_{-1}KQ_n(R_F)$ and ${}_{-1}KQ_n(\overline{R}_F)$ are finite. Thus ${}_{-1}KQ_n(\overline{R}_F)$ has order 2.

For $n \equiv 5 \pmod 8$, (16) implies that ${}_{-1}KQ_n(\overline{R}_F)$ is cyclic, whence by (15) there is an isomorphism ${}_{-1}KQ_n(\overline{R}_F) \cong \mathbb{Z}/2$.

For $n \equiv 3 \pmod 8$, we use the exact sequence (15) from ${}_{-1}KQ_{n+2}(\overline{R}_F)$ to $K_{n+5}(\mathbb{R})$. In view of the two previous results, this takes the form

$$\mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to {}_{-1}KQ_n(\overline{R}_F) \to \mathbb{Z}/t_n \to \mathbb{Z}.$$

Chasing this sequence from the left reveals that ${}_{-1}KQ_n(\overline{R}_F)$ is a finite group of order $2t_n$. Meanwhile, the exact sequence (16) obliges this group to be cyclic.

For $n \equiv 6 \pmod 8$, (15) implies that ${}_{-1}KQ_n(\overline{R}_F) \cong \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$. On the other hand, the exact sequence (16) shows that ${}_{-1}KQ_n(\overline{R}_F)$ is a subgroup of $\mathbb{Z}$.

Finally, if $n \equiv 7 \pmod 8$ then (15) produces an isomorphism between ${}_{-1}KQ_n(\overline{R}_F)$ and ${}_{-1}KQ_n(\mathbb{F}_q) \cong \mathbb{Z}/t_n$. □

We can now prove Corollary 1.3.

Corollary 5.2: *For a totally real number field F the following are equivalent.*

(i) F is 2-regular.

(ii) Theorem 1.1 holds for ${}_1\mathcal{KQ}(R_F)$.

(iii) The counterpart of Theorem 1.1 holds for $\mathcal{K}(R_F)$ (as in the cartesian square (2)).

Proof. Theorem 1.1 of course asserts that (i) implies (ii). In the other direction, the preceding proof shows that (ii) leads to the second column of Table 3. Since the hyperbolic map

$$\mathbb{Z} \cong K_0(R_F) \longrightarrow {}_1KQ_0(R_F) \cong \mathbb{Z} \oplus \mathbb{Z}^r \oplus \mathbb{Z}/2$$

is determined by ranks, it is a split injection. This makes its cokernel $W(R_F)$ isomorphic to $\mathbb{Z}^r \oplus \mathbb{Z}/2$, up to odd-order torsion. However, $W(R_F)$ injects into $W(F)$ [22, IV (3.3)], which has no odd-order torsion [22, III (3.10)]. Thus, $W(R_F) \cong \mathbb{Z}^r \oplus \mathbb{Z}/2$. Then by Proposition 2.1(1), (7), it follows that F is 2-regular.

Similarly, the K-theoretic theorem of [15], [24] and [26] in case (i) asserts that (i) implies (iii). Again, the computation $K_2(R_F)\{2\} \cong (\mathbb{Z}/2)^r$ follows from (iii). Here, Proposition 2.1(1), (2) yield that F is 2- regular. □

6 Proof of Theorem 1.4

As for Theorem 1.2, the splitting results proven in §4 show that in order to prove Theorem 1.4, it suffices to compute the groups ${}_\varepsilon V_n(\overline{R}_F)$ introduced in the same section.

Theorem 6.1: *Up to finite groups of odd order, the groups*

$${}_\varepsilon V_n(\overline{R}_F) := \pi_n({}_\varepsilon \mathcal{V}(\overline{R}_F))$$

are given in Table 4.

Table 4:

$n \geq 0$	${}_{-1}V_n(\overline{R}_F)$	${}_1V_n(\overline{R}_F)$
$8k$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}^2$
$8k+1$	0	$(\mathbb{Z}/2)^2$
$8k+2$	$\mathbb{Z}$	$(\mathbb{Z}/2)^2$
$8k+3$	0	0
$8k+4$	$\mathbb{Z}$	$\mathbb{Z}^2$
$8k+5$	$\mathbb{Z}/2$	0
$8k+6$	$\mathbb{Z} \oplus \mathbb{Z}/2$	0
$8k+7$	$\mathbb{Z}/2$	0

Proof. In the homotopy cartesian square

$$\begin{array}{ccc} {}_1\mathcal{V}(\overline{R}_F) & \longrightarrow & {}_1\mathcal{V}(\mathbb{R}) = \mathcal{K}(\mathbb{R}) \\ \downarrow & & \downarrow\sigma \\ {}_1\mathcal{V}(\mathbb{F}_q) & \longrightarrow & {}_1\mathcal{V}(\mathbb{C}) = \Omega^{-1}\mathcal{K}(\mathbb{R}) \end{array}$$

as noted in proving Lemma 4.4 above, the map σ is nullhomotopic. Therefore the upper horizontal fibration is homotopy trivial, with fiber identified in [12, Corollary 1.6] as $\mathcal{K}(\mathbb{R})$. Hence, the spectrum ${}_1\mathcal{V}(R_F)$ has the homotopy type of the wedge sum $\vee^{2r}\mathcal{K}(\mathbb{R})$.

In the homotopy cartesian square

$$\begin{array}{ccc} {}_{-1}\mathcal{V}(\overline{R}_F) & \longrightarrow & {}_{-1}\mathcal{V}(\mathbb{R}) = \Omega^2\mathcal{K}(\mathbb{R}) \\ \downarrow & & \downarrow\tau \\ {}_{-1}\mathcal{V}(\mathbb{F}_q) & \longrightarrow & {}_{-1}\mathcal{V}(\mathbb{C}) = \Omega^3\mathcal{K}(\mathbb{R}) \end{array}$$

from the proof of Lemma 4.4 again, the map τ is determined by the cup-product with the generator of $K_1(\mathbb{R})$. In the associated "horizontal" exact sequence

$$K_{n+4}(\mathbb{R}) \longrightarrow {}_{-1}V_n(\overline{R}_F) \longrightarrow K_{n+2}(\mathbb{R}) \longrightarrow K_{n+3}(\mathbb{R}),$$

the last map is trivial because it is given by the cup-product with the generator of $K_1(\mathbb{R})$ composed with $\Psi^q - 1$. By a general argument for cohomology theories (specifically, the identity map on $\Omega^2\mathcal{K}(\mathbb{R})$ is obliged to lift to ${}_{-1}\mathcal{V}(\overline{R}_F)$), we deduce that ${}_{-1}\mathcal{V}(\overline{R}_F)$ splits into the wedge $\Omega^4\mathcal{K}(\mathbb{R})\vee\Omega^2\mathcal{K}(\mathbb{R})$. The computation for $\varepsilon = -1$ follows immediately. ☐

We can now use Theorems 1.2 and 1.4 to determine the composition

$${}_\varepsilon KQ_n(R_F) \xrightarrow{F} K_n(R_F) \xrightarrow{H} {}_\varepsilon KQ_n(R_F)$$

of the homomorphisms induced by the forgetful and hyperbolic functors. From their respective induced homotopy fiber sequences

$${}_\varepsilon\mathcal{V}(R_F) \longrightarrow {}_\varepsilon\mathcal{KQ}(R_F) \longrightarrow \mathcal{K}(R_F)$$

and

$${}_\varepsilon\mathcal{U}(R_F) \longrightarrow \mathcal{K}(R_F) \longrightarrow {}_\varepsilon\mathcal{KQ}(R_F),$$

and the natural weak equivalence

$$ {}_{\varepsilon}\mathcal{V}(R_F) \simeq \Omega_{-\varepsilon}\mathcal{U}(R_F) $$

of [18], we have the exact sequences

$$ \cdots \to {}_{\varepsilon}V_n(R_F) \longrightarrow {}_{\varepsilon}KQ_n(R_F) \xrightarrow{F} K_n(R_F) \longrightarrow {}_{\varepsilon}V_{n-1}(R_F) \to \cdots , $$

and

$$ \cdots \to {}_{-\varepsilon}V_{n-1}(R_F) \longrightarrow K_n(R_F) \xrightarrow{H} {}_{\varepsilon}KQ_n(R_F) \longrightarrow {}_{-\varepsilon}V_{n-2}(R_F) \to \cdots . $$

Since all terms are now known (and many are zero), a routine computation gives the following.

Corollary 6.2: *For $n \geq 1$, the endomorphism HF of ${}_{\varepsilon}KQ_n(R_F)$ is multiplication by 2 when $n \equiv 3 \pmod 4$, has image of order 2 when both $\varepsilon = 1$ and $n \equiv 1, 2 \pmod 8$, and is zero otherwise.*

The relation between this endomorphism and the natural involution on ${}_{\varepsilon}KQ_n(R_F)$ is discussed in Appendix C below.

A similar computation affords the corresponding result for the other composition of F and H. From [19, pg. 230] we note that this endomorphism of $K_n(R_F)$ is the sum of the identity and the involution induced by the duality functor. Since this involution is independent of ε, we need consider only the simpler case $\varepsilon = -1$.

Corollary 6.3: *For $n \geq 1$, the endomorphism FH of $K_n(R_F)$ is (up to odd torsion) multiplication by 2 when $n \equiv 3 \pmod 4$, and is zero otherwise.*

Corollary 6.4: *The canonical involution on $K_n(R_F)$ is (up to odd torsion) the identity for $n = 0$ and $n \equiv 3 \pmod 4$, and the opposite of the identity otherwise.*

Remark 6.5. Concerning the odd torsion, the functors F and H induce bijections between the symmetric parts of the K- and KQ-groups of A. Here the involution on the K-groups is induced by the duality functor. Clearly the composition FH is the multiplication by 2 map on the symmetric part. The same result holds for HF on the symmetric part, while it is trivial on the antisymmetric part. We note that, even for the case of the rational numbers, it remains to compute the odd torsion part of ${}_{\varepsilon}KQ_n(R_F)$.

7 Proof of Theorem 1.5

Consider the naturally induced map:

$$\begin{array}{ccccccc} {}_{\varepsilon}\mathcal{KQ}(R_F) & \rightarrow & \bigvee^{r}{}_{\varepsilon}\mathcal{KQ}(\mathbb{R}) & & \mathcal{K}(R_F)^{h(\varepsilon\mathbb{Z}/2)} & \rightarrow & \bigvee^{r}\mathcal{K}(\mathbb{R})^{h(\varepsilon\mathbb{Z}/2)} \\ \downarrow & & \downarrow & \rightarrow & \downarrow & & \downarrow \\ {}_{\varepsilon}\mathcal{KQ}(\mathbb{F}_q) & \rightarrow & \bigvee^{r}{}_{\varepsilon}\mathcal{KQ}(\mathbb{C}) & & \mathcal{K}(\mathbb{F}_q)^{h(\varepsilon\mathbb{Z}/2)} & \rightarrow & \bigvee^{r}\mathcal{K}(\mathbb{C})^{h(\varepsilon\mathbb{Z}/2)} \end{array} \tag{17}$$

Theorem 1.1 and [15], [24] show that both the hermitian and algebraic K-theory squares are homotopy cartesian squares. By [4, Lemmas 7.3-7.5] the map

$$_{\varepsilon}\mathcal{KQ}(A) \rightarrow \mathcal{K}(\mathcal{A})^{h(\varepsilon\mathbb{Z}/2)}$$

in (17) is a weak equivalence for $A = \mathbb{F}_q, \mathbb{R}, \mathbb{C}$. It follows that the induced map of homotopy pullbacks is a weak equivalence. □

A K-theory background

In this appendix we deduce the homotopy cartesian square of K-theory spectra (2) using the space level results given in [15]. The results on étale K-theory of real number fields at the prime 2 quoted from [24] hold for algebraic K-theory on account of the Quillen-Lichtenbaum conjecture proven in [25]. Throughout we retain the assumptions and notations employed in the main body of the text.

According to the product decomposition of the K-theory space $K(R_F)$ given in [15] there is a homotopy cartesian square:

$$\begin{array}{ccc} K(R_F) & \rightarrow & \prod^{r} BO \\ \downarrow & & \downarrow \\ K(\mathbb{F}_q) & \rightarrow & \prod^{r} BU \end{array}$$

The Quillen-Lichtenbaum conjecture for totally real 2-regular number fields implies that $\mathcal{K}(R_F)$ is a (-1)-connective cover of its $K(1)$-localization $L_{K(1)}\mathcal{K}(R_F)$. Here $K(1)$ is the first Morava K-theory spectrum at the prime 2. Next we incorporate [6] which reduces questions about $K(1)$-local spectra to space level questions. That is, applying Bousfield's homotopy functor T from spaces to spectra yields now the desired conclusion since by *loc. cit.* $L_{K(1)}\mathcal{K}(R_F)$ identifies with $T\Omega^\infty\mathcal{K}(R_F)$. We refer to [23] for an overview of the stable homotopy theoretic interpretations of the Quillen-Lichtenbaum conjecture.

B Homology module maps

Most theories in this paper are modules over the graded ring ${}_\varepsilon KQ_*(R_\mathbb{Q})$ or ${}_\varepsilon KQ_*(\mathbb{R})$ (in the topological case). The framework for such considerations is laid out in [20, §3], using the description of algebraic K-theory in terms of flat "virtual" bundles. We shall explicate these module structures in order to clarify the arguments given in Section 4.

In the topological case when $A = \mathbb{R}$, $\mathbb{C}$, the module structures on $\mathcal{K}(A)$, ${}_\varepsilon\mathcal{KQ}(A)$, ${}_\varepsilon\mathcal{U}(A)$ and ${}_\varepsilon\mathcal{V}(A)$ allow simple descriptions. For clarity we discuss the example of ${}_1\mathcal{V}(\mathbb{R})$ along with the relative theory obtained by taking the homotopy fiber of the map

$$ {}_\varepsilon\mathcal{V}(\mathbb{R}) \to {}_\varepsilon\mathcal{V}(\mathbb{C}). $$

We start with a geometric viewpoint: the cohomology theory associated to the spectrum ${}_1\mathcal{KQ}(\mathbb{R})$ is constructed as the K-theory of real vector bundles equipped with nondegenerate quadratic forms. As shown in [17, Exercise 9.22], such a vector bundle E splits as a Whitney sum

$$ E = E^+ \oplus E^-, $$

where the quadratic form is positive on E^+ and negative on E^-. A bundle version of Sylvester's theorem tells us that the isomorphism classes of E^+ and E^- are independent of the sum decomposition. Hence, there is a splitting

$$ {}_1\mathcal{KQ}(\mathbb{R}) \simeq \mathcal{K}(\mathbb{R}) \vee \mathcal{K}(\mathbb{R}). $$

Recall that ${}_1\mathcal{V}(\mathbb{R})$ is the homotopy fiber of the forgetful map

$$ {}_1\mathcal{KQ}(\mathbb{R}) \to \mathcal{K}(\mathbb{R}). $$

The associated cohomology theory is a "relative" K-theory as explained for instance in [17, §2] (in a slightly different context). One considers homotopy classes of triples

$$ \tau = (E, F, \alpha), $$

where E and F are real vector bundles equipped with nondegenerate quadratic forms, and α is an isomorphism between the underlying real vector bundles. If G is another real vector bundle with a nondegenerate quadratic form, then its cup-product with τ is given as the triple

$$(G \otimes E, G \otimes F, \mathrm{id} \otimes \alpha).$$

This defines a ${}_1\mathcal{KQ}(\mathbb{R})$-module structure on ${}_1\mathcal{V}(\mathbb{R})$. By associating to every real vector bundle a metric, *i.e.* a positive quadratic form, we obtain a well-defined map up to homotopy $\mathcal{K}(\mathbb{R}) \to {}_1\mathcal{KQ}(\mathbb{R})$, which is a right inverse to the forgetful map. Therefore, every ${}_1\mathcal{KQ}(\mathbb{R})$-module acquires a naturally induced $\mathcal{K}(\mathbb{R})$-module structure.

We can identify ${}_1\mathcal{V}(\mathbb{R})$ with $\mathcal{K}(\mathbb{R})$ as modules over $\mathcal{K}(\mathbb{R})$: if E is a real vector bundle there is an associated triple (E_1, E_2, id) where E_1 is the bundle E equipped with a positive quadratic form, and likewise for E_2 but with a negative quadratic form.

In the same vein, the theory ${}_1\mathcal{V}(\mathbb{C})$ arises from triples (E, F, α), where E and F are real vector bundles and α is an isomorphism between the corresponding complex vector bundles. By the above considerations, this theory has a $\mathcal{K}(\mathbb{R})$-module structure.

Now, since ${}_1\mathcal{V}(\mathbb{R})$ is free of rank one as a $\mathcal{K}(\mathbb{R})$-module, the map

$${}_1\mathcal{V}(\mathbb{R}) \to {}_1\mathcal{V}(\mathbb{C})$$

is determined up to weak homotopy equivalence by its effect on the zeroth homotopy groups. In Section 4 we combined this fact with the vanishing of $K_{-1}(\mathbb{R})$ to conclude that the map is in fact nullhomotopic. A similar argument applies when $\varepsilon = -1$, in which case the map is nontrivial.

C The composition HF and the involution on ${}_\varepsilon KQ_n(R_F)$

The composition HF of the forgetful functor $F : {}_\varepsilon\mathcal{Q}(A) \to \mathcal{P}(A)$ and the hyperbolic functor $H : \mathcal{P}(A) \to {}_\varepsilon\mathcal{Q}(A)$ is in the abstract well-known (see [21, pp. 60-61] for instance). It associates to an ε-quadratic module (E, q) the direct sum of (E, q) and $(E, -q)$. In order to understand the composition HF, one has therefore to determine the involution $(E, q) \mapsto (E, -q)$, at least on the level of the groups ${}_\varepsilon KQ_n(A)$.

This involution can also be described in terms of ε-orthogonal groups. If $E = H(A^n)$ is a hyperbolic module with its usual ε-quadratic form q, we can write an automorphism of E as a 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $MM^* = M^*M = I$, where M^* is the adjoint of M defined by

$$M^* = \begin{bmatrix} {}^{\mathrm{t}}\bar{d} & \varepsilon {}^{\mathrm{t}}\bar{b} \\ \varepsilon {}^{\mathrm{t}}\bar{c} & {}^{\mathrm{t}}\bar{a} \end{bmatrix}.$$

We can identify (E, q) and $(E, -q)$ through the orthogonal transformation defined by the matrix

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, we see that the involution on ${}_{\varepsilon}O_{n,n}(A)$ induced by the previous functor is given by the inner transformation induced by T. In other words, it is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}.$$

An important case is when $\sqrt{-1}$ belongs to A. Then (E, q) is canonically isomorphic to $(E, -q)$ through the orthogonal transformation given by the multiplication by $i = \sqrt{-1}$. On the level of orthogonal groups, one just needs to remark that the above two matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$$

are conjugate through the inner automorphism defined by the ε-orthogonal matrix

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

For the case $A = R_F$, the rings we need to consider are $\mathbb{R}$ (the real numbers with the usual topology), $\mathbb{C}$ (the complex numbers with the usual topology) and the finite field $\mathbb{F}_q$.

For the case of $\mathbb{R}$, we know that the orthogonal group ${}_1O_{n,n}(\mathbb{R})$ has the homotopy type of $O(n) \times O(n)$. Therefore, if we identify the spectrum ${}_1\mathcal{KQ}(\mathbb{R})$ as $\mathcal{K}(\mathbb{R}) \times \mathcal{K}(\mathbb{R})$, the involution on ${}_1\mathcal{KQ}(\mathbb{R})$ switches the factors $\mathcal{K}(\mathbb{R})$ up to homotopy.

On the other hand, ${}_{-1}O_{n,n}(\mathbb{R})$ is the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ which has the homotopy type of $U(n)$. More precisely, the homotopy equivalence $U(n) \simeq \mathrm{Sp}_{2n}(\mathbb{R})$ is given by the map

$$a + ib \longmapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Therefore the involution on ${}_{-1}\mathcal{KQ}(\mathbb{R}) \simeq \mathcal{K}(\mathbb{C})$ is induced by complex conjugation.

For the case of $\mathbb{C}$, we apply the previous remark that $\sqrt{-1}$ belongs to $\mathbb{C}$. Therefore, the involutions on ${}_1\mathcal{KQ}(\mathbb{C})$ and ${}_{-1}\mathcal{KQ}(\mathbb{C})$ are homotopic to the identity.

For the case of the finite field $\mathbb{F}_q$, we distinguish two cases:

1) $q \equiv 1 \pmod 4$. Here, the square root of -1 belongs to $\mathbb{F}_q$ and so we can repeat the previous argument: the involutions on ${}_1\mathcal{KQ}(\mathbb{F}_q)$ and ${}_{-1}\mathcal{KQ}(\mathbb{F}_q)$ are homotopic to the identity.

2) $q \equiv 3 \pmod 4$. The involutions are not homotopic to the identity in general. For a counterexample, one may look at the group

$${}_1KQ_1(\mathbb{F}_q) \cong \mathbb{Z}/2 \times \mathbb{F}_q^*/(\mathbb{F}_q^*)^2$$

for instance, where the induced involution is not the identity.

A tool for dealing with this case is to pass to the algebraic closure $\bar{\mathbb{F}}_q$, inducing a map of fibrations

$$\begin{array}{ccccc} {}_1\mathcal{KQ}(\mathbb{F}_q) & \longrightarrow & {}_1\mathcal{KQ}(\bar{\mathbb{F}}_q) & \xrightarrow{\Psi - \mathrm{id}} & {}_1\mathcal{KQ}(\bar{\mathbb{F}}_q) \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ {}_1\mathcal{KQ}(\mathbb{F}_q) & \longrightarrow & {}_1\mathcal{KQ}(\bar{\mathbb{F}}_q) & \xrightarrow{\Psi - \mathrm{id}} & {}_1\mathcal{KQ}(\bar{\mathbb{F}}_q) \end{array}$$

where Ψ denotes the Frobenius map (see [4, Theorem 5.5]). From the fact that the involution on ${}_1\mathcal{KQ}(\bar{\mathbb{F}}_q)$ is homotopic to the identity (by the previous case), we in most cases may deduce that also the involution on ${}_1\mathcal{KQ}(\mathbb{F}_q)$ is homotopic to the identity.

These considerations enable another proof of Corollary 6.2 above.

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