Beilinson's Tate conjecture for K_2 and finiteness of torsion zero-cycles on elliptic surface

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1 Introduction

The Tate conjecture for Chow groups is one of the most important question in arithmetic geometry. It appears in various areas explicitly or implicitly and often plays a key role. Although there have been significant progresses on this conjecture (e.g., for abelian varieties), very few things are known about its analogue for higher K-theory raised by Beilinson (cf. [Ja], 5.19). In this paper we are mainly concerned with the Tate conjecture for K_2, which asserts that for a nonsingular variety U over a number field or a finite field F the étale chern class map

$$c_{\text{ét}} : K_2(U) \otimes \mathbb{Q}_p \longrightarrow H^2_{\text{ét}}(\overline{U}, \mathbb{Q}_p(2))^{G_F}$$

is surjective. Here $G_F := \mathrm{Gal}(\overline{F}/F)$ is the absolute Galois group, $\overline{U}$ denotes $U \times_F \overline{F}$ and the superscript G_F denotes the fixed part by G_F.

In this paper, we focus on elliptic surfaces $\pi : X \to C$ over a p-adic local field K which is absolutely unramified. Let $D = \sum D_i$ be the sum of the split multiplicative fibers of π, and put $U = X - D$. Assume that X and C have projective smooth models $\mathscr{X}$ and $\mathscr{C}$ over the integer ring R of K, respectively, and that the closure $\mathscr{D} \subset \mathscr{X}$ of D has normal crossings. Put $\mathscr{U} := \mathscr{X} - \mathscr{D}$. Then we introduce a space of *formal Eisenstein series*

$$\Phi(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p} \subset \Gamma(\mathscr{X}, \Omega^2_{\mathscr{X}/R}(\log \mathscr{D}))$$

(see §5.2 for details), where the right hand side is the space of global 2-forms with log poles along $\mathscr{D}$. One of our main results asserts that

$$\mathrm{Im}\big(H^2_{\text{syn}}(\mathscr{X}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2)) \to \Gamma(\mathscr{X}, \Omega^2_{\mathscr{X}/R}(\log \mathscr{D}))\big) \subset \Phi(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p}, \tag{1.0.1}$$

where $H^*_{\text{syn}}(\mathscr{X}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2))$ denotes the syntomic cohomology of $\mathscr{X}$ with log poles along $\mathscr{D}$ due to Kato and Tsuji. From this result, we will further deduce inequalities

$$\dim_{\mathbb{Q}_p} c_{\text{ét}}(K_2(U) \otimes \mathbb{Q}_p) \le \dim_{\mathbb{Q}_p} H^2_{\text{ét}}(\overline{U}, \mathbb{Q}_p(2))^{G_K} \le \mathrm{rank}_{\mathbb{Z}_p} \Phi(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p} \tag{1.0.2}$$

using the fact that the Fontaine-Messing map $H^2_{\text{syn}}(\mathscr{X}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2)) \to H^2_{\text{ét}}(\overline{U}, \mathbb{Q}_p(2))^{G_K}$ is surjective (§5.3). The inclusion (1.0.1) is an extension of Beilinson's theorem on Eisenstein symbols in the following sense. When $\pi : X \to X(\Gamma)$ is the universal family of elliptic curves over a modular curve $X(\Gamma)$, the space $\Phi(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p}$ consists of the (usual) Eisenstein series of weight 3 ([A2] §8.3). Beilinson proved that it is spanned by the dlog image of the Eisenstein symbols in $K_2(U)$ ([Be]), so that

$$\mathrm{Im}\big(K_2(U) \otimes \mathbb{Q}_p \to \Gamma(X, \Omega^2_{X/K}(D))\big) = \Phi(\mathscr{X}, \mathscr{D})_{\mathbb{Q}_p} = \mathbb{Q}_p^{\#\text{ of cusps}}$$

under our notation. Therefore we can regard (1.0.1) as a partial extension of Beilinson's theorem to arbitrary elliptic surfaces. A distinguished feature is that it gives a new upper bound of the rank of $H^2_{\text{ét}}(\overline{U}, \mathbb{Q}_p(2))^{G_F}$ in view of the fact that the rank of $\varPhi(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p}$ is strictly less than the number of split multiplicative fibers in some cases (cf. [ASat], §5). This is a key tool in our application to Beilinson's Tate conjecture for K_2. In fact, if one can construct enough elements in $K_2(U)$ (e.g. by symbols) so that the dimension of $c_{\text{ét}}(K_2(U))$ is equal to the upper bound, then the equalities hold in (1.0.2) and $c_{\text{ét}}$ is surjective. See [ASat], §5 for a number of examples. The proof of (1.0.1) is quite different from that of the theorem of Beilinson. A key ingredient is the p-adic Hodge theory, in particular a detailed computation on the syntomic cohomology of Tate curves over 2-dimensional complete local rings.

In this paper, we present another application of (1.0.1) to the finiteness of torsion 0-cycles. It is a folklore conjecture that $\mathrm{CH}^m(X)$ is finitely generated for a projective smooth variety X over a number field. Most people had believed that the finiteness of $\mathrm{CH}^m(X)_{\mathrm{tors}}$ remains true even for X over a p-adic local field, until couter-examples were found recently ([RSr], [ASai]). We have to note that all such examples are *not* defined over number fields. Therefore we are naturally lead to the following modified question:

> *If X is a projective nonsingular variety over a p-adic field which has a model over a number field, then is $\mathrm{CH}^m(X)_{\mathrm{tors}}$ finite?*

It is in fact a crucial question whether the p-primary torsion part $\mathrm{CH}^m(X)\{p\}$ is finite. When $m = 2$, the finiteness of $\mathrm{CH}^2(X)\{p\}$ is reduced to the study of the p-adic regulator on K_1 by a recent work of Saito and the second author [SS1], that is, $\mathrm{CH}^2(X)\{p\}$ is finite if the p-adic regulator map

$$\varrho : K_1(X)^{(2)} \otimes \mathbb{Q}_p \longrightarrow H^1_g(G_K, H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p(2)))$$

is surjective onto the g-part of Bloch-Kato [BK2], (3.7). When $H^2(X, \mathscr{O}_X) = 0$, this map is well-known to be surjective even when X is not defined over a number field (cf. [CTR1], [CTR2], [S], 3.6). However the question becomes more difficult when $H^2(X, \mathscr{O}_X) \neq 0$, and nobody found an affirmative or negative example so far. Using our main result (1.0.1), we will give an elliptic $K3$ surface X whose p-adic regulator map ϱ is surjective, which would be the first example that answers the above question affirmatively (Theorem 6.0.1, Corollary 6.0.3, §6.3).

This paper is organized as follows. In §2, we review and fix the notation for (log) syntomic cohomology and Tate curves. In §3, we state the main result on Tate curves (Theorem 3.2.3) and prove it admitting a key commutative key diagram. §4 is devoted to the proof of the key diagram. In §5 we prove the main results on elliptic surfaces over p-adic fields. In §6 we give an example of elliptic $K3$ surface over $\mathbb{Q}_p$ with finitely many torsion 0-cycles.

2 Preliminaries

For a scheme X over a ring A and an A-algebra B we write $X_B := X \times_A B$. For an integer n which is invertible on X, $\mathbb{Z}/n(1)$ denotes the étale sheaf μ_n of n-th roots of unity. We often write $\mathbb{Z}/n(m)$ ($m \in \mathbb{N}$) for the étale sheaf $\mu_n^{\otimes m}$. For a function $f \in \Gamma(X, \mathscr{O}_X)$ which is not a zero divisor, we put

$$X[f^{-1}] := \mathbf{Spec}(\mathscr{O}_X[T]/(fT-1)),$$

which is the maximal open subset of X where f is invertible.

2.1 Syntomic cohomology

Let p be a prime number. For a scheme T, we put

$$T_n := T \otimes \mathbb{Z}/p^n.$$

Definition 2.1.1 *Let T be a scheme.*

(1) *A morphism $\varphi : T \to T$ over $\mathbb{F}_p$ is called the absolute Frobenius endomorphism, if the underlying morphism of topological spaces is the identitiy map and the homomorphism $\varphi^* : \mathscr{O}_T \to \varphi_* \mathscr{O}_T = \mathscr{O}_T$ sends $x \mapsto x^p$. Here the equality $\varphi_* \mathscr{O}_T = \mathscr{O}_T$ means the natural identification.*

(2) *A morphism $\varphi : T \to T$ over $\mathbb{Z}_p$ is called a Frobenius endomorphism, if $\varphi \otimes \mathbb{Z}/p : T_1 \to T_1$ is the absolute Frobenius endomorphism.*

Let X be a scheme which is flat over $\mathbb{Z}_p$. Assume the following condition:

Condition 2.1.2 *There exists a closed immersion $X \hookrightarrow Z$ satisfying the following conditions* (*cf.* [Ka4], 2.4):

(1) *Z_n has p-bases over $\mathbb{Z}/p^n$ locally for any $n \geq 1$* (*loc. cit., Definition* 1.3).

(2) *Z has a Frobenius endomorphism, or more weakly, Z_n has a Frobenius endomorphism φ_n for each $n \geq 1$ and the morphism $\varphi_{n+1} \otimes_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n$ agrees with φ_n for any $n \geq 1$.*

(3) *Let D_n be the PD-envelope of X_n in Z_n compatible with the canonical PD-structure on the ideal $(p) \subset \mathbb{Z}/p^n$. For $i \geq 1$, let $J_{D_n}^{[i]} \subset \mathscr{O}_{D_n}$ be the i-th divided power of the ideal $J_{D_n} := \mathrm{Ker}(\mathscr{O}_{D_n} \to \mathscr{O}_{X_n})$. For $i \leq 0$, we put $J_{D_n}^{[i]} := \mathscr{O}_{D_n}$. Then the following sequence is exact for any $m, n \geq 1$ and any $i \geq 0$:*

$$J_{D_{m+n}}^{[i]} \xrightarrow{\times p^m} J_{D_{m+n}}^{[i]} \xrightarrow{\times p^n} J_{D_{m+n}}^{[i]} \longrightarrow J_{D_n}^{[i]} \longrightarrow 0. \tag{2.1.3}$$

We give some examples of Z which satisfy the condition (1):

Example 2.1.4 *Let $W = W(k)$ be the Witt ring of a perfect field k, and let $W[\![t]\!]$ be the formal power series ring over W with t an indeterminate. Then $W_n[\![t]\!] := W[\![t]\!]/(p^n)$ has a p-basis over $\mathbb{Z}/p^n$ for any $n \geq 1$. Indeed, $W_n[\![t]\!]$ is flat over $\mathbb{Z}/p^n$ and $t \in k[\![t]\!]$ is a p-basis of $k[\![t]\!]$ over $\mathbb{F}_p$. Therefore $t \in W_n[\![t]\!]$ is a p-basis over $\mathbb{Z}/p^n$ by* [Ka3], *Proposition* 1.4. *More generally, Z_n has p-bases locally over $\mathbb{Z}/p^n$ in the following cases*:

(i) *Z is smooth of finite type over $W[\![t]\!]$.*

(ii) *Z is flat of finite type over $W[\![t]\!]$ and Z_1 is a regular semistable family over $k[\![t]\!]$.*

The following fact provides a sufficient condition for $X \hookrightarrow Z$ to satisfy the condition (3):

Proposition 2.1.5 (Fontaine-Messing / Kato [Ka4], Lemma 2.1) *Let $X \hookrightarrow Z$ be a closed immersion satisfying the conditions* (1) *and* (2) *in Condition* 2.1.2. *Assume that Z is locally noetherian, and $\mathrm{Ker}(\mathscr{O}_{Z,x} \to \mathscr{O}_{X,x})$ is generated by an $\mathscr{O}_{Z,x}$-regular sequence for any $x \in X_1$. Then $X \hookrightarrow Z$ satisfies the condition* (3) *in Condition* 2.1.2.

For $0 \leq r \leq p-1$, we define the syntomic complex $\mathscr{S}_n(r)_{X,Z}$ (with respect to the embedding $X \hookrightarrow Z$) as follows. Let $\mathbb{J}^{[r]}_{n,X,Z}$ be the complex of sheaves on $(X_1)_{\text{ét}}$

$$J^{[r]}_{D_n} \xrightarrow{d} J^{[r-1]}_{D_n} \otimes_{\mathscr{O}_{Z_n}} \Omega^1_{Z_n} \xrightarrow{d} \cdots \xrightarrow{d} J^{[r-q]}_{D_n} \otimes_{\mathscr{O}_{Z_n}} \Omega^q_{Z_n} \xrightarrow{d} \cdots,$$

where $J^{[r]}_{D_n}$ is placed in degree 0. Put $\mathbb{E}_{n,X,Z} := \mathbb{J}^{[0]}_{n,X,Z}$. By the assumption that $0 \leq r \leq p-1$, the Frobenius endomorphism on Z_{n+r} induces a homomorphism of complexes

$$f_r := \overline{p^{-r} \cdot \varphi^*_{n+r}} : \mathbb{J}^{[r]}_{n,X,Z} \longrightarrow \mathbb{E}_{n,X,Z}$$

(see [Ka4], p. 411 for details).

Definition 2.1.6 *For $0 \leq r \leq p-1$, we define the complex $\mathscr{S}_n(r)_{X,Z}$ on $(X_1)_{\text{ét}}$ as the mapping fiber of*

$$1 - f_r : \mathbb{J}^{[r]}_{n,X,Z} \longrightarrow \mathbb{E}_{n,X,Z}.$$

More precisely, the degree q-part of $\mathscr{S}_n(r)_{X,Z}$ is

$$(J^{[r-q]}_{D_n} \otimes_{\mathscr{O}_{Z_n}} \Omega^q_{Z_n}) \oplus (\mathscr{O}_{D_n} \otimes_{\mathscr{O}_{Z_n}} \Omega^{q-1}_{Z_n})$$

and the differential operator is given by

$$(x, y) \longmapsto (dx, x - f_r(x) - dy).$$

We define the syntomic cohomology of X with coefficients in $\mathscr{S}_n(r)$ as the hypercohomology groups of this complex:

$$H^*_{\mathrm{syn}}(X, \mathscr{S}_n(r)) := \mathbb{H}^*_{\text{ét}}(X, \mathscr{S}_n(r)_{X,Z}).$$

This notation is well-defined, because the image of the complex $\mathscr{S}_n(r)_{X,Z}$ in the derived category is independent of $X \hookrightarrow Z$ as in Condition 2.1.2 ([Ka4], *p.* 412). *See also Remark* 2.2.8 *below.*

Proposition 2.1.7 *For $m, n \geq 1$, there is an exact sequence of complexes on $(X_1)_{\text{ét}}$*

$$\mathscr{S}_{m+n}(r)_{X,Z} \xrightarrow{\times p^m} \mathscr{S}_{m+n}(r)_{X,Z} \xrightarrow{\times p^n} \mathscr{S}_{m+n}(r)_{X,Z} \longrightarrow \mathscr{S}_n(r)_{X,Z} \longrightarrow 0.$$

Consequently, there is a short exact sequence

$$0 \longrightarrow \mathscr{S}_m(r)_{X,Z} \xrightarrow{\underline{p^n}} \mathscr{S}_{m+n}(r)_{X,Z} \longrightarrow \mathscr{S}_n(r)_{X,Z} \longrightarrow 0.$$

Proof. The assertion follows from the exactness of (2.1.3) and the fact that $\Omega^q_{Z_n}$ is free over $\mathbb{Z}/p^n$ ([Ka4], Lemma 1.8). $\square$

Remark 2.1.8 *Assume that the identity map $X \to X$ satisfies the conditions* (1)–(3) *in Condition* 2.1.2*. Then we have $D_n = X_n$, i.e.,*

$$\mathbb{J}^{[r]}_{n,X,X} = \Omega^{\bullet \geq r}_{X_n} \quad \textit{and} \quad \mathbb{E}_{n,X,X} = \Omega^{\bullet}_{X_n},$$

and there is a short exact sequence of complexes

$$0 \longrightarrow \mathscr{S}_n(r)_{X,X} \longrightarrow \Omega^{\bullet \geq r}_{X_n} \xrightarrow{1-f_r} \Omega^{\bullet}_{X_n} \longrightarrow 0. \tag{2.1.9}$$

Remark 2.1.10 *Syntomic complexes can be defined in a more general situation by a gluing argument in the derived category* (*see* [Ka1], *Remark* 1.8 *and* [Ka4], *Lemma* 2.2)*. However we will not need this fact in this paper.*

2.2 Syntomic cohomology with log poles

The notation remain as in §2.1. The aim of this subsection is to define of the syntomic cohomology of a regular scheme X with log poles along a simple normal crossing divisor $\mathscr{D}$ following Tsuji ([Ts2], §2), which is necessary to formulate Theorem 3.2.3 below. The case that $\mathscr{D}$ is empty corresponds to the syntomic cohomology defined in §2.1. To give precise arguments, we will use the terminology in log geometry [Ka3].

Definition 2.2.1 *Let (T, M_T) be a log scheme.*

(1) ([Ka3], *Definition* 4.7) *A morphism $\varphi : (T, M_T) \to (T, M_T)$ over $\mathbb{F}_p$ is called the absolute Frobenius endomorphism, if the underlying morphism $T \to T$ is the absolute Frobenius endomorphism in the sense of Definition* 2.1.1 (1) *and the homomorphism $\varphi^* : M_T \to \varphi_* M_T = M_T$ is the multiplication by p. Here the equality $\varphi_* M_T = M_T$ means the natural identification obtained from the fact that the underlying morphism of φ of topological spaces is the identity map.*

(2) *A morphism $\varphi : (T, M_T) \to (T, M_T)$ over $\mathbb{Z}_p$ is called a Frobenius endomorphism, if $\varphi \otimes \mathbb{Z}/p : (T_1, M_{T_1}) \to (T_1, M_{T_1})$ is the absolute Frobenius endomorphism. Here M_{T_1} denotes the inverse image log structure of M_T onto T_1* (*loc. cit.*, (1.4)).

Let (X, M) be a fine log scheme such that X is flat over $\mathbb{Z}_p$ (as a usual scheme). The main example we are concerned with is the following case:

Example 2.2.2 *Let X be a regular scheme which is flat over $\mathbb{Z}_p$. Let $\mathscr{D}$ be a simple normal crossing divisor on X, which may be empty. Put $U := X - \mathscr{D}$ and let $j : U \hookrightarrow X$ be the natural open immersion. We define the sheaf M of monoids on $X_{\text{ét}}$ as*

$$M := \mathscr{O}_X \cap j_* \mathscr{O}_U^\times.$$

The canonical map $M \to \mathscr{O}_X$ is a fine log structure on X, which we call the log structure associated with $\mathscr{D}$.

For $n \geq 1$, we write M_n for the inverse image log structure of M onto X_n. We assume the following condition, which is a logarithmic variant of Condition 2.1.2:

Condition 2.2.3 *There exist exact closed immersions $i_n : (X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})$ of log schemes for $n \geq 1$ which satisfy the following conditions for all $n \geq 1$ (cf.* [Ts2], §2):

(0) *(Z_n, M_{Z_n}) is fine. We have*

$$(Z_n, M_{Z_n}) \xrightarrow{\sim} (Z_{n+1}, M_{Z_{n+1}}) \otimes_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n$$

as log schemes, and the morphism $i_{n+1} \otimes_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n$ agrees with i_n.

(1) *(Z_n, M_{Z_n}) has p-bases locally over $\mathbb{Z}/p^n$ with the trivial log structure* (*loc. cit., Definition* 1.4).

(2) *(Z_n, M_{Z_n}) has a Frobenius endomorphism φ_n and the morphism $\varphi_{n+1} \otimes_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n$ agrees with φ_n.*

(3) *Let (D_n, M_{D_n}) be the PD-envelope of (X_n, M_n) in (Z_n, M_{Z_n}) which is compatible with the canonical PD-structure on the ideal $(p) \subset \mathbb{Z}/p^n$* ([Ka3], *Definition* (5.4)). *For $i \geq 1$, let $J_{D_n}^{[i]} \subset \mathscr{O}_{D_n}$ be the i-th divided power of the ideal $J_{D_n} = \operatorname{Ker}(\mathscr{O}_{D_n} \to \mathscr{O}_{X_n})$. For $i \leq 0$, we put $J_{D_n}^{[i]} := \mathscr{O}_{D_n}$. Then the following sequence is exact for any $m, n \geq 1$ and any $i \geq 0$:*

$$J_{D_{m+n}}^{[i]} \xrightarrow{\times p^m} J_{D_{m+n}}^{[i]} \xrightarrow{\times p^n} J_{D_{m+n}}^{[i]} \longrightarrow J_{D_n}^{[i]} \longrightarrow 0. \tag{2.2.4}$$

We give some examples of (Z_n, M_{Z_n}) which satisfy the condition (1):

Example 2.2.5 *Let $W[\![t]\!]$ be as in Example* 2.1.4. *We endow $\operatorname{Spec}(W[\![t]\!])$ with a pre-log structure $\mathbb{N} \to W[\![t]\!]$ by sending 1 to t, and write N for the associated log structure on $\operatorname{Spec}(W[\![t]\!])$. Then $(W_n[\![t]\!], N_n)$ has a p-basis over $\mathbb{Z}/p^n$ for any $n \geq 1$, which one can check in a similar way as in Example* 2.1.4. *More generally, a fine log scheme (Z_n, M_{Z_n}) which is log smooth over $(W_n[\![t]\!], N_n)$ has p-bases locally over $\mathbb{Z}/p^n$. Indeed, (Z_n, M_{Z_n}) has p-bases locally over $(W_n[\![t]\!], N_n)$ by* [Ts2], *Lemma* 1.5. *Hence (Z_n, M_{Z_n}) has p-bases locally over $\mathbb{Z}/p^n$ by loc. cit., Proposition* 1.6 (2).

The following fact is a logarithmic variant of Propostion 2.1.5:

Proposition 2.2.6 (Tsuji [Ts2], Corollary 1.9) *Let* $\{i_n : (X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})\}_{n \geq 1}$ *be a system of exact closed immersions satisfying the conditions* (0)–(2) *in Condition* 2.2.3. *Assume that* Z_n *is locally noetherian, and that* $\mathrm{Ker}(\mathscr{O}_{Z_n,x} \to \mathscr{O}_{X_n,x})$ *is generated by an* $\mathscr{O}_{Z_n,x}$*-regular sequence for any* $x \in X_1$. *Then* $\{i_n : (X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})\}_{n \geq 1}$ *satisfies the condition* (3) *in Condition* 2.2.3.

For $0 \leq r \leq p-1$, we define a log syntomic complex $\mathscr{S}_n(r)_{(X,M),(Z_*,M_{Z_*})}$ as follows. Let $\mathbb{J}^{[r]}_{n,(X,M),(Z_*,M_{Z_*})}$ be the complex of sheaves on $(X_1)_{\text{ét}}$

$$J^{[r]}_{D_n} \xrightarrow{d} J^{[r-1]}_{D_n} \otimes_{\mathscr{O}_{Z_n}} \omega^1_{(Z_n,M_{Z_n})} \xrightarrow{d} \cdots \xrightarrow{d} J^{[r-q]}_{D_n} \otimes_{\mathscr{O}_{Z_n}} \omega^q_{(Z_n,M_{Z_n})} \xrightarrow{d} \cdots,$$

where $J^{[r]}_{D_n}$ is placed in degree 0 and $\omega^*_{(Z_n,M_{Z_n})}$ denotes the differential module of (Z_n, M_{Z_n}) [Ka3], (1.7). The arrows d denote the derivations defined in [Ts2], Corollary 1.10. Put

$$\mathbb{E}_{n,(X,M),(Z_*,M_{Z_*})} := \mathbb{J}^{[0]}_{n,(X,M),(Z_*,M_{Z_*})}.$$

By the assumption that $0 \leq r \leq p-1$, the Frobenius endomorphism on $(Z_{n+r}, M_{Z_{n+r}})$ induces a homomorphism of complexes

$$f_r := \overline{p^{-r} \cdot \varphi^*_{n+r}} : \mathbb{J}^{[r]}_{n,(X,M),(Z_*,M_{Z_*})} \longrightarrow \mathbb{E}_{n,(X,M),(Z_*,M_{Z_*})}$$

(see [Ts2], p. 540 for details).

Definition 2.2.7 *For* $0 \leq r \leq p-1$*, we define the complex* $\mathscr{S}_n(r)_{(X,M),(Z_*,M_{Z_*})}$ *on* $(X_1)_{\text{ét}}$ *as the mapping fiber of*

$$1 - f_r : \mathbb{J}^{[r]}_{n,(X,M),(Z_*,M_{Z_*})} \longrightarrow \mathbb{E}_{n,(X,M),(Z_*,M_{Z_*})}$$

(*cf. Definition* 2.1.6). *We define the syntomic cohomology of* (X, M) *with coefficients in* $\mathscr{S}_n(r)$ *as the hypercohomology groups of this complex*:

$$H^*_{\text{syn}}((X,M), \mathscr{S}_n(r)) := \mathbb{H}^*_{\text{ét}}(X, \mathscr{S}_n(r)_{(X,M),(Z_*,M_{Z_*})}).$$

This notation is well-defined, because the image of the complex $\mathscr{S}_n(r)_{(X,M),(Z_*,M_{Z_*})}$ *in the derived category is independent of embedding systems as in Condition* 2.2.3 *by Remark* 2.2.8 *below. In the case of Example* 2.2.2*, we put*

$$H^*_{\text{syn}}(X(\mathscr{D}), \mathscr{S}_n(r)) := H^*_{\text{syn}}((X,M), \mathscr{S}_n(r))$$

for simplicity.

Remark 2.2.8 *Take another embedding system* $\{i'_n : (X_n, M_n) \hookrightarrow (Z'_n, M_{Z'_n})\}_{n \geq 1}$ *as in Condition* 2.2.3*, and consider the embedding system*

$$\left\{i_n \times i'_n : (X_n, M_n) \hookrightarrow (Z_n, M_{Z_n}) \times_{\mathbb{Z}/p^n} (Z'_n, M_{Z'_n}) =: (Z''_n, M_{Z''_n})\right\}_{n \geq 1}.$$

We define a Frobenius endomorphism on $(Z''_n, M_{Z''_n})$ $(n \geq 1)$ *as the fiber product of those of* (Z_n, M_{Z_n}) *and* $(Z'_n, M_{Z'_n})$*. Then this embedding system satisfies the conditions* (0)–(3) *in Condition* 2.2.3 *as well* (*see* [Ka4], *Proof of Lemma* 2.2, *and* [Ts2], *Proposition* 1.8), *and there are natural quasi-isomorphisms of complexes on* $(X_1)_{\text{ét}}$

$$\mathscr{S}_n(r)_{(X,M),(Z_*,M_{Z_*})} \xrightarrow{\text{qis}} \mathscr{S}_n(r)_{(X,M),(Z''_*,M_{Z''_*})} \xleftarrow{\text{qis}} \mathscr{S}_n(r)_{(X,M),(Z'_*,M_{Z'_*})}$$

by [Ts2], *Corollary* 1.11*. Hence the image of the complex* $\mathscr{S}_n(r)_{(X,M),(Z_*,M_{Z_*})}$ *in the derived category is independent of embedding systems. Moreover, this fact verifies that log syntomic cohomology groups are contravariantly functorial for morphisms* $(X, M) \to (X', M')$ *of log schemes which satisfy Condition* 2.2.3 (*see also* [Ka1], *p.* 212).

Definition 2.2.9 *For* $r, r' \geq 0$ *with* $r + r' \leq p - 1$, *we define a product structure*

$$\mathscr{S}_n(r)_{(X,M),(Z_*,M_{Z_*})} \otimes \mathscr{S}_n(r')_{(X,M),(Z_*,M_{Z_*})} \longrightarrow \mathscr{S}_n(r + r')_{(X,M),(Z_*,M_{Z_*})}$$

by

$$(x, y) \otimes (x', y') \longmapsto (xx', (-1)^q xy' + f_{r'}(x')y),$$

where

$$(x, y) \in \mathscr{S}_n(r)^q_{(X,M),(Z_*,M_{Z_*})} = (J^{[r-q]}_{D_n} \otimes_{\mathscr{O}_{Z_n}} \omega^q_{(Z_n,M_{Z_n})}) \oplus (\mathscr{O}_{D_n} \otimes_{\mathscr{O}_{Z_n}} \omega^{q-1}_{(Z_n,M_{Z_n})}),$$
$$(x', y') \in \mathscr{S}_n(r')^{q'}_{(X,M),(Z_*,M_{Z_*})} = (J^{[r'-q']}_{D_n} \otimes_{\mathscr{O}_{Z_n}} \omega^{q'}_{(Z_n,M_{Z_n})}) \oplus (\mathscr{O}_{D_n} \otimes_{\mathscr{O}_{Z_n}} \omega^{q'-1}_{(Z_n,M_{Z_n})}).$$

The following proposition follows from the same arguments as for Proposition 2.1.7 (use [Ts2], Corollary 1.9 instead of [Ka4], Lemma 1.8):

Proposition 2.2.10 *For* $\{i_n : (X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})\}_{n \geq 1}$ *as before and integers* $m, n \geq 1$, *there is a short exact sequence of complexes on* $(X_1)_{\text{ét}}$

$$0 \longrightarrow \mathscr{S}_m(r)_{(X,M),(Z_*,M_{Z_*})} \xrightarrow{\underline{p}^n} \mathscr{S}_{m+n}(r)_{(X,M),(Z_*,M_{Z_*})} \longrightarrow \mathscr{S}_n(r)_{(X,M),(Z_*,M_{Z_*})} \longrightarrow 0.$$

The following standard fact relates the syntomic cohomology with de Rham cohomology.

Proposition 2.2.11 *For* $0 \leq r \leq p - 1$ *and* $i \geq 0$, *there is a canonical map*

$$\epsilon^i : H^i_{\text{syn}}((X, M), \mathscr{S}_n(r)) \longrightarrow \mathbb{H}^i_{\text{ét}}\big(X_n, \omega^{\bullet \geq r}_{(X_n,M_n)}\big),$$

which is compatible with product structures and contravariantly functorial in (X, M).

Proof. Fix a system of embeddings $(X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})$ $(n \geq 1)$ as before. There are natural maps

$$\mathscr{S}_n(r)_{(X,M),(Z_*,M_{Z_*})} \longrightarrow \mathbb{J}^{[r]}_{n,(X,M),(Z_*,M_{Z_*})} \longrightarrow \omega^{\bullet \geq r}_{(X_n,M_n)},$$

where the first arrow arises from the definition of the syntomic complex, and the second arrow is induced by the natural maps

$$\mathscr{O}_{X_n} \otimes_{\mathscr{O}_{D_n}} \left(J_{D_n}^{[r-q]} \otimes_{\mathscr{O}_{Z_n}} \omega^q_{(Z_n, M_{Z_n})}\right) \longrightarrow \begin{cases} 0 & (q \leq r-1) \\ \omega^q_{(X_n, M_n)} & (q \geq r). \end{cases}$$

Let $\epsilon_{n,(X,M),(Z_*,M_{Z_*})}$ be the composite of the above natural maps of complexes, and define the desired map ϵ^i as that induced by $\epsilon_{n,(X,M),(Z_*,M_{Z_*})}$. One can easily check that the image of $\epsilon_{n,(X,M),(Z_*,M_{Z_*})}$ in the derived category is independent of the choice of an embedding system, by repeating the arguments in Remark 2.2.8. Thus we obtain the proposition. $\square$

Remark 2.2.12 *If the identity maps $(X_n, M_n) \to (X_n, M_n)\,(n \geq 1)$ satisfy the conditions* (0)–(3) *in Condition* 2.2.3*, then we have $(D_n, M_{D_n}) = (X_n, M_n)$, i.e.,*

$$\mathbb{J}^{[r]}_{n,(X,M),(X_*,M_*)} = \omega^{\bullet \geq r}_{(X_n,M_n)} \quad \textit{and} \quad \mathbb{E}_{n,(X,M),(X_*,M_*)} = \omega^{\bullet}_{(X_n,M_n)},$$

and there is a short exact sequence of complexes

$$0 \longrightarrow \mathscr{S}_n(r)_{(X,M),(X_*,M_*)} \xrightarrow{\epsilon_n} \omega^{\bullet \geq r}_{(X_n,M_n)} \xrightarrow{1-f_r} \omega^{\bullet}_{(X_n,M_n)} \longrightarrow 0. \qquad (2.2.13)$$

for $0 \leq r \leq p-1$.

We provide a spectral sequence computing syntomic cohomology with log poles, which will be used in §4.4 below.

Proposition 2.2.14 *Let $X, \mathscr{D}$ and M be as in Example* 2.2.2*, assume that $\mathscr{D}$ is flat over $\mathbb{Z}_p$. Let $\{\mathscr{D}_i\}_{i \in I}$ be the irreducible components of $\mathscr{D}$. Put $X^{(0)} := X$ and*

$$X^{(m)} := \coprod_{\{i_1, i_2, \cdots, i_m\} \subset I} \mathscr{D}_{i_1} \times_X \cdots \times_X \mathscr{D}_{i_m}$$

for $m \geq 1$, where for each subset $\{i_1, i_2, \cdots, i_m\} \subset I$, the indices are pairwise distinct. Assume the following conditions:

(i) *The identity maps $(X_n, M_n) \to (X_n, M_n)$ satisfy the conditions* (0)–(3) *in Condition* 2.2.3 *for all $n \geq 0$.*

(ii) *The identity maps $X^{(m)} \to X^{(m)}$ satisfy the conditions* (1)–(3) *in Condition* 2.1.2 *for all $m \geq 0$.*

(iii) *The given Frobenius endomorphisms on X_{n+r} and $(X^{(0)})_{n+r}$ are compatible under the canonical finite morphism $(X^{(0)})_{n+r} \to X_{n+r}$.*

(iv) *For any $m \geq 0$, the given Frobenius endomorphisms on $(X^{(m)})_{n+r}$ and $(X^{(m+1)})_{n+r}$ are compatible under the canonical finite morphism $(X^{(m+1)})_{n+r} \to (X^{(m)})_{n+r}$.*

Fix an ordering on the set I. Then for $0 \le r \le p-1$, there is a spectral sequence of syntomic cohomology groups

$$E_1^{a,b} = H^{2a+b}_{\mathrm{syn}}(X^{(-a)}, \mathscr{S}_n(a+r)) \Longrightarrow H^{a+b}_{\mathrm{syn}}(X(\mathscr{D}), \mathscr{S}_n(r)).$$

Proof. By (i) and (ii), the syntomic complexes $\mathscr{S}_n(r)_{(X,M),(X_*,M_*)}$ and $\mathscr{S}_n(r)_{X^{(m)},X^{(m)}}$ are defined for $0 \le r \le p-1$ and $m \ge 0$. These complexes are computed as in Remarks 2.1.8 and 2.2.12. By (iii) and (iv), there is a natural 'filtration' on $\mathscr{S}_n(r)_{(X,M),(X_*,M_*)}$ as follows:

$$\begin{array}{lr}
0 \longrightarrow \mathscr{S}_n(r)_{X,X} \xrightarrow{\alpha_0} \mathscr{S}_n(r)_{(X,M),(X_*,M_*)} \longrightarrow C^\bullet_{1,n} \longrightarrow 0 & \\
0 \longrightarrow \mathscr{S}_n(r)_{X^{(m)},X^{(m)}}[-m] \xrightarrow{\alpha_m} C^\bullet_{m,n} \longrightarrow C^\bullet_{m+1,n} \longrightarrow 0 & (m \ge 1) \\
C^\bullet_{m,n} = 0 & (\text{if } \ X^{(m)} = \emptyset),
\end{array}$$

where we have omitted the indication of the direct image of sheaves under the canonical finite morphisms $(X^{(m)})_1 \to X_1$. The arrow α_0 denotes the natural inclusion of complexes, and the arrow α_m for $m \ge 1$ is given by the alternate sum of the inverse of Poincaré residue mappings whose signs are determined by the fixed ordering on I. The spectral sequence in question is obtained from this filtration. $\square$

The following fact relates the syntomic cohomology with étale cohomology and plays an important role in this paper.

Theorem 2.2.15 (Tsuji [Ts1], §3.1) *Let $X, \mathscr{D}$ and M be as in Example* 2.2.2 *($\mathscr{D}$ may be empty). Assume that (X, M) satisfies Condition* 2.2.3 *and that there exists a henselian local ring $\mathscr{R}$ which is faithfully flat over $\mathbb{Z}_p$ and such that X is proper over $\mathscr{R}$. Put $U := X - \mathscr{D}$. Then for $0 \le r \le p-2$ and $i \ge 0$, there is a canonical homomorphism*

$$c^i : H^i_{\mathrm{syn}}(X(\mathscr{D}), \mathscr{S}_n(r)) \longrightarrow H^i_{\text{ét}}(U[p^{-1}], \mathbb{Z}/p^n(r)),$$

which is compatible with product structures and contravariantly functorial in (X, M).

Proof. Since X is regular and flat over $\mathbb{Z}_p$, we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_p\, t^{\{r\}} \longrightarrow Fil^r_p A_{\mathrm{crys}}(\overline{A^h}) \xrightarrow{1-\frac{\varphi}{p^r}} A_{\mathrm{crys}}(\overline{A^h}) \longrightarrow 0$$

for an affine open subset $\mathrm{Spec}(A) \subset X$ (see [Ts1], p. 245). We obtain a morphism

$$c : \mathscr{S}_n(r)_{(X,M)} \longrightarrow \iota^* Rj_* \mathbb{Z}/p^n(r) \qquad (X_1 \xrightarrow{\iota} X \xleftarrow{j} U[p^{-1}])$$

in the derived category of étale sheaves on X_1 by repeating the arguments in loc. cit., pp. 316–322, and obtain the desired map c^i by the proper base-change theorem:

$$c^i : H^i_{\mathrm{syn}}(X(\mathscr{D}), \mathscr{S}_n(r)) \xrightarrow{c} H^i_{\text{ét}}(X_1, \iota^* Rj_* \mathbb{Z}/p^n(r)) \xleftarrow{\sim} H^i_{\text{ét}}(U[p^{-1}], \mathbb{Z}/p^n(r)).$$

See [Ts2], pp. 544–545 for the functoriality. $\square$

2.3 Symbol maps

For a scheme Z, we write $\mathscr{O}(Z)$ for $\Gamma(Z, \mathscr{O}_Z)$, for simplicity. We first review étale symbol maps. Let X be a scheme and let n be a positive integer which is invertible on X. Then we have a short exact sequence on $X_{\text{ét}}$, called *the Kummer sequence*

$$0 \longrightarrow \mathbb{Z}/n(1) \longrightarrow \mathscr{O}_X^\times \xrightarrow{\times n} \mathscr{O}_X^\times \longrightarrow 0.$$

See the beginning of §2 for the definition of the étale sheaf $\mathbb{Z}/n(1)$. Taking étale cohomology groups, we get a connecting map

$$\mathscr{O}(X)^\times/n \hookrightarrow H^1_{\text{ét}}(X, \mathbb{Z}/n(1)). \tag{2.3.1}$$

We write $\{x\}^{\text{ét}}$ for the image of $x \in \mathscr{O}(X)^\times$ under this map. Taking cup products, we obtain a map

$$(\mathscr{O}(X)^\times)^{\otimes r}/n \longrightarrow H^r_{\text{ét}}(X, \mathbb{Z}/n(r)), \tag{2.3.2}$$

which sends $x_1 \otimes x_2 \otimes \cdots \otimes x_r$ (each $x_i \in \mathscr{O}(X)^\times$) to $\{x_1\}^{\text{ét}} \cup \{x_2\}^{\text{ét}} \cup \cdots \cup \{x_r\}^{\text{ét}}$. By an argument of Tate [T], Proposition 2.1, this map annihilates Steinberg relations in $(\mathscr{O}(X)^\times)^{\otimes r}$, i.e., the elements of the form

$$x_1 \otimes x_2 \otimes \cdots \otimes x_r \;\; \text{with} \;\; x_i + x_j = 0 \text{ or } 1 \;\; \text{for some } i \neq j$$

map to 0 under the map (2.3.2). Consequently, we get a map

$$K^M_r(\mathscr{O}(X))/n \longrightarrow H^r_{\text{ét}}(X, \mathbb{Z}/n(r)), \tag{2.3.3}$$

which we call *the étale symbol map*. When X is the spectrum of a field, we often call this map *the Galois symbol map*.

Remark 2.3.4 (1) *Since we have $H^1_{\text{ét}}(X, \mathscr{O}_X^\times) \simeq \mathrm{Pic}(X)$ by Hilbert's theorem 90, the map* (2.3.1) *is bijective if X is the spectrum of a UFD* (*e.g., a field*).

(2) *If $r = 2$ and $X = \mathrm{Spec}(F)$ with F a field, then the map* (2.3.3) *is bijective by the Merkur'ev-Suslin theorem* [MS].

We next review syntomic symbol maps ([FM], p. 205, [Ka1], Chapter I, §3, [Ts1], §2.2, [Ts2], p. 542). Let (X, M) be a log scheme which is flat over $\mathbb{Z}_p$ and satisfies Condition 2.2.3. Fix an embedding system $\{i : (X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})\}_{n \geq 1}$ as in Condition 2.2.3. We define the complex C_n as

$$C_n := \left(1 + J_{D_n} \longrightarrow M_{D_n}^{\text{gp}}\right) \qquad (1 + J_{D_n} \text{ is placed in degree } 0),$$

where for a sheaf of $\mathscr{M}$ of commutative monoids, $\mathscr{M}^{\text{gp}}$ denotes the associated sheaf of abelian groups. We define the map of complexes

$$s : C_{n+1} \longrightarrow \mathscr{S}_n(1)_{(X,M),(Z_*,M_{Z_*})}$$

as the map

$$s^0 : 1 + J_{D_{n+1}} \longrightarrow J_{D_n}, \quad a \longmapsto \log(a)$$

in degree 0, and the map

$$\begin{aligned} s^1 : M^{\mathrm{gp}}_{D_{n+1}} &\longrightarrow \big(\mathscr{O}_{D_n} \otimes_{\mathscr{O}_{Z_n}} \omega^1_{(Z_n, M_{Z_n})}\big) \oplus \mathscr{O}_{D_n} \\ b \quad &\longmapsto \big(d\log(b), p^{-1}\log(b^p\varphi_{n+1}(b)^{-1})\big) \end{aligned}$$

in degree 1. Here $\varphi_{n+1}(b)b^{-p}$ belongs to $1 + p\mathscr{O}_{D_{n+1}}$ and the logarithm $\log(b^p\varphi_{n+1}(b)^{-1})$ belongs to $p\mathscr{O}_{D_{n+1}}$. The notation 'p^{-1}' means the inverse image under the isomorphism

$$p : \mathscr{O}_{D_n} \xrightarrow{\sim} p\mathscr{O}_{D_{n+1}} \qquad \text{(cf. (2.2.4))}.$$

One can easily check that the maps s^0 and s^1 yield a map of complexes. Since there is a natural quasi-isomorphism $C_{n+1} \xrightarrow{\sim} M^{\mathrm{gp}}_{n+1}[1]$, the map s induces a morphism

$$M^{\mathrm{gp}}_{n+1}[1] \longrightarrow \mathscr{S}_n(1)_{(X,M),(Z_*,M_{Z_*})}$$

in the derived category, which is independent of the choice of embedding systems.

Now suppose further that $X, \mathscr{D}$ and M be as in Example 2.2.2, and let $j : U := X - \mathscr{D} \hookrightarrow X$ be the natural open immersion. Then we have $M^{\mathrm{gp}} = j_*\mathscr{O}_U^\times$, and obtain a map

$$\mathscr{O}(U_{n+1})^\times \longrightarrow H^1_{\mathrm{syn}}(X(\mathscr{D}), \mathscr{S}_n(1)). \tag{2.3.5}$$

We often write $\{x\}^{\mathrm{syn}} \in H^1_{\mathrm{syn}}(X(\mathscr{D}), \mathscr{S}_n(1))$ for the image of $x \in \mathscr{O}(U_{n+1})^\times$. This map and the product structure of syntomic cohomology (Definition 2.2.9) give rise to a map

$$K^M_r(\mathscr{O}(U_{n+1})) \longrightarrow H^r_{\mathrm{syn}}(X(\mathscr{D}), \mathscr{S}_n(r)) \tag{2.3.6}$$

for $0 \le r \le p-1$ (cf. [Ka1], Proposition 3.2), which we call *the syntomic symbol map*.

Remark 2.3.7 *When M is the trivial log structure $\mathscr{O}_X^\times$, we obtain a symbol map*

$$K^M_r(\mathscr{O}(X_{n+1})) \longrightarrow H^r_{\mathrm{syn}}(X, \mathscr{S}_n(r))$$

for $0 \le r \le p-1$. If the identity map $X \to X$ satisfies the conditions (1)–(3) *in Condition* 2.1.2*, then the connecting homomorphism induced by* (2.1.9)

$$H^0_{\mathrm{dR}}(X_n) \longrightarrow H^1_{\mathrm{syn}}(X, \mathscr{S}_n(1))$$

sends $a \in H^0_{\mathrm{dR}}(X_n)$ to $\{1+pa\}^{\mathrm{syn}}$, where $1+pa$ is well-defined in $\mathscr{O}(X_{n+1})^\times$. One can easily check this fact directly from the definition of the symbol map.

Theorem 2.3.8 (Tsuji [Ts1], Proposition 3.2.4) *Let $X, \mathscr{D}$ and M be as in Example* 2.2.2*, and assume that (X, M) satisfies the assumptions in Theorem* 2.2.15*. Put $U := X - \mathscr{D}$. Then there is a commutative diagram*

$$\begin{tikzcd} K^M_r(\mathscr{O}(U)) \arrow[r, two heads] \arrow[dr] & K^M_r(\mathscr{O}(U_{n+1})) \arrow[r, "(2.3.6)"] & H^r_{\mathrm{syn}}(X(\mathscr{D}), \mathscr{S}_n(r)) \arrow[d, "c^r"] \\ & K^M_r(\mathscr{O}(U)[p^{-1}]) \arrow[r, "(2.3.3)"] & H^r_{\mathrm{\acute{e}t}}(U[p^{-1}], \mathbb{Z}/p^n(r)) \end{tikzcd}$$

for $0 \le r \le p-2$, where c^r denotes the canonical map in Theorem 2.2.15*.*

Proof. The case $r = 1$ follows from the same arguments as in [Ts1], Proposition 3.2.4. The general case follows from the previous case by the compatibility of these arrows with product structures (Theorem 2.2.15). $\square$

We state a syntomic analogue of the facts in Remark 2.3.4, which will be used in §4.4 below.

Theorem 2.3.9 *Let R be a henselian discrete valuation ring whose fraction field L has characteristic zero and whose residue field F has characteristic p. Then the syntomic symbol map*

$$K_r^M(R)/p^n \longrightarrow H^r_{\mathrm{syn}}(R, \mathscr{S}_n(r))$$

is surjective for $r = 2$ (and $p \geq 5$), and bijective for $r = 1$ (and $p \geq 3$).

Proof. We prove only the case $r = 2$. The case $r = 1$ follows from a similar (and simpler) arguments as below and the details are left to the reader. We first show that the following sequence of Milnor K-groups is exact:

$$K_2^M(R) \longrightarrow K_2^M(L) \xrightarrow{\partial} K_1^M(F) \longrightarrow 0, \tag{2.3.10}$$

where the first arrow is the natural pull-back of symbols. The arrow ∂ is the boundary map of Milnor K-groups, which is obviously surjective. We show the exactness at $K_2^M(L)$. Indeed, we have a localization sequence of algebraic K-groups

$$K_2(R) \longrightarrow K_2(L) \xrightarrow{d} K_1(F)$$

and natural isomorphisms $K_2(L) \simeq K_2^M(L)$ and $K_1(F) \simeq F^\times$. The arrow d agrees with ∂ (up to a sign) under these isomorphisms. Moreover the natural map $K_2^M(R) \to K_2(R)$ is surjective ([Sr], p. 17, Remark). Hence the sequence (2.3.10) is exact at $K_2^M(L)$.

Put $\eta := \mathrm{Spec}(F)$, and consider the following commutative diagram with exact rows:

$$\begin{CD}
@. K_2^M(R)/p^n @>>> K_2^M(L)/p^n @>{\partial}>> F^\times/p^n @>>> 0\\
@. @VVV @V{\wr}V{(2.3.3)}V @V{\wr}V{\mathrm{dlog}}V \\
0 @>>> H^2_{\mathrm{syn}}(R, \mathscr{S}_n(2)) @>{c^2}>> H^2_{\text{ét}}(L, \mathbb{Z}/p^n(2)) @>{\partial'}>> H^0_{\text{ét}}(\eta, W_n\Omega^1_{\eta,\log}),
\end{CD}$$

where the upper row is the exact sequence (2.3.10) modulo p^r. The arrow ∂' is the boundary map of Galois cohomology ([Ka2], §1) and the lower row is exact by Kurihara [Ku]. The central vertical arrow is bijective by Remark 2.3.4 (2). The right vertical arrow is bijective by the short exact sequence

$$0 \longrightarrow \mathscr{O}_\eta^\times \xrightarrow{\times p^n} \mathscr{O}_\eta^\times \xrightarrow{\mathrm{dlog}} W_n\Omega^1_{\eta,\log} \longrightarrow 0$$

on $\eta_{\text{ét}}$ and Remark 2.3.4 (1). Thus we obtain the lemma. $\square$

We end this section with the following standard fact on Milnor K-groups, which will be used in the proof of Proposition 3.4.4 below:

Proposition 2.3.11 *Let R be a henselian discrete valuation ring whose residue field F has characteristic p. Let L be the fraction field of R. Assume that F is infinite and that $\Omega^2_{F,\log} = 0$. Then the natural map $K_2^M(R)/p^n \to K_2^M(L)/p^n$ is injective for any $n \geq 1$.*

Proof. Since F is infinite, we have $K_2^M(R) = K_2(R)$ by a theorem of van der Kallen ([Sr], p. 17, Remark). There is a localization exact sequence of algebraic K-groups

$$K_2(F) \longrightarrow K_2(R) \longrightarrow K_2(L) \longrightarrow K_1(F)(= F^\times).$$

The last arrow is surjective, as R is a principal ideal domain. We decompose this sequence into two exact sequences

$$\begin{gathered} K_2(F) \longrightarrow K_2(R) \longrightarrow M \longrightarrow 0, \\ 0 \longrightarrow M \longrightarrow K_2(L) \longrightarrow F^\times \longrightarrow 0. \end{gathered}$$

Since $F^\times$ is p-torsion-free, we have $M/p^n \hookrightarrow K_2(L)/p^n$. On the other hand, we have

$$K_2(F)/p^n = K_2^M(F)/p^n = 0$$

by the assumption that $\Omega^2_{F,\log} = 0$ ([BK1], Theorem 2.1) and $K_2(R)/p^n \simeq M/p^n$. Hence

$$K_2^M(R)/p^n = K_2(R)/p^n \simeq M/p^n \hookrightarrow K_2(L)/p^n = K_2^M(L)/p^n$$

as required. □

By Theorem 2.3.9 and Proposition 2.3.11, we obtain the following consequence, which will not be used in the rest of this paper:

Corollary 2.3.12 *Let R be as in Theorem* 2.3.9. *Assume $p \geq 5$ and that the residue field F satisfies the assumptions in Proposition* 2.3.11. *Then the symbol map $K_2^M(R)/p^n \to H^2_{\mathrm{syn}}(R, \mathscr{S}_n(2))$ is bijective.*

2.4 Tate curve

Let B be a noetherian complete local ring with $6^{-1} \in B$. Let $q \in B$ be an element which is contained in the maximal ideal of B and not nilpotent. Put

$$A := B[q^{-1}].$$

The Tate curve $E = E_q$ over A with period q is the projective completion in $\mathbb{P}^2_A$ of the affine curve

$$y^2 + xy = x^3 + a_4(q)x + a_6(q) \quad \text{on} \quad \mathrm{Spec}(A[x, y]),$$

where $a_4(q)$ and $a_6(q) \in B$ are defined as follows:

$$a_4(q) = -5\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \quad a_6(q) = -\sum_{n=1}^{\infty}\frac{(5n^3+7n^5)q^n}{12(1-q^n)}.$$

We review the Tate parameterization of E. Let $O \in E(A)$ be the infinite point. The series

$$x(\alpha) = \sum_{n\in\mathbb{Z}} \frac{q^n\alpha}{(1-q^n\alpha)^2} - 2\sum_{n\geq 1} \frac{q^n}{(1-q^n)^2}$$
$$y(\alpha) = \sum_{n\in\mathbb{Z}} \frac{(q^n\alpha)^2}{(1-q^n\alpha)^3} + \sum_{n\geq 1} \frac{q^n}{(1-q^n)^2}$$

converge for all $\alpha \in A^\times - q^{\mathbb{Z}}$. They induce an injective homomorphism

$$A^\times/q^{\mathbb{Z}} \longrightarrow E(A), \quad \alpha \longmapsto \begin{cases}(x(\alpha):y(\alpha):1) & (\alpha \notin q^{\mathbb{Z}})\\ O & (\alpha \in q^{\mathbb{Z}}),\end{cases}$$

which is bijective if B is a complete discrete valuation ring ([Si], V, Theorem 3.1). This assignment is not algebraic, but we have the following algebraic byproduct. Let u be an indeterminate, and put

$$\mathscr{B}_0 := \varprojlim_n B[u,u^{-1}]/(q^n) \quad \text{and} \quad \mathscr{A}_0 := \mathscr{B}_0[q^{-1}].$$

Proposition 2.4.1 *There is an injective homomorphism of B-algebras*

$$B[x,y]/(y^2+xy-x^3-a_4(q)x-a_6(q)) \longrightarrow \mathscr{B}_0[(1-u)^{-1}]$$

given by the 'q-adic presentation' of $(x(u),y(u))$:

$$x \longmapsto \frac{u}{(1-u)^2} + \sum_{d\geq 0}\sum_{m|d} m\big(u^m+u^{-m}-2\big)q^d$$
$$y \longmapsto \frac{u^2}{(1-u)^3} + \sum_{d\geq 0}\sum_{m|d} m\left(\frac{m-1}{2}\,u^m - \frac{m+1}{2}\,u^{-m}+1\right)q^d.$$

Moreover, this ring homomorphism induces a dominant morphism of schemes

$$\beta_0 : \mathrm{Spec}(\mathscr{A}_0) \to E.$$

Proof. The assertion follows from the same arguments as in [Si], p. 425, Proof of Theorem 3.1 (c). $\square$

Definition 2.4.2 (1) *We define the canonical invariant* 1*-form* $\omega_E \in \mathit{\Gamma}(E, \Omega^1_E)$ *as*

$$\omega_E := \frac{dx}{2y+x},$$

which maps to $u^{-1}du$ *under the pull-back map*

$$\beta_0^* : \mathit{\Gamma}(E, \Omega^1_E) \hookrightarrow \Omega^1_{\mathscr{A}_0}.$$

(2) *We define the theta function* $\theta(u,q) \in \mathscr{B}_0$ *as*

$$\theta(u,q) := (1-u)\prod_{n=1}^{\infty}(1-q^n u)(1-q^n u^{-1}), \tag{2.4.3}$$

which satisfies

$$\theta(qu,q) = \theta(u^{-1},q) = -u^{-1}\theta(u,q).$$

The following proposition follows from standard facts on theta functions and theta divisors, whose details are left to the reader.

Proposition 2.4.4 *Put* $K := \mathrm{Frac}(B)$*, and let* $K(E)$ *be the function field of* E*.*

(1) *A function* $f(u) \in \mathrm{Frac}(\mathscr{B}_0)$ *given by a finite product*

$$f(u) = c\prod_i \frac{\theta(\alpha_i u, q)}{\theta(\beta_i u, q)} \qquad (c, \alpha_i, \beta_i \in A^\times) \tag{2.4.5}$$

is q*-periodic, if* $\prod_i \ \alpha_i/\beta_i = 1$*.*

(2) *The ring homomorphism in Proposition* 2.4.1 *induces a natural inclusion*

$$\left\{ c\prod_i \frac{\theta(\alpha_i u, q)}{\theta(\beta_i u, q)} \,\middle|\, c, \alpha_i, \beta_i \in A^\times \text{ with } \prod_i \ \alpha_i/\beta_i = 1 \right\} \hookrightarrow K(E)^\times.$$

Proposition 2.4.6 *Let* f *be the structural morphism* $E[p^{-1}] \to \mathrm{Spec}(A[p^{-1}])$*. Then there is an exact sequence of étale sheaves on* $\mathrm{Spec}(A[p^{-1}])$

$$0 \longrightarrow \mathbb{Z}/p^n(1) \longrightarrow R^1 f_* \mathbb{Z}/p^n(1) \longrightarrow \mathbb{Z}/p^n \longrightarrow 0. \tag{2.4.7}$$

Proof. See [DR], VII.1.13. □

2.5 Frobenius endomorphism on Tate curves

Let p be a prime number at least 5, and let B, A and q be as in the beginning of §2.4. We assume here that B is a $\mathbb{Z}_p$-algebra, and that B has a Frobenius endomorphism ϕ. Here a Frobenius endomorphism means a ring endomorphism compatible with the absolute Frobenius endomorphism on $B/(p)$. Let $E_q = E_{q,A}$ be the Tate curve over A with period q. There is a canonical morphism

$$\mathrm{can} : E_{\phi(q)} \longrightarrow E_q$$

induced by the ring homomorphism

$$\begin{aligned} A[x,y]/(y^2 + xy - x^3 &- a_4(q)x - a_6(q)) \\ &\longrightarrow A[x,y]/(y^2 + xy - x^3 - a_4(\phi(q))x - a_6(\phi(q))) \end{aligned}$$

sending

$$x \mapsto x, \qquad y \mapsto y, \qquad a \mapsto \phi(a) \quad (a \in A).$$

We define $W : E_{\phi(q)} \to E_q$ as the composite

$$W : E_{\phi(q)} \xrightarrow{\mathrm{can}} E_q \xrightarrow{[p]} E_q,$$

where $[p]$ denotes the multiplication by p (with respect to the group structure on E_q).

Remark 2.5.1 *Under the map in Proposition* 2.4.1*, the map W corresponds to the endomorphism of* $\mathrm{Frac}(\mathscr{B}_0)$ *sending*

$$u \mapsto u^p, \qquad b \mapsto \phi(b) \quad (b \in B).$$

In what follows, we assume that

$$\phi(q) = q^p \quad \text{and} \quad \phi(a) = a \quad (a \in \mathbb{Z}_p).$$

There is an isogeny over $\mathrm{Spec}(A)$

$$\iota : E_{q^p} \longrightarrow E_q$$

corresponding to the identity map of $\mathrm{Frac}(\mathscr{B}_0)$, which is an analogue of the natural projection $\mathbb{C}^\times/q^{p\mathbb{Z}} \to \mathbb{C}^\times/q^{\mathbb{Z}}$. The morphism W factors through ι. Indeed, ι is surjective and we have

$$\mathrm{Ker}(\iota) \subset \mathrm{Ker}([p] : E_{q^p} \to E_{q^p})$$

as finite étale group schemes over $\mathrm{Spec}(A)$. Thus W gives rise to an endomorphism

$$\varphi : E_q \longrightarrow E_q. \tag{2.5.2}$$

We show that φ is a Frobenius endomorphism:

Lemma 2.5.3 *Put* $(E_q)_1 := E_q \otimes_A A/(p) = E_q \otimes \mathbb{F}_p$*. Then the morphism* $\varphi_1 : (E_q)_1 \to (E_q)_1$ *induced by* φ *is the absolute Frobenius endomorphism of* $(E_q)_1$.

Proof. Define the Frobenius endomorphism ϕ on $\mathbb{Z}_p((q)) := \mathbb{Z}_p[[q]][q^{-1}]$ as $\phi(q) = q^p$ and $\phi(a) = a$ for $a \in \mathbb{Z}_p$. The natural ring homomorphism $\mathbb{Z}_p((q)) \to A$ is compatible with the Frobenius endomorphisms. Note that $E_q = E_{q,A}$ is obtained from $E' = E_{q,\mathbb{Z}_p((q))}$ by scalar extension, and that we have

$$\varphi = \varphi' \otimes \phi$$

under the identification $E_q = E' \otimes_{\mathbb{Z}_p((q))} A$. Here φ' mean the map (2.5.2) defined for E'. Therefore it is enough to consider the case that $A = \mathbb{Z}_p((q))$. Then φ_1 agrees with φ'', the map (2.5.2) defined for $E'' = E_{q,\mathbb{F}_p((q))}$. Under the morphism β_0 in Proposition 2.4.1 with $B = \mathbb{F}_p[[q]]$, φ'' corresponds to the absolute Frobenius endomorphism of the field

$$\mathscr{A}_0 = \mathscr{B}_0[q^{-1}] = \left(\varprojlim_n \mathbb{F}_p[[q]][u, u^{-1}]/(q^n) \right) [q^{-1}]$$

(cf. Remark 2.5.1), which shows that φ'' is the absolute Frobenius morphism. $\square$

3 De Rham regulator of Tate curves

In this section, we assume $p \geq 5$. The main result of this section will be stated in Theorem 3.2.3 below.

3.1 Setting

Let R be a p-adic integer ring which is unramified over $\mathbb{Z}_p$, and let k be the residue field of R. Let q_0 be an indeterminate. We define the rings A and B as

$$B := R[\![q_0]\!], \qquad A := B[q_0^{-1}].$$

B is a 2-dimensional regular complete local domain, and A is a Dedekind domain. Let $\widehat{A}$ be the p-adic completion of A:

$$\widehat{A} := \varprojlim_{n \geq 1} A/p^n,$$

which is a complete discrete valuation ring whose maximal ideal is generated by p. Put

$$L := \widehat{A}[p^{-1}].$$

Let r be a positive integer prime to p, and put $q := q_0^r$. Let $E = E_q$ be the Tate curve over $\mathrm{Spec}(A)$ with period q. Let $\mathscr{E}'$ be the projective curve over $\mathrm{Spec}(B)$ defined by the following homogeneous equation in $\mathbb{P}^2_B = \mathrm{Proj}(B[x,y,z])$:

$$\mathscr{E}' : y^2z + xyz = x^3 + a_4(q)\,xz^2 + a_6(q)\,z^3$$

(see §2.1 for $a_4(q)$ and $a_6(q)$), which is a projective flat model of E over B. By blowing-up $\mathscr{E}'$ along the locus $\{x = y = q_0 = 0\}$ up to $(r-1)$-times, we get a regular scheme $\mathscr{E}$, which is a generalized elliptic curve in the sense of [DR], II.1.12. The divisor $\mathscr{D} := \{q_0 = 0\} \subset \mathscr{E}$ is the standard Néron r-gon over $\mathrm{Spec}(R)$, and the structural morphism $\mathscr{E} \to \mathrm{Spec}(B)$ is smooth outside of the intersection loci of two distinct irreducible components of $\mathscr{D}$. There is a cartesian diagram

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "\pi_E"'] \arrow[dr, phantom, "\square"] & \mathscr{E} \arrow[d, "\pi_{\mathscr{E}}"] \\ \mathrm{Spec}(A) \arrow[r, hook] & \mathrm{Spec}(B), \end{tikzcd}$$

where π_E is projective smooth and $\pi_{\mathscr{E}}$ is projective flat. The horizontal arrows are open embeddings.

Lemma 3.1.1 *Let M be the log structure on $\mathscr{E}$ associated with $\mathscr{D}$, as in Example* 2.2.2. *Then the identity maps $(\mathscr{E}_n, M_n) \to (\mathscr{E}_n, M_n)\,(n \geq 1)$ satisfy the conditions* (0)–(3) *in Condition* 2.2.3. *Consequently, the log syntomic cohomology $H^*_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_n(r))$ is defined for $0 \leq r \leq p-1$ (Definition* 2.2.7).

Proof. The condition (0) is obvious, and (3) follows from Proposition 2.2.6. The condition (1) follows from Example 2.2.5, and (2) follows from Lemma 4.1.3 below. $\square$

Remark 3.1.2 *Similarly, the identity map $\mathscr{E} \to \mathscr{E}$ satisfies the conditions* (1)–(3) *in Condition* 2.1.2*, by Example* 2.1.4 *and Lemma* 4.1.3 *below. Consequently, the syntomic cohomology $H^*_{\mathrm{syn}}(\mathscr{E}, \mathscr{S}_n(r))$ $(0 \le r \le p-1)$ is computed by the distinguished triangle*

$$0 \longrightarrow \mathscr{S}_n(r)_{\mathscr{E},\mathscr{E}} \longrightarrow \Omega^{\bullet \ge r}_{\mathscr{E}_n} \xrightarrow{1-f_r} \Omega^{\bullet}_{\mathscr{E}_n} \longrightarrow 0$$

by Remark 2.1.8.

We put

$$H^*_{\mathrm{syn}}(-, \mathscr{S}_{\mathbb{Z}_p}(r)) := \varprojlim_{n\ge 1} H^*_{\mathrm{syn}}(-, \mathscr{S}_n(r)), \qquad H^*_{\text{ét}}(-, \mathbb{Z}_p(r)) := \varprojlim_{n\ge 1} H^*_{\text{ét}}(-, \mathbb{Z}/p^n(r)).$$

We define the map of Kähler differential forms

$$\tau^{\mathrm{dR}}_{\infty} : \Gamma(E, \Omega^2_E) \longrightarrow \Omega^1_A$$

as the composite of natural isomorphisms

$$\Gamma(E, \Omega^2_E) \simeq \Gamma(E, \Omega^1_A \otimes_A \Omega^1_{E/A}) \simeq \Omega^1_A \otimes_A \Gamma(E, \Omega^1_{E/A})$$

and the map

$$\Omega^1_A \otimes_A \Gamma(E, \Omega^1_{E/A}) \longrightarrow \Omega^1_A, \quad \eta \otimes \omega_E \mapsto \eta \quad (\eta \in \Omega^1_A).$$

Here ω_E denotes the canonical invariant 1-form on E defined in Definition 2.4.2 (1), and we have used the fact that $\Gamma(E, \Omega^1_{E/A})$ is a free A-module generated by ω_E.

3.2 Main result on Tate curves

Let $\zeta_1, \dots, \zeta_d \in R$ be roots of unity which form a basis of R over $\mathbb{Z}_p$, whose existence is verified by the assumption that R is absolutely unramified. An arbitrary formal Laurant power series $f(q_0) \in R[\![q_0, q_0^{-1}]\!]$ is expanded into a power series of the form

$$f(q_0) = \sum_{j\le 0} b_j\, q_0^j + \sum_{j\ge 1} \left(\frac{a_{1j}\, \zeta_1\, q_0^j}{1 - \zeta_1\, q_0^j} + \cdots + \frac{a_{dj}\, \zeta_d\, q_0^j}{1 - \zeta_d\, q_0^j} \right) \quad (a_{ij} \in \mathbb{Z}_p,\ b_j \in R) \tag{3.2.1}$$

and the coefficients a_{ij} and b_j are uniquely determined by $f(q_0)$. We say that $f(q_0)$ is a *formal power series of Eisenstein type* (in $R[\![q_0, q_0^{-1}]\!]$) if

(E1) $c_k = 0$ for $k < 0$ and $c_0 \in \mathbb{Z}_p$,

(E2) $a^{(j)}_k \in k^2\mathbb{Z}_p$ for all j and $k \ge 1$.

The condition (E2) does not depend on the choice of ζ_i ([A2], Lemma 3.4). Moreover, $f(q_0) \in R[\![q_0, q_0^{-1}]\!]$ is of Eisenstein type if and only if so is it in $R'[\![q_0, q_0^{-1}]\!]$ for an p-adic integer ring R' which is unramified over R.

Remark 3.2.1 *The reason we call "Eisenstein" is the following fact. Suppose that $f(q_0)$ is the q_0-expansion of a modular form of weight 3 at a cusp. Then $f(q_0)$ is of Eisenstein type if and only if it is a linear combination of the usual Eisenstein series of weight 3* ([A2], §8.3).

The main result of this section deals with the image of a 'de Rham regulator map' from syntomic cohomology

$$\mathrm{reg}_{\mathrm{dR}} : H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2)) \xrightarrow{\epsilon^2} \varprojlim_{n\geq 1} \Gamma(E_n, \Omega^2_{E_n}) \xrightarrow{\tau^{\mathrm{dR}}_\infty} \Omega^1_{\widehat{A}} \xrightarrow{f\cdot \frac{dq_0}{q_0} \mapsto f} \widehat{A},$$

where $\tau^{\mathrm{dR}}_\infty$ denote the map defined in §3.1. See Proposition 2.2.11 for ϵ^2.

Theorem 3.2.3 *Assume that $f(q_0) \in \widehat{A}$ is contained in the image of $\mathrm{reg}_{\mathrm{dR}}$. Then it is a formal series of Eisenstein type.*

3.3 A commutative diagram

In the rest of this section, we prove Theorem 3.2.3 assuming a commutative diagram (3.3.1) below. For $n \geq 1$, let $\tau^{\text{ét}}_\infty$ be the composite of canonical maps of étale cohomology groups

$$\tau^{\text{ét}}_\infty : H^2_{\text{ét}}(E_L, \mathbb{Z}/p^n(2)) \longrightarrow H^1_{\text{ét}}(L, H^1_{\text{ét}}(E_{\overline{L}}, \mathbb{Z}/p^n(2))) \longrightarrow H^1_{\text{ét}}(L, \mathbb{Z}/p^n(1)).$$

Here the second arrow is induced by the map $H^1_{\text{ét}}(E_{\overline{L}}, \mathbb{Z}/p^n(1)) \to \mathbb{Z}/p^n$ in (2.4.7). The first arrow is obtained from the Hochschild-Serre spectral sequence

$$E_2^{a,b} = H^a_{\text{ét}}(L, H^b_{\text{ét}}(E_{\overline{L}}, \mathbb{Z}/p^n(2))) \Longrightarrow H^{a+b}_{\text{ét}}(E_L, \mathbb{Z}/p^n(2))$$

and the fact that

$$E_2^{0,2} \simeq H^0_{\text{ét}}(L, \mathbb{Z}/p^n(1)) = 0,$$

where we have used the assumption that R is unramified over $\mathbb{Z}_p$.

A key ingredient of the proof of Theorem 3.2.3 is the following commutative diagram:

$$\begin{tikzcd}
H^2_{\text{ét}}(E_L, \mathbb{Z}_p(2)) \arrow[r, "\tau^{\text{ét}}_\infty"] & H^1_{\text{ét}}(L, \mathbb{Z}_p(1)) \\
H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[u, "c^2"] \arrow[r, "\tau^{\mathrm{syn}}_\infty"] \arrow[d, "\epsilon^2"'] & H^1_{\mathrm{syn}}(A, \mathscr{S}_{\mathbb{Z}_p}(1)) \arrow[u, "c^1"'] \arrow[d, "d\log"] \\
\varprojlim_{n\geq 1} \Gamma(E_n, \Omega^2_{E_n}) \arrow[r, "\tau^{\mathrm{dR}}_\infty"] & \Omega^1_{\widehat{A}}.
\end{tikzcd} \tag{3.3.1}$$

Here c^1 and c^2 are canonical maps in Theorem 2.2.15, and $d\log$ is given by logarithmic differentials (see Theorem 2.3.9 for the isomorphism s):

$$H^1_{\mathrm{syn}}(A, \mathscr{S}_{\mathbb{Z}_p}(1)) \to H^1_{\mathrm{syn}}(\widehat{A}, \mathscr{S}_{\mathbb{Z}_p}(1)) \overset{s}{\simeq} \varprojlim_{n\geq 1} \widehat{A}^\times/p^n \xrightarrow{f \mapsto \frac{df}{f}} \varprojlim_{n\geq 1} \Omega^1_{\widehat{A}/p^n} = \Omega^1_{\widehat{A}}\ .$$

We will construct $\tau_\infty^{\rm syn}$ in §4.2 and prove the commutativity of the squares in §4.4 below.

As a preliminary of the proof of Theorem 3.2.3, we prove Lemma 3.3.2 below. For $n \geq 1$, let ν be the isomorphism

$$\nu : H^1_{\text{ét}}(L, \mathbb{Z}/p^n(1)) \simeq L^\times/p^n.$$

in Remark 2.3.4 (1).

Lemma 3.3.2 *For any $n \geq 0$, the composite map*

$$H^2_{\text{ét}}(E_L, \mathbb{Z}/p^n(2)) \xrightarrow{\nu\circ\tau_\infty^{\text{ét}}} L^\times/p^n \xrightarrow{a\mapsto\{a,q_0\}} K_2^M(L)/p^n$$

is zero.

Proof. Since $q = q_0^r$ and $(r, p) = 1$ by definition, we may replace the second arrow with the assignment $a \mapsto \{a, q\}$. We consider the following commutative diagram, whose top row is an exact sequence arising from (2.4.7):

$$\begin{tikzcd}
H^1_{\text{ét}}(L, H^1_{\text{ét}}(E_{\overline{L}}, \mathbb{Z}/p^n(2))) \arrow[r] & H^1_{\text{ét}}(L, \mathbb{Z}/p^n(1)) \arrow[r, "(1)"] \arrow[d, "\nu", "\wr"'] & H^2_{\text{ét}}(L, \mathbb{Z}/p^n(2)) \\
H^2_{\text{ét}}(E_L, \mathbb{Z}/p^n(2)) \arrow[u] \arrow[r, "\nu\circ\tau_\infty^{\text{ét}}"] & L^\times/p^n & K_2^M(L)/p^n. \arrow[u, "(2)"', "\wr"]
\end{tikzcd}$$

Here the arrow (2) is a Galois symbol map (2.3.3) (see also Remark 2.3.4). We claim that the arrow (1) maps $a \in H^1_{\text{ét}}(L, \mathbb{Z}/p^n(1))$ to $a \cup \{q\}^{\text{ét}}$ up to a sign. Indeed, the connecting homomorphism

$$\mathbb{Z}/p^n = H^0_{\text{ét}}(L, \mathbb{Z}/p^n) \longrightarrow H^1_{\text{ét}}(L, \mathbb{Z}/p^n(1))$$

associated with (2.4.7) sends 1 to $\{q\}^{\text{ét}}$, and the claim follows from a straight-forward computation on cup products. The lemma follows from these facts. $\square$

3.4 Proof of Theorem 3.2.3

We prove of Theorem 3.2.3, assuming the diagram (3.3.1). Assume that $f(q_0) \in \widehat{A}$ lies in the image of $\mathrm{reg}_{\mathrm{dR}}$. By the lower square of the diagram (3.3.1), there exists $h(q_0) \in \widehat{A}^\times$ such that

$$\frac{dh(q_0)}{h(q_0)} = f(q_0)\,\frac{dq_0}{q_0} \in \Omega^1_{\widehat{A}}\,. \tag{3.4.1}$$

We fix such an $h(q_0)$ in what follows. By Lemma 3.3.2 and the upper square of the diagram (3.3.1), $h(q_0)$ must satisfy

$$\{h(q_0), q_0\} = 0 \in K_2^M(L)/p^n \ \text{ for any } n \geq 0. \tag{3.4.2}$$

Now expand $f(q_0)$ into a series of the form (3.2.1):

$$f(q_0) = \sum_{j\leq 0} b_j\, q_0^j + \sum_{j\geq 1}\left(\frac{a_{1j}\,\zeta_1\, q_0^j}{1-\zeta_1\, q_0^j} + \cdots + \frac{a_{dj}\,\zeta_d\, q_0^j}{1-\zeta_d\, q_0^j}\right) \qquad (a_{ij} \in \mathbb{Z}_p,\ b_j \in R)$$

and expand $h(q_0)$ into an infinite product of the following form:

$$h(q_0) = c\, q_0^m \prod_{j\geq 1} (1-\zeta_1\, q_0^j)^{a'_{1j}} \cdots (1-\zeta_d\, q_0^j)^{a'_{dj}} \qquad (a'_{ij}\in \mathbb{Z}_p,\ c\in R^{\times},\ m\in\mathbb{Z}).$$

By (3.4.1), we see that

$$a_{ij} = -j a'_{ij}, \tag{3.4.3}$$

hence a_{ij} is divisible by j for any $j \geq 1$. We next write down what the equation (3.4.2) yields, using the explicit reciprocity law of higher-dimensional regular local rings due to Kato [Ka4]:

Proposition 3.4.4 *a'_{ij} is divisible by j in $\mathbb{Z}_p$ for any $j \geq 0$.*

Theorem 3.2.3 immediately follows from this proposition and (3.4.3).

Proof of Proposition 3.4.4. Note that the symbol $\{h(q_0), q_0\}$ is defined in $K_2^M\big(\widehat{A}\big)$. Since $\Omega^2_{k((q_0))} = 0$, we have

$$\{h(q_0), q_0\} = 0 \in K_2^M\big(\widehat{A}\big)/p^n \ \text{ for any } n\geq 1, \tag{3.4.5}$$

by (3.4.2) and Proposition 2.3.11. Let $\phi : \widehat{A} \to \widehat{A}$ be the Frobenius endomorphism of $\widehat{A}$ defined by the canonical Frobenius automorphism on R and the assignment $q_0 \mapsto q_0^p$. Define the function $\ell_\phi : \widehat{A}\to\widehat{A}$ as

$$\ell_\phi(a) := \frac{1}{p}\log\left(\frac{\phi(a)}{a^p}\right).$$

It follows from (3.4.5) and the explicit reciprocity law ([Ka4], Corollary 2.9), that

$$\ell_\phi(h(q_0))\frac{dq_0}{q_0} - \ell_\phi(q_0)\frac{dh(q_0)}{h(q_0)} \in d\widehat{A}.$$

Since $\ell_\phi(q_0)=0$ by definition, there exists $\alpha\in\widehat{A}$ satisfying

$$\ell_\phi(h(q_0)) = q_0\frac{d\alpha}{dq_0}. \tag{3.4.6}$$

Because $\ell_\phi(fg) = \ell_\phi(f)+\ell_\phi(g)$ and

$$\ell_\phi(1-rq_0) = \sum_{(n,p)=1}\frac{(rq_0)^n}{n} \qquad (r\in R)$$

by definition, we have

$$\ell_\phi(h(q_0)) = \ell_\phi(c) + \sum_{j\geq 1}\sum_{(n,p)=1}\left(a'_{1j}\frac{(\zeta_1\, q_0^j)^n}{n} + \cdots + a'_{dj}\frac{(\zeta_d\, q_0^j)^n}{n}\right).$$

Comparing these coefficients with those in the right hand side of (3.4.6), one can easily check that a'_{ij} is divisible by j in $\mathbb{Z}_p$. $\square$

Thus we obtained Theorem 3.2.3, assuming the diagram (3.3.1).

4 Construction of the key diagram

The notation remains as in §3.1. In this section, we construct the key homomorphism

$$\tau_\infty^{\mathrm{syn}} : H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2)) \to H^1_{\mathrm{syn}}(A, \mathbb{Z}_p(1))$$

to establish the commutative diagram (3.3.1).

4.1 Preliminary

We define the Frobenius map $\phi : B \to B$ as

$$\phi(q_0) = q_0^p \quad \text{and} \quad \phi(a) = \sigma(a) \quad (a \in R), \tag{4.1.1}$$

where σ denotes the canonical Frobenius automorphism of R. Let $\mathscr{B}$ (resp. $\mathscr{R}$) be the (q_0, p)-adic completion of $B[u, u^{-1}]$ (resp. p-adic completion of $R[u, u^{-1}]$):

$$\mathscr{B} := \varprojlim_n B[u, u^{-1}]/(q_0, p)^n, \qquad \mathscr{R} := \varprojlim_n R[u, u^{-1}]/(p^n).$$

We extend the Frobenius map ϕ on B to $\mathscr{B}$ by defining $\phi(u) := u^p$. We define the Frobenius map ϕ on $\mathscr{R}$ as $\phi(u) := u^p$ and $\phi(a) = \sigma(a)\,(a \in R)$. The ring homomorphism in Proposition 2.4.1 induces a morphism of schemes

$$\beta : \mathrm{Spec}(\mathscr{B}) \longrightarrow \mathscr{E}, \tag{4.1.2}$$

where $\mathscr{E}$ is as we defined in §3.1. For a scheme (or a ring) Z and $n \geq 1$, we put

$$Z_n := Z \otimes \mathbb{Z}/(p^n).$$

We have a Frobenius endomorphism $\varphi : E \to E$ by (4.1.1) and the construction in §2.5.

Lemma 4.1.3 *For any integer $n \geq 1$, the Frobenius endomorphism $\varphi_n : E_n \to E_n$ extends to a Frobenius endomorphism $\varphi_n : (\mathscr{E}_n, M_n) \to (\mathscr{E}_n, M_n)$. Furthermore, $\varphi_{n+1} \otimes \mathbb{Z}/p^n$ agrees with φ_n for any $n \geq 1$.*

This lemma has been used in Lemma 3.1.1.

Proof. Let Z be the singular locus of $\mathscr{D}$ (see §3.1 for the definition of $\mathscr{D}$). Note that $X := \mathscr{E} - Z$ is a commutative group scheme over $\mathrm{Spec}(B)$. In particular, the multiplication by p is defined on X with respect to its group structure, and there is an endomorphism

$$\varphi : X \longrightarrow X$$

defined in a similar way as for φ on E. By Lemma 2.5.3, φ is a Frobenius endomorphism of X, i.e., $\varphi_1 : X_1 \to X_1$ induced by φ is the absolute Frobenius of X_1, because X_1 and

$E_1 = (E_q)_1$ have the same function field. Since $\mathscr{D}$ is defined by q_0 and φ sends q_0 to q_0^p, the map $\varphi : X \to X$ induces a Frobenius endomorphism

$$\varphi : (X, M_X) \longrightarrow (X, M_X)$$

of log schemes, where M_X denotes the log structure on X associated with the regular divisor $\mathscr{D} - Z$ (see Example 2.2.2).

We start the proof of the lemma. Because a Frobenius endomorphism on $\mathscr{E}_n$ is the identity map on the underlying topological space, we define an endomorphism $\psi : \mathscr{O}_{\mathscr{E}_n} \to \mathscr{O}_{\mathscr{E}_n}$ which lifts the absolute endomorphism of $\mathscr{O}_{\mathscr{E}_1}$ and which is compatible with φ_n^* on $\mathscr{O}_{X_n}$. Let $\alpha : X \hookrightarrow \mathscr{E}$ and $\alpha_n : X_n \hookrightarrow \mathscr{E}_n$ be the natural open embeddings. We first show that the natural adjunction map

$$\mathscr{O}_{\mathscr{E}_n} \longrightarrow \alpha_{n*}\mathscr{O}_{X_n} \tag{4.1.4}$$

is bijective. Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_{\mathscr{E}} \arrow[r, "\times p"] \arrow[d, "\wr"'] & \mathscr{O}_{\mathscr{E}} \arrow[r] \arrow[d, "\wr"'] & \mathscr{O}_{\mathscr{E}_1} \arrow[r] \arrow[d, "\wr"'] & 0 & \\
0 \arrow[r] & \alpha_*\mathscr{O}_X \arrow[r, "\times p"] & \alpha_*\mathscr{O}_X \arrow[r] & \alpha_{1*}\mathscr{O}_{X_1} \arrow[r] & R^1\alpha_*\mathscr{O}_X \arrow[r, "\times p"] & R^1\alpha_*\mathscr{O}_X .
\end{tikzcd}$$

Here the vertical arrows are natural adjunction maps, which are bijective by the facts that $\mathscr{E}$ and $\mathscr{E}_1$ are regular and that the complement Z (resp. Z_1) has pure codimension 2 in $\mathscr{E}$ (resp. in $\mathscr{E}_1$). Hence $R^1\alpha_*\mathscr{O}_X$ is p-torsion-free and we obtain the bijectivity of (4.1.4) by repeating a similar argument using $\mathscr{O}_{\mathscr{E}_n}$ and $\alpha_{n*}\mathscr{O}_{X_n}$ instead of $\mathscr{O}_{\mathscr{E}_1}$ and $\alpha_{1*}\mathscr{O}_{X_1}$. We define $\psi : \mathscr{O}_{\mathscr{E}_n} \to \mathscr{O}_{\mathscr{E}_n}$ as the composite map

$$\mathscr{O}_{\mathscr{E}_n} \simeq \alpha_{n*}\mathscr{O}_{X_n} \xrightarrow{\alpha_{n*}(\varphi_n^*)} \alpha_{n*}(\mathscr{O}_{X_n}) \simeq \mathscr{O}_{\mathscr{E}_n}.$$

It is clear that ψ lifts the absolute Frobenius endomorphism of $\mathscr{O}_{\mathscr{E}_1}$. Thus we obtain a Frobenius endomorphism φ_n on $\mathscr{E}_n$ which extends φ_n on X_n, and it is easy to see that this induces a Frobenius endomorphism φ_n on $(\mathscr{E}_n, M_n)$. By the construction, this morphism is the only morphism that extends φ_n on (X_n, M_{X_n}), and the compatibility assertion $\varphi_{n+1} \otimes \mathbb{Z}/p^n = \varphi_n$ follows from the fact that φ_n on (X_n, M_{X_n}) is induced by φ on (X, M_X). $\square$

4.2 Syntomic residue mapping

To construct $\tau_\infty^{\mathrm{syn}}$, we first review the definition of residue mappings.

Definition 4.2.1 *Let m be a positive interger, and put $C := B/(q_0, p)^m$ and $C((u)) := C[\![u]\!][u^{-1}]$. We define residue mappings*

$$\mathrm{Res}_{u=0} : \Omega^r_{C((u))} \longrightarrow \Omega^{r-1}_C$$

as follows. For $r = 1$, we define $\mathrm{Res}_{u=0}$ as the composite map

$$\Omega^1_{C((u))} \xrightarrow{\mathrm{can}} \Omega^1_{C((u))/C} \simeq C((u)) \cdot du \longrightarrow C$$

where the last arrows sends $c_i u^i du$ $(c_i \in C)$ *to* 0 *if* $i \neq -1$, *and to* c_{-1} *if* $i = -1$. *For* $r \geq 2$, *we define* $\mathrm{Res}_{u=0}$ *as*

$$\Omega^r_{C(\!(u)\!)} \xrightarrow{\text{can}} \Omega^{r-1}_C \otimes \Omega^1_{C(\!(u)\!)/C} \xrightarrow{\mathrm{id}\otimes\mathrm{Res}_{u=0}} \Omega^{r-1}_C.$$

Since $\mathscr{A}$ has a Frobenius endomorphism ϕ (cf. §4.1), the syntomic cohomology groups $H^*_{\mathrm{syn}}(\mathscr{A}, \mathscr{S}_n(2))$ are computed by the complex

$$\mathscr{S}_n(2)_{\mathscr{A},\mathscr{A}} := \text{mapping fiber of } \left(1 - f_2 : \Omega^{\bullet \geq 2}_{\mathscr{A}_n} \longrightarrow \Omega^\bullet_{\mathscr{A}_n}\right)$$

(see Remark 2.1.8). Similarly, $H^*_{\mathrm{syn}}(A, \mathscr{S}_n(1))$ are computed by the complex

$$\mathscr{S}_n(1)_{A,A} := \text{mapping fiber of } \left(1 - f_1 : \Omega^{\bullet \geq 1}_{A_n} \longrightarrow \Omega^\bullet_{A_n}\right).$$

We define a residue mapping

$$\varrho^r_{\mathscr{B}} : \Omega^r_{\mathscr{B}} \longrightarrow \varprojlim_m \Omega^{r-1}_{B/(q_0,p)^m} = \Omega^{r-1}_B$$

as the projective limit, with respect to m, of the composite map:

$$\Omega^r_{\mathscr{B}} \xrightarrow{\text{canonical}} \Omega^r_{B(\!(u)\!)/(q_0,p)^m} \xrightarrow{\mathrm{Res}_{u=0}} \Omega^{r-1}_{B/(q_0,p)^m}.$$

The map $\varrho^r_{\mathscr{B}}$ induces a map

$$\varrho^r_{\mathscr{B},n} : \Omega^r_{\mathscr{B}_n} \longrightarrow \Omega^{r-1}_{B_n}. \tag{4.2.2}$$

Inverting q_0, we get a map

$$\varrho^r_n : \Omega^r_{\mathscr{A}_n} \longrightarrow \Omega^{r-1}_{A_n}. \tag{4.2.3}$$

The following lemma is straight-forward and left to the reader:

Lemma 4.2.4 *The following square is commutative*:

$$\begin{CD} \Omega^r_{\mathscr{A}_n} @>{f_2}>> \Omega^r_{\mathscr{A}_n} \\ @V{\varrho^r_n}VV @VV{\varrho^r_n}V \\ \Omega^{r-1}_{A_n} @>{f_1}>> \Omega^{r-1}_{A_n}. \end{CD}$$

By this lemma, the maps $\varrho^\bullet_n$ induce a homomorphism of complexes

$$\mathscr{S}_n(2)_{\mathscr{A},\mathscr{A}} \longrightarrow \mathscr{S}_n(1)_{A,A}[-1] \quad \text{for} \quad n \geq 1$$

and a residue map of syntomic cohomology

$$\widehat{\tau}^{\mathrm{syn}}_\infty : H^2_{\mathrm{syn}}(\mathscr{A}, \mathscr{S}_{\mathbb{Z}_p}(2)) \longrightarrow H^1_{\mathrm{syn}}(A, \mathscr{S}_{\mathbb{Z}_p}(1)). \tag{4.2.5}$$

We define the required arrow $\tau^{\mathrm{syn}}_\infty$ in the diagram (3.3.1) as the composite map

$$\tau^{\mathrm{syn}}_\infty : H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2)) \xrightarrow{\beta^*} H^2_{\mathrm{syn}}(\mathscr{A}, \mathscr{S}_{\mathbb{Z}_p}(2)) \xrightarrow{\widehat{\tau}^{\mathrm{syn}}_\infty} H^1_{\mathrm{syn}}(A, \mathscr{S}_{\mathbb{Z}_p}(1)),$$

where β^* denotes the pull-back by $\beta : \mathrm{Spec}(\mathscr{B}) \to \mathscr{E}$ in (4.1.2).

Remark 4.2.5 *By* (4.2.2)*, we obtain a residue map*

$$\widehat{\tau}_{\infty}^{\mathrm{syn}} : H^2_{\mathrm{syn}}(\mathscr{B}, \mathscr{S}_{\mathbb{Z}_p}(2)) \longrightarrow H^1_{\mathrm{syn}}(B, \mathscr{S}_{\mathbb{Z}_p}(1)).$$

By a similar construction, we obtain a residue map

$$\widehat{\tau}_{\infty}^{\mathrm{syn}} : H^2_{\mathrm{syn}}(\mathscr{R}, \mathscr{S}_{\mathbb{Z}_p}(2)) \longrightarrow H^1_{\mathrm{syn}}(R, \mathscr{S}_{\mathbb{Z}_p}(1)).$$

These maps will be used in §4.7 *below.*

4.3 Étale residue mapping

Let $R[u]_{(p)}$ be the localization of $R[u]$ at the prime ideal (p), and denote its p-adic completion by $\mathscr{R}^\flat$:

$$\mathscr{R}^\flat := \varprojlim_n R[u]_{(p)}/(p^n),$$

which is a complete discrete valuation ring with residue field $k(u)$. There is a natural embedding of $R[u, u^{-1}]$-algebras

$$\mathscr{R} \hookrightarrow \mathscr{R}^\flat.$$

Let K be the fraction field of R, and put

$$\mathscr{R}^\flat_K := \mathscr{R}^\flat \otimes_R K = \mathscr{R}^\flat[p^{-1}].$$

The following lemma will be used in Proposition 4.3.4 below.

Lemma 4.3.1 *For* $n \geq 1$ *and* $f \in (\mathscr{R}^\flat)^\times$ (*resp.* $f \in (\mathscr{R}^\flat_K)^\times$)*, there are* $f_0 \in R[u]^\times_{(p)}$ *and* $g \in \mathscr{R}^\flat$ (*resp.* $f_0 \in K(u)^\times$ *and* $g \in \mathscr{R}^\flat_K$) *with* $f = f_0(1 + p^n g)$*. Consequently the natural maps*

$$R[u]^\times_{(p)}/p^n \longrightarrow (\mathscr{R}^\flat)^\times/p^n, \quad K(u)^\times/p^n \longrightarrow (\mathscr{R}^\flat_K)^\times/p^n$$

are surjective for any $n \geq 1$.

Proof. Exercise. □

We construct here an auxiliary residue mapping

$$\widetilde{\tau}_{\infty}^{\mathrm{\acute{e}t}} : H^2_{\mathrm{\acute{e}t}}(\mathscr{R}^\flat_K, \mathbb{Z}/p^n(2)) \longrightarrow H^1_{\mathrm{\acute{e}t}}(K, \mathbb{Z}/p^n(1)),$$

which will be useful in §4.7 below. Let $\overline{K}$ be the algebraic closure of K, and put $\mathscr{R}^\flat_{\overline{K}} := \mathscr{R}^\flat_K \otimes_K \overline{K}$. We have a G_K-equivariant homomorphism

$$(\mathscr{R}^\flat_{\overline{K}})^\times/p^n \longrightarrow \mathbb{Z}/p^n, \qquad f \longmapsto \mathrm{Res}_{u=0}\, \frac{df}{f}, \tag{4.3.1}$$

where $\mathrm{Res}_{u=0}$ denotes a residue map defined in a similar way as for $\varrho^1_{\mathscr{B},n}$ in (4.2.2). Since $H^1_{\mathrm{\acute{e}t}}(\mathscr{R}^\flat_{\overline{K}}, \mathbb{Z}/p^n(1)) \simeq (\mathscr{R}^\flat_{\overline{K}})^\times/p^n$, this map induces a G_K-equivariant homomorphism

$$H^1_{\mathrm{\acute{e}t}}(\mathscr{R}^\flat_{\overline{K}}, \mathbb{Z}/p^n(1)) \longrightarrow \mathbb{Z}/p^n. \tag{4.3.2}$$

We define the desired map $\widetilde{\tau}^{\text{ét}}_{\infty}$ by the composition

$$\widetilde{\tau}^{\text{ét}}_{\infty} : H^2_{\text{ét}}(\mathscr{R}^{\flat}_K, \mathbb{Z}/p^n(2)) \longrightarrow H^1_{\text{ét}}(K, H^1_{\text{ét}}(\mathscr{R}^{\flat}_{\overline{K}}, \mathbb{Z}/p^n(2))) \xrightarrow{(4.3.2)} H^1_{\text{ét}}(K, \mathbb{Z}/p^n(1)),$$

where the first arrow is an edge homomorphism of the Hochschild-Serre spectral sequence

$$E_2^{a,b} = H^a_{\text{ét}}(K, H^b_{\text{ét}}(\mathscr{R}^{\flat}_{\overline{K}}, \mathbb{Z}/p^n(2))) \Longrightarrow H^{a+b}_{\text{ét}}(\mathscr{R}^{\flat}_K, \mathbb{Z}/p^n(2))$$

and we have used the following fact:

Proposition 4.3.4 *The natural restriction map*

$$H^2_{\text{ét}}(\mathscr{R}^{\flat}_K, \mathbb{Z}/p^n(2)) \longrightarrow E_2^{0,2} = H^2_{\text{ét}}(\mathscr{R}^{\flat}_{\overline{K}}, \mathbb{Z}/p^n(2))^{G_K}$$

is zero for any $n \geq 1$.

Proof. We show that the natural map

$$K_2^M(\mathscr{R}^{\flat}_K)/p^n \longrightarrow K_2^M(\mathscr{R}^{\flat}_{\overline{K}})/p^n$$

is zero, which implies the assertion by Remark 2.3.4 (2). Let $K(u)$ be the rational fuction field in u over K, which is a subfield of $\mathscr{R}^{\flat}_K$. By Lemma 4.3.1 the natural map

$$K_2^M(K(u))/p^n \longrightarrow K_2^M(\mathscr{R}^{\flat}_K)/p^n$$

is surjective. Hence it is enough to show that the natural map

$$K_2^M(K(u))/p^n \longrightarrow K_2^M(\overline{K}(u))/p^n$$

is zero. One can see $K_2^M(\overline{K}(u))/p^n = 0$ in the following way. Since

$$\{\overline{K}^{\times}, \overline{K}(u)^{\times}\} = 0 \quad \text{in} \quad K_2^M(\overline{K}(u))/p^n,$$

it is enough to show that $\{u-a, b-u\} = 0$. If $a = b$, it is clear. If $a \neq b$, one may replace $(u-a)$ with $(u-a)/(b-a)$ and $(b-u)$ with $(b-u)/(b-a)$. Then one has

$$\left\{\frac{u-a}{b-a}, \frac{b-u}{b-a}\right\} = 0 \quad \text{as} \quad \frac{u-a}{b-a} + \frac{b-u}{b-a} = 1,$$

which shows the assertion. □

The following proposition will be used in §4.7 below.

Proposition 4.3.5 *Let m be a positive integer, and let $s_m : B \to R$ be the homomorphism of R-algebras sending q_0 to p^m. Consider the following cartesian diagram of schemes*:

$$\begin{tikzcd}
\mathrm{Spec}(\mathscr{R}^{\flat}) \arrow[r, "\gamma"] & \mathrm{Spec}(\mathscr{B}_{(m)}) \arrow[r, "\alpha_m"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{E}_{(m)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Spec}(R) \arrow[d, "s_m"] \\
 & \mathrm{Spec}(\mathscr{B}) \arrow[r, "\beta"] & \mathscr{E} \arrow[r] & \mathrm{Spec}(B),
\end{tikzcd}$$

where $\mathscr{B}_{(m)}$ and $\mathscr{E}_{(m)}$ are defined by this diagram, and γ is induced by the natural inclusion $\mathscr{R} \hookrightarrow \mathscr{R}^\flat \simeq \mathscr{B}_{(m)}$. Then the following diagram is commutative:

$$\begin{tikzcd}
H^2_{\text{ét}}(E_{(m)}, \mathbb{Z}/p^n(2)) \arrow[rr, "\alpha_m^*"] \arrow[dr, "\tau_\infty^{\text{ét}}"'] & & H^2_{\text{ét}}(\mathscr{R}^\flat_K, \mathbb{Z}/p^n(2)) \arrow[dl, "\tilde{\tau}_\infty^{\text{ét}}"] \\
& H^1_{\text{ét}}(K, \mathbb{Z}/p^n(1)), &
\end{tikzcd}$$

where $\tau_\infty^{\text{ét}}$ is defined in a similar way as for $\tau_\infty^{\text{ét}}$ in the diagram (3.3.1).

Proof. Put $E_{\overline{K}} := E_{(m)} \otimes_K \overline{K}$, which is indepenet of m. By the construction of these maps, the assertion is reduced to the commutativity of a diagram of G_K-modules

$$\begin{tikzcd}
H^1_{\text{ét}}(E_{\overline{K}}, \mathbb{Z}/p^n(1)) \arrow[rr, "\gamma^* \circ \alpha_m^*"] \arrow[dr, "(2.4.7)"'] & & H^1_{\text{ét}}(\mathscr{R}^\flat_{\overline{K}}, \mathbb{Z}/p^n(1)) \arrow[dl, "(4.3.2)"] \\
& \mathbb{Z}/p^n, &
\end{tikzcd}$$

which has been shown in [A1], Lemma 4.2. $\square$

4.4 Commutativity of the key diagram, Step 0

We prove that the squares in (3.3.1) are commutative. The lower square of (3.3.1) commutes by the construction of τ_∞^{syn}, which is rather straight-forward and left to the reader. We prove the commutativity of the upper square of (3.3.1), which will be finished in §4.7 below.

We first reduce the problem to the case that the positive integer r fixed in §3.1 is at least 3. Put $q_1 := (q_0)^{1/3}$, and let E' be the Tate curve $E_{q,A[q_1]}$ over $A[q_1] = R((q_1))$. Let $\mathscr{E}' := \mathscr{E}_{q,B[q_1]}$ be the regular proper model of E' over $B' := B[q_1] = R[\![q_1]\!]$ defined in a similar way as for $\mathscr{E}$ (see §3.1). Let $\mathscr{D}' \subset \mathscr{E}'$ be the divisor defined by q_1. Consider the following diagram:

$$\begin{tikzcd}
H^2_{\text{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[rrr, "\tau_\infty^{\text{syn}}"] \arrow[dr, "\alpha"] \arrow[ddd, "c_2"'] & \arrow[r, phantom, "(1)"] & & H^1_{\text{syn}}(B, \mathscr{S}_{\mathbb{Z}_p}(1)) \arrow[dl, "\alpha"'] \arrow[ddd, "c^1"] \\
\arrow[d, phantom, "(2)"] & H^2_{\text{syn}}(\mathscr{E}'(\mathscr{D}'), \mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[r, "\tau_\infty^{\text{syn}}"] \arrow[d, "c^2"'] & H^1_{\text{syn}}(B', \mathscr{S}_{\mathbb{Z}_p}(1)) \arrow[d, "c^1"] & \arrow[d, phantom, "(3)"] \\
& H^2_{\text{ét}}(\mathscr{E}'[p^{-1}], \mathbb{Z}_p(2)) \arrow[r, "\tau_\infty^{\text{ét}}"] & H^1_{\text{ét}}(B'[p^{-1}], \mathbb{Z}_p(1)) & \\
H^2_{\text{ét}}(\mathscr{E}[p^{-1}], \mathbb{Z}_p(2)) \arrow[ur, "\alpha"] \arrow[rrr, "\tau_\infty^{\text{ét}}"] & \arrow[r, phantom, "(4)"] & & H^1_{\text{ét}}(B[p^{-1}], \mathbb{Z}_p(1)) \arrow[ul, hook, "\beta"']
\end{tikzcd}$$

Here the arrows α and β are pull-back maps, and β is injective by a standard norm argument. The squares (1) and (4) commute by the construction of τ_∞^{syn} and $\tau_\infty^{\text{ét}}$, respectively. The squares (2) and (3) commute by the functoriality of c^2 and c^1, respectively (Theorem 2.2.15).

Therefore the commutativity of the outer rectangle is reduced to that of the central square, i.e., we are reduced to the case that $r \geq 3$ (because we have $q = q_0^r = q_1^{3r}$ and $(p, 3r) = 1$ by the assumptions on p and r).

We may thus assume $r \geq 3$. Then we introduce an element $\xi_{\mathscr{E}} \in H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2))$ as follows. Let $Z_i \subset \mathscr{E}$ be the section defined by $u = q_0^i$. Put

$$\mathscr{D}_i := Z_i \cap \mathscr{D} \quad \text{and} \quad Z := \sum_{i=0}^{r-1} Z_i.$$

Noting $H^0_{\mathrm{syn}}(Z_i(\mathscr{D}_i), \mathscr{S}_{\mathbb{Z}_p}(1)) = 0$, we obtain, from Proposition 2.2.14, an exact sequence

$$0 \to H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2)) \to H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}+Z), \mathscr{S}_{\mathbb{Z}_p}(2)) \xrightarrow{\partial} \bigoplus_{i=0}^{r-1} H^1_{\mathrm{syn}}(Z_i(\mathscr{D}_i), \mathscr{S}_{\mathbb{Z}_p}(1)). \tag{4.4.1}$$

Let $0 < a < b < r$ be integers. Put

$$f(u) := \frac{\theta(q_0^a u)^r}{\theta(u)^{r-a}\theta(qu)^a} = (-u)^a \frac{\theta(q_0^a u)^r}{\theta(u)^r} \quad \text{and} \quad g(u) := \frac{\theta(q_0^b u)^r}{\theta(u)^{r-b}\theta(qu)^b} = (-u)^b \frac{\theta(q_0^b u)^r}{\theta(u)^r},$$

which are rational functions on $E = E_{q,A}$ by Proposition 2.4.4. Put

$$\xi'_{\mathscr{E}} := \left\{ \frac{f(u)}{f(q_0^{-b})}, \frac{g(u)}{g(q_0^{-a})} \right\}^{\mathrm{syn}} = \left\{ \frac{f(u)}{f(q_0^{-b})} \right\}^{\mathrm{syn}} \cup \left\{ \frac{g(u)}{g(q_0^{-a})} \right\}^{\mathrm{syn}} \in H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}+Z), \mathscr{S}_{\mathbb{Z}_p}(2)),$$

and the braces $\{-\}^{\mathrm{syn}}$ denote the syntomic symbol in (2.3.5) and (2.3.6) with $U := \mathscr{E} - \mathscr{D} - Z$. One can easily check that the boundary $\partial(\xi'_{\mathscr{E}})$ agrees with the syntomic symbol of the tame symbol of

$$\left\{ \frac{f(u)}{f(q_0^{-b})}, \frac{g(u)}{g(q_0^{-a})} \right\} \in K_2^M(\mathscr{O}(U)).$$

The tame symbol vanishes at each $Z_i - D_i$ by a straight-forward computation. Therefore one has $\partial(\xi'_{\mathscr{E}}) = 0$, and $\xi'_{\mathscr{E}}$ defines an element $\xi_{\mathscr{E}} \in H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2))$ by the exact sequence (4.4.1).

Proposition 4.4.2 *$H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2))$ is generated by the subgroups*

$$H^2_{\mathrm{syn}}(\mathscr{E}, \mathscr{S}_{\mathbb{Z}_p}(2)), \quad R_{\mathrm{dR}} \cup \{q_0\}^{\mathrm{syn}} \quad \textit{and} \quad \mathbb{Z}_p\, \xi_{\mathscr{E}},$$

where R_{dR} denotes the image of the natural map

$$R = \varprojlim_{n \geq 1} H^0_{\mathrm{dR}}(\mathscr{E}_n) \longrightarrow H^1_{\mathrm{syn}}(\mathscr{E}, \mathscr{S}_{\mathbb{Z}_p}(1))$$

induced by (2.1.3), *and the braces $\{-\}^{\mathrm{syn}}$ denote the syntomic symbol in* (2.3.5):

$$\{-\}^{\mathrm{syn}} : \Gamma(E, \mathscr{O}_E^{\times}) \longrightarrow H^1_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(1)).$$

This proposition will be proved in §4.5 below. We explain how to prove the commutativity of (3.3.1). It is enough, by Proposition 4.4.2, to check the commutativity for the elements of

$$R_{\mathrm{dR}} \cup \{q_0\}^{\mathrm{syn}}, \quad \mathbb{Z}_p\,\xi_{\mathscr{E}} \quad \text{and} \quad H^2_{\mathrm{syn}}(\mathscr{E}, \mathscr{S}_{\mathbb{Z}_p}(2)).$$

The commutativity for $R_{\mathrm{dR}} \cup \{q_0\}^{\mathrm{syn}}$ is clear. Indeed, we have

$$\tau^{\mathrm{syn}}_\infty(R_{\mathrm{dR}} \cup \{q_0\}^{\mathrm{syn}}) = 0 \quad \text{in} \quad H^1_{\mathrm{syn}}(A, \mathscr{S}_{\mathbb{Z}_p}(1))$$

by the construction of $\tau^{\mathrm{syn}}_\infty$. On the other hand, we have

$$\tau^{\text{ét}}_\infty \circ c^2(R_{\mathrm{dR}} \cup \{q_0\}^{\mathrm{syn}}) = \tau^{\text{ét}}_\infty(\{1+pR\}^{\text{ét}} \cup \{q_0\}^{\text{ét}}) = 0 \quad \text{in} \quad H^1_{\text{ét}}(L, \mathbb{Z}_p(1))$$

by Remark 2.3.7 and Theorem 2.3.8. We will show

$$c^1 \circ \tau^{\mathrm{syn}}_\infty(\xi_{\mathscr{E}}) = \tau^{\text{ét}}_\infty \circ c^2(\xi_{\mathscr{E}}) \quad \text{in } H^1_{\text{ét}}(L, \mathbb{Z}_p(1)) \simeq L^\times/p^n \tag{4.4.3}$$

in §4.6 below. As for the commutativity for the elements of $H^2_{\mathrm{syn}}(\mathscr{E}, \mathscr{S}_{\mathbb{Z}_p}(2))$, it is equivalent to the commutativity of the following square:

$$\begin{tikzcd}
H^2_{\mathrm{syn}}(\mathscr{E}, \mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[r, "\tau^{\mathrm{syn}}_\infty"] \arrow[d, "c^2"'] & H^1_{\mathrm{syn}}(B, \mathscr{S}_{\mathbb{Z}_p}(1)) \arrow[d, "c^1"] \\
H^2_{\text{ét}}(\mathscr{E}[p^{-1}], \mathbb{Z}_p(2)) \arrow[r, "\tau^{\text{ét}}_\infty"] & H^1_{\text{ét}}(B[p^{-1}], \mathbb{Z}_p(1)),
\end{tikzcd} \tag{4.4.4}$$

where the arrow $\tau^{\text{ét}}_\infty$ is defined by the composite map of sheaves on $B[p^{-1}]_{\text{ét}}$

$$R^1 f_{\mathscr{E}*}\mathbb{Z}/p^n(2) \longrightarrow j_*R^1 f_*\mathbb{Z}/p^n(2) \xrightarrow{(2.4.7)} j_*\mathbb{Z}/p^n(1) = \mathbb{Z}/p^n(1) \qquad (n \geq 1)$$

and similar arguments as for $\tau^{\text{ét}}_\infty$ in the diagram (3.3.1). Here $f_{\mathscr{E}}$, f and j are as follows:

$$\begin{tikzcd}
E[p^{-1}] \arrow[r, hook] \arrow[d, "f"'] & \mathscr{E}[p^{-1}] \arrow[d, "f_{\mathscr{E}}"] \\
\mathrm{Spec}(A[p^{-1}]) \arrow[r, hook, "j"] & \mathrm{Spec}(B[p^{-1}]).
\end{tikzcd}$$

We will prove the commutativity of (4.4.4) in §4.7 below.

4.5 Proof of Proposition 4.4.2

Put

$$\mathscr{E}^{(0)} := \mathscr{E}, \qquad \mathscr{E}^{(1)} := \mathscr{D}^{(1)}, \qquad \mathscr{E}^{(2)} := \mathscr{D}^{(2)},$$

where $\mathscr{D}^{(1)}$ denotes the disjoint union of the irreducible components of $\mathscr{D}$, and $\mathscr{D}^{(2)}$ denotes the disjoint union of the intersections of two distinct irreducible components of $\mathscr{D}$. Since $\mathscr{D}$ is the standard Néron r-gon over $\mathrm{Spec}(R)$, we have

$$\mathscr{E}^{(1)} = \coprod_{j=1}^{r} \mathbb{P}^1_R \quad \text{and} \quad \mathscr{E}^{(2)} = \coprod_{j=1}^{r} \mathrm{Spec}(R).$$

Fix an integer $n \geq 1$. One can easily check that $\mathscr{E}$ and $\mathscr{E}^{(m)}$ ($m = 0, 1, 2$) satisfy the conditions (i)–(iv) in Proposition 2.2.14, using Lemma 3.1.1 and Remark 3.1.2. Hence there is a spectral sequence

$$E_1^{a,b} = H^{2a+b}_{\mathrm{syn}}(\mathscr{E}^{(-a)}, \mathscr{S}_n(a+2)) \Longrightarrow H^{a+b}_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_n(2)). \tag{4.5.1}$$

Since we have

$$E_1^{a,b} = 0 \quad \text{unless } -2 \leq a \leq 0 \text{ and } 2a + b \geq 0, \tag{4.5.2}$$

the quotient group $C_n := H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_n(2))/E_\infty^{0,2}$ fits into an exact sequence

$$0 \longrightarrow E_2^{-1,3} \longrightarrow C_n \longrightarrow E_2^{-2,4} \longrightarrow E_2^{0,3}.$$

We will prove the following lemma:

Lemma 4.5.3 (1) *Put $R_n := R/(p^n)$. Then $E_2^{-1,3}$ agrees with the diagonal subgroup of*

$$E_1^{-1,3} = \bigoplus_{j=1}^{r} H^1_{\mathrm{syn}}(\mathbb{P}^1_R, \mathscr{S}_n(1)) \simeq \bigoplus_{j=1}^{r} R_n.$$

(2) *$E_2^{-2,4}$ agrees with the diagonal subgroup of*

$$E_1^{-2,4} = \bigoplus_{j=1}^{r} H^0_{\mathrm{syn}}(R, \mathscr{S}_n(0)) \simeq \bigoplus_{j=1}^{r} \mathbb{Z}/p^n.$$

Moreover, the edge homomorphism

$$H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_n(2)) \longrightarrow E_1^{-2,4}$$

sends $\xi_{\mathscr{E}}$ to $(r, r, \ldots, r)$ up to a sign.

We first prove Proposition 4.4.2, admitting this lemma. Consider the spectral sequence (4.5.1). Since $H^0_{\mathrm{syn}}(\mathbb{P}^1_R, \mathscr{S}_n(1)) = 0$ by (2.1.9), we have

$$E_1^{0,2} = E_2^{0,2} \overset{(4.5.2)}{=\!=} E_\infty^{0,2}$$

and hence a short exact sequence

$$0 \longrightarrow H^2_{\mathrm{syn}}(\mathscr{E}, \mathscr{S}_n(2))) \longrightarrow H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_n(2))) \longrightarrow C_n \longrightarrow 0.$$

Taking projective limit with respect to $n \geq 1$, we get an exact sequence

$$0 \longrightarrow H^2_{\mathrm{syn}}(\mathscr{E}, \mathscr{S}_{\mathbb{Z}_p}(2))) \longrightarrow H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2))) \longrightarrow \varprojlim_{n \geq 1} C_n.$$

By Lemma 4.5.3, there is a short exact sequence

$$0 \longrightarrow R_n \longrightarrow C_n \longrightarrow \mathbb{Z}/p^n \longrightarrow 0, \tag{4.5.4}$$

which shows that $\varprojlim_{n\geq 1} C_n$ is generated by elements which lift to either

$$R_{\mathrm{dR}} \cup \{q_0\}^{\mathrm{syn}} \quad \text{or} \quad \mathbb{Z}_p\,\xi_{\mathscr{E}} \quad \big(\subset H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_{\mathbb{Z}_p}(2))\big).$$

Thus we obtain the proposition. We prove Lemma 4.5.3 in what follows.

Proof of Lemma 4.5.3. The assertion on the the edge homomorphism $H^2_{\mathrm{syn}}(\mathscr{E}(\mathscr{D}), \mathscr{S}_n(2)) \to E_1^{-2,4}$ follows from computations on symbols (cf. [A1], Remark 5.5). Thus $E_2^{-2,4}$ contains the diagonal subgroup of $E_1^{-2,4}$, by the assumption that p is prime to r. Similarly, it is easy to see that $E_2^{-1,3}$ contains the diagonal subgroup of $E_1^{-1,3}$. It remains to show that $E_2^{-1,3}$ and $E_2^{-2,4}$ are contained in the diagonal subgroups of $E_1^{-1,3}$ and $E_1^{-2,4}$, respectively.

Recall that $\mathscr{D}$ is the standard Néron r-gon over $\mathrm{Spec}(R)$. Let $\mathscr{D}_1, \dots, \mathscr{D}_r$ be the irreducible components of $\mathscr{D}$, which are all isomorphic to $\mathbb{P}^1_R$. Changing the ordering of these components if necessary, we suppose that $\mathscr{D}_1$ meets $\mathscr{D}_r$ and $\mathscr{D}_2$, and that $\mathscr{D}_j$ meets $\mathscr{D}_{j-1}$ and $\mathscr{D}_{j+1}$ for $j = 2, \dots, r-1$ (hence $\mathscr{D}_r$ meets $\mathscr{D}_{r-1}$ and $\mathscr{D}_1$). Put

$$T_j := \mathscr{D}_j \times_{\mathscr{E}} \mathscr{D}_{j+1} \quad (j = 1, \dots, r-1), \qquad T_r := \mathscr{D}_r \times_{\mathscr{E}} \mathscr{D}_1,$$

which are all isomorphic to $\mathrm{Spec}(R)$. Let

$$i : \mathscr{E}^{(1)} = \coprod_{j=1}^{r} \mathscr{D}_j \longrightarrow \mathscr{E}$$

be the natural finite morphism. Since $\mathscr{D}_j \simeq \mathbb{P}^1_R$ for each $1 \leq j \leq r$, we have isomorphisms

$$\begin{aligned}
&H^1_{\mathrm{syn}}(\mathscr{D}_j, \mathscr{S}_n(1)) \simeq H^0_{\mathrm{dR}}(\mathscr{D}_{j,n}) \simeq R_n,\\
&H^2_{\mathrm{syn}}(\mathscr{D}_j, \mathscr{S}_n(1)) \simeq \mathrm{Ker}(1-\sigma : R_n \to R_n) = \mathbb{Z}/p^n,\\
&H^3_{\mathrm{syn}}(\mathscr{D}_j, \mathscr{S}_n(2)) \simeq H^2_{\mathrm{dR}}(\mathscr{D}_{j,n}) \simeq R_n \qquad \text{(trace isomorphism)},
\end{aligned}$$

by (2.1.9). Lemma 4.5.3 follows from the following claims (i) and (ii):

(i) *The composite map*

$$E_1^{-1,3} = \bigoplus_{j=1}^{r} R_n = \bigoplus_{j=1}^{r} H^0_{\mathrm{dR}}(\mathscr{D}_{j,n}) \xrightarrow{i_*} H^2_{\mathrm{dR}}(\mathscr{E}_n) \xrightarrow{i^*} \bigoplus_{j=1}^{r} H^2_{\mathrm{dR}}(\mathscr{D}_{j,n}) \simeq \bigoplus_{j=1}^{r} R_n$$

is given by the $r \times r$ *matrix*

$$\begin{pmatrix}
-2 & 1 & 0 & \cdots & 0 & 1\\
1 & -2 & 1 & \ddots & \vdots & 0\\
0 & 1 & -2 & \ddots & 0 & \vdots\\
\vdots & \ddots & \ddots & \ddots & 1 & 0\\
0 & \cdots & 0 & 1 & -2 & 1\\
1 & 0 & \cdots & 0 & 1 & -2
\end{pmatrix}.$$

Consequently, $E_2^{-1,3}$ is contained in the diagonal subgroup of $E_1^{-1,3}$ by the assumption that r is prime to p (note also that the above composite map agrees with the composite map $E_1^{-1,3} \to E_1^{0,3} \to \bigoplus_{j=1}^r H^3_{\mathrm{syn}}(\mathscr{D}_j, \mathscr{S}_n(2)) \simeq \bigoplus_{j=1}^r R_n$).

(ii) *The edge homomorphism*

$$E_1^{-2,4} = \bigoplus_{j=1}^{r} \mathbb{Z}/p^n[T_j] \longrightarrow E_1^{-1,4} = \bigoplus_{j=1}^{r} H^2_{\mathrm{syn}}(\mathscr{D}_j, \mathscr{S}_n(1)) \simeq \mathbb{Z}/p^n[D_j]$$

sends $[T_j]$ to

$$\begin{cases} [D_j] - [D_{j+1}] & (1 \leq j \leq r-1) \\ [D_1] - [D_r] & (j = r), \end{cases}$$

where the signs arise from the construction of the spectral sequence (4.5.1). *Consequently, $E_2^{-2,4}$ is contained in the diagonal subgroup of $E_1^{-2,4}$.*

These claims follows from standard facts on intersection theory of divisors. The details are straight-forward and left to the reader. This completes the proof of Lemma 4.5.3 and Proposition 4.4.2.

4.6 Commutativity of the key diagram, Step 1

This subsection is devoted to showing (4.4.3). More precisely, we will prove

$$\begin{aligned} c^1 \circ \tau_\infty^{\mathrm{syn}}(\xi_{\mathscr{E}}) &= (-1)^{a(r-b)} q_0^{a(b-a)(b-r)} \left(\frac{\theta(q_0^b)^b}{\theta(q_0^{b-a})^{b-a}\theta(q_0^a)^a} \right)^r \left(\frac{S(q_0^b)}{S(q_0^{b-a})S(q_0^a)} \right)^{r^2} \\ &= \tau_\infty^{\text{ét}} \circ c^2(\xi_{\mathscr{E}}) \end{aligned}$$

in $L^\times/p^n$, where we put

$$S(\alpha) := \prod_{k=1}^{\infty} \left(\frac{1-\alpha q^k}{1-\alpha^{-1}q^k} \right)^k.$$

Since the equality for $\tau_\infty^{\text{ét}}$ has been shown in [A1], Lemma 7.4, we prove the equality for $\tau_\infty^{\mathrm{syn}}$. It follows from the construction of the syntomic residue map (§4.2) that we may truncate $\theta(u)$ with respect to q_0. Therefore we can calculate $\tau_\infty^{\mathrm{syn}}$ by factorizing the infinite products. It is enough to check the following:

$$\widehat{\tau}_\infty^{\mathrm{syn}}\{a, b\} = 1, \qquad a,\ b \in A^\times \tag{4.6.1}$$

$$\widehat{\tau}_\infty^{\mathrm{syn}}\{a, u\} = a, \qquad a \in A^\times \tag{4.6.2}$$

$$\widehat{\tau}_\infty^{\mathrm{syn}}\{a, g(u)\} = 1, \qquad a \in A^\times,\ g(u) \in R[\![q_0, u]\!]^\times \tag{4.6.3}$$

$$\widehat{\tau}_\infty^{\mathrm{syn}}\{u, g(u)\} = g(0)^{-1}, \qquad g(u) \in R[\![q_0, u]\!]^\times \tag{4.6.4}$$

$$\widehat{\tau}_\infty^{\mathrm{syn}}\{u, 1 - bu^{-1}\} = 1, \qquad b \in (p, q_0) \tag{4.6.5}$$

$$\widehat{\tau}_\infty^{\mathrm{syn}}\{g(u), h(u)\} = 1, \qquad g(u), h(u) \in R[\![q_0, u]\!]^\times \tag{4.6.6}$$

$$\widehat{\tau}_\infty^{\mathrm{syn}}\{g(u), 1-bu^{-1}\} = g(0)^{-1}g(b), \qquad g(u) \in R[\![q_0, u]\!]^\times,\ b \in (p, q_0) \tag{4.6.7}$$
$$\widehat{\tau}_\infty^{\mathrm{syn}}\{1-bu^{-1}, 1-cu^{-1}\} = 1, \qquad b, c \in (p, q_0) \tag{4.6.8}$$

where $\widehat{\tau}_\infty^{\mathrm{syn}}$ is the residue map (4.2.5). We prove only (4.6.4) and (4.6.7). The other equalities are simpler and left to the reader. By the definition of syntomic symbols (cf. §2.3), we have

$$\{u\}^{\mathrm{syn}} = \left(\frac{du}{u}, 0\right), \quad \{g(u)\}^{\mathrm{syn}} = \left(\frac{dg(u)}{g(u)}, \frac{1}{p}\log\frac{\varphi(g(u))}{g(u)^p}\right).$$

By the definition of the product of syntomic cohomology (cf. Definition 2.2.9), we have

$$\begin{aligned}\{u, g(u)\}^{\mathrm{syn}} &= \left(\frac{du}{u}\frac{dg(u)}{g(u)}, -\frac{1}{p}\log\frac{\varphi(g(u))}{g(u)^p}\frac{du}{u}\right)\\ &= \left(-\frac{dg(u)}{g(u)}\frac{du}{u}, -\frac{1}{p}\log\frac{\varphi(g(u))}{g(u)^p}\frac{du}{u}\right).\end{aligned}$$

Therefore

$$\hat{\tau}_\infty^{\mathrm{syn}}[u, g(u)] = \left(-\frac{dg(0)}{g(0)}, -\frac{1}{p}\log\frac{\varphi(g(0))}{g(0)^p}\right) = -[g(0)],$$

which completes the proof of (4.6.4). One can reduce (4.6.7) to (4.6.4) in the following way. It is enough to show $\widehat{\tau}_\infty^{\mathrm{syn}}\{g(u), u-b\} = g(b)$. Consider an endomorphism $w : \mathscr{A} \to \mathscr{A}$ of topological ring such that $w(u) = u + b$. Then we have $\widehat{\tau}_\infty^{\mathrm{syn}} \circ w = \widehat{\tau}_\infty^{\mathrm{syn}}$ as $\varrho_n^r \circ w = \varrho_n^r$ (cf. (4.2.3)). Thus one has

$$\widehat{\tau}_\infty^{\mathrm{syn}}\{g(u), u-b\} = \widehat{\tau}_\infty^{\mathrm{syn}} w\{g(u), u-b\} = \widehat{\tau}_\infty^{\mathrm{syn}}\{g(u+b), u\} = g(b).$$

as required.

4.7 Commutativity of the key diagram, Step 2

We prove the commutativity of (4.4.4). Let $\mathscr{R}^\flat$ and $\mathscr{R}_K^\flat$ be as we defined in §4.3. Since B is a regular local ring, $B[p^{-1}]$ is a UFD. By Remark 2.3.4 (1), we have

$$H^1_{\text{ét}}(B[p^{-1}], \mathbb{Z}_p(1)) \simeq \varprojlim_n B[p^{-1}]^\times/p^n.$$

Lemma 4.7.1 *For $m \geq 1$, let $s_m : B \to R$ be the R-homomorphism sending q_0 to p^m. Then the map*

$$\prod_{m\geq 1} s_m \ : \ \varprojlim_n B[p^{-1}]^\times/p^n \longrightarrow \prod_{m\geq 1}\varprojlim_n K^\times/p^n$$

is injective.

Proof. Exercise (left to the reader). □

We consider the following cartesian squares of schemes for each $m \geq 1$:

$$\begin{tikzcd} \mathrm{Spec}(\mathscr{R}^\flat) \arrow[r, "\gamma"] & \mathrm{Spec}(\mathscr{B}_{(m)}) \arrow[r, "\alpha_m"] \arrow[d, "\sigma_m"'] \arrow[dr, phantom, "\square"] & \mathscr{E}_{(m)} \arrow[r] \arrow[d, "t_m"'] \arrow[dr, phantom, "\square"] & \mathrm{Spec}(R) \arrow[d, "s_m"'] \\ & \mathrm{Spec}(\mathscr{B}) \arrow[r, "\beta"] & \mathscr{E} \arrow[r] & \mathrm{Spec}(B), \end{tikzcd}$$

where $\mathscr{B}_{(m)}$ and $\mathscr{E}_{(m)}$ are defined by this diagram, and γ is induced by the natural inclusion $\mathscr{R} \hookrightarrow \mathscr{R}^\flat \simeq \mathscr{B}_{(m)}$. See §4.1 for $\mathscr{B}$ and β. We defined $\mathscr{E}_{(m)}$ (resp. $\mathrm{Spec}(\mathscr{B}_{(m)})$) by the lower (resp. upper) cartesian square. By Lemma 4.7.1, the commutativity of (4.4.4) is reduced to showing that of the following diagram for all $m \geq 1$:

$$\begin{tikzcd} H^2_{\mathrm{syn}}(\mathscr{E}, \mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[r, "t_m^* \circ c^2"] \arrow[d, "\tau_\infty^{\mathrm{syn}}"'] & H^2_{\text{ét}}(\mathscr{E}_{(m)}[p^{-1}], \mathbb{Z}_p(2)) \arrow[d, "\tau_\infty^{\text{ét}}"] \\ H^1_{\mathrm{syn}}(B, \mathscr{S}_{\mathbb{Z}_p}(1)) \arrow[r, "s_m^* \circ c^1"] & H^1_{\text{ét}}(K, \mathbb{Z}_p(1)). \end{tikzcd} \tag{4.7.2}$$

Lemma 4.7.3 *The diagram* (4.7.2) *factors as follows*:

$$\begin{tikzcd} H^2_{\mathrm{syn}}(\mathscr{E}, \mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[r, "c^2"] \arrow[d, "\beta^*"'] \arrow[dr, phantom, "\bigcirc"] & H^2_{\text{ét}}(\mathscr{E}[p^{-1}], \mathbb{Z}_p(2)) \arrow[r, "t_m^*"] \arrow[d, "\beta^*"'] \arrow[dr, phantom, "\bigcirc"] & H^2_{\text{ét}}(\mathscr{E}_{(m)}[p^{-1}], \mathbb{Z}_p(2)) \arrow[d, "\alpha_m^*"] \\ H^2_{\mathrm{syn}}(\mathscr{B}, \mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[r, "c^2"] \arrow[d, "\widehat{\tau}_\infty^{\mathrm{syn}}"'] & H^2_{\text{ét}}(\mathscr{B}[p^{-1}], \mathbb{Z}_p(2)) \arrow[r, "\sigma_m^*"] & H^2_{\text{ét}}(\mathscr{B}_{(m)}[p^{-1}], \mathbb{Z}_p(2)) \arrow[d, "\widehat{\tau}_\infty^{\text{ét}}"] \\ H^1_{\mathrm{syn}}(B, \mathscr{S}_{\mathbb{Z}_p}(1)) \arrow[rr, "s_m^* \circ c^1"] & & H^1_{\text{ét}}(K, \mathbb{Z}_p(1)), \end{tikzcd}$$

and the upper squares are commutative. Here $\widehat{\tau}_\infty^{\mathrm{syn}}$ denotes the residue map constructed in §4.2 *and $\widehat{\tau}_\infty^{\text{ét}}$ is defined as the projective limit, with respect to $n \geq 1$, of the composite map*

$$\widehat{\tau}_\infty^{\text{ét}} : H^2_{\text{ét}}(\mathscr{B}_{(m)}[p^{-1}], \mathbb{Z}/p^n(2)) \xrightarrow{\gamma^*} H^2_{\text{ét}}(\mathscr{R}_K^\flat, \mathbb{Z}/p^n(2)) \xrightarrow{\widetilde{\tau}_\infty^{\text{ét}}} H^1_{\text{ét}}(K, \mathbb{Z}/p^n(1)).$$

See §4.3 *for $\widetilde{\tau}_\infty^{\text{ét}}$.*

Proof. The factorization $\tau_\infty^{\mathrm{syn}} = \widehat{\tau}_\infty^{\mathrm{syn}} \circ \beta^*$ follows from the construction of these maps, and the factorization $\tau_\infty^{\text{ét}} = \widehat{\tau}_\infty^{\text{ét}} \circ \alpha_m^*$ follows from Proposition 4.3.5. The second assertion follows from the functoriality of c^2 and that of étale cohomology. $\square$

Lemma 4.7.4 *In the diagram of Lemma* 4.7.3*, the lower square factors as follows*:

$$\begin{tikzcd} H^2_{\mathrm{syn}}(\mathscr{B}, \mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[r, "\sigma_m^*"] \arrow[d, "\widehat{\tau}_\infty^{\mathrm{syn}}"'] \arrow[dr, phantom, "\bigcirc"] & H^2_{\mathrm{syn}}(\mathscr{B}_{(m)}, \mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[r, "c^2"] \arrow[d, "\widehat{\tau}_\infty^{\mathrm{syn}}"'] & H^2_{\text{ét}}(\mathscr{B}_{(m)}[p^{-1}], \mathbb{Z}_p(2)) \arrow[d, "\widehat{\tau}_\infty^{\text{ét}}"] \\ H^1_{\mathrm{syn}}(B, \mathscr{S}_{\mathbb{Z}_p}(1)) \arrow[r, "s_m^*"] & H^1_{\mathrm{syn}}(R, \mathscr{S}_{\mathbb{Z}_p}(1)) \arrow[r, "c^1"] & H^1_{\text{ét}}(K, \mathbb{Z}_p(1)), \end{tikzcd}$$

and the left square commutes.

Proof. The first assertion follows from the functoriality of c^2 and c^1. The commutativity of the left square follows from the construction of $\widehat{\tau}_{\infty}^{\mathrm{syn}}$'s (see §4.2). $\square$

Lemma 4.7.5 *In the diagram of Lemma* 4.7.4*, the right square factors as follows*:

$$\begin{tikzcd} H^2_{\mathrm{syn}}(\mathscr{B}_{(m)}, \mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[r, "c^2"] \arrow[d, "\gamma^*"'] \arrow[dr, phantom, "\bigcirc"] & H^2_{\text{ét}}(\mathscr{B}_{(m)}[p^{-1}], \mathbb{Z}_p(2)) \arrow[d, "\gamma^*"] \\ H^2_{\mathrm{syn}}(\mathscr{R}^\flat, \mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[r, "c^2"] \arrow[d, "\widetilde{\tau}_{\infty}^{\mathrm{syn}}"'] & H^2_{\text{ét}}(\mathscr{R}^\flat_K, \mathbb{Z}_p(2)) \arrow[d, "\widetilde{\tau}_{\infty}^{\text{ét}}"] \\ H^1_{\mathrm{syn}}(R, \mathscr{S}_{\mathbb{Z}_p}(1)) \arrow[r, "c^1"] & H^1_{\text{ét}}(K, \mathbb{Z}_p(1)), \end{tikzcd}$$

and the upper square commutes. Here $\widetilde{\tau}_{\infty}^{\mathrm{syn}}$ *denotes the syntomic residue map defined in a similar way as for* $\widehat{\tau}_{\infty}^{\mathrm{syn}}$. *Note also that* $\mathscr{R}^\flat_K = \mathscr{R}^\flat[p^{-1}]$.

Proof. The first assertion follows from the construction of $\widehat{\tau}_{\infty}^{\mathrm{syn}}$ and $\widehat{\tau}_{\infty}^{\text{ét}}$. The commutativity of the upper square follows from the functoriality of c^2 (Theorem 2.2.15). $\square$

By Lemmas 4.7.3–4.7.5, the commutativity of (4.7.2) is reduced to that of the lower square of the diagram in Lemma 4.7.5, which is further reduced to that of the following square for all $n \geq 1$:

$$\begin{tikzcd} H^2_{\mathrm{syn}}(\mathscr{R}^\flat, \mathscr{S}_n(2)) \arrow[r, "c^2"] \arrow[d, "\widetilde{\tau}_{\infty}^{\mathrm{syn}}"'] & H^2_{\text{ét}}(\mathscr{R}^\flat_K, \mathbb{Z}/p^n(2)) \arrow[d, "\widetilde{\tau}_{\infty}^{\text{ét}}"] \\ H^1_{\mathrm{syn}}(R, \mathscr{S}_n(1)) \arrow[r, "c^1"] & H^1_{\text{ét}}(K, \mathbb{Z}/p^n(1)). \end{tikzcd} \tag{4.7.6}$$

By Theorem 2.3.8 and Theorem 2.3.9, the diagram (4.7.6) is written as

$$\begin{tikzcd} K_2^M(\mathscr{R}^\flat)/p^n \arrow[r] \arrow[d, "\tau^{\mathrm{syn}}"'] & K_2^M(\mathscr{R}^\flat_K)/p^n \arrow[d, "\tau^{\text{ét}}"] \\ R^\times/p^n \arrow[r] & K^\times/p^n. \end{tikzcd} \tag{4.7.7}$$

where the horizontal arrows are the natural maps and τ^{syn} (resp. $\tau^{\text{ét}}$) is induced from $\widetilde{\tau}_{\infty}^{\mathrm{syn}}$ (resp. $\widetilde{\tau}_{\infty}^{\text{ét}}$). Thus the commutativity of (4.7.6) is reduced to the explicit calculations of τ^{syn} and $\tau^{\text{ét}}$. By Lemma 4.3.1 we may replace $K_2^M(\mathscr{R}^\flat)/p^n$ with $K_2^M(R[u]_{(p)})/p^n$ and $K_2^M(\mathscr{R}^\flat_K)/p^n$ with $K_2^M(K(u))/p^n$. Then the explicit formula of $\tau^{\text{ét}}$ has been shown in [A1] Thm.4.4. Therefore it is enough to show that the formula of τ^{syn} agrees with it. However this follows from (4.6.1), $\cdots$, (4.6.8). This finishes the proof of the commutativity of the diagram (4.7.7) and hence the diagrams (4.7.6), (4.4.4) and (3.3.1). This completes the proof of Theorem 3.2.3.

5 Main results on Elliptic surfaces over p-adic fields

We mean by an *elliptic surface* over a commutative ring A a projective flat morphism $\pi : X \to C$ with a section $e : C \to X$ such that X and C are projective smooth over $\mathrm{Spec}(A)$ of relative dimension 2 and 1, respectively and such that the general fiber of π is an elliptic curve.

For a field F, we denote the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ by G_F. We often write $H^*(F, -)$ for the continuous Galois cohomology $H^*_{\text{cont}}(G_F, -)$.

5.1 Split multiplicative fiber

Let

$$\widetilde{T}_{\mathbb{Z}} := \coprod \mathbb{P}^1_{\mathbb{Z}}$$

be the disjoint union of copies of $\mathbb{P}^1_{\mathbb{Z}}$ indexed by $\mathbb{Z}/m$. Gluing the section $0 : \mathrm{Spec}(\mathbb{Z}) \hookrightarrow \mathbb{P}^1_{\mathbb{Z}}$ of the i-th $\mathbb{P}^1_{\mathbb{Z}}$ with $\infty : \mathrm{Spec}(\mathbb{Z}) \hookrightarrow \mathbb{P}^1_{\mathbb{Z}}$ of the $(i+1)$-st $\mathbb{P}^1_{\mathbb{Z}}$, we obtain a connected proper curve $T_{\mathbb{Z}}$ over $\mathrm{Spec}\mathbb{Z}$ whose normalization is $\widetilde{T}_{\mathbb{Z}} \to T_{\mathbb{Z}}$. We call $T_{\mathbb{Z}} \otimes_{\mathbb{Z}} A$ the *Néron polygon* or the *Néron m-gon* over a ring A (cf. [DR], II.1.1). Let $\pi : X \to C$ be an elliptic surface over a ring A. For an A-valued point $P \in C(A)$, we call the fiber $\pi^{-1}(P)$ *split multiplicative of type* I_m over A, if there is a closed subscheme $D^\dagger \subset \pi^{-1}(P)$ which is isomorphic to the Néron m-gon $T_{\mathbb{Z}} \times_{\mathbb{Z}} A$. We call $\pi^{-1}(P)$ *multiplicative*, if $\pi^{-1}(P) \otimes_A \overline{k}$ is split multiplicative over $\overline{k}$ for any geometric point $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(A)$. A singular fiber which is not multiplicative is called an *additive* fiber. When $A = \overline{F}$ is an algebraically closed field, all the singular fibers are classified by Kodaira and Néron (cf. [Si] IV §8). In particular, we note

$$\pi^{-1}(P) \text{ is multiplicative } \iff H^1_{\text{ét}}(\pi^{-1}(P), \mathbb{Z}_p) \neq 0 \; (\Longrightarrow H^1_{\text{ét}}(\pi^{-1}(P), \mathbb{Z}_p) = \mathbb{Z}_p).$$

5.2 $\Phi(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p}$ and $\Phi(\mathscr{X}, \mathscr{D})_{\mathbb{F}_p}$

Let $p \geq 5$ be a prime number. Let K be a finite *unramified* extension of $\mathbb{Q}_p$ and let R be its integer ring. Let $\pi_R : \mathscr{X} \to \mathscr{C}$ be an elliptic surface over R. Let $\Sigma = \{P_1, \cdots, P_s\}$ be the set of all R-rational points for which the fiber $\mathscr{D}_i := \pi^{-1}(P_i)$ are split multiplicative fibers of type I_{r_i}. Put $\mathscr{D} := \sum_{i=1}^{s} \mathscr{D}_i$ and $\mathscr{U} := \mathscr{X} - \mathscr{D}$. We assume

$$p \not| 6 r_1 \cdots r_s.$$

Let t_i be a uniformizer of $\mathscr{O}_{\mathscr{C}, P_i}$. Let $\iota_i : \mathrm{Spec}(R((t_i))) \to \mathscr{C} - \{P_i\}$ be the punctured neighborhood of P_i, and let X_i be the fiber:

$$\begin{tikzcd}
X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{U} \arrow[d, "\pi_R"] \\
\mathrm{Spec}(R((t_i))) \arrow[r, "\iota_i"] & \mathscr{C} - \{P_i\}.
\end{tikzcd}$$

Then X_i is isomorphic to a Tate elliptic curve over $R((t_i))$. More explicitly, let $q \in R((t_i))$ be the unique power series such that $\mathrm{ord}_{t_i}(q) = r_i$ and

$$j(X_i) = \frac{1}{q} + 744 + 196884q + \cdots . \tag{5.2.1}$$

Then there is an isomorphism of $R((t_i))$-schemes

$$X_i \simeq E_q, \tag{5.2.2}$$

which is unique up to the translation and the involution $u \mapsto u^{-1}$ (cf. [DR], VII.2.6). Put

$$a_i := t_i^{r_i} \cdot j(X_i)\big|_{t_i=0} = \frac{t_i^{r_i}}{q}\bigg|_{t_i=0} \in R^\times.$$

Put $K_i := K(\sqrt[r_i]{a_i})$, and let R_i be its integer ring. Note that K_i is also unramified over $\mathbb{Q}_p$ as r_i is prime to p. There is $q_i \in R_i((t_i))$ such that $q_i^{r_i} = q$, and we have $R_i((t_i)) = R_i((q_i))$. Let κ_i be the composition of natural maps

$$\begin{aligned} \kappa_i : \Gamma(\mathscr{X}, \Omega^2_{\mathscr{X}/R}(\log \mathscr{D})) \longrightarrow \Gamma(X_i, \Omega^2_{X_i/R}) \simeq \Gamma(E_q, \Omega^2_{E_q/R}) \\ \longrightarrow R_i((q_i))\frac{dq_i}{q_i}\frac{dx}{2y+x} \simeq R_i((q_i)), \end{aligned} \tag{5.2.3}$$

where the isomorphism in the middle is induced by (5.2.2). We define a $\mathbb{Z}_p$-submodule $\Phi(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p}$ of $\Gamma(\mathscr{X}, \Omega^2_{\mathscr{X}/R}(\log \mathscr{D}))$ by

$$\omega \in \Phi(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p} \Longleftrightarrow \kappa_i(\omega) \textit{ is a formal series of Eisenstein type for each } i \text{ (§3.2)}.$$

Moreover, we define $\Phi^{(n)}(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p} \subset \Gamma(\mathscr{X}, \Omega^2_{\mathscr{X}/R}(\log \mathscr{D}))$ as the $\mathbb{Z}_p$-submodule consisting of elements ω such that each $\kappa_i(\omega)$ ($1 \le i \le s$) satisfies (E1) (see §3.2) and

(E2)$_n$ $\quad a_k^{(j)} \in k^2\mathbb{Z}_p$ for all j and $1 \le k \le n$.

Obviously we have

$$\Phi(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p} = \bigcap_{n \ge 1} \Phi^{(n)}(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p}.$$

Let

$$\partial_{\mathrm{dR}} : \Gamma(\mathscr{X}, \Omega^2_{\mathscr{X}/R}(\log \mathscr{D})) \longrightarrow \bigoplus_{i=1}^{s} R[\mathscr{D}_i]$$

be the Poincare residue map. By the condition (E1), one has

$$\partial_{\mathrm{dR}}\big(\Phi^{(n)}(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p}\big) \subset \bigoplus_{i=1}^{s} \mathbb{Z}_p[\mathscr{D}_i].$$

Let $f_p : \bigoplus_{i=1}^s \mathbb{Z}_p[\mathscr{D}_i] \to \bigoplus_{i=1}^s \mathbb{F}_p[\mathscr{D}_i]$ be the natural map taking the residue class modulo p. We define

$$\Phi^{(n)}(\mathscr{X},\mathscr{D})_{\mathbb{F}_p} := f_p \circ \partial_{\mathrm{dR}}\big(\Phi^{(n)}(\mathscr{X},\mathscr{D})_{\mathbb{Z}_p}\big) \subset \bigoplus_{i=1}^s \mathbb{F}_p[\mathscr{D}_i]$$
$$\Phi(\mathscr{X},\mathscr{D})_{\mathbb{F}_p} := f_p \circ \partial_{\mathrm{dR}}\big(\Phi(\mathscr{X},\mathscr{D})_{\mathbb{Z}_p}\big) \subset \bigoplus_{i=1}^s \mathbb{F}_p[\mathscr{D}_i].$$

Note that we have

$$\Phi(\mathscr{X},\mathscr{D})_{\mathbb{F}_p} = \bigcap_{n\geq 1} \Phi^{(n)}(\mathscr{X},\mathscr{D})_{\mathbb{F}_p}.$$

5.3 Main results on elliptic surfaces

The boundary maps on étale, syntomic and de Rham cohomology induce a commutative diagram

$$\begin{tikzcd} H^2_{\text{ét}}(\overline{U},\mathbb{Z}_p(2))^{G_K} \arrow[r, "\partial_{\text{ét}}"] & \bigoplus_{i=1}^s \mathbb{Z}_p[\mathscr{D}_i] \arrow[d, equal] \\ H^2_{\mathrm{syn}}(\mathscr{X}(\mathscr{D}),\mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[u, "a"] \arrow[r, "\partial_{\mathrm{syn}}"] \arrow[d, "b"] & \bigoplus_{i=1}^s \mathbb{Z}_p[\mathscr{D}_i] \arrow[d, hook] \\ \Gamma(\mathscr{X},\Omega^2_{\mathscr{X}/R}(\log\mathscr{D})) \arrow[r, "\partial_{\mathrm{dR}}"] & \bigoplus_{i=1}^s R[\mathscr{D}_i]. \end{tikzcd} \tag{5.3.1}$$

Here $\partial_{\text{ét}}$ denotes the composite map

$$H^2_{\text{ét}}(\overline{U},\mathbb{Z}_p(2)) \longrightarrow H^3_{\overline{D}}(\overline{X},\mathbb{Z}_p(2)) \simeq \bigoplus_{i=1}^s \mathbb{Z}_p[\mathscr{D}_i]$$

and the first arrow is the boundary localization sequence of the etale cohomology. The last isomorphism is obtained from the Poincare-Lefschetz duality. See Proposition 2.2.14 for ∂_{syn}.

Lemma 5.3.2 *The map a in* (5.3.1) *is surjective.*

Proof. Put $\mathscr{V} := \mathscr{C} - \Sigma$. Let $\mathscr{U}_s$ and $\mathscr{V}_s$ denote the fibers of $\mathscr{U} \to \mathrm{Spec}(R)$ and $\mathscr{V} \to \mathrm{Spec}(R)$ over the closed point s of $\mathrm{Spec}(R)$. By a result of Tsuji [Ts2], Theorem 5.1, there is an exact sequence

$$H^2_{\mathrm{syn}}(\mathscr{X}(\mathscr{D}),\mathscr{S}_{\mathbb{Z}_p}(2)) \to H^2_{\text{ét}}(U,\mathbb{Z}_p(2)) \xrightarrow{\delta_U} \varprojlim_n \mathscr{O}(\mathscr{U}_s)^\times/p^n \simeq \varprojlim_n \mathscr{O}(\mathscr{V}_s)^\times/p^n, \tag{5.3.3}$$

where the last isomorphism follows from the fact that $\mathscr{U}_s \to \mathscr{V}_s$ is proper with geometrically connected fibers. On the other hand, we assert that there is an exact sequence

$$0 \longrightarrow H^2_{\text{ét}}(S,\mathbb{Z}_p(2)) \xrightarrow{\pi_K^*} H^2_{\text{ét}}(U,\mathbb{Z}_p(2)) \longrightarrow H^2_{\text{ét}}(\overline{U},\mathbb{Z}_p(2))^{G_K} \longrightarrow 0, \tag{5.3.4}$$

where we put $S := \mathscr{V} \otimes_R K$. Indeed, the Hochschild-Serre spectral sequences

$$E_{2,U}^{p,q} = H^p(K, H^q_{\text{ét}}(\overline{U}, \mathbb{Z}_p(2))) \Longrightarrow E_U^{p+q} = H^{p+q}(U, \mathbb{Z}_p(2)) \tag{5.3.5}$$

$$E_{2,S}^{p,q} = H^p(K, H^q_{\text{ét}}(\overline{S}, \mathbb{Z}_p(2))) \Longrightarrow E_S^{p+q} = H^{p+q}(S, \mathbb{Z}_p(2)) \tag{5.3.6}$$

and the isomorphisms $H^q_{\text{ét}}(\overline{S}, \mathbb{Z}_p(2)) \simeq H^q_{\text{ét}}(\overline{U}, \mathbb{Z}_p(2))$ for $q \leq 1$ (cf. Lemma 5.4.1 below) yield a short exact sequence

$$0 \longrightarrow H^2_{\text{ét}}(S, \mathbb{Z}_p(2)) \xrightarrow{\pi_K^*} H^2_{\text{ét}}(U, \mathbb{Z}_p(2)) \longrightarrow E_{\infty,U}^{0,2} \longrightarrow 0.$$

It remains to show $E_{2,U}^{0,2} = E_{\infty,U}^{0,2}$. Since $\mathrm{cd}(K) = 2$, it is enough to show that the edge homomorphism $E_{2,U}^{0,2} \to E_{2,U}^{2,1}$ is zero. There is a commutative diagram with exact rows

$$\begin{tikzcd}
E_{2,U}^{0,2} \arrow[r, "\alpha"] & E_{2,U}^{2,1} \arrow[r, "\beta"] & E_U^3 \\
E_{2,S}^{0,2} \arrow[u] \arrow[r, "\gamma"] & E_{2,S}^{2,1} \arrow[u, "\wr"] \arrow[r, "\delta"] & E_S^3. \arrow[u, hook]
\end{tikzcd}$$

Therefore $\alpha = 0 \Longleftrightarrow \beta$ is injective $\Longleftrightarrow \delta$ is injective $\Longleftrightarrow \gamma = 0$. The last condition follows from $E_{2,S}^{0,2} = 0$, which completes the proof of the exact sequence (5.3.4).

Finally we show that a is surjective. By (5.3.3) and (5.3.4), it is enough to show

$$\mathrm{Im}(H^2_{\text{ét}}(U, \mathbb{Z}/p^n(2)) \xrightarrow{\delta_U} \mathscr{O}(\mathscr{U}_s)^\times/p^n) = \mathrm{Im}(H^2_{\text{ét}}(S, \mathbb{Z}/p^n(2)) \xrightarrow{\delta_S} \mathscr{O}(\mathscr{V}_s)^\times/p^n).$$

The inclusion '$\supset$' follows from the fact $\pi_s^* \circ \delta_U = \delta_S \circ \pi_K^*$, and '$\subset$' follows from the fact $e_s^* \circ \delta_U = \delta_S \circ e_K^*$, where $e_R : \mathscr{C} \to \mathscr{X}$ denotes the section, and π_F and e_F ($F = K, s$) denote $\pi_R \otimes_R F$ and $e_R \otimes_R F$, respectively. □

Theorem 5.3.7 *Let $\mathscr{X} \to \mathscr{C}$ and $\mathscr{D}$ be as above. Assume $p \nmid 6r_1 \cdots r_s$. Then we have*

$$\partial_{\text{ét}}\big(H^2_{\text{ét}}(\overline{U}, \mathbb{Z}_p(2))^{G_K}\big) \subset \partial_{\text{dR}}\big(\varPhi(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p}\big) \quad \textit{in} \quad \bigoplus_{i=1}^s \mathbb{Z}_p[\mathscr{D}_i]. \tag{5.3.8}$$

Note that $\partial_{\text{ét}}(H^2_{\text{ét}}(\overline{U}, \mathbb{Q}_p(2))^{G_K}) \simeq H^2_{\text{ét}}(\overline{U}, \mathbb{Q}_p(2))^{G_K}$ because $\partial_{\text{ét}} \otimes \mathbb{Q}$ is injective.

Proof. By a commutative diagram

$$\begin{tikzcd}
H^2_{\text{syn}}(X_i, \mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[d] & H^2_{\text{syn}}(\mathscr{U}, \mathscr{S}_{\mathbb{Z}_p}(2)) \arrow[l] \arrow[d, "b"] \\
\varprojlim_n R_i((q_i))/p^n \dfrac{dq_i}{q_i} \dfrac{dx}{2y+x} & \varGamma(\mathscr{X}, \Omega^2_{\mathscr{X}/R}(\log \mathscr{D})), \arrow[l]
\end{tikzcd}$$

the inclusion (5.3.8) is an immediate consequence of the main result on Tate curve (Theorem 3.2.3) and Lemma 5.3.2. □

It is in general impossible to compute the rank of $\partial_{\mathrm{dR}}(\varPhi(\mathscr{X},\mathscr{D})_{\mathbb{Z}_p})$ even though one can compute $\varPhi^{(n)}(\mathscr{X},\mathscr{D})_{\mathbb{Z}_p}$ for a particular n. On the other hand, $\varPhi^{(n)}(\mathscr{X},\mathscr{D})_{\mathbb{F}_p}$ is actually computable. Our second result is to give an upper bound of $\partial_{\text{ét}}(H^2_{\text{ét}}(\overline{U},\mathbb{Z}_p(2))^{G_K})$ by those.

Theorem 5.3.9 *Let $\pi_R:\mathscr{X}\to\mathscr{C}$ and $\mathscr{D}$ be as before. Assume $p\not|\,6r_1\cdots r_s$ and further the following conditions*:

(A) $H^3_{\text{ét}}(\overline{X},\mathbb{Z}_p)$ *is torsion-free.*

(B) $H^2_{\text{ét}}(\overline{X},\mathbb{Z}/p(2))^{G_K}=0$.

Then the quotient group

$$\mathbb{Z}_p^{\oplus s}\big/\partial_{\text{ét}}\big(H^2_{\text{ét}}(\overline{U},\mathbb{Z}_p(2))^{G_K}\big) \tag{5.3.10}$$

is torsion-free. In particular

$$\dim_{\mathbb{Q}_p}\partial_{\text{ét}}(H^2_{\text{ét}}(\overline{U},\mathbb{Q}_p(2))^{G_K})\le\dim_{\mathbb{F}_p}\varPhi(\mathscr{X},\mathscr{D})_{\mathbb{F}_p}. \tag{5.3.11}$$

Proof. Put $V_{\mathbb{Z}_p}:=H^2_{\text{ét}}(\overline{X},\mathbb{Z}_p(2))/\langle D\rangle\otimes\mathbb{Z}_p(1)$ where $\langle D\rangle$ denotes the subgroup of the cycle classes of irreducible components of $\overline{D}$. There is an exact sequence

$$0\longrightarrow V_{\mathbb{Z}_p}\longrightarrow H^2_{\text{ét}}(\overline{U},\mathbb{Z}_p(2))\xrightarrow{\partial_{\text{ét}}}\mathbb{Z}_p^{\oplus s}\longrightarrow 0 \tag{5.3.12}$$

under the condition **(A)** by Lemma 5.4.7 below. Moreover $H^2_{\text{ét}}(\overline{X},\mathbb{Z}_p(2))$ is torsion-free by Lemma 5.4.3 below and the natural surjective map

$$H^2_{\text{ét}}(\overline{X},\mathbb{Z}_p(2))\longrightarrow V_{\mathbb{Z}_p}$$

has a G_K-equivariant splitting (cf. Lemma 5.4.4 below). We thus obtain an exact sequence

$$0\longrightarrow H^2_{\text{ét}}(\overline{U},\mathbb{Z}_p(2))^{G_K}\xrightarrow{\partial_{\text{ét}}}\mathbb{Z}_p^{\oplus s}\longrightarrow H^1(K,H^2_{\text{ét}}(\overline{X},\mathbb{Z}_p(2))). \tag{5.3.13}$$

The condition **(B)** together with an exact sequence

$$0\longrightarrow H^2_{\text{ét}}(\overline{X},\mathbb{Z}_p(2))\xrightarrow{p}H^2_{\text{ét}}(\overline{X},\mathbb{Z}_p(2))\longrightarrow H^2_{\text{ét}}(\overline{X},\mathbb{Z}/p(2))\longrightarrow 0$$

implies that $H^1(K,H^2_{\text{ét}}(\overline{X},\mathbb{Z}_p(2)))$ is torsion-free. Hence so is the quotient (5.3.10). □

Finally we give the following useful criterion for the conditions **(A)** and **(B)**.

Proposition 5.3.14 *Consider the following conditions:*

(A') *the elliptic surface $\pi:\overline{X}\to\overline{C}$ has at least one additive (singular) fiber,*

(B') *Let $Y=\mathscr{X}_s$ be the special fiber of $\mathscr{X}$ over the closed point $s\in\operatorname{Spec}(R)$, and let $C:\varGamma(Y,\Omega^2_Y)\to\varGamma(Y,\Omega^2_Y)$ be the Cartier operator. Then $C-\mathrm{id}$ is injective.*

Then **(A')** $\Longrightarrow$ **(A)**, *and* **(B')** $\Longrightarrow$ **(B)**.

Proof. See Lem.5.4.6 below for the first assertion, and see [ASat], §4.4 for the second. □

5.4 Auxiliary results on Betti cohomology groups

In the previous section, we used several results on étale cohomology groups of elliptic surfaces over $\overline{K}$, which we prove here. We take the base-change by a fixed embedding $\overline{K} \hookrightarrow \mathbb{C}$ and then replace the étale cohomology with the Betti cohomology H^*. For simplicity, we denote $X \otimes_K \mathbb{C}$, $S \otimes_K \mathbb{C}$ and $U \otimes_K \mathbb{C}$ by X, S and U, respectively.

Lemma 5.4.1 $H^q(S, \mathbb{Z}) \simeq H^q(U, \mathbb{Z})$ *for* $q \leq 1$.

Proof. The case $q = 0$ is clear. As for the case $q = 1$, the Leray spectral sequence

$$E_2^{pq} = H^p(S, R\pi_* \mathbb{Z}_U) \Longrightarrow H^{p+q}(U, \mathbb{Z}) \tag{5.4.2}$$

yields an exact sequence

$$0 \longrightarrow H^q(S, \mathbb{Z}) \longrightarrow H^q(U, \mathbb{Z}) \longrightarrow \Gamma(S, R^1\pi_* \mathbb{Z}).$$

Since $\Gamma(S, R^1\pi_* \mathbb{Z}) \hookrightarrow H^1(X_t, \mathbb{Z})^{\pi_1(S)} = 0$, the assertion follows. □

Lemma 5.4.3 *Assume that* $H^3(X, \mathbb{Z})$ *is* p*-torsion-free. Then* $H^2(X, \mathbb{Z})$ *is* p*-torsion-free as well.*

Proof. By the universal coefficient theorem $H^1(X, M) \simeq \mathrm{Hom}(H_1(X, \mathbb{Z}), M)$ for an abelian group M. Therefore if $H_1(X, \mathbb{Z}) = H^3(X, \mathbb{Z})$ is p-torsion-free, then the map $H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{Z}/p)$ is surjective. Equivalently, the multiplication by p is injective on $H^2(X, \mathbb{Z})$. □

Lemma 5.4.4 *Let* $\langle D \rangle \subset H^2(X, \mathbb{Z})$ *be the subgroup generated by the cycle classes of irreducible components of* $D = \sum_{i=1}^s D_i$*. Assume* $p \geq 5$*. Then the inclusion map*

$$\langle D \rangle \otimes \mathbb{Z}_p \longrightarrow H^2(X, \mathbb{Z}_p)$$

has a natural splitting.

Proof. Let us first assume that X is minimal (hence D_i is a Néron polygon). Let $D_i^{(k)}$ ($k \geq 1$) be the irreducible components of D_i, and let

$$\bigoplus_{i,k} \mathbb{Z}\,[D_i^{(k)}] \longrightarrow H^2(X, \mathbb{Z})$$

be the map sending $[D_i^{(k)}]$ to its cycle class. Let

$$H^2(X, \mathbb{Z}) \longrightarrow \bigoplus_{i,k} H^2(D_i^{(k)}, \mathbb{Z}) \simeq \bigoplus_{i,k} \mathbb{Z}\,[D_i^{(k)}] \quad \text{and} \quad H^2(X, \mathbb{Z}) \xrightarrow{e^*} H^2(C, \mathbb{Z}) \simeq \mathbb{Z}[C]$$

be the natural pull-back maps. Then the composite map

$$\bigoplus_{i,k} \mathbb{Z}[D_i^{(k)}] \longrightarrow H^2(X,\mathbb{Z}) \longrightarrow \bigoplus_{i,k} \mathbb{Z}[D_i^{(k)}] \oplus \mathbb{Z}[C] \tag{5.4.5}$$

is described as follows

$$[D_i^{(k)}] \longmapsto \sum_{j,l} (D_i^{(k)}, D_j^{(l)})[D_j^{(l)}] + (D_i^{(k)}, e(C))[C]$$

where $(-,-)$ denotes the intersection pairing. Therefore by an elementary computation on the intersection matrix, we see that the composite map (5.4.5) induces an injective map

$$\langle D\rangle \simeq \bigoplus_{i,k\geq 1} \mathbb{Z}[D_i^{(k)}] \Big/ \langle [D_i]-[D_j]; i<j\rangle \quad\hookrightarrow\quad \bigoplus_{i\geq 1,k\geq 2} \mathbb{Z}[D_i^{(k)}] \oplus \mathbb{Z}[C],$$

whose matrix has determinant prime to 6.

Next we consider a general X. Take the minimal model $\mu : X \to X_0$. Then $H^2(X,\mathbb{Z})$ is a direct sum of $H^2(X_0,\mathbb{Z})$ and the cycle classes of exceptional divisors. Since $\langle \mu(D)\rangle \otimes \mathbb{Z}_p$ is a direct summand of $H^2(X_0,\mathbb{Z}_p)$, $\langle D\rangle \otimes \mathbb{Z}_p$ is a direct summand of $H^2(X,\mathbb{Z}_p)$. $\square$

Lemma 5.4.6 *Assume that there is at least one additive singular fiber E. Then $H_1(X,\mathbb{Z}) \simeq H_1(C,\mathbb{Z})$. In particular $H_1(X,\mathbb{Z}) = H^3(X,\mathbb{Z})$ is torsion-free.*

Proof. We show that $e_* : H_1(C,\mathbb{Z}) \to H_1(X,\mathbb{Z})$ is surjective. Let $S^0 \subset S$ be a Zariski open (non-empty) subset such that $U^0 := \pi^{-1}(S^0) \to S^0$ is a smooth fibration. By the fibration exact sequence $1 \to \pi_1(X_t) \to \pi_1(U^0) \to \pi_1(S^0) \to 1$, which is split by e, one has a surjection $\pi_1(X_t) \times \pi_1(S_0)^{\mathrm{ab}} \to \pi_1(U_0)^{\mathrm{ab}}$. Since $\pi_1(U_0)^{\mathrm{ab}} \to \pi_1(X)^{\mathrm{ab}} = H_1(X,\mathbb{Z})$ is surjective, it is enough to show that the map $\pi_1(X_t) = H_1(X_t,\mathbb{Z}) \to H_1(X,\mathbb{Z})$ is zero, which follows from the fact that it factors through $H_1(E,\mathbb{Z}) = 0$. $\square$

Lemma 5.4.7 *Let $\partial : H^2(U,\mathbb{Z}) \to H^3_D(X,\mathbb{Z}) = \mathbb{Z}^{\oplus s}$ be the boundary map on the Betti cohomology arising from the localization exact sequence*

$$\cdots \longrightarrow H^2(U,\mathbb{Z}) \xrightarrow{\partial} \mathbb{Z}^{\oplus s} \longrightarrow H^3(X,\mathbb{Z}) \longrightarrow H^3(U,\mathbb{Z}) \longrightarrow \cdots .$$

Then $\partial \otimes \mathbb{Q}$ is surjective. In particular, if $H^3(X,\mathbb{Z})$ is p-torsion-free, then the cokernel of ∂ is p-torsion-free as well.

Proof. We show that $H^3(X,\mathbb{Q}) \longrightarrow H^3(U,\mathbb{Q})$ is injective. The case $s=0$ (i.e., $X=U$) is obvious. Assume $s>0$. One has $H^3(U,\mathbb{Q}) = H^1(S, R^2\pi_*\mathbb{Q}_U) = H^1(S,\mathbb{Q})$ from the Leray spectral sequence (5.4.2). Since

$$H^2(C, R^1\pi_*\mathbb{Q}_X) \xleftarrow{\simeq} H^2_c(S, R^1\pi_*\mathbb{Q}_U) = H^0(S, R^1\pi_*\mathbb{Q}_U)^\vee = 0,$$

one also has $H^3(X,\mathbb{Q}) = H^1(C,\mathbb{Q})$ from the Leray spectral sequence for $\pi : X \to C$. Thus the map $H^3(X,\mathbb{Q}) \to H^3(U,\mathbb{Q})$ is identified with the natural restriction map $H^1(C,\mathbb{Q}) \to H^1(S,\mathbb{Q})$, which is clearly injective. $\square$

6 An elliptic $K3$ surface over $\mathbb{Q}_p$ with finitely many torsion zero-cycles

There is an application of our main results to Beilinson's Tate conjecture for K_2. See the introduction for a precise contents of this conjecture, and see [ASat] for details and constructions of non-trivial examples. In this section, all cohomology groups of schemes are taken over the étale topology.

Here we discuss another application to p-adic regulator map on K_1

$$\varrho : K_1(X)^{(2)} \otimes \mathbb{Q}_p \longrightarrow H^1_g(K, H^2(\overline{X}, \mathbb{Q}_p(2))).$$

The main result of this section is

Theorem 6.0.1 *Let $\pi : X_0 \to \mathbb{P}^1$ be the elliptic K3 surface over $\mathbb{Q}$ whose general fiber $\pi^{-1}(t)$ is an elliptic curve given by*

$$3Y^2 + X^3 + (3X + 4t^4)^2 = 0.$$

Put $X := X_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Then the p-adic regulator

$$\varrho : K_1(X)^{(2)} \otimes \mathbb{Q}_p \longrightarrow H^1_g(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_p(2))) \tag{6.0.2}$$

is surjective when $p \equiv 3$ mod 4 and $7 \leq p \leq 31$.

Theorem 6.0.1 immediately implies the following finiteness result by [SS1], Theorem 3.1.1.

Corollary 6.0.3 *The p-primary torsion subgroup of $\mathrm{CH}_0(X)$ is finite.*

See §6.3 below for the finiteness of the full torsion part $\mathrm{CH}_0(X)_{\mathrm{tors}}$. The proof of Theorem 6.0.1 is divided into two parts, i.e. surjectivity onto H^1_g/H^1_f and H^1_f (see §6.2). To show the former, we shall reduce it to the theorem of Flach and Mildenhall. To show the latter, we shall construct a new indecomposable element in $K_1(\mathscr{X})^{(2)}$. One needs a new techniques to show the non-vanishing in H^1_f and Theorem 5.3.9 plays an essential role in the proof (cf. Claim 6.2.3).

Remark 6.0.4 *In* [ASat], *we constructed a number of indecomposable elements which do not vanish in the f-part of Galois cohomology. However there is no example of elliptic surface in loc. cit. for which we could prove the surjectivity onto H^1_f* (*Theorem* 6.0.1 *is the first example*).

Remark 6.0.5 *The elliptic surface defined by $3Y^2 + X^3 + (3X + 4t)^2 = 0$ ($t \in \mathbb{P}^1$) is the universal family of elliptic curves over the modular curve $X_1(3)$. The elliptic K3 in Theorem* 6.0.1 *is a finite covering of this surface, but it is no longer modular.*

Remark 6.0.6 *It might be possible to extend the above results for $p > 31$. We will give, in Remark* 6.2.7 *below, a sufficient condition for Theorem* 6.0.1 *and Corollary* 6.0.3 *to hold for more general p's. We do not know how to check the condition for infinitely many p's. Moreover we do not know whether Theorem* 6.0.1 *or* Corollary 6.0.3 *remains true when $p \equiv 1$ mod 4.*

6.1 Preliminary facts

Before proving the theorem, we recall some standard facts on the X, which are easy to check.

Fact 1. X has good reduction at $p \geq 5$. For $p \geq 5$, the reduction Y_p of X at p is ordinary if and only if $p \equiv 1$ mod 4, and super-singular if and only if $p \equiv 3$ mod 4.

Fact 2. The functional j-invariant is $27(9-8t^4)^3/((1-t^4)t^{12})$. There are 5 multiplicative fibers over $t = 0,\ \pm 1,\ \pm\sqrt{-1}$, and one additive fiber over $t = \infty$:

t	0	$\pm 1, \pm\sqrt{-1}$	∞
$\pi^{-1}(t)$	I_{12}	I_1	IV^*

Fact 3. The Néron-Severi group $\mathrm{NS}(X_{\overline{\mathbb{Q}}})$ has rank 20. Explicitely it is generated by the following irreducible curves:

(i) irreducible components of $\pi^{-1}(0)$ (12-components),

(ii) irreducible components of $\pi^{-1}(\infty)$ (7-components),

(iii) the section at infinity $e(\mathbb{P}^1) = E$,

(iv) the section C defined by $x = -4t^4/3$ and $y = 8t^6/9$.

Note $\pi^{-1}(0) \equiv \pi^{-1}(\infty)$ in $\mathrm{NS}(X_{\overline{\mathbb{Q}}})$ and it is the unique numerical relation among the above.

Fact 4. Put $\overline{Y_p} := Y_p \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$. The Néron-Severi group $\mathrm{NS}(\overline{Y_p})$ has rank 20 if $p \equiv 1$ mod 4, and rank 22 if $p \equiv 3$ mod 4. In the former case, one has $\mathrm{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} = \mathrm{NS}(\overline{Y_p}) \otimes \mathbb{Q}$. In the latter case, we will describe curves on $\overline{Y_p}$ which do not come from $X_{\overline{\mathbb{Q}}}$ in §6.2.1 below.

6.2 Proof of Theorem 6.0.1

We prove the surjectivity of (6.0.2) assuming $p \equiv 3$ mod 4 and $7 \leq p \leq 31$. Let $\mathscr{X} \to \mathrm{Spec}(\mathbb{Z}_p)$ be a projective smooth model of X, and let $Y \to \mathrm{Spec}(\mathbb{F}_p)$ be its special fiber. Since the decomposable part of $K_1(X)^{(2)} \otimes \mathbb{Q}_p$ is onto $H^1(\mathbb{Q}_p, \mathrm{NS}(\overline{X}) \otimes \mathbb{Q}_p(1))$ (cf. [S], Lemma 3.6), we may replace the target with $H^1_g(\mathbb{Q}_p, V)$, where we put

$$V := H^2(\overline{X}, \mathbb{Q}_p(2))/\mathrm{NS}(\overline{X}) \otimes \mathbb{Q}_p(1).$$

We shall prove it in the following steps.

Step 1. $K_1(X)^{(2)} \otimes \mathbb{Q}_p \to H^1_g(\mathbb{Q}_p, V)/H^1_f(\mathbb{Q}_p, V)$ is surjective for any $p \geq 5$.

Step 2. $K_1(\mathscr{X})^{(2)} \otimes \mathbb{Q}_p \to H^1_f(\mathbb{Q}_p, V)$ is surjective for $p \equiv 3$ mod 4 and $7 \leq p \leq 31$.

6.2.1 Proof of Step 1

By the Tate conjecture for Y ([ArSw]) and the same arguments as in [LS], Theorem 5.1, there is a canonical isomorphism

$$H^1_g(\mathbb{Q}_p, V)/H^1_f(\mathbb{Q}_p, V) \simeq \mathrm{NS}(Y)/\mathrm{NS}(X) \otimes \mathbb{Q}_p$$

and the composition

$$K_1(X)^{(2)} \otimes \mathbb{Q}_p \to H^1_g(\mathbb{Q}_p, V)/H^1_f(\mathbb{Q}_p, V) \simeq \mathrm{NS}(Y)/\mathrm{NS}(X) \otimes \mathbb{Q}_p$$

is given by the boundary map $K_1(X)^{(2)} \to \mathrm{NS}(Y) \otimes \mathbb{Q}$ arising from the localization exact sequence in K-theory.

Claim 6.2.1 *Let C_i $(i = 1, 2)$ be elliptic curves over $\mathbb{Q}$ defined by equations*

$$3y^2 + x^4 + 1 = 0 \quad \textit{and} \quad u^2 + 4v^4 + 3 = 0,$$

respectively. Then there is a dominant rational map

$$f : C_1 \times C_2 \dashrightarrow X_0$$

of degree 8 given by $(x, y) \times (u, v) \mapsto (X, Y, t) = ((ux)^4, u^6x^4y, uvx)$.

Proof. Straight-forward. □

Note that C_1 and C_2 are isomorphic to each other up to twist. Let $S \to C_1 \times C_2$ be a birational transformation such that f is extended to a morphism $\bar{f} : S \to X$. Put

$$V(-) := H^2(-, \mathbb{Q}_p(2))/\mathrm{NS}(-) \otimes \mathbb{Q}_p(1).$$

Then one has

$$V(\overline{X}) \hookrightarrow V(\overline{S}) \xleftarrow{\sim} V(\overline{C_1 \times C_2}).$$

We show that the first map is bijective. Indeed, we have $\dim V(\overline{X}) = 2$ by **Fact 3**. On the other hand, since C_1 and C_2 are isomorphic to a CM elliptic curve up to twist, $V(\overline{C_1 \times C_2})$ is also 2-dimensional. Hence $V(\overline{X}) = V(\overline{S})$.

In order to show **Step 1** it is enough to show that

$$K_1(C_1 \times C_2)^{(2)} \otimes \mathbb{Q}_p \to H^1_g(\mathbb{Q}_p, V(\overline{C_1 \times C_2}))/H^1_f(\mathbb{Q}_p, V(\overline{C_1 \times C_2}))$$

is surjective. We may replace $\mathbb{Q}_p$ with arbitrary finite extension $K/\mathbb{Q}_p$ by a standard norm argument. Fix K such that $C_{1,K} \simeq C_{2,K} (=: C)$ with smooth reduction $\mathscr{C}_s$ and such that

$$\mathrm{End}(C) = \mathrm{End}(\overline{C}) \quad \text{and} \quad \mathrm{End}(\mathscr{C}_s) = \mathrm{End}(\overline{\mathscr{C}_s}).$$

We show that

$$\begin{aligned} K_1(C \times C)^{(2)} \otimes \mathbb{Q}_p \longrightarrow{} & H^1_g(K, V(\overline{C \times C}))/H^1_f(K, V(\overline{C \times C})) \\ & \simeq \mathrm{End}(\overline{\mathscr{C}_s}) \otimes \mathbb{Q}_p/\mathrm{End}(\overline{C}) \otimes \mathbb{Q}_p \end{aligned}$$

is surjective. If $\mathscr{C}_s$ is ordinary (i.e., $p \equiv 1$ mod 4), the target is zero. If $\mathscr{C}_s$ is super-singular (i.e., $p \equiv 3$ mod 4), it is generated by the Frobenius endomorphism and its composition with the CM endomorphism. The surjectivity then follows from [F], Proposition 2.1 or [M], Theorem 5.8. This completes the proof of **Step 1**.

6.2.2 Proof of Step 2

This is the most important step.

We first note that $H^1_f(\mathbb{Q}_p, V) \simeq H^3_{\mathrm{syn}}(\mathscr{X}(\mathscr{D}), \mathscr{S}_{\mathbb{Q}_p}(2))/K_1^{\mathrm{dec}}(\mathscr{X})^{(2)} \otimes \mathbb{Q}_p \simeq H^2(\mathscr{O}_X)$ is 1-dimensional. We shall show that there is an element $\xi \in K_1(\mathscr{X})^{(2)}$ such that $\varrho(\xi) \neq 0$ in $H^1_f(\mathbb{Q}_p, V)$.

Let $\mathscr{D}_1 := \pi^{-1}(1)$ and $\mathscr{D}_2 := \pi^{-1}(-1)$ be the multiplicative fiber over $\mathbb{Z}_p$ which are Néron 1-gon (**Fact 2**). Put $\mathscr{D} := \mathscr{D}_1 + \mathscr{D}_2$ and $\mathscr{U} := \mathscr{X} - \mathscr{D}$. We consider rational functions

$$u_1 := \frac{Y-(X+4)}{Y+(X+4)}\Big|_{\mathscr{D}_1}, \qquad u_2 := \frac{Y-(X+4)}{Y+(X+4)}\Big|_{\mathscr{D}_2}$$

on $\mathscr{D}_1$ and $\mathscr{D}_2$ respectively. They define elements $\xi'_i \in K'_1(\mathscr{D}_i)^{(1)}$ by Quillen's localization exact sequence

$$0 \longrightarrow K'_1(\mathscr{D}_i)^{(1)} \longrightarrow K'_1(\mathscr{D}_i^{\mathrm{reg}})^{(1)} \xrightarrow{j_*\mathrm{div}} \mathbb{Q}.$$

Here $\mathscr{D}_i^{\mathrm{reg}}$ denotes the smooth locus of $\mathscr{D}_i$, which is isomorphic to $\mathbb{G}_{m,\mathbb{Z}_p}$, and $j : \widetilde{\mathscr{D}}_i \to \mathscr{D}_i$ denotes the normalization. We denote by $\xi_i \in K_1(\mathscr{X})^{(2)}$ the image of ξ'_i via the natural map $K'_1(\mathscr{D}_i)^{(1)} \to K_1(\mathscr{X})^{(2)}$. We shall prove

$$\varrho(\xi_1 - \xi_2) \neq 0 \ \text{ in } \ H^1_f(\mathbb{Q}_p, V). \tag{6.2.2}$$

Claim 6.2.3 *If $p = 7, 11, 19, 23$ or 31, then $H^2(\overline{U}, \mathbb{Q}_p(2))^{G_{\mathbb{Q}_p}} = 0$.*

Proof. First we check the conditions **(A)'** and **(B)'** in Proposition 5.3.14. **(A)'** holds by **Fact 2**. Since $p \equiv 3 \bmod 4$ by assumption, Y is super-singular by **Fact 1** and the Cartier operator C is zero on $H^0(Y, \Omega^2_Y)$. Thus **(B)'** holds.

We apply Theorem 5.3.9 to show $H^2(\overline{U}, \mathbb{Q}_p(2))^{G_{\mathbb{Q}_p}} = 0$. We will show

$$\Phi^{(p^2)}(\mathscr{X}, \mathscr{D})_{\mathbb{F}_p} = 0$$

for $p = 7, 11, 19, 23, 31$. The space $\Gamma(\mathscr{X}, \Omega^2_{\mathscr{X}/\mathbb{Z}_p}(\log \mathscr{D}))$ is a free $\mathbb{Z}_p$-module of rank 3 generated by

$$\omega_0 := dt\frac{dX}{Y}, \quad \omega_1 := \frac{dt}{t-1}\frac{dX}{Y}, \quad \omega_2 := \frac{dt}{t+1}\frac{dX}{Y},$$

which satisfies

$$\partial_{\mathrm{dR}}(\omega_0) = 0, \quad \partial_{\mathrm{dR}}(\omega_1) = \mathscr{D}_1, \quad \partial_{\mathrm{dR}}(\omega_2) = \mathscr{D}_2.$$

Let $\mathscr{X}^*$ be the 'tubular neighborhood' of $\mathscr{D}_1$:

$$\begin{tikzcd}
\mathscr{X}^* \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{X} - \mathscr{D}_1 \arrow[d] \\
\mathrm{Spec}(\mathbb{Z}_p((t-1))) \arrow[r] & \mathbb{P}^1_{\mathbb{Z}_p} - \{1\}.
\end{tikzcd}$$

We define an isomorphism between $\mathscr{X}^*$ and the Tate elliptic curve

$$E_q : y^2 + xy = x^3 + a_4(q)x + a_6(q) \quad \text{over} \quad \mathbb{Z}_p((q))$$

as follows:

$$x + \frac{1}{12} = -\frac{E_1^2 \cdot (X+3)}{12}, \qquad y + \frac{1}{2}x = \frac{E_1^3 \cdot Y}{24}, \qquad t^4 = \frac{E_{3,a}}{(E_1)^3} = \frac{E_{3,a}}{E_{3,a} + 27E_{3,b}},$$

$$\left(\implies \qquad j(q) = \frac{27(9-8t^4)^3}{(1-t^4)t^{12}}, \qquad j(q^3) = \frac{27(1+8t^4)^3}{(1-t^4)^3 t^4} \right)$$

where E_1, $E_{3,a}$ and $E_{3,b}$ are the q-expansions of the Eisenstein series for $\Gamma_1(3)$ at the cusp $t = 1$, i.e.,

$$E_1 = 1 + 6 \sum_{k=1}^{\infty} \left(\frac{q^{3k-2}}{1-q^{3k-2}} - \frac{q^{3k-1}}{1-q^{3k-1}} \right)$$

$$E_{3,a} = 1 - 9 \sum_{k=1}^{\infty} \left(\frac{(3k-2)^2 q^{3k-2}}{1-q^{3k-2}} - \frac{(3k-1)^2 q^{3k-1}}{1-q^{3k-1}} \right)$$

$$E_{3,b} = \sum_{k=1}^{\infty} \frac{k^2(q^k - q^{2k})}{1-q^{3k}}.$$

Let ι be the composite map

$$\iota : \mathit{\Gamma}(\mathscr{X}, \Omega^2_{\mathscr{X}}(\log \mathscr{D})) \to \mathit{\Gamma}(\mathscr{X}^*, \Omega^2_{\mathscr{X}^*}) \simeq \mathit{\Gamma}(E_q, \Omega^2_{E_q}) \to \mathbb{Z}_p((q)) \frac{du}{u} \frac{dq}{q}$$

(u denotes the Tate parameter of E_q). We note

$$\iota\left(\frac{dt}{t}\frac{dX}{Y}\right) = -\frac{27}{4} E_{3,b} \frac{du}{u}\frac{dq}{q}, \qquad \iota\left(\frac{d(t^4-1)}{t^4-1}\frac{dX}{Y}\right) = E_{3,a}\frac{du}{u}\frac{dq}{q}.$$

Put

$$f_1(q)\frac{du}{u}\frac{dq}{q} = \iota(\omega_1), \quad f_2(q)\frac{du}{u}\frac{dq}{q} = \iota(\omega_2), \quad g(q)\frac{du}{u}\frac{dq}{q} = \iota(\omega_0)$$

$$\left(\iff f_1(q) = -\frac{27t}{4(t-1)} E_{3,b}, \quad f_2(q) = -\frac{27t}{4(t+1)} E_{3,b}, \quad g(q) = -\frac{27t}{4} E_{3,b} \right).$$

Express

$$f_1(q) = 1 + \sum_{i=1}^{\infty} \frac{a_i q^i}{1-q^i}, \quad f_2(q) = \sum_{i=1}^{\infty} \frac{b_i q^i}{1-q^i}, \quad g(q) = \sum_{i=1}^{\infty} \frac{c_i q^i}{1-q^i}$$

with $a_i, b_i, c_i \in \mathbb{Z}_p$. By definition, $\alpha_1\omega_1 + \alpha_2\omega_2 + \beta\omega_0 \in \mathit{\Phi}^{(p^2)}(\mathscr{X}, \mathscr{D})_{\mathbb{Z}_p}$ if and only if

$$\alpha_1 a_i + \alpha_2 b_i + \beta c_i \equiv 0 \mod i^2\mathbb{Z}_p \tag{6.2.4}$$

for all $i \le p^2$ and $p|i$. With the aid of computer one can compute $a_i,\ b_i$ and c_i explicitly

$$a_1 = \frac{9}{8}, \quad b_1 = -\frac{27}{8}, \quad c_1 = -\frac{27}{4}, \quad a_2 = -\frac{45}{64}, \quad b_2 = \frac{297}{64}, \quad c_2 = \frac{513}{16}, \cdots .$$

For example, when $p = 7$ one has

$$\begin{aligned} ((a_{7i} \bmod 7^2)_{1\le i\le 6},\ a_{49} \bmod 7^4) &= (12, 27, 44, 37, 11, 30, 392), \\ ((b_{7i} \bmod 7^2)_{1\le i\le 6},\ b_{49} \bmod 7^4) &= (23, 36, 40, 19, 38, 26, -147), \\ ((c_{7i} \bmod 7^2)_{1\le i\le 6},\ c_{49} \bmod 7^4) &= (19, 20, 2, 9, 11, 2, -686). \end{aligned}$$

On the other hand, a direct calculation shows that if α_i and β satisfy (6.2.4), then α_1 and α_2 must be divided by 7. This implies $\Phi^{(p^2)}(\mathscr{X}, \mathscr{D})_{\mathbb{F}_p} = 0$ when $p = 7$. The proof for $p = 11, 19, 23, 31$ is similar. This completes the proof of Claim 6.2.3. □

We turn to the proof of (6.2.2). Write by $\langle D\rangle \subset \mathrm{NS}(\overline{X})$ the subgroup generated by the cycle classes of $\overline{D_1}$ and $\overline{D_2}$. Put $V' := H^2(\overline{X}, \mathbb{Q}_p(2))/\langle D\rangle \otimes \mathbb{Q}_p(1)$. Then the localization sequence in étale cohomology yields a short exact sequence

$$0 \longrightarrow V' \longrightarrow H^2(\overline{U}, \mathbb{Q}_p(2)) \longrightarrow \mathbb{Q}_p[D_1] \oplus \mathbb{Q}_p[D_2] \longrightarrow 0$$

(cf. Lemma 5.4.7). Taking continuous Galois cohomology, one obtains a long exact sequence fitting into a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^2(\overline{U}, \mathbb{Q}_p(2))^{G_{\mathbb{Q}_p}} & \longrightarrow & \bigoplus_{i=1}^2 \mathbb{Q}_p[D_i] & \xrightarrow{\ \delta\ } & H^1(\mathbb{Q}_p, V') \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \varrho} \\ & & K_2(U)^{(2)} & \longrightarrow & \bigoplus_{i=1}^2 K_1'(D_i)^{(1)} & \longrightarrow & K_1(X)^{(2)}. \end{array} \tag{6.2.5}$$

By Claim 6.2.3, the map δ is injective. Therefore

$$\varrho(\xi_1 - \xi_2) = \delta([D_1] - [D_2]) \ne 0 \text{ in } H^1(\mathbb{Q}_p, V'). \tag{6.2.6}$$

Let $\mathrm{NS}^\perp \subset H^2(\overline{X}, \mathbb{Q}_p(1))$ be the orthogonal complement of $\mathrm{NS}(\overline{X})_{\mathbb{Q}_p}$ with respect to the cup-product pairing on H^2. Since the intersection form on $\mathrm{NS}(\overline{X})_{\mathbb{Q}_p}$ is non-degenerate, one has

$$\mathrm{NS}^\perp \oplus \mathrm{NS}(\overline{X})_{\mathbb{Q}_p} \xrightarrow{\ \simeq\ } H^2(\overline{X}, \mathbb{Q}_p(1))$$

and hence $\mathrm{NS}^\perp \otimes \mathbb{Q}_p(1) \simeq V$. Therefore in order to show (6.2.2), it is enough to show that $\varrho(\xi_1 - \xi_2)$ is contained in

$$\text{the image of } \ H^1(\mathbb{Q}_p, \mathrm{NS}^\perp \otimes \mathbb{Q}_p(1)) \to H^1(\mathbb{Q}_p, V').$$

To prove this, we may replace $\mathbb{Q}_p$ with a finite extension K such that $\mathrm{NS}(X) = \mathrm{NS}(\overline{X})$ by a standard norm argument. Then we want to show that $\varrho(\xi_1 - \xi_2)$ goes to zero via the pull-back for each generator Z of $\mathrm{NS}(X) \otimes \mathbb{Q}$,

$$v_Z : H^1(K, H^2(\overline{X}, \mathbb{Q}_p(2))) \longrightarrow H^1(K, H^2(\widetilde{Z}, \mathbb{Q}_p(2))) = \mathbb{Q} \otimes \varprojlim_n K^\times/p^n,$$

where $\widetilde{Z}$ denotes the normalization of Z. If Z is a curve in (i) or (ii) (see Fact 3 in §6.1), one clearly has $v_Z(\varrho(\xi_i)) = 1$. If $Z = E$ is the section of infinity, then one has

$$\frac{Y-(X+4)}{Y+(X+4)}\Big|_{D_i\cap E} = 1$$

and hence $v_Z(\varrho(\xi_i)) = 1$. If $Z = C$ is the section in (iv), then one has

$$\frac{Y-(X+4)}{Y+(X+4)}\Big|_{D_i\cap C} = \frac{8/9-(-4/3+4)}{8/9+(-4/3+4)} = -\frac{1}{2}.$$

Hence $v_Z(\varrho(\xi_1)) = v_Z(\varrho(\xi_2)) = -1/2$ and $v_Z(\varrho(\xi_1-\xi_2)) = 1$. This completes the proof of $\varrho(\xi_1-\xi_2) \in \operatorname{Im} H^1(\mathbb{Q}_p, \mathrm{NS}^\perp)$ and hence (6.2.2).

Remark 6.2.7 *Here is a shorter* (*but essentially the same*) *proof of* (6.2.6) (*and* (6.2.2)). *In order to prove* (6.2.6), *it is enough to show*

$$[\mathscr{D}_1] - [\mathscr{D}_2] \notin \Phi^{(p^2)}(\mathscr{X}, \mathscr{D})_{\mathbb{F}_p}. \tag{6.2.8}$$

This condition can be written down more explicitly as follows. Let $t \in \mathbb{Z}[2^{-1}]((q))$ *satisfy* $t^4 = E_{3,a}/(E_1)^3$ *as in the proof of Claim* 6.2.3. *Let* c_i *and* d_i *satisfy*

$$g(q) = -\frac{27t}{4}E_{3,b} = \sum_{i=1}^{\infty} \frac{c_i q^i}{1-q^i},$$

$$f_1(q) - f_2(q) = \frac{27t}{2(t^2-1)}E_{3,b} = \frac{t\,(t^2+1)}{2}(E_{3,a}+27E_{3,b}) = 1 + \sum_{i=1}^{\infty}\frac{d_i q^i}{1-q^i}.$$

Then (6.2.8) *is equivalent to that there is no solution* $n \in \mathbb{Z}_p$ *to the congruent relations*

$$d_{ip} + nc_{ip} \equiv 0 \mod p^2\mathbb{Z}_p \quad (1 \le \forall i < p), \quad d_{p^2} + nc_{p^2} \equiv 0 \mod p^4\mathbb{Z}_p, \tag{6.2.9}$$

which we could check for $p = 7, 11, 19, 23$ *and* 31 *by computer. Moreover, if one can show* (6.2.9) *for more general* p*'s, then he will obtain the same results as Theorem* 6.0.1 *and Corollary* 6.0.3 *for such* p*'s.*

6.3 Finiteness of torsion in $\mathrm{CH}_0(X)$

We end this paper by showing that the torsion part of $\mathrm{CH}_0(X)$ is finite. Since we have proved the finiteness of the p-primary torsion part, it remains to show that the ℓ-primary torsion part is finite for any $\ell \ne p$ and zero for almost all $\ell \ne p$. In view of the isomorphism (6.3.3) below, Bloch's exact sequence (cf. [CTR1], (2.1))

$$0 \longrightarrow H^1_{\mathrm{zar}}(X, \mathscr{K}_2) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow N^1H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \longrightarrow \mathrm{CH}_0(X)\{\ell\} \longrightarrow 0$$

and the isomorphism $H^1_{\mathrm{zar}}(X, \mathscr{K}_2) \otimes \mathbb{Q} \simeq K_1(X)^{(2)}$ (cf. [So]), it is enough to show that the regulator map

$$K_1(X)^{(2)} \otimes \mathbb{Q}_\ell \longrightarrow H^1(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_\ell(2))) \tag{6.3.1}$$

is surjective for any $\ell \neq p$ and that $H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ is divisible for almost all $\ell \neq p$.

Assume $\ell \neq p$ in what follows. Put $\overline{Y} := Y \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$. There are isomorphisms

$$H^1(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_\ell(2))) \simeq H^0(\mathbb{F}_p, H^2(\overline{Y}, \mathbb{Q}_\ell(1))) \simeq \mathrm{NS}(Y)_{\mathbb{Q}_\ell}$$

by the Tate conjecture for Y ([ArSw]) and a similar argument as for (6.3.4) below. The map (6.3.1) is identified with the boundary map

$$K_1(X)^{(2)} \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{NS}(Y)_{\mathbb{Q}_\ell}. \tag{6.3.2}$$

This map is surjective by **Step 1**, which shows that (6.3.1) is surjective.

Next we show that $H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ is divisible for almost all ℓ. Since $\overline{X}$ is a $K3$ surface, we have $H^1(\overline{X}, \mathbb{Z}_\ell) = H^3(\overline{X}, \mathbb{Z}_\ell) = 0$ and $H^2(\overline{X}, \mathbb{Z}_\ell)$ is torsion-free for any ℓ. Hence we have

$$H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = H^2(\overline{X}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \quad \text{and} \quad H^1(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = H^3(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0,$$

and moreover

$$H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = H^1(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \tag{6.3.3}$$

by a Hochschild-Serre spectral sequence. We compute the right hand side as follows. There is a short exact sequence

$$\begin{aligned} 0 \to H^1(\mathbb{F}_p, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \to H^1(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \\ \to H^2(\overline{Y}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))^{G_{\mathbb{F}_p}} \to 0. \end{aligned}$$

By the divisibility of $H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ and a standard argument on weights (cf. [CTSS], §2, [D]), the first term is zero. Thus we have

$$H^1(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \simeq H^2(\overline{Y}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))^{G_{\mathbb{F}_p}}, \tag{6.3.4}$$

and we are reduced to showing that the right hand side is divisible for almost all ℓ. Put

$$N_\ell := H^2(\overline{Y}, \mathbb{Z}_\ell(1))^{G_{\mathbb{F}_p}},$$

and note the following fact due to Deligne [D]:

($*$) *the characteristic polynomial of the geometric Frobenius φ acting on $H^2(\overline{Y}, \mathbb{Q}_\ell(1))$ is independent of $\ell (\neq p)$.*

Since $H^2(\overline{Y}, \mathbb{Z}_\ell(1))$ is torsion-free, it is easy to see that $H^2(\overline{Y}, \mathbb{Z}_\ell(1))/N_\ell$ is torsion-free as well, for any ℓ. Hence there is a short exact sequence

$$0 \longrightarrow N_\ell \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow H^2(\overline{Y}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \longrightarrow (H^2(\overline{Y}, \mathbb{Z}_\ell(1))/N_\ell) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow 0$$

for any ℓ. By ($*$), we have

$$((H^2(\overline{Y}, \mathbb{Z}_\ell(1))/N_\ell) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{G_{\mathbb{F}_p}} = 0$$

for almost all ℓ. For such ℓ, we have

$$H^2(\overline{Y}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))^{G_{\mathbb{F}_p}} = N_\ell \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

which is divisible. This completes the proof of the finiteness of $\mathrm{CH}_0(X)_{\mathrm{tors}}$.

Remark 6.3.5 *Here is a more systematic* (*but essentially the same*) *proof of the finiteness result in this subsection. By the surjectivity of* (6.3.2) *and a result of Spiess* [Sp], *Proposition* 4.3 (*see also* [SS2] *for a generalization*), *we have*

$$\mathrm{CH}_0(X)_{\mathrm{tors}} \simeq \mathrm{CH}_0(Y)_{\mathrm{tors}} \quad \textit{up to the } p\textit{-primary torsion part.}$$

The right hand side is finite by Colliot-Thélène–Sansuc–Soulé [CTSS].

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