

# GROTHENDIECK—SERRE CONJECTURE FOR ADJOINT GROUPS OF TYPES $E_6$ AND $E_7$

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ABSTRACT. Assume that  $R$  is a semi-local regular ring containing an infinite perfect field, or that  $R$  is a semi-local ring of several points on a smooth scheme over an infinite field. Let  $K$  be the field of fractions of  $R$ . Let  $H$  be a strongly inner adjoint simple algebraic group of type  $E_6$  or  $E_7$  over  $R$ . We prove that the kernel of the map

$$H_{\acute{e}t}^1(R, H) \rightarrow H_{\acute{e}t}^1(K, H)$$

induced by the inclusion of  $R$  into  $K$  is trivial. This continues the recent series of papers [PaSV], [Pa] on the Grothendieck—Serre conjecture [Gr, Rem. 1.11].

In what follows we use the notation and terminology of [PS]. Our numbering of vertices of Dynkin diagrams follows [B].

**Lemma 1.** *Let  $R$  be a regular semi-local domain and let  $K$  be the field of fractions of  $R$ . Let  $H$  be a simple group scheme of inner type over  $R$  such that  $H \times_{\text{Spec } R} \text{Spec } K$  is of strongly inner type. Then  $H$  is of strongly inner type.*

*Proof.* Let  $A$  be a Tits algebra of  $H$ . By the condition, the class  $[A \otimes_R K]$  is trivial in  $\text{Br}(K)$ . By [Gr, Corollaire 1.10]  $[A]$  is trivial in  $\text{Br}(R)$ . So all Tits algebras of  $H$  are trivial, and therefore  $H$  is strongly inner.  $\square$

**Lemma 2.** *Let  $R$  be a semi-local domain, and let  $H$  be a strongly inner simply connected simple group scheme of type  $E_6$  (resp.,  $E_7$ ) over  $R$ . There exists an inner simply connected simple group scheme  $G$  of type  $E_7$  (resp.,  $E_8$ ) over  $R$ , together with a maximal parabolic subgroup  $P$  of type  $\{1, 2, 3, 4, 5, 6\}$  (resp.,  $\{1, 2, 3, 4, 5, 6, 7\}$ ), such that  $H$  is isomorphic to the derived subgroup of a Levi subgroup of  $P$ . Such a group scheme  $G$  is unique up to an isomorphism.*

*Proof.* Let  $H_0$  be a split simply-connected algebraic group over  $R$  of the same type as  $H$ , and let  $G_0$  be a split simply-connected algebraic group over  $R$  of type  $E_7$  (respectively,  $E_8$ ) if  $H$  is of type  $E_6$  (respectively,  $E_7$ ). Let  $P_0$  be a standard maximal parabolic subgroup of  $G_0$  corresponding to the 7th (respectively, the 8th) vertex of the Dynkin diagram of  $G_0$ . Then  $H_0$  is isomorphic to the derived subgroup of a standard Levi subgroup  $L_0$  of  $P_0$ . By [PS, Th. 2 (2)] for any strongly inner form  $H$  of  $H_0$  there exist an inner form  $G$  of  $G_0$ , a parabolic subgroup  $P$  of  $G$  of the same type as  $P_0$  in  $G_0$ , and a Levi subgroup  $L$  of  $P$ , such that  $H$  is isomorphic to the derived subgroup of  $L$ . By [PS, Th. 2 (3)] if  $\text{Pic}(R) = 0$ , which is our case, such a  $G$  is unique up to an isomorphism.  $\square$

**Theorem 1.** *Let  $R$  be a semi-local domain. Assume moreover that  $R$  is regular and contains a infinite perfect field  $k$ , or that  $R$  is a semi-local ring of several points on a  $k$ -smooth scheme over an infinite field  $k$ . Let  $K$  be the field of fractions of  $R$ . Let  $H$  be an adjoint strongly inner simple group scheme of type  $E_6$  or  $E_7$  over  $R$ . Then the map*

$$H_{\acute{e}t}^1(R, H) \rightarrow H_{\acute{e}t}^1(K, H)$$

*induced by the inclusion of  $R$  into  $K$  has trivial kernel.*

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*Proof.* Let  $H_0$  be a split simply-connected algebraic group over  $R$  of the same type as  $H$ . The elements of  $H_{\acute{e}t}^1(R, H)$  parametrize isomorphism classes of inner forms of  $H_0$ . Let  $H_1$  and  $H_2$  be two inner forms of  $H_0$ , and assume that

$$H_1 \times_{\mathrm{Spec} R} \mathrm{Spec} K \cong H_2 \times_{\mathrm{Spec} R} \mathrm{Spec} K.$$

Assume that  $H_1$  is of strongly inner type. Then  $H_2$  is also of strongly inner type by Lemma 1. Now by Lemma 2 there exist two simply connected  $R$ -group schemes  $G_i$ ,  $i = 1, 2$ , of inner type  $E_7$  (resp.,  $E_8$ ), if  $H$  is of type  $E_6$  (resp.,  $E_7$ ), together with parabolic subgroups  $P_i$ ,  $i = 1, 2$ , of type  $\{1, 2, 3, 4, 5, 6\}$  (resp.,  $\{1, 2, 3, 4, 5, 6, 7\}$ ), such that  $H_i$ ,  $i = 1, 2$ , is isomorphic to the derived subgroup of a Levi subgroup  $L_i$  of  $P_i$ . Since  $H_1 \times_{\mathrm{Spec} R} \mathrm{Spec} K \cong H_2 \times_{\mathrm{Spec} R} \mathrm{Spec} K$ , the uniqueness part of Lemma 2 applied to  $R = K$  implies that

$$G_1 \times_{\mathrm{Spec} R} \mathrm{Spec} K \cong G_2 \times_{\mathrm{Spec} R} \mathrm{Spec} K.$$

Since the groups  $G_i$ ,  $i = 1, 2$ , are isotropic, and are inner forms of each other, by [Pa, Th. 1.0.1, Th. 1.0.2] they are isomorphic over  $R$ . We can assume  $G_1 = G_2$ . Since  $R$  is semi-local, by [SGA, Exp. XXVI Cor. 5.5 (iv)] there exists  $g \in G_1(R)$  such that  $gP_1g^{-1} = P_2$  and  $gL_1g^{-1} = L_2$ . Then the derived subgroups of  $L_1$  and  $L_2$  are isomorphic, that is,  $H_1 \cong H_2$  over  $R$ .  $\square$

**Remark.** Actually, the proof is also valid for adjoint (semi)simple algebraic groups of classical type.

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