K-THEORY OF CONES OF SMOOTH VARIETIES

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ABSTRACT. Let R be the homogeneous coordinate ring of a smooth projective variety X over a field k of characteristic 0. We calculate the K-theory of R in terms of the geometry of the projective embedding of X. In particular, if X is a curve then we calculate $K_0(R)$ and $K_1(R)$, and prove that $K_{-1}(R) = \oplus H^1(C, \mathcal{O}(n))$. The formula for $K_0(R)$ involves the Zariski cohomology of twisted Kähler differentials on the variety.

Let $R = k \oplus R_1 \oplus \cdots$ be the homogeneous coordinate ring of a smooth projective variety X over a field k of characteristic 0. In this paper we compute the lower K-theory ($K_i(R)$, $i \leq 1$) in terms of the Zariski cohomology groups $H^*(X, \mathcal{O}(t))$ and $H^*(X, \Omega^*_X(t))$, where $\mathcal{O}(1)$ is the ample line bundle of the embedding and Ω^*_X denotes the Kähler differentials of X relative to $\mathbb{Q}$. We also obtain computations of the higher K-groups $K_n(R)/K_n(k)$, especially for curves. A complete calculation for the conic $xy = z^2$ is given in Theorem 4.3. These calculations have become possible thanks to the new techniques introduced in [1], [2] and [4].

Here, for example, is part of Theorem 2.1; R^+ is the seminormalization of R.

Theorem. *Let R be the homogeneous coordinate ring of a smooth d-dimensional projective variety X in $\mathbb{P}^N_k$. Then* $\mathrm{Pic}(R) \cong (R^+/R)$ *and*

$$K_0(R) \cong \mathbb{Z} \oplus \mathrm{Pic}(R) \oplus \bigoplus_{i=1}^{d} \bigoplus_{t=1}^{\infty} H^i(X, \Omega^i_X(t)), \quad \textit{and}$$
$$K_{-m}(R) \cong \bigoplus_{i=0}^{d-m} \bigoplus_{t=1}^{\infty} H^{m+i}(X, \Omega^i_X(t)), \quad m > 0.$$

We have $K_{-m}(R) = 0$ for $m > d$, and $K_{-d}(R) = \bigoplus_{t \geq 1} H^d(X, \mathcal{O}(t))$.

If k has finite transcendence degree over $\mathbb{Q}$ then $K_0(R)/\mathbb{Z}$ and each $K_{-m}(R)$ are finite-dimensional k-vector spaces.

For example, if $X = \mathrm{Proj}(R)$ is a smooth curve over k which is definable over a number field contained in k, we show that $\Omega^1_k \otimes \mathcal{O}(t) \to \Omega^1_X(t)$ induces:

$$K_0(R) = \mathbb{Z} \oplus \mathrm{Pic}(R) \oplus \left(\Omega^1_k \otimes K_{-1}(R)\right), \quad K_{-1}(R) \cong \oplus_{t=1}^{\infty} H^1(X, \mathcal{O}_X(t)). \tag{0.1}$$

We also have $K_n^{(n+2)}(R) \cong \Omega^{n+1}_k \otimes K_{-1}(R)$ for all $n \geq 1$. (See Proposition 3.2(d).)

When R is normal, (0.1) implies that $K_0(R) = \mathbb{Z}$ holds if and only if either (a) k is algebraic over $\mathbb{Q}$, or (b) $K_{-1}(R) = 0$. Case (a) was discovered by Krishna and Srinivas [9, 1.2], while parts of case (b) were discovered in [23]. By Riemann-Roch, the vanishing of $K_{-1}(R)$ is equivalent to the vanishing of the vector spaces $H^0(X, \Omega^1_{X/k}(-t))$ for $t > 0$, which is a delicate arithmetic question (unless, for

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example, the embedding has degree $d \geq 2g-2$). Note that case (b) clarifies Srinivas' theorem in [15] that when $k = \mathbb{C}$ and $H^1(X, \mathcal{O}(1)) \neq 0$ we have $K_0(R) \neq \mathbb{Z}$.

Still assuming that X is a curve, suppose in addition that k is a number field; then $\Omega^1_k = 0$ and hence $K_0(R) = \mathbb{Z} \oplus (R^+/R)$. We also establish (in 1.17 and 2.11) the previously unknown calculations that

$$K_1(R) = k^\times \oplus \left[\bigoplus_{t=1}^{\infty} H^0(X, \Omega^1_{X/k}(t))\right] / \Omega^1_{R/k}, \quad K_2(R) = K_2(k) \oplus \operatorname{tors} \Omega^1_{R/k}, \tag{0.2}$$

$$K_n(R) = K_n(k) \oplus HC_{n-1}(R)/HC_{n-1}(k), \qquad n \geq 3. \tag{0.3}$$

The K_1 formula (0.2) is a clarification of a result of Srinivas [17]. When k is not algebraic over $\mathbb{Q}$, formulas (0.1), (0.2) and (0.3) need to be altered to involve the arithmetic Gauss-Manin connection; see Proposition 3.5 and Example 3.6.

For any smooth d-dimensional variety X, $K_0(R)/\mathbb{Z}$ is the direct sum of the eigenspaces $K_0^{(i)}(R)$ of the Adams operation, $1 \leq i \leq d+1 = \dim R$, and we give a formula for these eigenspaces. For example, the top eigenspace, $K_0^{(d+1)}(R)$, may be identified with the Chow group of smooth zero-cycles in $\operatorname{Spec}(R)$; we show that

$$K_0^{(d+1)}(R) \cong \bigoplus\nolimits_{t=1}^{\infty} H^d(X, \Omega^d_X(t)).$$

As pointed out in [9], the normal domain $R_k = k[x,y,z]/(x^n+y^n+z^n)$ has $K_0(R_\mathbb{Q}) = \mathbb{Z}$ but if $n \geq 4$ then $H^1(X, \mathcal{O}(1))$ is nonzero while $K_0(R_\mathbb{C})/\mathbb{Z}$ is a very big $\mathbb{C}$-vector space; by (0.1), it is the direct sum of the $\Omega^1_\mathbb{C} \otimes H^1(X, \mathcal{O}(t))$, $t \geq 1$.

We also obtain reasonably nice formulas for the eigenspaces $K_n^{(i)}(R)$ when $n > 0$ and $i \geq n$; see Theorem 1.13. To illustrate the range of our cohomological results, consider $K_1(R)$ when X is a smooth curve and R is normal; we have $K_1(R) = k^\times \oplus K_1^{(2)}(R) \oplus K_1^{(3)}(R)$, where

$$K_1^{(2)}(R) \cong \Omega^1_{\mathrm{cdh}}(R)/\Omega^1_R = \left(\bigoplus\nolimits_{t=1}^{\infty} H^0(X, \Omega^1_X(t))\right)/\Omega^1_R, \qquad \text{and}$$

$$K_1^{(3)}(R) = \bigoplus_{t=1}^{\infty} \operatorname{coker}\left\{\Omega^1_k \otimes H^0(X, \Omega^1_{X/k}(t)) \xrightarrow{\nabla} \Omega^2_k \otimes H^1(X, \mathcal{O}_X(t))\right\}. \tag{0.4}$$

The map ∇ in (0.4) is a twisted Gauss-Manin connection (see Lemma 3.4). In Section 3, we prove that if $n \geq 1$ then $K_n^{(n+1)}(R)$ contains $\Omega^{n-1}_k \otimes_\mathbb{Q} k^{d+g-1}$ as a direct summand provided that either

(a) X has genus g and is embedded in $\mathbb{P}^N_k$ by a complete linear system of degree d, with $d \geq 2g-1$, or
(b) X is induced by base change to k from a curve defined over a number field contained in k.

(See Theorem 3.8 and Example 3.9.) In particular $K_1^{(2)}(R) \neq 0$, and in general, $K_n^{(n+1)}(R) \neq 0$ if $n-1 \leq \operatorname{tr.deg}(k/\mathbb{Q})$. Observe that the case $n=1$ improves the result of Srinivas in [17, §1] that there is a surjection from $\widetilde{K}_1(R) = K_1(R)/K_1(k)$ to $H^0(X, \Omega^1_{X/k}(1))$ and hence that $\widetilde{K}_1(R) \neq 0$ if $d \geq 2g+1$.

Finally, in Theorem 4.3 we give a complete calculation of the K-theory of the homogeneous coordinate ring of the plane conic, $R = k[x,y,z]/(xy-z^2)$.

This paper is organized as follows. In Section 1, we reduce the calculation of $K_n(R)$ to a *cdh*-cohomology computation and knowledge of $HC_{n-1}(R)$. This relies on the basic observation that cones are $\mathbb{A}^1$-contractible, so that the reduced K-theory $\widetilde{K}_n(R) = K_n(R)/K_n(k)$ can be calculated in terms of $NK_n(R)$, making our previous calculations (see [1], [2], [4]) applicable. Several of the formulas we obtain are valid for general graded algebras of the form $R = k \oplus R_1 \oplus \cdots$. We also specialize these formulas to the case when $\dim R = 2$, and obtain an expression for $K_n(R)$ in terms of *cdh* cohomology and cyclic homology ($n \geq 1$).

In Section 2 we compute the *cdh* terms in the formulas of the previous sections for the case when R is the affine cone of a smooth variety. In Section 3, we return to the case when the graded coordinate ring has dimension 2, that is, we investigate cones over smooth projective curves. Finally, in Section 4 we apply the techniques of this paper to completely determine the K-theory of $R = k[x, y, z]/(xy - z^2)$.

Notations: Throughout this paper we consider (commutative, unital) algebras over a fixed ground field k, which we assume has characteristic zero. Undecorated tensor products $\otimes$ and differential forms Ω^* are taken over $\mathbb{Q}$; we write $\otimes_k$ and $\Omega^*_{/k}$ for tensor product and forms relative to k. Similarly, cyclic homology is always taken over $\mathbb{Q}$. If F is a functor defined on schemes over k, we will write $F(R)$ for $F(\mathrm{Spec}(R))$. If R is an augmented k-algebra (for example, the homogeneous coordinate ring of a variety), and F is a functor from rings to some abelian category, then we write $\widetilde{F}(R)$ for the (split) quotient $F(R)/F(k)$.

1. K-THEORY OF GRADED ALGEBRAS

Throughout this section, we let $R = R_0 \oplus R_1 \oplus \cdots$ be a finitely generated graded algebra over a field k of characteristic 0 such that R_0 is a local, artinian k-algebra whose residue field is isomorphic to k. These conditions ensure that the map $K_n(R) \to K_n(k)$ induced by the composition of $R \twoheadrightarrow R_0 \twoheadrightarrow k$ is a split surjection. For example, R_0 might be k itself, and indeed for most of the calculations in this paper, one may as well assume $R_0 = k$. Let $\mathfrak{m}_R$ denote the unique graded maximal ideal of R; that is, $\mathfrak{m}_R$ is the kernel of the split surjection $R \twoheadrightarrow k$.

We let R_{red} denote the reduced ring associated to R. It is a graded ring whose degree 0 piece is the field k. We let $\widetilde{R}$ denote the normalization of R_{red} (i.e., the integral closure of R_{red} in its ring of total quotients). It is well known that $\widetilde{R} = \widetilde{R}_0 \oplus \widetilde{R}_1 \oplus \cdots$ is graded, that $\widetilde{R}_0$ is a product of fields, and that $\mathrm{Pic}(\widetilde{R}) = 0$.

We let R^+ denote the semi-normalization of R_{red}, that is, the maximal extension of R_{red} inside its total quotient ring Q such that for all $x \in Q$, $x^2, x^3 \in R^+$ implies $x \in R^+$; see [18]. Alternatively, $\mathrm{Spec}(R^+) \to \mathrm{Spec}(R_{red})$ is a universal homeomorphism.

We are interested in computing the kernel $\widetilde{K}_n(R)$ of the split surjection $K_n(R) \to K_n(k)$, for $n = 1, 0, -1, \ldots, 1 - d$. (By [1], $K_n(R) = NK_n(R) = 0$ for $n \leq -d$.) In general, for any graded ring $R = R_0 \oplus R_1 \oplus R_2 \oplus \ldots$, the groups $\widetilde{K}_n(R)$ are known to be R_0-modules (see [20]), and hence (since R_0 contains $\mathbb{Q}$) they are uniquely divisible as abelian groups. Thus there is a decomposition $\widetilde{K}_n(R) \cong \bigoplus_i \widetilde{K}_n^{(i)}(R)$ according to the eigenvalues k^i of the Adams operations ψ^k.

Remark 1.1. Suppose that the punctured spectrum, $\mathrm{Spec}(R_{red}) \setminus \{\mathfrak{m}_R\}$, is non-singular. Then the conductor $\mathfrak{c}$ to the normalization $\widetilde{R}$ of R_{red} is $\mathfrak{m}_R$-primary. An easy calculation shows that the seminormalization of R_{red} is

$$R^+ = k \oplus \widetilde{R}_1 \oplus \widetilde{R}_2 \oplus \cdots ,$$

with $\widetilde{R}/R^+ = \widetilde{R}_0/k$ and $R^+/R_{red} = \widetilde{R}/(\widetilde{R}_0 + R_{red})$. Then $K_n^{(i)}(R) \cong \widetilde{K}_n^{(i)}(R)$ for $n \le 1$, with two exceptions: $\widetilde{K}_0^{(0)}(R) = 0$, and $\widetilde{K}_1^{(1)}(R) \cong \operatorname{nil}(R)/\operatorname{nil}(R_0)$. The problem of computing $\widetilde{R}/R_{red}$ (and hence R^+/R_{red}) is hard.

The main results of this section, Theorems 1.2 and 1.13, are formulated in terms of the *cdh* cohomology groups $H^*_{\mathrm{cdh}}(R, \Omega^i)$ introduced in [1] and [2], where the Kähler differentials, $\Omega^i = \Omega^i_{-/\mathbb{Q}}$, are taken relative to the base field $\mathbb{Q}$. By [4, 2.5], we have that $H^0_{\mathrm{cdh}}(R, \mathcal{O}) = R^+$. For simplicity, we write $H^m_{\mathrm{cdh}}(R, \Omega^i)/dH^m_{\mathrm{cdh}}(R, \Omega^{i-1})$ for the cokernel of the map $d : H^m_{\mathrm{cdh}}(R, \Omega^{i-1}) \to H^m_{\mathrm{cdh}}(R, \Omega^i)$ induced by the Kähler differential. Theorem 1.2 will follow from Proposition 1.5 and Theorem 1.12 below.

Theorem 1.2. *Let $R = R_0 \oplus R_1 \oplus \cdots$ be a finitely generated graded algebra over a field k of characteristic 0. Assume R_0 is local artinian with residue field k. Then the Adams operations induce an eigenspace decomposition:*

$$K_0(R) = \mathbb{Z} \oplus R^+/R_{red} \oplus \bigoplus_{i=1}^{\dim R-1} H^i_{\mathrm{cdh}}(R, \Omega^i)/d\,H^i_{\mathrm{cdh}}(R, \Omega^{i-1}).$$

The negative K-groups are given by

$$K_{-m}(R) = H^m_{\mathrm{cdh}}(R, \mathcal{O}) \oplus \bigoplus_{i=1}^{\dim R-m-1} H^{m+i}_{\mathrm{cdh}}(R, \Omega^i)/d\,H^{m+i}_{\mathrm{cdh}}(R, \Omega^{i-1}).$$

for $m > 0$. The groups indexed by i are $K_0^{(i+1)}$ and $K_{-m}^{(i+1)}(R)$, respectively.

By [21, 1.2] we have $KH_*(R) \cong KH_*(R_0) \cong K_*(k)$, and thus by [2, 1.6], we have

$$\widetilde{K}_n(R) \cong \pi_n \mathcal{F}_K(R) \cong \pi_{n-1}\mathcal{F}_{HC}(R) \quad \text{for all } n. \tag{1.3}$$

Here, $\mathcal{F}_{HC}(R) = \mathcal{F}_{HC}(R/\mathbb{Q})$ is the homotopy fiber of $HC(R) \to \mathbb{H}_{\mathrm{cdh}}(R, HC)$, with cyclic homology taken relative to the subfield $\mathbb{Q}$ of k, so that there is a long exact sequence

$$\cdots \to HC_n(R) \to \mathbb{H}^{-n}_{\mathrm{cdh}}(R, HC) \to \widetilde{K}_n(R) \to HC_{n-1}(R) \to \cdots .$$

These groups all have λ-decompositions and the maps in this sequence are compatible with these decompositions (see [3]), but there is a weight shift in that $\widetilde{K}_n^{(i)}(R)$ maps to $HC_{n-1}^{(i-1)}(R)$. We have $\widetilde{K}_n^{(0)}(R) = 0$ for all n because $\mathcal{F}_{HC}^{(-1)} \simeq 0$. Moreover, by [2, 2.2] we have $\mathbb{H}^m_{\mathrm{cdh}}(R, HC^{(i)}) \cong \mathbb{H}^{2i+m}_{\mathrm{cdh}}(R, \Omega^{\le i})$, so the long exact sequence becomes

$$\cdots HC_n^{(i-1)}(R) \to \mathbb{H}^{2i-n-2}_{\mathrm{cdh}}(R, \Omega^{<i}) \to \widetilde{K}_n^{(i)}(R) \to HC_{n-1}^{(i-1)}(R) \cdots . \tag{1.4}$$

The general picture is given by the following proposition.

Proposition 1.5. *Let $R = R_0 \oplus R_1 \oplus \cdots$ be as in Theorem 1.2. Then $\widetilde{K}_n^{(0)}(R) = 0$ for all n. For $n \le 0$, or for $n > 0$ and $i \ge n+2$, we have*

$$\widetilde{K}_n^{(i)}(R) \cong \mathbb{H}^{2i-n-2}_{\mathrm{cdh}}(R, \Omega^{<i}), \quad \textit{except for} \quad (n,i) = (0,1),$$

In the exceptional case, $\widetilde{K}_0^{(1)}(R) = \operatorname{Pic}(R) = R^+/R_{red}$.

Proof. The group $HC_n(R)$ vanishes for $n < 0$ and is R for $n = 0$. Similarly, $HC_n^{(i)}(R)$ vanishes for $i > n > 0$. The proposition now follows from (1.4) and the fact that $H^0_{cdh}(R, \mathcal{O}) = R^+$ by [4, 2.5]. □

To go further, it is useful to invoke the following trick, using the standard $\mathbb{A}^1$-contraction of a cone to its vertex.

Standard Trick 1.6. If R is a positively graded algebra, there is an algebra map $\nu : R \to R[t]$ sending $r \in R_n$ to rt^n. If F is a functor on algebras, then the composition of ν with evaluation at $t = 0$ factors as $R \to R_0 \to R$, so $F(R) \xrightarrow{\nu} F(R[t]) \xrightarrow{t=0} F(R)$ is zero on the kernel $\widetilde{F}(R)$ of $F(R) \to F(R_0)$. Similarly, the composition of ν with evaluation at $t = 1$ is the identity. That is, ν maps $\widetilde{F}(R)$ isomorphically onto a summand of $NF(R)$, and $\widetilde{F}(R)$ is in the image of the map $(t = 1) : NF(R) \to F(R)$.

The following technical result is crucial for our calculations; it asserts that many SBI sequences decompose into split short exact sequences. We write $\mathcal{F}_{HH}$ and $\mathcal{F}_{HC}$ for the homotopy fibers of $HH(R) \to \mathbb{H}_{\mathrm{cdh}}(R, HH)$ and $HC(R) \to \mathbb{H}_{\mathrm{cdh}}(R, HC)$, respectively. Then we have distinguished cohomological triangles

$$\mathcal{F}_{HC}[-1] \xrightarrow{S} \mathcal{F}_{HC}[1] \xrightarrow{B} \mathcal{F}_{HH} \xrightarrow{I} \mathcal{F}_{HC},$$
$$\mathbb{H}_{\mathrm{cdh}}(R, HC)[-1] \xrightarrow{S} \mathbb{H}_{\mathrm{cdh}}(R, HC)[1] \xrightarrow{B} \mathbb{H}_{\mathrm{cdh}}(R, HH) \xrightarrow{I} \mathbb{H}_{\mathrm{cdh}}(R, HC).$$

Lemma 1.7. *If $R = R_0 \oplus R_1 \oplus \cdots$ is a graded algebra then for each m the map $\pi_m\mathcal{F}_{HC}(R) \xrightarrow{S} \pi_{m-2}\mathcal{F}_{HC}(R)$ is zero, and there is a split short exact sequence:*

$$0 \to \pi_{m-1}\mathcal{F}_{HC}(R) \xrightarrow{B} \pi_m\mathcal{F}_{HH}(R) \xrightarrow{I} \pi_m\mathcal{F}_{HC}(R) \to 0.$$

Similarly, there are split short exact sequences:

$$0 \to \widetilde{\mathbb{H}}^{m+1}_{\mathrm{cdh}}(R, HC) \xrightarrow{B} \widetilde{\mathbb{H}}^{m}_{\mathrm{cdh}}(R, HH) \xrightarrow{I} \widetilde{\mathbb{H}}^{m}_{\mathrm{cdh}}(R, HC) \to 0.$$

and

$$0 \to \widetilde{\mathbb{H}}^{n-1}_{\mathrm{cdh}}(R, \Omega^{<i}) \xrightarrow{B} \widetilde{H}^{n-i}_{\mathrm{cdh}}(R, \Omega^{i}) \xrightarrow{I} \widetilde{\mathbb{H}}^{n}_{\mathrm{cdh}}(R, \Omega^{\le i}) \to 0.$$

Proof. The third sequence is obtained from the second one by taking the i^{th} component in the Hodge decomposition, described in [2, 2.2], and setting $n = 2i + m$. For the first two sequences to split, it suffices to show that I is onto and split.

By [2, 2.4], $\mathcal{F}_{HH}(k) = \mathcal{F}_{HC}(k) = 0$, so $\widetilde{\mathcal{F}}_{HH} = \mathcal{F}_{HH}$ and $\widetilde{\mathcal{F}}_{HC} = \mathcal{F}_{HC}$. By the standard trick 1.6, it suffices to show that the maps $N\pi_m\mathcal{F}_{HH}(R) \to N\pi_m\mathcal{F}_{HC}(R)$ and $N\mathbb{H}^m_{\mathrm{cdh}}(R, HH) \to N\mathbb{H}^m_{\mathrm{cdh}}(R, HC)$ are onto and split. But they are split surjections, as is evident from the respective decompositions of their terms in [4, 3.2] and [4, 2.2], such as: $\mathbb{H}_{\mathrm{cdh}}(R, NHC^{(i)}) \simeq H_{\mathrm{cdh}}(R, \Omega^i)[i] \otimes_R tR[t]$, $N\mathcal{F}^{(i)}_{HC}(R) \simeq \mathcal{F}^{(i)}_{HH}(R) \otimes_R tR[t]$ and $N\mathcal{F}_{HC} \simeq \mathcal{F}_{HH} \otimes_R tR[t]$. (Note that for any presheaf F, $H_{\mathrm{cdh}}(-, NF) = NH_{\mathrm{cdh}}(-, F)$.) □

Splicing the final sequences of Lemma 1.7 together, we see that the de Rham complexes are exact in *cdh*-cohomology:

Proposition 1.8. *The following sequences are exact:*

$$0 \to k \to R^+ \xrightarrow{d} \widetilde{H}^0_{\mathrm{cdh}}(R, \Omega^1) \xrightarrow{d} \widetilde{H}^0_{\mathrm{cdh}}(R, \Omega^2) \to \cdots \tag{1.8a}$$

$$0 \to H^m_{\mathrm{cdh}}(R, \mathcal{O}) \xrightarrow{d} H^m_{\mathrm{cdh}}(R, \Omega^1) \xrightarrow{d} H^m_{\mathrm{cdh}}(R, \Omega^2) \to \cdots, \qquad m > 0. \tag{1.8b}$$

Note that the first complex is the cdh reduced *de Rham complex.*

An analogous exact sequence

$$\cdots \to \pi_{m-1}\mathcal{F}_{HH}(R) \xrightarrow{d} \pi_m\mathcal{F}_{HH}(R) \xrightarrow{d} \pi_{m+1}\mathcal{F}_{HH}(R) \to \cdots$$

is obtained by splicing the other sequences in 1.7. Using the interpretation of their Hodge components, described in [4, 3.4], produces two more exact sequences:

Proposition 1.9. *The following sequences are exact:*

$$0 \to \mathrm{nil}(R) \to \mathrm{tors}\,\Omega^1_R \to \mathrm{tors}\,\Omega^2_R \to \mathrm{tors}\,\Omega^3_R \to \cdots \tag{1.9a}$$

$$0 \to (R^+/R) \to \Omega^1_{\mathrm{cdh}}(R)/\Omega^1_R \to \Omega^2_{\mathrm{cdh}}(R)/\Omega^2_R \to \cdots . \tag{1.9b}$$

Here we have used the following notation

$$\Omega^i_{\mathrm{cdh}}(R) = H^0_{\mathrm{cdh}}(R, \Omega^i) \tag{1.10}$$

$$\mathrm{tors}\,\Omega^i_R = \ker(\Omega^i_R \to \Omega^i_{\mathrm{cdh}}(R)) \tag{1.11}$$

If R is reduced then $\mathrm{tors}\,\Omega^i_R$ is the usual torsion submodule, by [4, 5.4].

We can now make the calculations necessary to deduce Theorem 1.2.

Theorem 1.12. *Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded algebra, finitely generated over a field k of characteristic 0. Assume R_0 is local artinian with residue field k. Then we have*

$$\mathbb{H}^{q+i}_{\mathrm{cdh}}(R, \Omega^{\le i}) = \begin{cases} H^{q+i}_{dR}(k), & q < 0; \\ \mathrm{coker}\big\{H^q_{\mathrm{cdh}}(R, \Omega^{i-1}) \xrightarrow{d} H^q_{\mathrm{cdh}}(R, \Omega^i)\big\}, & q \ge 0; \\ 0, & q \ge \dim(R). \end{cases}$$

Proof. The Cartan-Eilenberg spectral sequence for $\Omega^{\le i}$ is

$${}^IE_1^{p,q} = H^q_{\mathrm{cdh}}(R, \Omega^p) \Longrightarrow \mathbb{H}^{p+q}_{\mathrm{cdh}}(R, \Omega^{\le i}) \qquad (0 \le p \le i,\ q \ge 0).$$

(See [22, 5.7.9].) Since $H^0_{\mathrm{cdh}}(R, \Omega^p) = \Omega^p_k \oplus \widetilde{H}^0_{\mathrm{cdh}}(R, \Omega^p)$, the row $q = 0$ is the brutal truncation of the direct sum of the de Rham complex of k over $\mathbb{Q}$ and the complex (1.8a). which is acylic by Proposition 1.8. Since $H^q_{\mathrm{cdh}}(R, \Omega^p) = \widetilde{H}^q_{\mathrm{cdh}}(R, \Omega^p)$ for $q > 0$, the other rows on the E_1-page are the truncations of the complex (1.8b), which is also acylic by 1.8. Hence the spectral sequence degenerates at E_2, yielding the calculation. Note that the last possible nonzero group is $\mathbb{H}^{i+\dim R-1}_{\mathrm{cdh}}(R, \Omega^{\le i}) = H^{\dim R-1}_{\mathrm{cdh}}(R, \Omega^i)$ by the cohomological bound in [2, 2.6]. □

Proof of Theorem 1.2. Simply plug the calculations of Theorem 1.12 into those of Proposition 1.5 to get the asserted result. □

We conclude the section with a calculation of the higher K-theory of R in terms of Kähler differentials, the cyclic homology of R and the *cdh*-cohomology of $\mathrm{Spec}(R)$. In the next section, we will reinterpret Theorems 1.13 and 1.15 in terms of the Zariski cohomology of $X = \mathrm{Proj}(R)$.

Theorem 1.13. *Let $R = R_0 \oplus R_1 \oplus \cdots$ be a finitely generated graded algebra over a field k of characteristic 0. Assume R_0 is local artinian with residue field k. Then for $n \ge 1$ we have:*

(a) $K^{(i)}_n(R) \cong HC^{(i-1)}_{n-1}(R)$ whenever $0 < i < n$;

(b) $\widetilde{K}^{(n)}_n(R) \cong \mathrm{tors}\,\Omega^{n-1}_R/d\,\mathrm{tors}\,\Omega^{n-2}_R$. In particular, $\widetilde{K}^{(1)}_1(R) \cong \mathrm{nil}(R)$ and

$$\widetilde{K}^{(2)}_2(R) \cong \mathrm{tors}\,\Omega^1_R/d\,\mathrm{nil}(R).$$

(c) $K_n^{(n+1)}(R) \cong \operatorname{coker}\{\Omega^{n-1}_{\mathrm{cdh}}(R) \xrightarrow{d} \Omega^n_{\mathrm{cdh}}(R)/\Omega^n_R\}$.

(d) $K_n^{(i)}(R) \cong \operatorname{coker}\{H^{i-(n+1)}_{\mathrm{cdh}}(R,\Omega^{i-2}) \xrightarrow{d} H^{i-(n+1)}_{\mathrm{cdh}}(R,\Omega^{i-1})\}$ *when* $i \geq n+2$.

Proof. By Theorem 1.12, we have $\widetilde{\mathbb{H}}^m_{\mathrm{cdh}}(R,\Omega^{\leq i}) = 0$ whenever $m < i$ (*i.e.*, $q < 0$). Substituting this into (1.4) gives assertion (a), because $HC_n^{(i)}(k) \xrightarrow{\simeq} H^{2i-n}_{dR}(k/\mathbb{Q})$ also holds. Taking $m = i$, it also gives exactness of the top row in the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{K}_n^{(n)}(R) \arrow[r] \arrow[d] & \widetilde{HC}^{(n-1)}_{n-1}(R) \arrow[r] \arrow[d, "B"', "\text{into}"] & \widetilde{\Omega}^{n-1}_{\mathrm{cdh}}(R)/d\Omega^{n-2}_{\mathrm{cdh}}(R) \arrow[d, "B"', "\text{into}"] \\
0 \arrow[r] & \operatorname{tors}\Omega^n_R \arrow[r] \arrow[d, "d"'] & \Omega^n_R/\Omega^n_k \arrow[r] \arrow[d, "d"'] & \Omega^n_{\mathrm{cdh}}(R)/\Omega^n_k \arrow[d, "d"'] \\
0 \arrow[r] & \operatorname{tors}\Omega^{n+1}_R \arrow[r] & \Omega^{n+1}_R/\Omega^{n+1}_k \arrow[r] & \Omega^{n+1}_{\mathrm{cdh}}(R)/\Omega^{n+1}_k.
\end{tikzcd}$$

The other two rows are exact by definition, see (1.11). The two right columns are exact by [22, 9.9.1] and (1.8a), respectively. By a diagram chase, $\widetilde{K}_n^{(n)}(R)$ is the kernel of $\operatorname{tors}\Omega^n_R \to \operatorname{tors}\Omega^{n+1}_R$. Part (b) now follows from (1.9a).

Part (c) is immediate from (1.4), given the following information: $\mathbb{H}^n_{\mathrm{cdh}}(R,\Omega^{\leq n})$ is the cokernel of $d : \Omega^{n-1}_{\mathrm{cdh}}(R) \to \Omega^n_{\mathrm{cdh}}(R)$ by Theorem 1.12, $HC_n^{(n)}(R) = \Omega^n_R/d\Omega^n_R$ and $HC^{(n)}_{n-1}(R) = 0$. Part (d) follows from Theorem 1.12 and the formula $\widetilde{K}_n^{(i)}(R) \cong \mathbb{H}^{2i-n-2}_{\mathrm{cdh}}(R,\Omega^{<i})$ for $i \geq n+2$, which is Proposition 1.5. ☐

Corollary 1.14. *If* $i > n$, *the map* $K_n^{(i)}(R) \to K_n^{(i)}(R^+)$ *is an isomorphism.*

If the dimension of R is 2 (for example, if R is the cone over a projective curve), then the calculations of Theorem 1.13 apply to compute the higher K-groups of R, but here the more dominant role is played by Kähler differentials. As in (1.10), we write $\Omega^i_{\mathrm{cdh}}(R)$ for $H^0_{\mathrm{cdh}}(R,\Omega^i)$.

Theorem 1.15. *Assume* $\dim(R) = 2$. *Then we have:*

(1) $K_1(R) = k^\times \oplus K_1^{(2)}(R) \oplus K_1^{(3)}(R)$ *with* $K_1^{(i)}(R) = 0$ *for all* $i \geq 4$, *with:*

$$K_1^{(2)}(R) \cong \Omega^1_{\mathrm{cdh}}(R)/(\Omega^1_R + d(R^+)), \quad \textit{and}$$

$$K_1^{(3)}(R) \cong \mathbb{H}^3_{\mathrm{cdh}}(R,\Omega^{\leq 2}) \cong \operatorname{coker}\{H^1_{\mathrm{cdh}}(R,\Omega^1) \xrightarrow{d} H^1_{\mathrm{cdh}}(R,\Omega^2)\};$$

(2) $K_2(R) \cong K_2(k) \oplus \operatorname{tors}\Omega^1_R \oplus K_2^{(3)}(R) \oplus K_2^{(4)}(R)$ *with*

$$K_2^{(3)}(R) \cong \Omega^2_{\mathrm{cdh}}(R)/(\Omega^2_R + d\Omega^1_{\mathrm{cdh}}(R)) \quad \textit{and}$$

$$K_2^{(4)}(R) \cong \operatorname{coker}\{H^1_{\mathrm{cdh}}(R,\Omega^2) \xrightarrow{d} H^1_{\mathrm{cdh}}(R,\Omega^3)\};$$

(3) *For all* $n \geq 3$, $K_n(R) \cong K_n(k) \oplus \bigoplus_{i=2}^{n+2} \widetilde{K}_n^{(i)}(R)$, *where*

$$\widetilde{K}_n^{(i)}(R) = \begin{cases} \widetilde{HC}^{(i-1)}_{n-1}(R), & i < n, \\ \operatorname{tors}\Omega^{n-1}_R/d\operatorname{tors}\Omega^{n-2}_R, & i = n, \\ \operatorname{coker}\{\Omega^{n-1}_{\mathrm{cdh}}(R) \xrightarrow{d} \Omega^n_{\mathrm{cdh}}(R)/\Omega^n_R\}, & i = n+1, \\ \operatorname{coker}\{H^1_{\mathrm{cdh}}(R,\Omega^n) \xrightarrow{d} H^1_{\mathrm{cdh}}(R,\Omega^{n+1})\}, & i = n+2. \end{cases}$$

Proof. For $n = 1$ we see from Remark 1.1 that $K_1^{(1)}(R) = \mathrm{nil}(R) = 0$, and from Theorem 1.13(c) that $K_1^{(2)}(R)$ is the cokernel of $d : R \to \Omega^1_{\mathrm{cdh}}(R)/\Omega^1_R$. Since $R \to \Omega^1_{\mathrm{cdh}}(R)$ factors through Ω^1_R, the description of $K_1^{(2)}(R)$ follows. From (1.4), we have $K_1^{(3)}(R) \cong \mathbb{H}^3_{\mathrm{cdh}}(R, \Omega^{\leq 2})$, which is described by 1.12, and $K_1^{(i)}(R) = \mathbb{H}^{2i-3}_{\mathrm{cdh}}(R, \Omega^{<i})$ for $i \geq 4$, which vanishes because $\mathbb{H}^m_{\mathrm{cdh}}(R, \Omega^{<i}) = 0$ for $m \geq 1+i$ by Theorem 1.12.

For $n \geq 2$, $K_n^{(i)}(R)$ was described in Proposition 1.5 and Theorem 1.13. ☐

Lemma 1.16. *Assume that $R = k \oplus R_1 \oplus \cdots$ is graded and $\dim(R) = 2$. Then for all $i \geq 2$:*

$$\Omega^i_{R/k}/d(\Omega^{i-1}_{R/k}) \cong \mathrm{tors}\,\Omega^i_{R/k}/d(\mathrm{tors}\,\Omega^{i-1}_{R/k}).$$

Proof. For $i \geq 3$ the R-module $\Omega^i_{R/k}$ is torsion because $\Omega^i_{\mathrm{cdh}}(R/k) = 0$. For $i = 2$ we simply chase the diagram

$$\begin{tikzcd}
\mathrm{tors}\,\Omega^1_{R/k} \arrow[r] \arrow[d, "\mathrm{into}"] & \mathrm{tors}\,\Omega^2_{R/k} \arrow[r] \arrow[d, "\mathrm{into}"] & \mathrm{tors}\,\Omega^3_{R/k} \arrow[r] \arrow[d, equal] & \mathrm{tors}\,\Omega^4_{R/k} \arrow[d, equal] \\
\Omega^1_{R/k} \arrow[r, "d"] & \Omega^2_{R/k} \arrow[r, "d"] & \Omega^3_{R/k} \arrow[r, "d"] & \Omega^4_{R/k},
\end{tikzcd}$$

comparing the exact sequence for $\mathrm{tors}\,\Omega^*_{R/k}$, analogous to (1.9a), to the (exact) de Rham sequence for $\Omega^*_{R/k}$. ☐

Proposition 1.17. *If k is algebraic over $\mathbb{Q}$ and $R = k \oplus R_1 \oplus \cdots$ is seminormal of dimension 2, then:*
a) $K_1(R) \cong k^\times \oplus \Omega^1_{\mathrm{cdh}}(R)/\Omega^1_R$;
b) $K_2(R) \cong K_2(k) \oplus \mathrm{tors}\,\Omega^1_R$;
c) $K_n(R) \cong K_n(k) \oplus \widetilde{HC}_{n-1}(R)$, $\quad n \geq 3$.

Proof. These assertions are special cases of Theorem 1.15. Using Lemma 1.16 for $n \geq 3$ we have

$$\widetilde{K}_n^{(n)}(R) \cong \mathrm{tors}\,\Omega^{n-1}_R/d\,\mathrm{tors}\,\Omega^{n-2}_R \cong \Omega^{n-1}_R/d\Omega^{n-2}_R = HC^{(n-1)}_{n-1}(R).$$

By (1.8a), $K_n^{(n+1)}(R)$ is a subquotient of $\Omega^{n+1}_{\mathrm{cdh}}(R)$ and vanishes for $n \geq 2$; by (1.8b), $K_n^{(n+2)}(R)$ is a subgroup of $H^1_{\mathrm{cdh}}(R, \Omega^{n+2})$ and vanishes for $n \geq 1$. ☐

We conclude this section with two classical examples for which $\mathrm{Spec}(R)$ has a smooth affine *cdh* cover, so that Ω^*_{cdh} is easy to determine.

Example 1.18. The cusp $R = k[t^2, t^3]$ has $R^+ = k[t]$ and $K_1^{(2)}(R) = \Omega^1_{\mathrm{cdh}}/d(R^+) = \Omega_k$ (cf. [10, 12.1]). The computation of $K_n(R)$ for $n \geq 2$ is also easily derived from Theorem 1.13, and stated explicitly in [6, 6.7].

Example 1.19. The seminormal ring $R = k[x_1, x_2, y_1, y_2]/(\{x_i y_j\})$ is the homogeneous coordinate ring of a pair of skew lines in $\mathbb{P}^3_k$. Its normalization is $\widetilde{R} = k[x_1, x_2] \times k[y_1, y_2]$, and $\mathrm{Spec}(\widetilde{R}) \to \mathrm{Spec}(R)$ is a *cdh* cover. It is easy to see that $H^1_{\mathrm{cdh}}(R, \Omega^i) = 0$, and $\Omega^i_R \to \Omega^i_{\mathrm{cdh}}(R)$ is onto for $i \neq 0$. Applying Theorem 1.15, we see that $K_0(R) = \mathbb{Z}$, $K_1(R) = k^\times$ and $K_{-1}(R) = 0$. This recovers a classic result of Murthy in [13]. If k is algebraic over $\mathbb{Q}$ then we also have $\mathrm{tors}\,\Omega^1_R \cong k^4$ (on the $x_i dy_j$), $\mathrm{tors}\,\Omega^2_R \cong k^4$ (on the $dx_i dy_j$) and $\Omega^3_R = 0$, so by Proposition 1.17 we have

$$K_2(R) = K_2(k) \oplus k^4, \quad \text{while} \quad \widetilde{K}_n(R) = \widetilde{HC}_{n-1}(R) \quad \text{for all } n \geq 3.$$

2. Affine cones of Varieties

Let X be a smooth projective variety in $\mathbb{P}^N_k$, and let $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ be the associated homogeneous coordinate ring. We will write L for the pullback to X of the ample bundle $\mathcal{O}(1)$ on $\mathbb{P}^N_k$, and if $\mathcal{F}$ is a quasi-coherent sheaf on X, we write $\mathcal{F}(t)$ for $\mathcal{F} \otimes_{\mathcal{O}_X} L^t$. In this section we compute the *cdh* cohomology of $\mathrm{Spec}(R)$ and use it to compute the K-theory of R, via Proposition 1.5. The main result is the theorem below, computing the non-positive K-groups of R. Later in this section, we give partial calculations of the positive K-groups.

Recall from Proposition 1.5 that $K^{(0)}_{-m}(R) = 0$ for all $m > 0$ and $\widetilde{K}^{(0)}_n(R) = 0$ for $n \geq 0$. Thus we are interested in $K^{(i+1)}_{-m}(R)$ for $i \geq 0$.

Theorem 2.1. *Let X be a smooth projective variety in $\mathbb{P}^N_k$ with homogeneous coordinate ring R. Then*

$$K^{(1)}_0(R) \cong R^+/R = \bigoplus_{t=1}^{\infty} H^0(X, \mathcal{O}_X(t))/R_t, \quad \textit{and}$$

$$K^{(i+1)}_0(R) \cong \bigoplus_{t=1}^{\infty} H^i(X, \Omega^i_X(t)), \quad \textit{for all } i \geq 1.$$

For any $m > 0$, and all $i \geq 0$, we have:

$$K^{(i+1)}_{-m}(R) \cong \bigoplus_{t=1}^{\infty} H^{m+i}(X, \Omega^i_X(t)).$$

If k has finite transcendence degree over $\mathbb{Q}$ then each vector space $K_0(R)/\mathbb{Z}$ and $K_{-m}(R)$ is finite-dimensional.

A few parts of Theorem 2.1 are easy to prove. The formula $K^{(1)}_0(R) = R^+/R$ is given in Proposition 1.5. Since $\mathrm{Spec}(R) \setminus \{\mathfrak{m}_R\}$ is regular, we see from Remark 1.1 that R^+ agrees with the normalization $\widetilde{R}$ of R in degrees $t > 0$, and it is well known that $\widetilde{R} = \bigoplus_{t=0}^{\infty} H^0(X, \mathcal{O}(t))$; see [5, Ex. II.5.14]. This yields the first display. The final assertion, when $\mathrm{tr.deg.}(k/\mathbb{Q}) < \infty$, follows from the fact that each Ω^i_X is a coherent sheaf; for each $q > 0$ the $H^q(X, \Omega^i_X(t))$ are finite-dimensional, and only finitely many are nonzero, by Serre's Theorem B ([5, III.5.2]).

The proof of the rest of the theorem will be given in Corollary 2.5 and Proposition 2.10, building upon several intermediate results.

To compute the *cdh* cohomology of $\mathrm{Spec}(R)$, we will use the blowup Y of $\mathrm{Spec}(R)$ at the origin (i.e., at $\mathfrak{m}_R$). The following description of Y is well known.

Lemma 2.2. *The exceptional fiber of $\pi : Y \to \mathrm{Spec}(R)$ is isomorphic to X and there is a projection $p : Y \to X$ identifying Y with the geometric line bundle $\mathrm{Spec}_X(\mathrm{Sym}(L))$ over X, with sheaf of sections L^*. Moreover, the inclusion of the exceptional fiber X into Y is the zero section of the bundle $p : Y \to X$.*

Proof. The exceptional fiber is Proj of the Rees algebra $\bigoplus \mathfrak{m}^i/\mathfrak{m}^{i+1}$, which is just R, and $X = \mathrm{Proj}(R)$ by construction. For each $x \in R_1$, the affine open $D_+(x)$ of X is $\mathrm{Spec}(A)$, where $R[1/x] = A[x, 1/x]$, and the line bundle L^n restricts to the A-submodule $x^n A$ of $R[1/x]$.

We now consider $Y = \mathrm{Proj}(R[\mathfrak{m}t])$. For $x \in R_1$, and $xt \in R_1 t$, the affine open $D_+(xt)$ in Y is $\mathrm{Spec}(B)$, where $R[\mathfrak{m}t][1/xt] = B[xt, 1/xt]$. The graded map $R \cong \oplus R_i t^i \to R[\mathfrak{m}t]$ induces a projection $Y \to X$ as well as an inclusion of $A[x]$ in B. This is onto, since B is generated by elements of the form $rt^m/(xt)^m = (r/x^n)x^{n-m}$ for $r \in R_n$, $n \geq m$. Hence $B = A[x]$. This shows that Y is the geometric line bundle over X, associated to the locally free sheaf L (see [5, Ex. II.5.18]). □

By [1] and [2], we have split exact sequences

$$\begin{aligned} &0 \to H^0_{\mathrm{cdh}}(R,\mathcal{F}) \to H^0_{\mathrm{zar}}(Y,\mathcal{F}) \oplus \mathcal{F}(k) \to H^0_{\mathrm{zar}}(X,\mathcal{F}) \to 0, \\ &0 \to H^m_{\mathrm{cdh}}(R,\mathcal{F}) \to H^m_{\mathrm{zar}}(Y,\mathcal{F}) \to H^m_{\mathrm{zar}}(X,\mathcal{F}) \to 0, \quad \text{for } m>0, \end{aligned} \tag{2.3}$$

when $\mathcal{F}$ is one of the *cdh* sheaves $\mathcal{O}$ or Ω^i, or a complex of *cdh* sheaves of the form $\Omega^{\leq i}$. Thus the calculation of $H^*_{\mathrm{cdh}}(R,\mathcal{F})$ is reduced to the calculation of $H^*_{\mathrm{zar}}(Y,\mathcal{F})$.

Lemma 2.4. *We have* $H^0_{\mathrm{cdh}}(R,\mathcal{O}) = R^+$ *and* $H^m_{\mathrm{cdh}}(R,\mathcal{O}) = \bigoplus_{t=1}^{\infty} H^m(X,\mathcal{O}_X(t))$ *for* $m>0$*.*

Proof. Since p is affine, $H^*_{\mathrm{zar}}(Y,\mathcal{O}_Y) = H^*_{\mathrm{zar}}(X,p_*\mathcal{O}_Y)$, and $p_*\mathcal{O}_Y = \mathrm{Sym}(L)$ by Lemma 2.2. Hence $H^m(Y,\mathcal{O}) = \bigoplus_{t=0}^{\infty} H^m_{\mathrm{zar}}(X,\mathcal{O}(t))$ for all m; if $m=0$, this equals R^+. Now apply (2.3). ☐

From Proposition 1.5 and 2.4 we deduce the case $K^{(1)}_*$ of Theorem 2.1. For comparison, recall that $K^{(1)}_0(R) = \mathrm{Pic}(R)$, $K^{(1)}_1(R) = R^\times = k^\times$ and $K^{(1)}_n(R) = 0$ for all $n \geq 2$ by Soulé [14].

Corollary 2.5. *For* $m>0$ *we have*

$$K^{(1)}_{-m}(R) = H^m_{\mathrm{cdh}}(R,\mathcal{O}) = \bigoplus_{t=1}^{\infty} H^m(X,\mathcal{O}_X(t)).$$

Remark 2.6. This clarifies results of Srinivas in [15, Thm. 3], [16] and Weibel [23], which observed (when X is a curve) that the right side of the display in Corollary 2.5 is an obstruction to the vanishing of $K_0(R)$ and $K_{-1}(R)$.

There is an exact sequence $0 \to p^*\Omega^1_X \to \Omega^1_Y \to \Omega^1_{Y/X} \to 0$ of sheaves on Y. The relative sheaf $\Omega^1_{Y/X}$ is the line bundle p^*L, and so we deduce exact sequences for all $i \geq 1$:

$$0 \to p^*\Omega^i_X \to \Omega^i_Y \to p^*(\Omega^{i-1}_X \otimes L) \to 0.$$

Since $p_*p^*\mathcal{F} = \mathcal{F} \otimes \mathrm{Sym}(L)$, applying p_* yields (graded) exact sequences of sheaves on X for all $i \geq 1$:

$$0 \to \Omega^i_X \otimes \mathrm{Sym}(L) \to p_*\Omega^i_Y \to \Omega^{i-1}_X \otimes L \otimes \mathrm{Sym}(L) \to 0. \tag{2.7}$$

Lemma 2.8. *There are graded split exact sequences*

$$0 \to \bigoplus_{t=0}^{\infty} H^*_{\mathrm{zar}}(X,\Omega^i_X(t)) \to H^*_{\mathrm{zar}}(Y,\Omega^i_Y) \to \bigoplus_{t=1}^{\infty} H^*_{\mathrm{zar}}(X,\Omega^{i-1}_X(t)) \to 0,$$

and for $t \geq 1$ *the composition*

$$H^*_{\mathrm{zar}}(X,\Omega^i_X(t)) \to H^*_{\mathrm{zar}}(Y,\Omega^i_Y) \xrightarrow{d} H^*_{\mathrm{zar}}(Y,\Omega^{i+1}_Y) \to H^*_{\mathrm{zar}}(X,\Omega^i_X(t))$$

is an isomorphism.

Proof. It follows from (2.7) that we have a (graded) exact sequence

$$\cdots \xrightarrow{\partial} \bigoplus_{t=0}^{\infty} H^*_{\mathrm{zar}}(X,\Omega^i_X(t)) \xrightarrow{p^*} H^*_{\mathrm{zar}}(Y,\Omega^i_Y) \to \bigoplus_{t=1}^{\infty} H^*_{\mathrm{zar}}(X,\Omega^{i-1}_X(t)) \xrightarrow{\partial} \cdots$$

Therefore, the second assertion implies the first. Referring to the maps of (2.7), it suffices to show that the composition

$$\Omega^i_X \otimes \mathrm{Sym}(L) \to p_*\Omega^i_Y \xrightarrow{d} p_*\Omega^{i+1}_Y \to \Omega^i_X \otimes L \otimes \mathrm{Sym}(L)$$

is the evident graded surjection, with kernel Ω^i_X. But, in the notation of the proof of Lemma 2.2, it suffices to look on the affine $D_+(x) = \operatorname{Spec}(A)$ of X, and here this is the map $\Omega^*_A \otimes_A A[x] \to \Omega^*_A \otimes_A \Omega^1_{A[x]/A}$ sending $\omega \otimes x^n$ to $\omega \otimes nx^{n-1}\,dx$. ☐

Example 2.8.1. In particular, $0 \to H^0(X, \Omega^1_X(t)) \to H^0(Y, \Omega^1_Y)_t \to R_t \to 0$ is exact for $t \geq 1$, and the composition $R_t \xrightarrow{d} H^0(Y, \Omega^1_Y)_t \to R_t$ is an isomorphism.

Corollary 2.9. *For $i \geq 1$ and $m \geq 1$ we have:*

$$\Omega^i_{\mathrm{cdh}}(R) \cong \Omega^i_k \oplus \bigoplus_{t=1}^{\infty} H^0_{\mathrm{zar}}(X, \Omega^i(t)) \oplus H^0_{\mathrm{zar}}(X, \Omega^{i-1}(t));$$

$$H^m_{\mathrm{cdh}}(R, \Omega^i) \cong \bigoplus_{t=1}^{\infty} H^m_{\mathrm{zar}}(X, \Omega^i(t)) \oplus H^m_{\mathrm{zar}}(X, \Omega^{i-1}(t)).$$

The cokernel of $\Omega^{i-1}_{\mathrm{cdh}}(R) \xrightarrow{d} \Omega^i_{\mathrm{cdh}}(R)$ *is* $\Omega^i_k / d\Omega^{i-1}_k \oplus \bigoplus_{t=1}^{\infty} H^0_{\mathrm{zar}}(X, \Omega^i_X(t))$*, and the cokernel of* $H^m_{\mathrm{cdh}}(R, \Omega^{i-1}) \xrightarrow{d} H^m_{\mathrm{cdh}}(R, \Omega^i)$ *is the summand* $\bigoplus_{t=1}^{\infty} H^m_{\mathrm{zar}}(X, \Omega^i_X(t))$.

Proof. The first assertions follow from Lemma 2.8 and (2.3). The cokernel assertions follow from this using (1.8a), (1.8b) and induction on i. ☐

We may now deduce the remaining cases of Theorem 2.1, the main theorem of this section. Recall that $K^{(1)}_{-m}(R)$ is $\oplus_t H^m(X, \mathcal{O}(t))$ by Corollary 2.5.

Proposition 2.10. *For $i \geq 1$, we have*

$$K^{(i+1)}_{-m}(R) \cong \mathbb{H}^{m+2i}_{\mathrm{cdh}}(R, \Omega^{\leq i}) \cong \bigoplus_{t=1}^{\infty} H^{m+i}(X, \Omega^i_X(t)), \qquad m \geq 0.$$

Proof. The first assertion is the case $m = 0$ of the displayed equation. The first displayed isomorphism is 1.5. The second isomorphism is established in Lemma 2.8, using the isomorphism $\mathbb{H}^{m+2i}_{\mathrm{cdh}}(R, \Omega^{\leq i}) \cong \operatorname{coker}\{H^{m+i}_{\mathrm{cdh}}(R, \Omega^{i-1}) \xrightarrow{d} H^{m+i}_{\mathrm{cdh}}(R, \Omega^i)\}$ of Theorem 1.12. ☐

The proof of Theorem 2.1 is now complete. We now deduce partial information about the groups $K_n(R)$ for $n \geq 1$.

Proposition 2.11. *Let X be a smooth projective variety in $\mathbb{P}^N_k$ with homogeneous coordinate ring R. Then for all $n \geq 1$ we have graded isomorphisms:*

$$K^{(n+1)}_n(R) \cong \operatorname{coker}\left\{ \Omega^n_R / d\Omega^{n-1}_R \to \bigoplus_{t=1}^{\infty} H^0(X, \Omega^n_X(t)) \right\};$$

$$K^{(i)}_n(R) \cong \bigoplus_{t=1}^{\infty} H^{i-n-1}(X, \Omega^{i-1}_X(t)), \qquad i \geq n+2.$$

The graded decomposition of $K^{(n+1)}_n(R) = \bigoplus_{t=1}^{\infty} K^{(n+1)}_n(R)_t$ *is:*

$$K^{(n+1)}_n(R)_t \cong \operatorname{coker}\left\{ \left(\Omega^n_R / d\Omega^{n-1}_R\right)_t \to H^0(X, \Omega^n_X(t)) \right\}.$$

Proof. By Theorem 1.13(c),

$$\begin{aligned} K^{(n+1)}_n(R) &\cong \Omega^n_{\mathrm{cdh}}(R) / (\Omega^n_R + d\Omega^{n-1}_{\mathrm{cdh}}(R)) \\ &= \operatorname{coker}\left(\Omega^n_R / d\Omega^{n-1}_R \to \Omega^n_{\mathrm{cdh}}(R) / d\Omega^{n-1}_{\mathrm{cdh}}(R)\right). \end{aligned}$$

Since $\widetilde{H}^0_{\mathrm{cdh}}(R,\Omega^n) = \Omega^n_{\mathrm{cdh}}(R)/\Omega^n_k$ and $\Omega^n_k \subset \Omega^n_R$, we see from Corollary 2.9 that this is the cokernel of $\Omega^n_R/d\Omega^{n-1}_R \to \bigoplus_{t=1}^{\infty} H^0(X,\Omega^1_X(t))$, as claimed. □

Remark 2.12. When $X=\mathbb{P}^r_k$ is embedded in $\mathbb{P}^N_k$ as a subvariety of degree $d>r$, our $L^t=\mathcal{O}_X(t)$ agrees with $\mathcal{O}_{\mathbb{P}^r_k}(d\cdot t)$, because it is the pullback of $\mathcal{O}_{\mathbb{P}^N_k}(t)$ to $X=\mathbb{P}^r_k$. Similarly, the terms written as $\Omega^i_X(t)$ in Proposition 2.11 should be read as $\Omega^i_{\mathbb{P}^r_k}\otimes\mathcal{O}_{\mathbb{P}^r_k}(d\cdot t)$.

3. Cones over smooth curves

In this section, we focus on the case when X is a curve (i.e., a smooth projective variety of dimension one, embedded in $\mathbb{P}^N_k$), and apply the results of Sections 1 and 2 in this case. Recall from Theorem 2.1 that $K_{-m}(R)=0$ for $m>1$.

The simplest case is when k is algebraic over $\mathbb{Q}$. In this case, we know from Proposition 1.17 that $\widetilde{K}_2(R)\cong \operatorname{tors}\Omega^1_R$ and if $n\ge 3$ then $\widetilde{K}_n(R)\cong\widetilde{HC}_{n-1}(R)$. It remains to describe the situation when $-1\le n\le 1$.

Lemma 3.1. *Suppose that k is algebraic over $\mathbb{Q}$ and that R is the homogeneous coordinate ring of a smooth curve X over k. Then $K_{-1}(R)=\oplus_{t=1}^{\infty}H^1(X,\mathcal{O}(t))$, $K_0(R)=\mathbb{Z}\oplus(R^+/R)$ and $\widetilde{K}_1(R)=\oplus_{t=1}^{\infty}H^0(X,\Omega^1_{X/k}(t))/\Omega^1_{R/k}$.*

Proof. By Theorem 2.1, $K_0^{(i)}(R)=0$ for $i\ge 3$ and $K_{-1}^{(i)}(R)$ is zero for $i\ge 2$, while $K_{-1}^{(1)}(R)$ is the sum of the $H^1(X,\mathcal{O}(t))$ by 2.5. By Serre Duality, $K_0^{(2)}(R)$ is the sum of the $H^1(X,\Omega^1_{X/k}(t))=H^0(X,\mathcal{O}_X(-t))^*$, which are zero for all $t>0$.

The formula for $\widetilde{K}_1(R)$ is immediate from Propositions 1.17 and 2.11. □

Proposition 3.2. *Suppose that R is the homogeneous coordinate ring of a smooth curve X over a number field F contained in k. Then for $R_k=R\otimes_F k$:*

(a) For $i<n$ we have $\widetilde{K}^{(i)}_n(R_k)\cong\oplus_{p=0}^{i}\,\Omega^p_k\otimes_F\widetilde{K}^{(i-p)}_{n-p}(R)$.

(b) For all $n\ge 2$, $\widetilde{K}^{(n)}_n(R_k)\cong\oplus_{p=0}^{n-2}\,\Omega^p_k\otimes_F\widetilde{K}^{(n-p)}_{n-p}(R)$.

(c) For all $n\ge 1$, $K^{(n+1)}_n(R_k)\cong\Omega^{n-1}_k\otimes_F K^{(2)}_1(R)$.

(d) For all $n\ge 0$, $K^{(n+2)}_n(R_k)\cong\Omega^{n+1}_k\otimes_F K_{-1}(R)\cong\Omega^{n+1}_k\otimes_k K_{-1}(R_k)$.

Proof. Write $\otimes$ for $\otimes_F$. Part (a) is immediate from Theorem 1.13(a) and Kassel's basechange formula $\widetilde{HC}_*(R_k)\cong\Omega^*_k\otimes\widetilde{HC}_*(R)$. (See [7, (3.2)].)

For (b), recall that $\widetilde{K}^{(n)}_n(R_k)\cong\operatorname{tors}\Omega^{n-1}_{R_k}/d\operatorname{tors}\Omega^{n-2}_{R_k}$ by Theorem 1.13(b). By the Künneth formula, $\operatorname{tors}\Omega^n_{R_k}=\oplus_{p+q=n}\Omega^p_k\otimes\operatorname{tors}\Omega^q_R$. Filtering by $p\ge 0$ yields a 2-diagonal spectral sequence computing the kernel and cokernel of $d:\operatorname{tors}\Omega^{n-1}_{R_k}\to\operatorname{tors}\Omega^n_{R_k}$, with $E_0^{p,-p}=\Omega^p_k\otimes\operatorname{tors}\Omega^{n-p}_R$ and $E_0^{p,-1-p}=\Omega^p_k\otimes\operatorname{tors}\Omega^{n-p-1}_R$. By (1.9a), we have $E_1^{p,-p}=\Omega^p_k\otimes\widetilde{K}^{(n+1)}_{n+1}(R)$ and $E_1^{p,-1-p}=\Omega^p_k\otimes d\ \operatorname{tors}\Omega^{n-p-2}_R$. Given α in Ω^p_k and $d\tau$ in $d\ \operatorname{tors}\Omega^{n-p-2}_R$, $d(\alpha\otimes d\tau)=d\alpha\otimes d\tau=d(d\alpha\otimes\tau)$ in $\operatorname{tors}\Omega^n_{R_k}$, which shows that $d^1=0$ and establishes (b).

By the Künneth formula and Proposition 2.11, $K^{(n+1)}_n(R_k)$ is the direct sum over $p+q=n$ of the cokernels of the maps

$$\Omega^p_k\otimes\Omega^q_R\to\Omega^p_k\otimes H^0(Y,\Omega^q_Y)\to\Omega^p_k\otimes\oplus_t H^0(X,\Omega^q_X(t)).$$

For $q = 0$, the composite is the identity map of $\Omega^n_k \otimes R$. For $q = 1$, the composite is Ω^{n-1}_k tensored with the map $\Omega^1_R \to \oplus_t H^0(X, \Omega^1_X(t))$ defining $K^{(2)}_1(R)$. For $q \geq 2$, the right side is zero. This establishes part (c).

Since $K_{-1}(R) \cong H^1(X, \mathcal{O}(t))$, part (d) is just Proposition 2.11, together with the Künneth formula that $H^1(X_k, \Omega^n_{X_k}(t))$ is the direct sum of $\Omega^n_k \otimes H^1(X, \mathcal{O}(t))$ and $\Omega^{n-1}_k \otimes H^1(X, \Omega^1_X(t))$, which is zero for $t > 0$ by Serre Duality. ☐

When $k/\mathbb{Q}$ is transcendental, we will use a variant of the arithmetic Gauss-Manin connection $H^1_{dR}(X/k) \to \Omega^1_k \otimes H^1_{dR}(X/k)$, or rather its ($k$-linear) filtered piece

$$\nabla : H^0(X, \Omega^1_{X/k}) \to \Omega^1_k \otimes H^1(X, \mathcal{O}_X)$$

as described in [8, Thm. 2] and [11, 3.2]. When $k = \mathbb{C}$, this can be interpreted in terms of the Hodge filtration as a map $H^{1,0}(X, \mathbb{C}) \to \Omega^1_{\mathbb{C}/k} \otimes H^{0,1}(X, \mathbb{C})$.

It is known (see [8]) that ∇ is the cohomology boundary map associated to the fundamental short exact sequence $0 \to \Omega^1_k \otimes \mathcal{O}_X \to \Omega^1_X \to \Omega^1_{X/k} \to 0$. Twisting this short exact sequence by $\mathcal{O}(t)$ yields a twisted version $\nabla_t : H^0(X, \Omega^1_{X/k}(t)) \to \Omega^1_k \otimes H^1(X, \mathcal{O}(t))$. We see from Lemma 2.8 that the direct sum of the ∇_t is a component of the cohomology boundary map associated to $0 \to \Omega^1_k \otimes \mathcal{O}_Y \to \Omega^1_Y \to \Omega^1_{Y/k} \to 0$; it follows that $\oplus \nabla_t$ is R-linear.

Since $\Omega^2_{X/k} = 0$, we have fundamental exact sequences for each i:

$$0 \to \Omega^i_k \otimes \mathcal{O}_X(t) \to \Omega^i_X(t) \to \Omega^{i-1}_k \otimes \Omega^1_{X/k}(t) \to 0. \tag{3.3}$$

The cohomology boundary maps are the k-linear homomorphisms

$$\Omega^{i-1}_k \otimes H^0(X, \Omega^1_{X/k}(t)) \xrightarrow{\nabla_t} \Omega^i_k \otimes H^1(X, \mathcal{O}(t)).$$

The sum of the ∇_t is again R-linear, as the sum of the sequences (3.3) is R-linear. Alternatively, we can use the fact that the arithmetic Gauss-Manin connection can be extended via the usual formula $\nabla_t(\omega \otimes x) = d\omega \otimes x + (-1)^{i-1}\omega \wedge \nabla_t(x)$, and the first term vanishes because it is in a lower part of the Hodge filtration.

Lemma 3.4. *If X is a smooth curve and $i \geq 1$, there is a graded exact sequence of R-modules, the sum over $t > 0$ of the exact sequences*

$$0 \to \Omega^i_k \otimes R_t \to H^0(X, \Omega^i_X(t)) \to \Omega^{i-1}_k \otimes H^0(X, \Omega^1_{X/k}(t)) \xrightarrow{\nabla_t}$$
$$\Omega^i_k \otimes H^1(X, \mathcal{O}_X(t)) \to H^1(X, \Omega^i_X(t)) \to 0.$$

Moreover, we have the identity

$$\nabla_t(\omega \otimes x) = \omega \wedge \nabla_t(x), \qquad \textit{for } \omega \in \Omega^{i-1}_k \textit{ and } x \in H^0(X, \Omega^1_{X/k}(t)).$$

Proof. This is just the cohomology exact sequence for (3.3), together with Serre Duality, which says that $H^1(X, \Omega^1_{X/k}(t)) = H^0(X, \mathcal{O}_X(-t)) = 0$ for all $t > 0$.

To prove that the boundary map is ∇, let $\mathcal{U}$ be a cover of X by affine open subschemes and consider the exact sequence of Čech complexes associated to (3.3). We have $\check{C}(\mathcal{U}, \Omega^i_k \otimes \mathcal{O}_X(t)) = \Omega^i_k \otimes \check{C}(\mathcal{U}, \mathcal{O}_X(t))$ and

$$\check{C}(\mathcal{U}, \Omega^{i-1}_k \otimes \Omega^1_X(t)) = \Omega^{i-1}_k \otimes \check{C}(\mathcal{U}, \Omega^1_{X/k}(t)).$$

Let $\omega \in \Omega^{i-1}_k$ and $x \in H^0(X, \Omega^1_{X/k}(t)) = H^0\check{C}(\mathcal{U}, \Omega^1_{X/k}(t))$. If $y \in \check{C}(\mathcal{U}, \Omega^1_X(t))$ maps to x, then $\delta(y)$ is in $\Omega^1_k \otimes \check{C}(\mathcal{U}, \mathcal{O}(t))$ and represents $\nabla(x)$. Since $\omega \wedge y$ lifts

$\omega\otimes x$, $\nabla(\omega\otimes x)$ is the class of $\delta(\omega\wedge y)$ in $\Omega^i_k\otimes H^1(X,\Omega^1_{X/k})$. Since ω is globally defined, we have $\delta(\omega\wedge y)=\omega\wedge\delta(y)$. □

Proposition 3.5. *If X is a smooth curve, we have graded exact sequences*

$$0\to K_1^{(2)}(R)\to\frac{\oplus_t H^0(X,\Omega^1_{X/k}(t))}{\text{image }\Omega^1_{R/k}}\xrightarrow{\nabla}\Omega^1_k\otimes\left(\oplus_t H^1(X,\mathcal{O}(t))\right)\to K_0^{(2)}(R)\to 0;$$

$$0\to K_{n+1}^{(n+2)}(R)\to\frac{\Omega^n_k\otimes\left[\oplus_t H^0(X,\Omega^1_{X/k}(t))\right]}{\text{image }\Omega^{n+1}_R}\xrightarrow{\nabla}\Omega^{n+1}_k\otimes\left(\oplus_t H^1(X,\mathcal{O}_X(t)\right)$$
$$\to K_n^{(n+2)}(R)\to 0,\qquad n\ge 1.$$

The direct sums are taken from $t=1$ to ∞.

Proof. This follows from the exact sequence of Lemma 3.4, using the formulas $K_n^{(n+2)}(R)_t\cong H^1(X,\Omega^{n+1}_X(t))$ and $K_{n+1}^{(n+2)}(R)_t\cong H^0(X,\Omega^{n+1}_X(t))/\text{im}\left(\Omega^{n+1}_R\right)_t$ of Propositions 2.10 and 2.11, once we observe that the first map of Lemma 3.4 factors through Ω^i_R. This is because it is a quotient of $\Omega^i_k\otimes R\to\pi_*(\Omega^i_Y)=H^0(Y,\Omega^i_Y)$, which factors as $\Omega^i_k\otimes R\to\Omega^i_R\to\pi_*(\Omega^i_Y)$. □

Example 3.6. If X is a curve definable over a number field contained in k, then the Fundamental Sequence (3.3) (with $i=1$ and $t=0$) splits as $\Omega^1_X\cong\Omega^1_{X/k}\oplus\Omega^1_k\otimes\mathcal{O}_X$, by the Künneth formula. This implies that $\Omega^i_X\cong\left(\Omega^{i-1}_k\otimes\Omega^1_{X/k}\right)\oplus\left(\Omega^i_k\otimes\mathcal{O}_X\right)$, so the Gauss-Manin connection ∇ of Lemma 3.4 vanishes and therefore:

$$K_n^{(n+1)}(R)=\frac{\Omega^{n-1}_k\otimes\left[\oplus_t H^0(X,\Omega^1_{X/k}(t))\right]}{\text{image }\Omega^n_R},\quad n\ge 1;$$
$$K_n^{(n+2)}(R)=\Omega^{n+1}_k\otimes\left[\oplus_t H^1(X,\mathcal{O}_X(t))\right]\cong\Omega^{n+1}_k\otimes K_{-1}(R),\quad n\ge 0.$$

Of course, the formula for $K_n^{(n+1)}(R)$ reduces to that of Proposition 3.2(c).

The formula for $K_0^{(2)}(R)$ clarifies the examples given by Srinivas in [15]. There it was shown that if X is definable over a number field, then $K_0(R)$ maps onto $\Omega^1_k\otimes H^1(X,\mathcal{O}_X(1))$ (see page 264). From this Srinivas deduced that if $k=\mathbb{C}$ and $H^1(X,\mathcal{O}_X(1))\ne 0$ then $\widetilde{K}_0(R)\ne 0$.

Lemma 3.7. *For any graded algebra $R=k\oplus R_1\oplus\cdots$, the degree 1 part of Ω^i_R decomposes as*

$$(\Omega^i_R)_1\cong(R_1\otimes\Omega^i_k)\oplus(\Omega^{i-1}_k\otimes R_1).$$

The inclusions of $R_1\otimes\Omega^i_k$ and $\Omega^{i-1}_k\otimes R_1$ are given by $r\otimes\omega\mapsto r\omega$ and $\omega\otimes r\mapsto\omega\wedge dr$, respectively.

Proof. We may suppose for simplicity that $N=\dim(R_1)$ is finite, so that the polynomial ring $S=k[x_1,\dots,x_N]$ maps to R, and $S\to R$ is an isomorphism in degree 1. For every subfield ℓ of k, $\Omega^1_{R/\ell}$ is the cokernel of the Hochschild boundary $R^{\otimes 3}\to R\otimes_\ell R$; thus the map $\Omega^1_{S/\ell}\to\Omega^1_{R/\ell}$ is an isomorphism in degree 1, and therefore so is $\Omega^i_{S/\ell}\to\Omega^i_{R/\ell}$. Since $\Omega^1_S\cong(\Omega^1_k\otimes S)\oplus\Omega^1_{S/k}$, it is easy to check that the degree 1 part of Ω^i_S is $(\Omega^i_k\otimes S_1)\oplus\Omega^{i-1}_k\otimes S_1$, via the given formulas. □

Theorem 3.8. *Let X be a curve of genus g, embedded in $\mathbb{P}^N_k$ by a complete linear system of degree $d > 1$. Assume that the twisted Gauss-Manin connection $\nabla : H^0(X, \Omega^1_{X/k}(1)) \longrightarrow \Omega^1_k \otimes H^1(X, \mathcal{O}_X(1))$ is zero. Then $K_1^{(2)}(R)_1 \cong k^{d+g-1} \neq 0$, and*

$$K_n^{(n+1)}(R)_1 \cong \Omega^{n-1}_k \otimes_{\mathbb{Q}} k^{d+g-1} \qquad (n \geq 1).$$

In particular, $K_n^{(n+1)}(R) \neq 0$ for all n with $1 \leq n < tr. deg.(k/\mathbb{Q})$.

Proof. By Proposition 2.11, the degree 1 part of $K_n^{(n+1)}(R)$ is

$$K_n^{(n+1)}(R)_1 = \operatorname{coker}((\Omega^n_R / d\Omega^{n-1}_R)_1 \to H^0(X, \Omega^n_X(1))).$$

By Lemmas 3.7 and 3.4, and our hypothesis, we have morphisms of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega^n_k \otimes R_1 \arrow[r] \arrow[d, "\mathrm{id}"] & (\Omega^n_R)_1 \arrow[r, "\omega\wedge dr\mapsto\omega\otimes r"] \arrow[d] & \Omega^{n-1}_k \otimes R_1 \arrow[r] \arrow[d, "1\otimes d"] & 0 \\
0 \arrow[r] & \Omega^n_k \otimes R_1 \arrow[r] \arrow[d, "\mathrm{id}"] & H^0(Y, \Omega^n_Y)_1 \arrow[r] \arrow[d] & \Omega^{n-1}_k \otimes H^0(Y, \Omega^1_{Y/k})_1 \arrow[r, "\partial"] \arrow[d] & {} \\
0 \arrow[r] & \Omega^n_k \otimes R_1 \arrow[r] & H^0(X, \Omega^n_X(1)) \arrow[r] & \Omega^{n-1}_k \otimes H^0(X, \Omega^1_{X/k}(1)) \arrow[r, "\nabla"] & 0
\end{tikzcd}$$

where the bottom vertical maps are given in Lemma 2.8 as the quotients by $dH^0(X, \Omega^{n-1}(1))$ and $\Omega^{n-1}_k \otimes dR_1$. It follows that the right vertical composite is zero. Hence $K_n^{(n+1)}(R)_1$, which is the cokernel of the middle vertical composite, is isomorphic to $\Omega^{n-1}_k \otimes H^0(X, \Omega^1_{X/k}(1))$. Finally, $\dim H^0(X, \Omega^1_{X/k}(1))) = d + g - 1$ by Riemann-Roch. ☐

Example 3.9. Here are two cases in which the hypotheses of Theorem 3.8 above are satisfied:

(a) X is embedded in $\mathbb{P}^N_k$ by a complete linear system of degree $d \geq 2g - 1$. In this case $\deg(\Omega^1_{X/k}(-1)) < 0$, so $H^1(X, \mathcal{O}(t)) = 0$ for all $t \geq 1$ by Serre duality. Theorem 3.8 improves the result of Srinivas in [17] that if $d \geq 2g+1$ then $\widetilde{K}_1(R) \neq 0$.

(b) X is definable over a number field contained in k.

4. K-THEORY OF THE PLANE CONIC

We conclude with a classical example: X is the plane conic with homogeneous coordinate ring $R = k[x, y, z]/(z^2 - xy)$. This curve is a degree 2 embedding of $\mathbb{P}^1_k$ in $\mathbb{P}^2_k$; as pointed out in Remark 2.12, our line bundle $\mathcal{O}_X(t)$ is the usual $\mathcal{O}_{\mathbb{P}^1_k}(2t)$.

Murthy observed long ago, in [13, 5.3], that $K_0(R) = \mathbb{Z}$ and $K_{-1}(R) = 0$; this also follows from our Theorem 2.1. Srinivas proved in [17] that $\widetilde{K}_1(R)$ surjects onto k. Theorem 4.3 below gives a complete calculation of $K_*(R)$, or rather, $\widetilde{K}_*(R) = K_*(R)/K_*(k)$.

Lemma 4.1. *For $R = k[x, y, z]/(z^2 - xy)$, $\Omega^1_{R/k}$ is a torsionfree R-module, and the map $\Omega^1_{R/k} \to H^0(Y, \Omega^1_{Y/k})$ is a graded injection with cokernel k in degree $t = 1$.*

Proof. As R is a normal complete intersection, a theorem of Vasconcelos ([19, 2.4]) says that Ω^1_R is a torsionfree R-module. As such, it is a graded submodule of $\Omega^1_{R[1/x]}$. From the factorization $\mathrm{Spec}(R[1/x]) \to Y \to \mathrm{Spec}(R)$, we see that the graded map $\Omega^1_{R/k} \to H^0(Y, \Omega^1_{Y/k})$ is an injection. Since $R/k \xrightarrow{d} \Omega^1_{R/k} \to H^0(Y, \Omega^1_{Y/k})$ is an injection with cokernel $\oplus_t H^0(X, \Omega^1_{X/k}(t))$ by Lemma 2.8, we are reduced to comparing the Hilbert functions of both sides.

It is easy to show that $\dim(R_t) = 2t + 1$ for all $t \geq 0$. From the resolution $0 \to R(-2) \xrightarrow{dF} R(-1)^3 \to \Omega^1_{R/k} \to 0$, we compute that $\dim(\Omega^1_{R/k})_t$ is 3 for $t = 1$ and $4t$ for $t \geq 2$. By Riemann-Roch, we have $\dim H^0(X, \Omega^1_{X/k}(t)) = 2t - 1$ for $t > 0$. By Lemma 2.8, this yields:

$$\dim\, H^0(Y, \Omega^1_{Y/k})_t = \dim\, H^0(X, \Omega^1_{X/k}(t)) + \dim\, R_t = (2t-1) + (2t+1) = 4t.$$

This shows that $(\Omega^1_{R/k})_t \cong R_t \oplus H^0(X, \Omega^1_{X/k}(t))$ when $t \geq 2$, as desired. □

Remark 4.1.1. Since Ω^1_R is torsionfree, the exact sequence (1.9a) shows that $d : \mathrm{tors}\,\Omega^2_{R/k} \cong \Omega^3_{R/k} \cong k$. In fact, the 2-form $\tau = z\,dx \wedge dy + 2y\,dx \wedge dz$ has $x\tau = y\tau = z\tau = 0$ and $d\tau = dx \wedge dy \wedge dz$.

Lemma 4.2. *For $R = \mathbb{Q}[x, y, z]/(z^2 - xy)$ and $n \geq 2$, $\widetilde{HC}^{(i)}_n(R)$ is $\mathbb{Q}$ if $n = 2i - 2$ and zero otherwise. For $R_k = R \otimes k$, $\widetilde{HC}^{(i)}_n(R_k)$ is Ω^p_k, where $p = 2i - n - 2$.*

Proof. The calculation of $HC^{(i)}_n(R)$ is taken from [12, Thms. 2–3], using the elementary calculation that $\Omega^3_R \cong \mathbb{Q}$ for $n > 3$ and exactness of the augmented Poincaré complex $\mathbb{Q} \to \Omega^*_R$ for $n = 2, 3$. The second sentence follows using the basechange formula of [7, (3.2)]. □

Theorem 4.3. *For $R_k = k[x, y, z]/(z^2 - xy)$ and all n, we have*

$$\widetilde{K}_n(R_k) \cong \Omega^{n-1}_k \oplus \Omega^{n-3}_k \oplus \Omega^{n-5}_k \oplus \cdots .$$

In particular, $K_1(R_k) \cong K_1(k) \oplus k$ and $K_2(R_k) \cong K_2(k) \oplus \Omega^1_k$.

Proof. By Proposition 3.2(a) and Lemma 4.2, we see that $\widetilde{K}^{(n-j)}_n(R)$ is Ω^{n-2j-3}_k for all $j > 0$. By Theorem 1.15 and Remark 4.1.1, we have $\widetilde{K}^{(3)}_3(R_{\mathbb{Q}}) \cong k$ and $\widetilde{K}^{(n)}_n(R_{\mathbb{Q}}) = 0$ for $n \neq 3$. By Proposition 3.2(b) this implies that $\widetilde{K}^{(n)}_n(R_k) \cong \Omega^{n-3}_k$ for all $n \neq 3$. By Proposition 3.5 and Lemma 4.1, we have $K^{(2)}_1(R_k) = k$. By Proposition 3.2, this implies that $K^{(n+1)}_n(R) \cong \Omega^{n-1}_k$ for all $n \geq 1$. Finally, by Proposition 2.11 we have $K^{(n+2)}_n(R_k)_t = H^1(X_k, \Omega^{n+1}_{X_k}(t))$, which vanishes for all $n, t \geq 1$ as it is the sum of $\Omega^n_k \otimes H^1(X, \Omega^1_X(t))$, which vanishes by Serre Duality, and $\Omega^{n+1}_k \otimes H^1(X, \mathcal{O}_X(t))$, which vanishes as $X = \mathbb{P}^1_k$. □

Remark 4.3.1. When k is algebraic over $\mathbb{Q}$, the formulas in Theorem 4.3 reduce to: $\widetilde{K}_n(R_k) = \mathbb{Q}$ for $n \geq 1$ odd, and $\widetilde{K}_n(R_k) = 0$ otherwise.

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