THE NORM RESIDUE ISOMORPHISM THEOREM

C. WEIBEL

Abstract. We provide a patch to complete the proof of the Voevodsky-Rost Theorem, that the norm residue map is an isomorphism. (This settles the motivic Bloch-Kato conjecture).

Introduction

The purpose of this paper is to patch up the sketched proof in [MC/l] of the "Rost-Voevodsky Theorem" that the norm residue map $K_n^M(k)/\ell \longrightarrow H^n_{\text{ét}}(k, \mu_\ell^{\otimes n})$ is an isomorphism, for any prime $\ell > 2$ and for any field k such that $1/\ell \in k$. This result is sometimes called the "Bloch-Kato conjecture."

It is based upon Voevodsky's 2003 preprint [MC/l]. That preprint gave a proof modulo three unproven results (Lemmas 2.2, 2.3 and Theorem 6.3). The third of these (6.3) asserts the existence of a "Rost variety" for $\underline{a}$ (as defined in Definition 1.1 below). This is not really a problem; Markus Rost constructed a variety $X_{\underline{a}}$ for any $\underline{a}$ in his 1998 "Chain Lemma" preprint [17], and proved a Norm Principle for $X_{\underline{a}}$; the details are now available in [4]. The proof that $X_{\underline{a}}$ is a Rost variety was published by Suslin and Joukhovitski in [21], using the Norm Principle.

The main innovation in the present paper is that we consider *integral to modular* cohomology operations from $H^{2n,n}(X, \mathbb{Z})$ to $H^{p,q}(X, \mathbb{Z}/\ell)$ in order to provide substitutes for the two missing lemmas 2.2 and 2.3 (see Theorem 6.1 and Proposition 4.13 below). Our substitute for 2.2 is inspired by the theorem of Henri Cartan [2] that in ordinary topology all such operations $\phi(x)$ are polynomials in the standard Steenrod operations $P^I(\bar{x})$, where (in the notation of [20]) $I = (\epsilon_0, s_1, \epsilon_1, \dots, s_k)$ is admissible (and hence $s_k \geq 1$ and $\epsilon_k = 0$).

Following the viewpoint presented in [26, 2.1], we shall consider the Morel-Voevodsky $\mathbb{A}^1$-homotopy category of pointed "spaces" (of [12]), and in particular the pointed spaces K_n which represent integral motivic cohomology in the sense that $\widetilde{H}^{2n,n}(X, \mathbb{Z}) \cong [X, K_n]_{\mathbb{A}^1}$ for pointed X (see 2.13). It follows that the set of (unstable) cohomology operations ϕ from $H^{2n,n}(-, \mathbb{Z})$ to $H^{p,q}(-, R)$ is in 1–1 correspondence with the elements of the group $\widetilde{H}^{p,q}(K_n, R) = [K_n, K(R(q), p)]$. All cohomology operations ϕ satisfy $\phi(0) = 0$, from naturality in $X_+ \to *$.

Example 0.1. When $n = 0$, K_0 is the J-indexed wedge of copies of S^0, $J = \mathbb{Z} - 0$. Hence cohomology operations $\phi : H^{0,0}(-, \mathbb{Z}) \to H^{p,q}(-, R)$ correspond to elements $\{\phi_j\}$ of $\widetilde{H}^{p,q}(\bigvee S^0, R) = \prod_{j \neq 0} H^{p,q}(k, R)$. If X is connected, then $\phi : \mathbb{Z} \to H^{p,q} \to H^{p,q}(X, R)$ is $\phi(j) = \phi_j$.

The usual cohomology operations in topology are a special case.

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Example 0.2. $H^{n,0}(-,A)$ is represented by the classical Eilenberg-Mac Lane space $K(A,n)$ (viewed as a simplicial set and hence as a pointed space). Thus cohomology operations from $H^{n,0}(-,A)$ to $H^{p,q}(-,R)$ correspond to elements of $H^{p,q}(K(A,n),R)$. In particular, motivic cohomology operations $H^{n,0}(-,A) \to H^{p,0}(-,R)$ are the same as the usual (unstable) topological cohomology operations described for example in [2]. A useful case is $n=1$ and $A=\mathbb{Z}$; since $K(\mathbb{Z},1)=S^1$ there are no cohomology operations $H^{1,0}(-,\mathbb{Z}) \to H^{p,0}(-,R)$ for $p \neq 1$.

We will be also be interested in the *Lefschetz* motives $\mathbb{L}^n_R = R_{\mathrm{tr}}(\mathbb{A}^n/\mathbb{A}^n - 0)$, which represent integral motivic cohomology in the triangulated category $\mathbf{DM}^{\mathrm{eff}}(R)$ of effective motives or its full subcategory $\mathbf{DM}^{\mathrm{eff},-}$ (see [23], [24] or [11].) To explain the connection between these approaches, recall that there is a functor $\mathbb{L}R_{\mathrm{tr}}$ from the Morel-Voevodsky $\mathbb{A}^1$-homotopy category to $\mathbf{DM}^{\mathrm{eff}}(R)$, which is left adjoint to the underlying (forgetful) functor $\mathbb{R}u$. We will describe this functor in section 2; other references are [10], [32], [28] or [16]. Following [26, p. 3], we define $K_n = u\mathbb{L}^n_{\mathbb{Z}}$, and have $H^{2n,n}(X,\mathbb{Z}) = \mathrm{Hom}_{\mathbf{DM}}(\mathbb{Z}_{\mathrm{tr}}(X),\mathbb{L}^n) \cong [X_+,K_n]_{\mathbb{A}^1}$.

More precisely, there is an adjunction (R_{tr},u) between (simplicial) pointed radditive functors and (positive chain complexes of) presheaves with transfers; see (2.5). It is a Quillen adjunction with respect to the $\mathbb{A}^1$-local model structures, and $(\mathbb{L}R_{\mathrm{tr}},\mathbb{R}u)$ is the derived adjunction.

This paper is laid out in stages. In the first section, we show that the existence of *Rost motives* implies the Rost-Voevodsky theorem (that the norm residue is an isomorphism). This reduction is taken from [33], which in turn is a reworking of the final portion of [23] and [MC/l].

The second stage begins in section 2, where we introduce the model structures used to work with $\mathbb{A}^1$-homotopy theory; this is equivalent to the Morel Voevodsky model [12], but is different because we need to work with normal varieties. In §3 we introduce the symmetric products $S^i_{\mathrm{tr}}\mathbb{L}^n$ as a device for studying cohomology operations; this parallels the approach of Dold and Steenrod in topology; see [19]. Section 4 is an exposé of certain new results of Voevodsky on symmetric products, which are taken from [30]. These allow us to prove the Künneth formula in 4.13, bypassing the unproven Lemma 2.3 in [MC/l].

The third and final stage begins in Section 5, where we introduce a $\mathbb{Z}/(\ell-1)$-grading on cohomology operations by "scalar weight." This grading is used to prove a uniqueness result (6.6) for the cohomology operation βP^n on $H^{2n+1,n}(-,\mathbb{Z})$; our presentation is based upon [MC/l, §2]. In sections 6–8, we adopt other arguments of [MC/l] to show that Rost motives exist, completing the proof.

Notation. The integer n and the prime $\ell > 2$ will be fixed. We will work over a fixed field k in which ℓ is invertible. The integer d will always be $\ell^{n-1}-1$ and b will always be $d/(\ell-1) = 1+\cdots+\ell^{n-2}$.

We fix the sequence of units $\underline{a} = (a_1,\dots,a_n)$, and $X_{\underline{a}}$ will always denote a d-dimensional Rost variety relative to $\underline{a}$, satisfying Axioms 1.3.

We will work in the triangulated category of motives $\mathbf{DM}^{\mathrm{eff},-}$ described in [11], usually inside the full subcategory generated by simplicial presheaves with transfers. The Lefschetz motive is $\mathbb{L} = \mathbb{Z}(1)[2]$. Unless explicitly stated otherwise, motivic cohomology will always be taken with coefficients $\mathbb{Z}_{(\ell)}$. The notation $H^{p,q}(-,R)$ refers to the Nisnevich motivic cohomology group $H^p_{\mathrm{nis}}(-,R(q))$.

1. The outer shell of the proof

We begin by setting the scene for the later sections, introducing the notion of a Rost motive (in 1.3) and showing (in Theorem 1.4) that it suffices to construct a Rost motive for every $\underline{a}$. Our candidate for a Rost motive is given in 1.13.

The material in this section is largely taken from my paper [33]. We will inductively assume that the norm residue maps $K_i^M(L)/\ell \to H^i_{\text{ét}}(L, \mu_\ell^{\otimes i})$ are isomorphisms for $i < n$ and all fields L containing $1/\ell$. From now on, k will denote a field of characteristic zero, and $\underline{a} = \{a_1, ..., a_n\}$ will be a nonzero element of $K_n^M(k)/\ell$.

Recall [21, 1.20] that a *ν_{n-1}-variety* over a field k is a smooth projective variety X of dimension $d = \ell^{n-1} - 1$, with $\deg s_d(X) \not\equiv 0 \pmod{\ell^2}$. Here $s_d(X)$ is the characteristic class of the tangent bundle T_X corresponding to the symmetric polynomial $\sum t_j^d$ in the Chern roots t_j of T_X; see [26, 14.3].

Definition 1.1. A *Rost variety* for a sequence $\underline{a} = (a_1, ..., a_n)$ of units in k is a ν_{n-1}-variety $X = X_{\underline{a}}$ such that: $\{a_1, ..., a_n\}$ vanishes in $K_n^M(k(X))/\ell$; for each $i < n$ there is a ν_i-variety mapping to X; and the motivic homology sequence

$$H_{-1,-1}(X^2) \xrightarrow{\pi_0^* - \pi_1^*} H_{-1,-1}(X) \to H_{-1,-1}(k) \quad (= k^\times). \tag{1.2}$$

is exact. As mentioned above, Rost varieties exist for every $\underline{a}$ by [17], [21] and [4].

Let $\mathfrak{X}$ denote the simplicial Čech scheme $\check{C}(X_{\underline{a}}) : p \mapsto X_{\underline{a}}^{p+1}$; see [MC/2, 9.1]. By abuse of notation, we will regard $\mathfrak{X}$ as a chain complex, and hence as an element of $\mathbf{DM}^{\text{eff}}(k, \mathbb{Z}_{(\ell)})$. The formulas $\mathfrak{X} \otimes \mathfrak{X} \simeq \mathfrak{X}$ and $\mathfrak{X} \otimes X \simeq X$ are well known, so the structure map $X \to \operatorname{Spec}(k)$ is equivalent to a structure map $X \to \mathfrak{X}$. Recall that $b = d/(\ell - 1) = 1 + \ell + \cdots + \ell^{n-2}$.

Definition 1.3. A *Rost motive* for $\underline{a}$ is a motive M satisfying the following axioms:
(a) M is a direct summand of a Rost variety $X_{\underline{a}}$, defined over $\mathbb{Z}_{(\ell)}$.
(b) The evident duality map $M^* \otimes \mathbb{L}^d \to X^* \otimes \mathbb{L}^d \cong X \to M$ is an isomorphism.
(c) There is a motive D, related to the structure map $y : M \to \mathfrak{X}$ and its twisted dual $y^D : \mathfrak{X} \otimes \mathbb{L}^d \to \mathbb{L}^d \xrightarrow{y^* \otimes 1} M^* \otimes \mathbb{L}^d \cong M$ by two distinguished triangles:

$$D \otimes \mathbb{L}^b \to M \xrightarrow{y} \mathfrak{X} \to, \tag{1.3.1}$$

$$\mathfrak{X} \otimes \mathbb{L}^d \xrightarrow{y^D} M \to D \to . \tag{1.3.2}$$

Note that D is determined up to isomorphism by (1.3.1): $D \cong \mathfrak{X} \otimes \tilde{M} \otimes \mathbb{L}^{-b}$, where $\tilde{M}$ is the geometric motive defined as the fiber of the structure map $M \to \mathbb{Z}$. Although D may not be a geometric motive, so that D^* may not be defined, the motive $D^\dagger = \mathfrak{X} \otimes \tilde{M}^* \otimes \mathbb{L}^b$ is well defined. We will see in 7.4 that triangle (1.3.2) is equivalent to the assertion that $D^\dagger \otimes \mathbb{L}^{d-b} \cong D$ in $\mathbf{DM}^{\text{eff}}$.

In Section 7 we will introduce the notion of "$\mathfrak{X}$-duality" on the subcategory of motives in $\mathbf{DM}^{\text{eff}}$ of the form $\mathfrak{X} \otimes N$, where N is a geometric motive: $(\mathfrak{X} \otimes N)^\dagger$ is $\mathfrak{X} \otimes N^*$. The definition of $D^\dagger$ in the previous paragraph is a special case. By (7.3), the map y^D defined in 1.3(c) is the same as the tensor product with $\mathbb{L}^d$ of the $\mathfrak{X}$-dual map $\mathfrak{X} \xrightarrow{y^\dagger} \mathfrak{X} \otimes M^* \to M^*$, followed by the isomorphism $M^* \otimes \mathbb{L}^d \cong M$.

When $\ell = 2$, triangle (1.3.1) was constructed in [MC/2, 4.4] with $D = \mathfrak{X}$, and (1.3.2) is its dual; axioms 1.3(a–b) hold by a result of Rost (cited as [MC/2, 4.3]).

The following theorem, which summarizes the contents of section 6 of [MC/l], was the main theorem in my paper [33].

Theorem 1.4. *Let n and ℓ be such that the norm residues maps are isomorphisms for all $i < n$. Suppose that a Rost motive exists for every $\underline{a}$ in $K_n^M(k)/\ell$.*

Then the norm residue map $K_n^M(k)/\ell \to H^n_{ét}(k, \mu_\ell^{\otimes n})$ is an isomorphism.

In the rest of this section we define a candidate for a Rost motive, following Voevodsky [MC/l], which we shall dub the "symmetric Rost motive" (see 1.13). As observed in [MC/l, 6.6], it follows from [MC/2, 6.9(1)] that:

$$H^{p,q}(\Sigma\mathfrak{X};\mathbb{Z}/\ell) = 0 \text{ when } (p,q) \text{ is in the region } q < n,\ p \le 1 + q. \tag{1.5}$$

Lemma 1.6. *If $\mathfrak{X}$ is the simplicial Čech scheme of a $\nu_{\le n-1}$-variety, then for all $i < n$ the Margulis homology sequence is exact:*

$$\cdots \xrightarrow{Q_i} H^{p,q}(\Sigma\mathfrak{X},\mathbb{Z}/\ell) \xrightarrow{Q_i} \cdots .$$

Proof. This is proven in [MC/2, 3.2]. The proof is repeated in [MC/l, 4.3]. □

Lemma 1.7. *The motivic cohomology groups $H^{*,*}(\Sigma\mathfrak{X},\mathbb{Z})$ have exponent ℓ.*

Lemma 1.7 is implicit in [MC/2, 9.3], and is proven in [33, 2.3], using [21]. It implies that the integral cohomology $H^{p,q}(\Sigma\mathfrak{X},\mathbb{Z})$ may be identified with the kernel of the Bockstein $\beta : H^{p,q}(\Sigma\mathfrak{X},\mathbb{Z}/\ell) \to H^{p+1,q}(\Sigma\mathfrak{X},\mathbb{Z}/\ell)$.

We now consider the cohomology operation Q_i, which has bidegree $(2\ell^i{-}1, \ell^i{-}1)$. Since Q_i anticommutes with β, it follows that Q_i sends the subgroup $H^{*,*}(\Sigma\mathfrak{X},\mathbb{Z})$ of $H^{*,*}(\Sigma\mathfrak{X},\mathbb{Z}/\ell)$ to itself. (This was observed in [MC/2, 7.2].)

Lemma 1.8. *The operations $Q = Q_{n-2}\cdots Q_0 : H^{n,n-1}(\mathfrak{X};\mathbb{Z}/\ell) \to H^{2b+1,b}(\mathfrak{X};\mathbb{Z}/\ell)$ and $Q_{n-1} : H^{2b+1,b}(\mathfrak{X};\mathbb{Z}/\ell) \to H^{2b\ell+2,b\ell+1}(\mathfrak{X};\mathbb{Z}/\ell)$ are injections.*

Proof. (Voevodsky) Using Lemma 1.6 and (1.5), it is routine to check that $Q_0 = \beta$ is injective on $H^{n+1,n-1}(\Sigma\mathfrak{X},\mathbb{Z}/\ell)$, and Q_i is injective on the group $H^{p,q}(\Sigma\mathfrak{X};\mathbb{Z}/\ell)$ containing $Q_{i-1}\cdots Q_0 H^{n+1,n-1}(\Sigma\mathfrak{X};\mathbb{Z}/\ell)$ when $i < n$, because the preceding term in 1.6 is zero by (1.5). Since $H^{p,q}(\mathfrak{X}) \cong H^{p+1,q}(\Sigma\mathfrak{X})$ for $p > q$, we are done. □

If $\underline{a} \ne 0$ in $K_n^M(k)/\ell$, Voevodsky shows in [MC/l, 6.5] that its norm residue symbol in $H^n_{ét}(k,\mu_\ell^{\otimes n})$ lifts to a nonzero element $\delta \in H^{n,n-1}(\mathfrak{X};\mathbb{Z}/\ell)$ via the canonical injection $H^{n,n-1}(\mathfrak{X};\mathbb{Z}/\ell) \hookrightarrow H^n_{ét}(k,\mu_\ell^{\otimes n})$. It follows from 1.8 and 1.7 that $\beta(\delta) \in H^{n+1,n-1}(\mathfrak{X};\mathbb{Z})$ and $\mu = Q(\delta) \in H^{2b+1,b}(\mathfrak{X},\mathbb{Z})$ are nonzero.

The following device allows us to regard μ as a morphism from $R_{\mathrm{tr}}(\mathfrak{X})$ to $R_{\mathrm{tr}}(\mathfrak{X})(b)[2b+1] = R_{\mathrm{tr}}(\mathfrak{X}) \otimes \mathbb{L}^b[1]$. First, $H^{2b+1,b}(\mathfrak{X},R) \cong \mathrm{Hom}_{\mathbf{DM}}(R_{\mathrm{tr}}(\mathfrak{X}),\mathbb{L}^b[1])$. Next, since $R_{\mathrm{tr}}(\mathfrak{X}) \cong R_{\mathrm{tr}}(\mathfrak{X}) \otimes R_{\mathrm{tr}}(\mathfrak{X})$, the correspondence $f \mapsto \mathfrak{X} \otimes f$ induces an isomorphism

$$\mathrm{Hom}_{\mathbf{DM}}(R_{\mathrm{tr}}(\mathfrak{X}),\mathbb{L}^b[1]) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{DM}}(R_{\mathrm{tr}}(\mathfrak{X}), R_{\mathrm{tr}}(\mathfrak{X}) \otimes \mathbb{L}^b[1]).$$

We define the motive A associated to $\underline{a}$ by the distinguished triangle:

$$R_{\mathrm{tr}}(\mathfrak{X}) \otimes \mathbb{L}^b \xrightarrow{x} A \xrightarrow{y} R_{\mathrm{tr}}(\mathfrak{X}) \xrightarrow{\mu} R_{\mathrm{tr}}(\mathfrak{X}) \otimes \mathbb{L}^b[1]. \tag{1.9}$$

We now suppose that R is either $\mathbb{Z}_{(\ell)}$ or $\mathbb{Z}/\ell$, so that $(\ell-1)!$ is invertible in R.

Definition 1.10. If $m < \ell$, the symmetrizing idempotent $e = \sum\{\sigma \in \Sigma_m\}/m!$ of $R[\Sigma_m]$ acts on $M^{\otimes m}$ for each object M of $\mathbf{DM}$, and we set $S^m(M) = e \cdot M^{\otimes m}$.

For each (p,q), Voevodsky constructs a motivic cohomology operation ϕ_V from $H^{2p+1,q}(-,R)$ to $H^{2p\ell+2,q\ell}(-,R)$ in [MC/l, 3.1–3.2] (we use $m=\ell-1$). In particular, the element $\phi_V(\mu)$ is computed from (1.9) as follows. Consider the transfer map for 1.10 associated to $\Sigma_{m-1}\subset\Sigma_m$ acting on $A^{\otimes m}$ (see [31, 6.7.16]):

$$S^m(A)\to S^{m-1}(A)\otimes A,\quad (a_1\otimes\cdots)\mapsto\sum(\cdots\otimes\hat{a}_j\otimes\cdots)\otimes a_j. \tag{1.11}$$

Because $A\cong R_{\mathrm{tr}}(\mathfrak{X})\otimes A$ and $S^i(A)\cong R_{\mathrm{tr}}(\mathfrak{X})\otimes S^i(A)$, composing (1.11) with $1\otimes y$ gives a map $u:S^m(A)\to S^{m-1}(A)$, and composing $1\otimes x$ with the corestriction map $S^{m-1}(A)\otimes A\to S^m(A)$ gives a map $v:S^{m-1}(A)\otimes\mathbb{L}^b\to S^m(A)$.

Voevodsky uses the slice filtration to prove the following result in [MC/l, 3.1].

Lemma 1.12. *For $m<\ell$, there exist unique morphisms r and s fitting into distinguished triangles:*

$$S^{m-1}(A)\otimes\mathbb{L}^b\xrightarrow{v}S^m(A)\xrightarrow{S^m y}R_{tr}(\mathfrak{X})\xrightarrow{s}S^{m-1}(A)\otimes\mathbb{L}^b[1];$$

$$R_{tr}(\mathfrak{X})\otimes\mathbb{L}^d\xrightarrow{S^m x}S^m(A)\xrightarrow{u}S^{m-1}(A)\xrightarrow{r}R_{tr}(\mathfrak{X})\otimes\mathbb{L}^d[1].$$

By 1.12, $M=S^{\ell-1}(A)$ and $D=S^{\ell-2}(A)$ satisfy axiom 1.3(c). The point of [MC/l], and of this paper, is that $S^{\ell-1}(A)$ is a Rost motive in the sense of 1.3, so that Theorem 1.4 applies to prove the Voevodsky-Rost Theorem. This motivates:

Definition 1.13. The *symmetric Rost motive* associated to $\underline{a}$ is defined to be the symmetric product $S^{\ell-1}(A)$ of A.

2. Motivic Model Structures

In this section, we explain the passage back and forth between the categories Sm = Sm/k and Norm = Norm/k of smooth and normal varieties over k, the Morel-Voevodsky categories of *spaces* on them, and the categories of presheaves with transfers on them.

Following Voevodsky [28], we begin a little more generally. Let $\mathcal{C}$ be any small category and let Pshv($\mathcal{C}$) denote the category of its presheaves (contravariant functors from $\mathcal{C}$ to **Sets**), and $\Delta^{\mathrm{op}}\mathrm{Pshv}(\mathcal{C})$ the category of simplicial objects in Pshv($\mathcal{C}$).

Definition 2.1. A morphism $X\to Y$ of simplicial presheaves is called a *global weak equivalence* (resp., a *projective fibration*) if $X(C)\to Y(C)$ is a simplicial weak equivalence (resp., a Kan fibration) for every C in $\mathcal{C}$.

Quillen showed that these determine the structure of a proper simplicial model category on $\Delta^{\mathrm{op}}\mathrm{Pshv}(\mathcal{C})$, called the *projective model structure*; the *projective cofibrations* are determined by the left lifting property, as usual. It is well known [5, 11.6–7] [6] that the projective model structure is cofibrantly generated.

Voevodsky prefers to work with the sub-model category of radditive presheaves, defined whenever $\mathcal{C}$ has an initial object $\emptyset$ and finite coproducts. By definition, a presheaf X is *radditive* if $X(\emptyset)$ is a point and $X(C\coprod C')\to X(C)\times X(C')$ is a bijection. The inclusion $i_{\mathrm{rad}}:\mathrm{rad}(\mathcal{C})\subset\mathrm{Pshv}(\mathcal{C})$ has a left adjoint and the universal map $X\to X^{\mathrm{rad}}$ is an isomorphism for radditive X; see [28, 3.11]. The global weak equivalences and projective fibrations between radditive functors determine a simplicial model structure on $\Delta^{\mathrm{op}}\mathrm{rad}(\mathcal{C})$ by [28, 3.27]. Since i_{rad} preserves equivalences and fibrations, the pair $(-^{\mathrm{rad}},i_{\mathrm{rad}})$ defines a Quillen adjunction $\Delta^{\mathrm{op}}\mathrm{Pshv}(\mathcal{C})\to\Delta^{\mathrm{op}}\mathrm{rad}(\mathcal{C})$.

The following trick [28, 3.5] allows us to pass back and forth between the pointed and unpointed cases: if $\mathcal{C}_+ \subset \mathcal{C}$ denotes the full subcategory of objects of the form $C \coprod *$, then $\mathrm{rad}(\mathcal{C}_+)$ is equivalent to the category of pointed objects in $\mathrm{rad}(\mathcal{C})$.

Here are two useful facts which are immediate from [28, 3.28]:

Lemma 2.2. *Let A and A' be simplicial radditive presheaves. (1) a morphism $A \to A'$ is a projective cofibration in $\Delta^{op} Pshv(\mathcal{C})$ if and only if it is a projective cofibration in $\Delta^{op} rad(\mathcal{C})$; (2) if $A \rightarrowtail B$ is a projective cofibration in $\Delta^{op} Pshv(\mathcal{C})$ then B is in $\Delta^{op} rad(\mathcal{C})$.*

Example 2.3. (1) Suppose that $\mathcal{C}$ is either Sm or Norm, so that there is a final object, and let $\mathcal{C}_0$ denote the full subcategory of nonempty connected varieties in $\mathcal{C}$. The restriction has a right adjoint $\mathrm{Pshv}(\mathcal{C}_0) \hookrightarrow \mathrm{Pshv}(\mathcal{C})$ whose image is the category $\mathrm{rad}(\mathcal{C})$. Thus any Zariski sheaf on $\mathcal{C}$ is a radditive presheaf, while weak equivalences are closed under finite coproducts and form a $\bar{\Delta}$-closed class by [28, 3.59].

(2) Let $\mathrm{Cor}(\mathcal{C})$ be the category of (finite) correspondences on either $\mathcal{C} = \mathrm{Sm}$ or $\mathcal{C} = \mathrm{Norm}$, as defined in [11, §1 and 1A], with coefficients in a ring R. We write $\mathrm{PST}(\mathcal{C})$ for the (abelian) category of additive functors $\mathrm{Cor}(\mathcal{C}) \to \mathrm{Ab}$; such functors are called *presheaves with transfers*. By [28, 6.1], $\mathrm{PST}(\mathcal{C})$ is the same as $\mathrm{rad}(\mathrm{Cor}(\mathcal{C}))$; by [11, 8.1], projective objects in $\mathrm{PST}(\mathcal{C})$ are summands of direct sums of representable presheaves. Since fibrations are just surjections, projective cofibrant objects in $\Delta^{\mathrm{op}}\mathrm{PST}(\mathcal{C})$ are just termwise projective objects, and a projective cofibration $A \to A'$ is just an injection whose cokernel is termwise projective.

Proposition 2.4. *Suppose that $\mathcal{C}$ and $\mathcal{C}'$ have finite coproducts. Given a functor $f : \mathcal{C} \to \mathcal{C}'$ which commutes with finite coproducts, then (f^*, f_*) is a Quillen adjunction from $\Delta^{op} rad(\mathcal{C})$ to $\Delta^{op} rad(\mathcal{C}')$, given by the inverse and direct image functors.*

Recall that $(f_*Y)(C) = Y(fC)$ and $(f^*X)(C') = \mathrm{colim}_{C' \to fC} X(C)$.

Proof. By [28, 3.16], f^* sends radditive presheaves to radditive presheaves, and is right adjoint to f_* by [28, 3.18]. It is trivial (see [28, 3.36]) that f_* respects equivalences and fibrations, so we may invoke the usual criterion [5, 8.5.3]. □

Proposition 2.4 applies to the embeddings $\mathrm{Sm} \subset \mathrm{Norm}$ and $\mathrm{Sm}_+ \subset \mathrm{Norm}_+$, as well as to the embedding $\mathrm{Cor}(\mathrm{Sm}) \subset \mathrm{Cor}(\mathrm{Norm})$. Hence there are Quillen adjunctions from $\Delta^{\mathrm{op}}\mathrm{rad}(\mathrm{Sm}_+)$ to $\Delta^{\mathrm{op}}\mathrm{rad}(\mathrm{Norm}_+)$ and from $\Delta^{\mathrm{op}}\mathrm{PST}(\mathrm{Sm})$ to $\Delta^{\mathrm{op}}\mathrm{PST}(\mathrm{Norm})$.

The Yoneda embedding of $\mathcal{C} = \mathrm{Sm}$ into $\mathrm{Cor}(\mathcal{C})$ commutes with coproducts, so by 2.4 it induces a Quillen adjunction (R_{tr}, u) from $\Delta^{\mathrm{op}}\mathrm{rad}(\mathrm{Sm}_+)$ to $\Delta^{\mathrm{op}}\mathrm{PST}(\mathrm{Sm})$, and similarly for Norm. The right adjoint $u : \mathrm{PST}(\mathcal{C}) \to \mathrm{rad}(\mathcal{C}_+)$ takes a presheaf with transfers to its underlying presheaf of pointed sets. The inverse image functor R_{tr} satisfies $R_{\mathrm{tr}}(X_+) = R_{\mathrm{tr}}(X)$, is discussed in [10], [32] and [16], and underlies the viewpoint presented in [26, 2.1]. By naturality, we have a commutative diagram of projective model categories and Quillen adjunctions:

$$\begin{tikzcd}
\Delta^{\mathrm{op}}\mathrm{rad}(\mathrm{Sm}_+) \arrow[r, "(i^*, i_*)"] \arrow[d, "(R_{\mathrm{tr}}, u)"'] & \Delta^{\mathrm{op}}\mathrm{rad}(\mathrm{Norm}_+) \arrow[d, "(R_{\mathrm{tr}}, u)"] \\
\Delta^{\mathrm{op}}\mathrm{PST}(\mathrm{Sm}) \arrow[r, "(i^*, i_*)"] & \Delta^{\mathrm{op}}\mathrm{PST}(\mathrm{Norm}).
\end{tikzcd} \tag{2.5}$$

Remark. Voevodsky defines a derived resolution functor 'Lres' in [28, §3.3]. It follows from the explicit construction in [28, 3.43] that $\mathrm{Lres}(X)$ is degeneracy-free in the sense of [28, 3.38] and hence by [28, 3.42] that it is projectively cofibrant.

When X is radditive, $\mathrm{Lres}(X) \to X$ is a global equivalence by [28, 3.52] and hence Lres is equivalent to the usual cofibrant replacement. For $f : \mathcal{C} \to \mathcal{C}'$, it follows that Voevodsky's $f^*(\mathrm{Lres}X)$ is equivalent to the usual left derived functor $\mathbb{L}f^*(X)$ on the projective homotopy categories.

We now consider localizations of the projective model structure, suitable for (Nisnevich) sheaf theory and $\mathbb{A}^1$-homotopy theory. Recall that we are restricting our attention to the case where $\mathcal{C}$ is Sm or Norm, and $\mathcal{C}_+$ is the category of varieties with a disjoint basepoint. The Nisnevich topology on $\mathcal{C}$ is defined in terms of "upper distinguished" squares; see [12, 3.1] or [11, 12.5].

Definition 2.6. A morphism $X \to Y$ in $\Delta^{\mathrm{op}}\mathrm{Pshv}(\mathcal{C}_+)$ or $\Delta^{\mathrm{op}}\mathrm{PST}(\mathcal{C})$ is called a *Nisnevich (local) equivalence* if it induces an isomorphism on the (Nisnevich) sheaves of homotopy groups. It is called a *local projective fibration* if it has the right lifting property with respect to projective cofibrations (2.1) which are Nisnevich equivalences.

Blander showed in [1, 1.5] that the Nisnevich equivalences, projective cofibrations and local projective fibrations are part of a cofibrantly generated proper simplicial model structure on $\Delta^{\mathrm{op}}\mathrm{Pshv}(\mathcal{C}_+)$, called the *local projective* model structure. By [1, 2.1] the same definitions yield a model structure on the category $\Delta^{\mathrm{op}}\mathrm{Sheaves}(\mathcal{C})$ of simplicial sheaves.

Theorem 2.7. *The Nisnevich equivalences and projective cofibrations determine a proper simplicial closed model structure in $\Delta^{op}\mathit{rad}(\mathcal{C})$; a morphism in $\Delta^{op}\mathit{rad}(\mathcal{C})$ is a local projective fibration if and only if it is a local projective fibration in $\Delta^{op}\mathit{Pshv}(\mathcal{C})$. Moreover, we have Quillen equivalences of local projective model categories:*

$$\Delta^{op}\mathit{Pshv}(\mathcal{C}) \xrightarrow{\simeq} \Delta^{op}\mathit{rad}(\mathcal{C}) \xrightarrow{\simeq} \Delta^{op}\mathit{Sheaves}(\mathcal{C}).$$

Proof. Suppose first that $X \xrightarrow{f} Y$ is a map between simplicial radditive presheaves. If f is a local projective fibration in $\Delta^{\mathrm{op}}\mathrm{Pshv}(\mathcal{C})$, then f has the right lifting property for acyclic cofibrations in $\Delta^{\mathrm{op}}\mathrm{rad}(\mathcal{C})$ by Lemma 2.2. Conversely if f has this lifting property in $\Delta^{\mathrm{op}}\mathrm{rad}(\mathcal{C})$ and $A \overset{\sim}{\rightarrowtail} B$ is an acyclic cofibration in $\Delta^{\mathrm{op}}\mathrm{Pshv}(\mathcal{C})$, then $A^{\mathrm{rad}} \overset{\sim}{\rightarrowtail} B^{\mathrm{rad}}$ is also an acyclic cofibration, and $A \to A^{\mathrm{rad}}$ is a Nisnevich equivalence. The right lifting property now shows that f is a local projective fibration in $\Delta^{\mathrm{op}}\mathrm{Pshv}(\mathcal{C})$:

$$\begin{tikzcd}
A \arrow[r, "\sim"] \arrow[d, "\sim"'] & A^{\mathrm{rad}} \arrow[r] \arrow[d, "\sim"'] & X \arrow[d, "f"] \\
B \arrow[r, "\sim"] & B^{\mathrm{rad}} \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

Given this, the proper model structure for $\Delta^{\mathrm{op}}\mathrm{rad}(\mathcal{C})$ will follow from the proper model structure for $\Delta^{\mathrm{op}}\mathrm{Pshv}(\mathcal{C})$ once we check the "trivial cofibration/fibration" factorization axiom. If A and C are simplicial radditive then any factorization $A \overset{\sim}{\rightarrowtail} B \twoheadrightarrow C$ in $\Delta^{\mathrm{op}}\mathrm{Pshv}(\mathcal{C})$ is a factorization in $\Delta^{\mathrm{op}}\mathrm{rad}(\mathcal{C})$ by Lemma 2.2. The cofibration/trivial fibration lifting axiom follows immediately from the corresponding axiom for presheaves, together with Lemma 2.2. The final assertion is immediate from [1, 2.2]. □

The corresponding results for $\Delta^{\rm op}{\rm PST}(\mathcal{C})$ are proven in the same way. By [7, 2.2], the Nisnevich equivalences and projective cofibrations determine a proper simplicial model structure on the category $\mathcal{A}$ of simplicial presheaves of abelian groups on ${\rm Cor}(\mathcal{C})$, and on the category $\mathcal{A}_{\rm nis}$ of simplicial sheaves. Since Nisnevich equivalences of simplicial sheaves of abelian groups are quasi-isomorphisms of their associated chain complexes, the homotopy category for Nisnevich equivalences is a full subcategory of the derived category of sheaves with transfers.

Let ${\rm NST}(\mathcal{C})$ denote the category of Nisnevich sheaves with transfers (sheaves which are also presheaves with transfers). The Nisnevich equivalences and projective cofibrations determine a similar model structure on the category ${\rm NST}(\mathcal{C})$. The sheafification $a_{\rm nis}$ of a presheaf with transfers is a sheaf with transfers by [11, 13.1], and $a_{\rm nis}$ is left adjoint to the forgetful functor.

Theorem 2.8. *The Nisnevich equivalences and projective cofibrations determine a proper simplicial model structure in $\Delta^{op}PST(\mathcal{C})$, and sheafification is a Quillen equivalence $\Delta^{op}PST(\mathcal{C}) \xrightarrow{a_{nis}} \Delta^{op}NST(\mathcal{C})$.*

Moreover, both (equivalent) homotopy categories embed as a full subcategory of the derived category of Nisnevich sheaves with transfers.

Proof. The radditive objects in $\mathcal{A}$ are just the presheaves with transfers by [28, 6.1]. The proof of Theorem 2.7 now goes through in this context, using 2.3(2), to show that a morphism in $\Delta^{\rm op}{\rm PST}(\mathcal{C})$ is a local projection exactly when it is so in the ambient category $\mathcal{A}$. The embedding into the derived category follows (as above) from the observation that a Nisnevich equivalence between simplicial sheaves is the same as a quasi-isomorphism between the associated chain complexes. (This embedding is a special case of [28, 6.14] and [30, 2.26].) □

It is easy to see that the square (2.5) is also a commutative diagram of Nisnevich local model categories and Quillen adjunctions.

Remark. An object L of $\Delta^{\rm op}{\rm Pshv}(\mathcal{C}_+)$ or $\Delta^{\rm op}{\rm PST}(\mathcal{C})$ is said to be *Nisnevich local* if L is projectively fibrant and converts upper distinguished squares in $\mathcal{C}$ to homotopy pullback squares. In fact, by [28, 2.18], a simplicial presheaf with transfers L is Nisnevich local in $\Delta^{\rm op}{\rm PST}(\mathcal{C})$ if and only if uL is Nisnevich local in $\Delta^{\rm op}{\rm Pshv}(\mathcal{C}_+)$. By [1, 4.1], these are exactly the objects which are fibrant for the local projective model structure for $\Delta^{\rm op}{\rm Pshv}(\mathcal{C}_+)$, $\Delta^{\rm op}{\rm rad}(\mathcal{C}_+)$ and $\Delta^{\rm op}{\rm PST}(\mathcal{C})$. By [30, 2.11], our definition that $X \to Y$ is a Nisnevich equivalence is the same as the more formal criterion in [28, 4.8] that for any Nisnevich local L and any simplicial set K there is a bijection of projective homotopy classes of maps $[Y \boxtimes K, L]_s \to [X \boxtimes K, L]_s$.

The following definition is taken from [28, 4.8–10] and [30, §2.2]).

Definition 2.9. Let L be a Nisnevich local object of $\Delta^{\rm op}{\rm Pshv}(\mathcal{C}_+)$ or $\Delta^{\rm op}{\rm PST}(\mathcal{C})$. We say that L is $\mathbb{A}^1$*-local* if $L(X) \to L(X \times \mathbb{A}^1)$ is a weak equivalence for all X. In fact, a simplicial presheaf with transfers L is $\mathbb{A}^1$-local in $\Delta^{\rm op}{\rm PST}(\mathcal{C})$ if and only if uL is $\mathbb{A}^1$-local in $\Delta^{\rm op}{\rm Pshv}(\mathcal{C}_+)$ by [28, 2.18].

A Nisnevich equivalence $X \to Y$ is called an $\mathbb{A}^1$*-local equivalence* if for any $\mathbb{A}^1$-local L there is a bijection of projective homotopy classes of maps $[Y, L]_s \to [X, L]_s$.

By Blander [1, 3.1], the $\mathbb{A}^1$-local weak equivalences and projective cofibrations determine a cofibrantly generated proper simplicial model category on $\Delta^{\rm op}{\rm Pshv}(\mathcal{C})$, where fibrations are defined by the right lifting property. (It is the left Bousfield localization; see [5, 3.31].) By [3, 2.13], the $\mathbb{A}^1$-local objects are exactly the fibrant objects in the $\mathbb{A}^1$-local projective model structure. The following proposition is immediate from this, given Theorem 2.7.

Proposition 2.10. *The $\mathbb{A}^1$-local equivalences and projective cofibrations determine a proper simplicial closed model structure in $\Delta^{op}rad(\mathcal{C})$; a morphism in $\Delta^{op}rad(\mathcal{C})$ is an $\mathbb{A}^1$-local projective fibration if and only if it is an $\mathbb{A}^1$-local projective fibration in $\Delta^{op}Pshv(\mathcal{C})$. Moreover, we have Quillen equivalences of $\mathbb{A}^1$-local projective model categories:*

$$\Delta^{op}Pshv(\mathcal{C}) \xrightarrow{\simeq} \Delta^{op}rad(\mathcal{C}) \xrightarrow{\simeq} \Delta^{op}Sheaves(\mathcal{C}).$$

It follows that the homotopy categories in 2.10 are all equivalent, and agree with the $\mathbb{A}^1$-homotopy category $H_{\mathbb{A}^1}(\mathcal{C})$ of Morel and Voevodsky [12]. For the sake of definiteness, we will write $H_{\mathbb{A}^1}(\mathcal{C})$ for the homotopy category of $\Delta^{\rm op}\mathrm{rad}(\mathcal{C})$ with respect to $\mathbb{A}^1$-local equivalence. We write $[X,Y]_{\mathbb{A}^1}$ for the morphisms from X to Y in $H_{\mathbb{A}^1}(\mathrm{Norm}_+)$. When X and Y are both smooth, $[X,Y]_{\mathbb{A}^1}$ agrees with the morphisms in $H_{\mathbb{A}^1}(\mathrm{Sm}_+)$ because $\mathbb{L}i^* : H_{\mathbb{A}^1}(\mathrm{Sm}_+) \to H_{\mathbb{A}^1}(\mathrm{Norm}_+)$ is a full embedding by [30, 2.30].

Similarly, the $\mathbb{A}^1$-local weak equivalences and projective cofibrations determine a model structure on $\Delta^{\rm op}\mathrm{PST}(\mathcal{C})$; we omit the details (cf. [15]). Following Voevodsky, we write $H_{\rm tr}(\mathcal{C})$ for the $\mathbb{A}^1$-local homotopy category of $\Delta^{\rm op}\mathrm{PST}(\mathcal{C})$. By [30, 2.27], the $\mathbb{A}^1$-local homotopy category $H_{\rm tr}(\mathrm{Sm})$ embeds as a full subcategory of $\mathbf{DM}^{\rm eff}$; this is a localization of the embedding in Theorem 2.8.

It is easy to see that (i^*, i_*) and $(R_{\rm tr}, u)$ are Quillen adjunctions for the $\mathbb{A}^1$-local model structures, so that (2.5) is also a commutative diagram of $\mathbb{A}^1$-local model categories and Quillen adjunctions. The corresponding diagram of $\mathcal{A}^1$-local homotopy categories is:

$$\begin{tikzcd} H_{\mathbb{A}^1}(\mathrm{Sm}_+) \arrow[r, "(\mathbb{L}i^*, i_*)"] \arrow[d, "(\mathbb{L}R_{\rm tr}, Ru)"'] & H_{\mathbb{A}^1}(\mathrm{Norm}_+) \arrow[d, "(\mathbb{L}R_{\rm tr}, Ru)"] \\ H_{\rm tr}(\mathrm{Sm}) \arrow[r, "(\mathbb{L}i^*, i_*)"] & H_{\rm tr}(\mathrm{Norm}). \end{tikzcd} \tag{2.11}$$

Since $R_{\rm tr}$ preserves cofibration sequences, such as the simplicial suspension sequence $V \to \mathrm{cone}(V) \to \Sigma V$, uM is a simplicial group object (so its classifying object $B(uM)$ exists) and since u preserves fibration sequences, such as $M \to \mathrm{cone}(M) \to M[1]$, we see that

$$R_{\rm tr}(\Sigma V) \simeq R_{\rm tr}(V)[1] \quad \text{and} \quad u(M[1]) \simeq B(uM). \tag{2.12}$$

At times, it will be useful to pass back and forth between the $\mathbb{A}^1$-local homotopy categories via i_* and the derived functor $\mathbb{L}i^* : H_{\mathbb{A}^1}(\mathrm{Sm}_+) \to H_{\mathbb{A}^1}(\mathrm{Norm}_+)$.

In one direction, $\mathbb{L}i^*$ is a full embedding by [30, 2.32]. For the other direction, suppose that $\mathrm{char}(k) = 0$. If V is any simplicial radditive presheaf on Norm_+, the natural transformation $R_{\rm tr}(i_*V) \to i_*(R_{\rm tr}V)$ is an $\mathbb{A}^1$-homotopy equivalence of presheaves with transfers on Sm by [30, 2.33], so $R_{\rm tr}(i_*V)$ and $i_*(R_{\rm tr}V)$ are identified in $\mathbf{DM}^{\rm eff}$.

For example, if V is in $\Delta^{\rm op}\mathrm{Norm}_+$ we can consider the motivic cohomology

$$H^{p,q}(i_*V, R) = \mathrm{Hom}_{\mathbf{DM}}(R_{\rm tr} i_* V, R(q)[p]) \cong \big[V, i^*uR(q)[p]\big]_{\mathbb{A}^1}.$$

Definition 2.13. For $n \geq 0$ we set $\mathbb{L}^n_R = R_{\mathrm{tr}}(\mathbb{A}^n/(\mathbb{A}^n - 0)) = R_{\mathrm{tr}}\mathbb{A}^n/R_{\mathrm{tr}}(\mathbb{A}^n - 0)$. These correspond to the Lefschetz motives in $\mathbf{DM}$ in the sense that (by [11, 15.14])

$$H^{2n,n}(X,R) \cong \mathrm{Hom}_{\mathbf{DM}}(R_{\mathrm{tr}}X, \mathbb{L}^n_R).$$

For simplicity, we will often write $\mathbb{L}^n$ for $\mathbb{L}^n_{\mathbb{Z}}$.

When $R = \mathbb{Z}$ we set $K_n = u\mathbb{L}^n_{\mathbb{Z}}$ in $\Delta^{\mathrm{op}}\mathrm{rad}(\mathrm{Sm}_+)$. By the Quillen adjunction (R_{tr}, u) we also have $H^{2n,n}(X,\mathbb{Z}) \cong [X_+, K_n]_{\mathbb{A}^1}$. By (2.12) we also have

$$H^{2n+1,n}(X,\mathbb{Z}) \cong \mathrm{Hom}_{\mathbf{DM}}(\mathbb{Z}_{\mathrm{tr}}X, \mathbb{L}^n_{\mathbb{Z}}[1]) \cong [X_+, BK_n]_{\mathbb{A}^1}.$$

Remark 2.14. By Yoneda, cohomology operations $H^{2n,n}(X,\mathbb{Z}) \to H^{p,q}(X,R)$ are in 1–1 correspondence with elements of $\mathrm{Hom}_{\mathbf{DM}}(R_{\mathrm{tr}}K_n, R(q)[p]) = \big[K_n, uR(q)[p]\big]_{\mathbb{A}^1}$. Similarly, cohomology operations $H^{2n+1,n}(X,\mathbb{Z}) \to H^{p,q}(X,R)$ are in 1–1 correspondence with elements of $\mathrm{Hom}_{\mathbf{DM}}(R_{\mathrm{tr}}(BK_n), R(q)[p]) = \big[BK_n, uR(q)[p]\big]_{\mathbb{A}^1}$.

3. Symmetric Products

If X is a normal quasiprojective variety over k, the symmetric products $S^m(X) = X^m/\Sigma_m$ are also normal. Thus S^m is a functor from the category Norm of normal quasiprojective varieties to itself. (If $X \neq \emptyset$, then $S^0(X) = \mathrm{Spec}(k)$.) We have:

$$S^m(X \amalg Y) = \amalg\ S^i(X) \times S^j(Y). \tag{3.1}$$

Since $R_{\mathrm{tr}}(X)$ makes sense for any scheme X of finite type ([11, 2.11]), it makes sense to write $S^m_{\mathrm{tr}}R_{\mathrm{tr}}(X)$ for $R_{\mathrm{tr}}(S^mX)$. Thus $S^m_{\mathrm{tr}}R_{\mathrm{tr}}$ is a functor from Norm to presheaves with transfers on Norm (or, by restriction, on Sm). If X is a simplicial object in Norm, we define $S^m(X)$ and $S^m_{\mathrm{tr}}R_{\mathrm{tr}}(X)$ degreewise.

It is easy to see that $S^m_{\mathrm{tr}}R_{\mathrm{tr}}$ extends to finite correspondences between normal varieties, so S^m_{tr} is a functor on the category Cor(Norm) of finite correspondences.

If $(\ell-1)!$ is invertible in R and $i < \ell$, a transfer argument for $X^i \to S^i(X)$ shows that $S^i_{\mathrm{tr}}R_{\mathrm{tr}}(X)$ is the symmetric power $S^i(R_{\mathrm{tr}}(X)) = e \cdot R_{\mathrm{tr}}(X)^{\otimes i}$ of 1.10. The following lemma is immediate from (3.1).

Lemma 3.2. *If $M = R_{tr}(X)$ and $N = R_{tr}(Y)$ for normal X and Y, we have*

$$S^m_{tr}(M \oplus N) \cong \oplus_{i+j=m} S^i_{tr}(M) \otimes S^j_{tr}(N).$$

If V is based, there is a canonical map $S^{m-1}(V) \to S^m(V)$, and we write $S^\infty(V)$ for the colimit of the $S^m(V)$. When the basepoints are disjoint, the filtration by the $S^m(V)$ splits, and the filtration of $S^m_{\mathrm{tr}}R_{\mathrm{tr}}(V)$ splits naturally, as we now show.

We write X_+ for the disjoint union of X and a basepoint $* = \mathrm{Spec}(k)$, considered as a based space. Then the decomposition $S^m(X_+) = \amalg_{i=0}^m S^iX$ of (3.1) yields a split sequence of pointed spaces

$$S^{m-1}(X_+) \to S^m(X_+) \to \widetilde{S}^m(X_+), \tag{3.3}$$

where $\widetilde{S}^m$ is the functor $\widetilde{S}^m(X_+) = (S^mX)_+$ on Norm_+. By naturality, (3.3) extends to a degreewise split sequence of simplicial objects in Norm_+.

As X_+ is a based space, $R_{\mathrm{tr}}(X_+) = R_{\mathrm{tr}}(X \amalg *)/R = R_{\mathrm{tr}}(X)$ is well defined and we have $S^m_{\mathrm{tr}}R_{\mathrm{tr}}(X_+) = S^m_{\mathrm{tr}}R_{\mathrm{tr}}(X) = R_{\mathrm{tr}}(S^mX) = R_{\mathrm{tr}}(\widetilde{S}^m(X_+))$.

The following result is the analogue of a formula for augmented simplicial R-modules discovered by Steenrod in [19].

Lemma 3.4. *(Steenrod's Formula) Applying R_{tr} to* (3.3) *yields a naturally split exact sequence for every $V = X_+$ in Norm$_+$:*

$$0 \to R_{tr}S^{m-1}(V) \to R_{tr}S^m(V) \to S^m_{tr}R_{tr}(V) \to 0.$$

Proof. By (3.3), the sequence is split exact as a functor of X. To split it functorially in X_+, we have to consider maps like $(X \amalg Y)_+ \to X_+$. Recall that, as in (1.11), for each X and $i < m$ the transfer maps for $S^i(X) \times X^{m-i} \to S^m(X)$ and the structure map $\pi_X : X \to *$ induce maps $R_{\mathrm{tr}}S^m(X) \to R_{\mathrm{tr}}S^i(X)$.

The alternating sum (over i) of the transfer maps defines a map

$$R_{\mathrm{tr}}\widetilde{S}^m(X_+) = R_{\mathrm{tr}}S^m(X) \to \oplus_{i=1}^m S^i_{\mathrm{tr}}R_{\mathrm{tr}}(X) = S^m_{\mathrm{tr}}R_{\mathrm{tr}}(X_+).$$

We claim that this map is natural in X_+. To see this, it suffices to consider $(X \amalg Y)_+ \to X_+$ arising from π_Y, and show that for $j > 0$ and $a < i$ the composition from the summand $S^i_{\mathrm{tr}}R_{\mathrm{tr}}(X) \otimes S^j_{\mathrm{tr}}R_{\mathrm{tr}}(Y)$ of $R_{\mathrm{tr}}S^m(X \amalg Y)$ to $S^a_{\mathrm{tr}}R_{\mathrm{tr}}(X)$ is zero. This composition factors through $S^a_{\mathrm{tr}}R_{\mathrm{tr}}(X) \otimes S^b_{\mathrm{tr}}R_{\mathrm{tr}}(Y)$ for $b = 0, ..., j$, and the result follows from $\sum_{b=0}^{j}(-1)^b\binom{j}{b} = 0$. □

Since $R_{\mathrm{tr}}S^0(V) = 0$, it follows from 3.4 that for any simplicial object V of Norm$_+$ we have a natural isomorphism

$$R_{\mathrm{tr}}(S^\infty V) \cong \oplus_{i=1}^\infty R_{\mathrm{tr}}(\widetilde{S}^i V) = \oplus_{i=1}^\infty S^i_{\mathrm{tr}}R_{\mathrm{tr}}(V). \tag{3.5}$$

Theorem 3.6. *Let V be a simplicial object of Norm$_+$ such that the simplicial set* $\mathrm{Hom}(X, V)$ *is connected for any X. Then the morphism $S^\infty(V) \to u\,\mathbb{Z}_{tr}(V)$ is a global weak equivalence: $S^\infty(V)(X) \to u\mathbb{Z}_{tr}(V)(X)$ is an equivalence for all X.*

Remark 3.6.1. The theorem remains valid (with the same proof) if the connected hypothesis is replaced by the assumption that the abelian monoids $\pi_0 S^\infty(V)(X)$ are abelian groups for all X.

Proof. Let $\mathbb{N}_{\mathrm{tr}}(V)(X)$ be the simplicial free abelian *monoid* generated in degree p by the elementary correspondences from X to V_p; the group completion of $\mathbb{N}_{\mathrm{tr}}(V)(X)$ is $\mathbb{Z}_{\mathrm{tr}}(V)(X)$. Then [22, 6.8] says that $u\mathbb{N}_{\mathrm{tr}}(V) \cong S^\infty(V)$ is an isomorphism of (simplicial) presheaves. The theorem follows from this because each simplicial H-space $S^\infty(V)(X)$ is connected, and so has a homotopy inverse map. This implies that $S^\infty(V)(X)$ is already group-complete. □

We now apply these general considerations to the objects $\mathbb{L}^n = \mathbb{L}^n_{\mathbb{Z}}$ and $K_n = u\mathbb{L}^n$ representing motivic cohomology; see 2.13. The example $V = S^0$, $S^\infty(V) = \mathbb{N}$ and $K_0 = u\mathbb{Z}_{tr}(V) = u\mathbb{Z}$ shows that we need to restrict to $n > 0$.

Theorem 3.7. *For $n \geq 1$, there are $\mathbb{A}^1$-weak equivalences*

$$R_{tr}(K_n) \simeq S^\infty_{tr}(\mathbb{L}^n_{\mathbb{Z}}) \simeq \bigoplus_{i=1}^{\infty} S^i_{tr}(\mathbb{L}^n_{\mathbb{Z}}), \text{ and } R_{tr}(BK_n) \simeq S^\infty_{tr}(\mathbb{L}^n_{\mathbb{Z}}[1]) \simeq \bigoplus_{i=1}^{\infty} S^i_{tr}(\mathbb{L}^n_{\mathbb{Z}}[1]).$$

Proof. Let $\check{C}(\mathbb{A}^n)$ be the simplicial Čech scheme on $\mathbb{A}^n$ and let V be the cone of $(\mathbb{A}^n - 0) \to \check{C}(\mathbb{A}^n)$, given by $(\mathbb{A}^n - 0) \subset \mathbb{A}^n$. Then V is an object of $\Delta^{\mathrm{op}}\mathrm{Sm}_+$ and each $\mathrm{Hom}(X, V)$ is connected because $\check{C}(\mathbb{A}^n) \to \mathrm{Spec}(k)$ is a weak equivalence. By Theorem 3.6, $S^\infty(V) \to u\mathbb{Z}_{\mathrm{tr}}(V)$ is a global weak equivalence. But $\mathbb{A}^n/(\mathbb{A}^n - 0)$ is globally weak equivalent to cone$((\mathbb{A}^n - 0) \to \mathbb{A}^n)$, which is $\mathbb{A}^1$-weak equivalent to V, so $\mathbb{L}^n$ is $\mathbb{A}^1$-weak equivalent to $\mathbb{Z}_{\mathrm{tr}}(V)$. Thus $K_n = u\mathbb{L}^n$ is $\mathbb{A}^1$-weak equivalent to $uR_{\mathrm{tr}}(V)$ and hence (by 3.6) to $S^\infty(V)$. Now use (3.5) and $S^i_{\mathrm{tr}}R_{\mathrm{tr}}(V) \simeq S^i_{\mathrm{tr}}\mathbb{L}^n$.

Similarly, $BK_n = u(\mathbb{L}^n[1]) \simeq u\mathbb{Z}_{\mathrm{tr}}(\Sigma V)$. In this case, the decomposition arises from (3.5) and $R_{\mathrm{tr}}(BK_n) \simeq R_{\mathrm{tr}}S^\infty(\Sigma V)$. □

As mentioned in Remark 2.14, cohomology operations on $H^{2n,n}(-,\mathbb{Z})$ correspond to elements of $\widetilde{H}^{*,*}(K_n, R)$. The canonical element in $\widetilde{H}^{2n,n}(K_n, R)$ corresponds to the change of coefficients $H^{2n,n}(X,\mathbb{Z}) \to H^{2n,n}(X, R)$.

Example 3.8. The i-th power operation $x \mapsto x^i$ from $H^{2n,n}(X,\mathbb{Z})$ to $H^{2ni,ni}(X,\mathbb{Z})$ (and thence to $H^{2ni,ni}(X,R)$) has a natural interpretation in terms of the decomposition in 3.7. In particular, if $i < \ell$ then $S^i_{\mathrm{tr}}(\mathbb{L}^n) \cong \mathbb{L}^{ni}$ by 1.10 and [11, 15.8], and the power map corresponds to the generator of $\mathrm{Hom}(S^i_{\mathrm{tr}}(\mathbb{L}^n), \mathbb{L}^{ni}) = \mathbb{Z}$.

Consider the diagonal $V \to V^i \to S^i(V)$ for $V = \check{C}(\mathbb{A}^n)/(\mathbb{A}^n - 0)$. Applying $\mathbb{Z}_{\mathrm{tr}}$ yields the motivic power map $\mathbb{L}^n \to \mathbb{L}^{ni}$ (see [11, 3.11]), and the symmetrizing map $\mathbb{L}^{ni} \to S^i_{\mathrm{tr}}(\mathbb{L}^n)$, up to the $\mathbb{A}^1$-weak equivalences $\mathbb{L}^{ni}_{\mathbb{Z}} \simeq \mathbb{Z}_{\mathrm{tr}}(V^i)$. Applying u to $\mathbb{L}^n \to \mathbb{L}^{ni}$ yields the map $K_n \to K_{ni}$ representing the power operation. By Theorem 3.7, the adjoint map $\mathbb{Z}_{\mathrm{tr}}(K_n) \to \mathbb{L}^{ni}$ may be identified with the map $S^\infty_{\mathrm{tr}}(\mathbb{L}^n) \to S^i_{\mathrm{tr}}(\mathbb{L}^n) \to \mathbb{L}^{ni}$.

Corollary 3.9. *When $n \geq 1$, cohomology operations $H^{2n,n}(-,\mathbb{Z}) \xrightarrow{\phi} H^{p,q}(-,R)$ are in 1–1 correspondence with elements of*

$$H^{p,q}(K_n, R) \cong \mathrm{Hom}_{\mathbf{DM}}(\oplus_1^\infty S^i_{tr}(\mathbb{L}^n), R(q)[p]) = \prod_{i=1}^\infty \mathrm{Hom}_{\mathbf{DM}}(S^i_{tr}(\mathbb{L}^n), R(q)[p]).$$

$$H^{p,q}(BK_n, R) \cong \mathrm{Hom}_{\mathbf{DM}}(\oplus_1^\infty S^i_{tr}(\mathbb{L}^n[1]), R(q)[p]) = \prod_{i=1}^\infty \mathrm{Hom}_{\mathbf{DM}}(S^i_{tr}(\mathbb{L}^n[1]), R(q)[p]).$$

Example 3.9.1. We will see in Theorem 4.6 and 4.10.1 that $(\mathbb{Z}/\ell)_{\mathrm{tr}}(K_n)$ is a sum of Tate motives, and that the only summands of weight $< n+\ell$ are $\mathbb{L}^{n+\ell-1}$, $\mathbb{L}^{n+\ell-1}[1]$ and the $S^i_{\mathrm{tr}}\mathbb{L}^n = \mathbb{L}^{ni}$ for $i < 1 + \ell/n$. It is easy to see from 3.8 and [26] that these correspond to the usual cohomology operations sending x to $P^1(\bar{x})$, $\beta P^1(\bar{x})$ and $\bar{x}^i$, where β is the Bockstein and P^1 is the first Steenrod operation (see [2, 20, 26]).

Corollary 3.10. *Cohomology operations $\phi : H^{2,1}(-,\mathbb{Z}) \to H^{p,q}(-,R)$ are in 1–1 correspondence with homogeneous polynomials $f(t) = \sum_{i>0} a_i t^i$ of bidegree (p,q) in $H^{*,*}(k,R)[t]$, where t has bidegree $(2,1)$. The operation corresponding to f is $\phi(x) = \sum a_i \bar{x}^i$.*

Proof. It is well known that $R_{\mathrm{tr}}(\mathbb{P}^1) \cong R \oplus \mathbb{L}^1$ and $S^i\mathbb{P}^1 \cong \mathbb{P}^i$. Using Lemma 3.2, we obtain $S^i_{\mathrm{tr}}(\mathbb{L}^1) \cong \mathbb{L}^i$. By 3.9, the classifying space $K_1 = u\mathbb{L}^1$ has cohomology:

$$H^{p,q}(K_1, R) = \prod_{i=1}^\infty \mathrm{Hom}_{\mathbf{DM}}(\mathbb{L}^i, R(q)[p]) = \prod_{i=1}^\infty H^{p-2i,q-i}(k,R).$$

In fact, $K_1 \simeq \mathbb{P}^\infty$ by [12, 3.8]. The need for $a_0 = 0$ follows from $\phi(0) = 0$. □

Remark. If X is a smooth variety, $H^{2n,n}(X,\mathbb{Z})$ is the Chow group $CH^n(X)$. Since $H^{2i,i}(k,\mathbb{Z}) = 0$ for $i > 0$, it follows that the power operations $x \to x^m$ are the only natural maps $CH^1(X) \to CH^m(X)$ coming from cohomology operations. In contrast, note that the cohomology operation $P^1 : CH^2(X) \to CH^{\ell+1}(X)/\ell$ of [26] is nontrivial; if $x, y \in CH^1(X)$ then $P^1(xy) = xy^\ell + x^\ell y$ by the Cartan formula.

4. Symmetric Powers of Tate motives

This section is devoted to an exposé of those results in Voevodsky's paper [30] that we shall need in order to establish the Künneth formula 4.13 for $H^{*,*}(K_n\wedge\cdots\wedge K_n,R)$. These are: a description (in 4.1) of S^m_{tr} in terms of S^ℓ_{tr}; a description (in 4.6) of $S^\ell_{\mathrm{tr}}(\mathbb{L}^n)$, and the structure theorem 4.10 that each $S^m_{\mathrm{tr}}(\mathbb{L}^n)$ is a sum of proper Tate motives (in the sense of 4.4 below).

Let G be any subgroup of the symmetric group Σ_m, and H any subgroup of Σ_n. For simplicial normal X, we set $S^G(X)=X^m/G$ and $S^G_{\mathrm{tr}}R_{\mathrm{tr}}(X_+)=R_{\mathrm{tr}}(S^G(X_+))$. The usual embedding $\Sigma_m\times\Sigma_n\subset\Sigma_{m+n}$ induces

$$S^{G\times H}(X_+)=S^G(X_+)\times S^H(X_+)\quad\text{and}\quad S^{G\times H}_{\mathrm{tr}}(M)=S^G_{\mathrm{tr}}(M)\otimes S^H_{\mathrm{tr}}(M),$$

where $M=R_{\mathrm{tr}}(X_+)$. Similarly, the wreath product

$$G\wr H=G^n\rtimes H\subset\Sigma_{mn}$$

acts on $\{1,\dots,mn\}$ by decomposing the set into n blocks of m elements, with H permuting the blocks and G acting inside the blocks. It is easy to see that

$$S^H(S^G(X_+))=S^{G\wr H}(X_+).$$

Using the ℓ-adic expansion $m=m_0+m_1\ell+\cdots+m_r\ell^r$ with $0\le m_i<\ell$, it is well known and easy to verify that the group

$$G=\Sigma_{m_0}\times(\Sigma_\ell\wr\Sigma_{m_1})\times(\Sigma_\ell\wr\Sigma_\ell)\wr\Sigma_{m_2}\times\cdots\times(\Sigma_\ell^{\wr r})\wr\Sigma_{m_r}\tag{4.0a}$$

contains a Sylow ℓ-subgroup of Σ_m. By the above remarks, if $M=R_{\mathrm{tr}}(V)$ then $S^G_{\mathrm{tr}}(M)$ equals:

$$S^{m_0}_{\mathrm{tr}}(M)\otimes S^{m_1}_{\mathrm{tr}}(S^\ell_{\mathrm{tr}}(M))\otimes S^{m_2}_{\mathrm{tr}}(S^\ell_{\mathrm{tr}}(S^\ell_{\mathrm{tr}}(M)))\otimes\cdots\otimes S^{m_r}_{\mathrm{tr}}((S^\ell_{\mathrm{tr}})^r(M)).\tag{4.0b}$$

Proposition 4.1. *If R is $\mathbb{Z}_{(\ell)}$ or $\mathbb{Z}/\ell$, and $M=R_{tr}(V)$ for a simplicial based V, then $S^m_{tr}(M)$ is a direct summand of the $S^G_{tr}(M)$ displayed in* (4.0b).

Proof. (Voevodsky) The display is just $S^G_{\mathrm{tr}}(M)$, and $S^G(V)=V^m/G$ is a ramified covering of $S^m(V)=V^m/\Sigma_m$ of degree $[\Sigma_m:G]$ prime to ℓ. Its inverse is a finite correspondence, and the composition is multiplication by $[\Sigma_m:G]$ on $S^m_{\mathrm{tr}}(V)$. □

Corollary 4.2. *For $r\ge1$, $S^{\ell^{r+1}}_{tr}(M)$ is a direct summand of $S^\ell_{tr}(S^{\ell^r}_{tr}(M))$. If $0<a<\ell$, $S^{a\ell^r}_{tr}(M)$ is a direct summand of $S^a_{tr}(S^{\ell^r}_{tr}(M))$.*

Proof. Set $m=\ell^r$ and $G=\Sigma_\ell^{\wr r}$ as in (4.0a). We have inclusions $G\wr\Sigma_\ell\subset\Sigma_m\wr\Sigma_\ell\subset\Sigma_{\ell m}$. By (4.0a), $G\wr\Sigma_\ell$ contains a Sylow ℓ-subgroup of $\Sigma_{\ell m}$. Hence $[\Sigma_{\ell m}:\Sigma_m\wr\Sigma_\ell]$ is prime to ℓ, and $S^{\ell m}(M)$ is a direct summand of $S^{\Sigma_m\wr\Sigma_\ell}(M)=S^\ell_{\mathrm{tr}}(S^m_{\mathrm{tr}}(M))$.

The second assertion is similar; we have inclusions $G\wr\Sigma_a\subset\Sigma_m\wr\Sigma_a\subset\Sigma_{am}$, and $G\wr\Sigma_a$ contains a Sylow ℓ-subgroup of Σ_{am}. □

4.3. Let C be the cyclic group $\mathbb{Z}/\ell$, set $A=\mathrm{Aut}(C)=(\mathbb{Z}/\ell)^\times$ and identify $G=C\rtimes A$ with the affine subgroup of Σ_ℓ. Since $[\Sigma_\ell:G]=(\ell-2)!$, standard transfer arguments show that $S^\ell_{\mathrm{tr}}(\mathbb{L}^n)$ is a summand of $S^G_{\mathrm{tr}}(\mathbb{L}^n)$, which in turn is $(S^C_{\mathrm{tr}}(\mathbb{L}^n))^A$.

We briefly recall the computation of $H^{*,*}(B_{\mathrm{gm}}\mu_\ell)$ and $H^{*,*}(B_{\mathrm{gm}}\Sigma_\ell)$ in [26, §6]. Suppose that k has ℓ-th roots of unity, so that we may identify C with the algebraic group μ_ℓ. In [26, 6.10], Voevodsky showed that (for odd ℓ):

$$H^{*,*}(B_{\mathrm{gm}}\mu_\ell,R)\cong H^{*,*}[[u,v]]/(u^2),\quad v=\beta(u),$$

where $H^{*,*}$ denotes $H^{*,*}(k,R)$ and u and v have bidegrees $(1,1)$ and $(2,1)$, respectively. The group A acts by algebra maps, and $a\in A$ satisfies: $a\cdot u = au$, $a\cdot v = av$; see [26, 6.11]. Thus for $c = uv^{\ell-2}$ and $d = -v^{\ell-1}$ we have:

$$H^{*,*}(B_{\mathrm{gm}}G,R)\cong H^{*,*}(B_{\mathrm{gm}}\mu_\ell,R)^A\cong H^{*,*}[[c,d]]/(c^2),\quad d=\beta(c).$$

By [26, 6.13-6.14], c and d lift to $H^{*,*}(B_{\mathrm{gm}}\Sigma,R)$. This implies that the canonical map $H^{*,*}(B_{\mathrm{gm}}\Sigma,R)\to H^{*,*}(B_{\mathrm{gm}}G,R) = H^{*,*}(B_{\mathrm{gm}}C,R)^A$ is an isomorphism.

Definition 4.4. By a *proper Tate motive* we mean a direct sum of motives of the form $\mathbb{L}^a[b]$ with $a\geq 0$ and $b\geq 0$.

When R is a field, the full subcategory of proper Tate motives is idempotent complete. (This is easy to see, and pointed out in [30, 3.62].)

The following two theorems should be compared to the formulas announced in the 1996 preprint [25, 3.4.4–3.4.6]. They say that $S^{C\rtimes A}_{\mathrm{tr}}(\mathbb{L}^n)$ and $S^{\ell}_{\mathrm{tr}}(\mathbb{L}^n)$ are proper Tate motives over $R=\mathbb{Z}/\ell$.

Theorem 4.5. *When $R=\mathbb{Z}/\ell$, $G=C\rtimes A$ and $n>0$, $S^G_{tr}(\mathbb{L}^n)$ is $\mathbb{A}^1$-equivalent to*

$$\mathbb{L}^{n\ell}\oplus\bigoplus\nolimits_{i=1}^{n-1}\big\{\mathbb{L}^{n+i(\ell-1)}\oplus\mathbb{L}^{n+i(\ell-1)}[1]\big\}.$$

Proof. Voevodsky proves this in [30, 4.47 and (41)]; we sketch his proof here. (A different proof is given by Nie in [14].) Without loss of generality, k has ℓ-th roots of unity. Voevodsky first proves in [30, 3.41] that $S^C_{\mathrm{tr}}(\mathbb{L}^n)\cong\mathbb{L}^n\otimes R_{\mathrm{tr}}(V-0)/C[1]$, where V is the direct sum of n copies of the reduced regular representation of C.

The map $(V-0)/C\to B_{\mathrm{gm}}\mu_\ell$ of [12, §4.2] induces a map from $H^{*,*}(B_{\mathrm{gm}}\mu_\ell,R)$ to $H^{*,*}((V-0)/C,R)$; by [26, 6.1], it is an isomorphism in cohomology in weight up to $n(\ell-1)$. Voevodsky proves in [30, 3.43] that $R_{\mathrm{tr}}((V-0)/C)$ is a sum of Tate motives with u and $v=\beta(u)$ corresponding to $\mathbb{L}[-1]$ and $\mathbb{L}$, respectively, so that:

$$H^{*,*}((V-0)/C,R) = H^{*,*}[u,v]/(u^2,v^{n(\ell-1)}).$$

Since $a\in A=(\mathbb{Z}/\ell)^\times$ acts by $u\mapsto au$, $v\mapsto av$, it follows that $H^{*,*}((V-0)/G,R) = H^{*,*}((V-0)/C,R)^A$ equals $H^{*,*}[c,d]/(c^2,d^n)$, with $c=uv^{\ell-2}$ and $d=v^{\ell-1}$. Translating this into motivic language proves that $R_{\mathrm{tr}}(V-0/G)$ is the sum of the $\mathbb{L}^{i(\ell-1)}[-1]$ and $\mathbb{L}^{i(\ell-1)}$, and the theorem follows. □

By 3.10, $S^{\ell}_{\mathrm{tr}}(\mathbb{L}^1)\cong\mathbb{L}^\ell$. This is the case $n=1$ of the following calculation.

Theorem 4.6. *When $R=\mathbb{Z}/\ell$ and $n>0$, the natural map $S^G_{tr}(\mathbb{L}^n)\to S^{\ell}_{tr}(\mathbb{L}^n)$ is an isomorphism. Thus $S^{\ell}_{tr}(\mathbb{L}^n)$ is given by the display in Theorem 4.5.*

Proof. By the usual transfer argument, $S^{\ell}_{\mathrm{tr}}(\mathbb{L}^n)$ is a direct summand of $S^G_{\mathrm{tr}}(\mathbb{L}^n)$. Since the category of proper Tate motives is idempotent complete, it suffices to show that there is an isomorphism in motivic cohomology.

The classifying map $(V-0)/\Sigma_\ell\to B_{\mathrm{gm}}\Sigma_\ell$ induces a map from $H^{*,*}(B\Sigma_\ell)$ to the cohomology of $(V-0)/\Sigma_\ell$, fitting into the commutative diagram:.

$$\begin{tikzcd}
H^{*,*}(B_{\mathrm{gm}}\Sigma_\ell,R) \arrow[r] \arrow[d, "\cong"'] & H^{*,*}((V-0)/\Sigma_\ell,R) \arrow[r] \arrow[d, "\text{into}"] & \mathrm{Hom}(S^{\ell}_{\mathrm{tr}}(\mathbb{L}^n),R(*)[*]\otimes\mathbb{L}^n[-1]) \arrow[d, "\text{into}"] \\
H^{*,*}(B_{\mathrm{gm}}G,R) \arrow[r, "\text{onto}"] & H^{*,*}((V-0)/G,R) \arrow[r, "\cong"] & \mathrm{Hom}(S^{G}_{\mathrm{tr}}(\mathbb{L}^n),R(*)[*]\otimes\mathbb{L}^n[-1]).
\end{tikzcd}$$

We saw in 4.3 that $H^{*,*}(B_{\mathrm{gm}}G,R)\cong H^{*,*}[[c,d]]/(c^2)$ and that the left vertical map is an isomorphism We saw in the proof of Theorem 4.5 that the bottom map is a surjection. Since the right vertical maps are injections by a transfer argument, they are isomorphisms by a diagram chase. ☐

Remark 4.6.1. A slightly different proof of 4.6 is given in [30, 3.48], by checking that the composite $S^G_{\mathrm{tr}}(\mathbb{L}^n) \to S^{\Sigma_\ell}_{\mathrm{tr}}(\mathbb{L}^n) \to S^G_{\mathrm{tr}}(\mathbb{L}^n)$ is also an isomorphism.

The following result is proven by Voevodsky in [30, 3.34], using the filtration of S^m_{tr} for the cone of a map. Part (a) is immediate from $S^i(R_{\mathrm{tr}}(X)) = e \cdot R_{\mathrm{tr}}(X^i)$; we omit the proof of (b).

Proposition 4.7. *Let T be a motive for which the switch involution τ on $T \otimes T$ is ± 1. Then (a) for $i < \ell$ the projection $\pi : T^{\otimes i} \to S^i_{tr}(T)$ of 1.10 is an isomorphism for $\tau = +1$ and zero for $\tau = -1$;*
(b) for $\ell > 2$ there is an distinguished triangle:

$$T^{\otimes \ell}[1] \xrightarrow{\delta[1]} S^\ell_{tr}(T)[1] \to S^\ell_{tr}(T[1]) \to T^{\otimes \ell}[2], \quad \tau = +1,$$

$$T^{\otimes \ell}[\ell-1] \xrightarrow{\delta[1]} S^\ell_{tr}(T)[1] \to S^\ell_{tr}(T[1]) \to T^{\otimes \ell}[\ell], \quad \tau = -1.$$

Now there is a topological realization functor, sending $\mathbb{A}^n/\mathbb{A}^n - 0$ to the based sphere S^{2n}, $\mathbb{L}^a[b]$ to $\tilde{C}_*(S^{2a+b}, R)$ and $S^m_{\mathrm{tr}}(\mathbb{L}^a[b])$ to $\tilde{C}_*(\widetilde{S}^m(S^{2a+b}), R)$. (See [12, p. 120].) Thus we can refer to topology to calculate the maps in 4.7.

Example 4.7.1. When $T = R = \mathbb{Z}/\ell$, the map $R \xrightarrow{\delta} R$ in 4.7 is an isomorphism and we get $S^\ell_{\mathrm{tr}}(R[1]) = 0$, and hence $S^\ell_{\mathrm{tr}}(R[2]) \cong R[2\ell]$. These reflect the cohomology of the topological spaces $S^1 = K(\mathbb{Z}, 1)$ and $BS^1 = K(\mathbb{Z}, 2)$.

Example 4.7.2. For $T = R[b]$ and $b \geq 1$, we can compare with the cohomology of the Eilenberg-Mac Lane spaces $K(\mathbb{Z}, b)$. Recall that by Dold-Thom we have a homotopy equivalence $K(\mathbb{Z}, b) \simeq S^\infty(S^b)$, so we have $H^*(K(\mathbb{Z}, b), R) \cong \oplus \widetilde{H}^*(\widetilde{S}^i(S^b), R)$ by (3.5). The topological realization of 4.7 yields triangles in $D^-(R)$:

$$\xrightarrow{\delta_{\mathrm{top}}} \tilde{C}_*(\widetilde{S}^\ell(S^b), R)[1] \to \tilde{C}_*(\widetilde{S}^\ell(S^{b+1}), R) \longrightarrow \begin{cases} \tilde{C}_*(S^{b\ell+2}, R) \\ \tilde{C}_*(S^{b\ell+\ell}, R) \end{cases} \xrightarrow{\delta_{\mathrm{top}}} .$$

Now take $R = \mathbb{Z}/\ell$. On cohomology, the map $\widetilde{H}^{*+1}(\widetilde{S}^\ell(S^b)) \to \widetilde{H}^*(\widetilde{S}^\ell(S^{b+1}))$ is an injection, because it is a summand of the suspension map $\widetilde{H}^{*+1}(K(\mathbb{Z}, b)) \to \widetilde{H}^*(K(\mathbb{Z}, b+1))$, which Cartan showed was an injection in [2]. It follows that the map δ_{top} is zero on cohomology, and hence $\delta_{\mathrm{top}} = 0$ in the semisimple triangulated category $D^-(\mathbb{Z}/\ell)$. Thus the triangles split and we have the formula in $D^-(\mathbb{Z}/\ell)$:

$$\tilde{C}_*(\widetilde{S}^\ell(S^b), R) \cong \begin{cases} \bigoplus_{i=1}^{(b-1)/2} R[b + 2i(\ell-1)] \otimes (R \oplus R[1]), & b \text{ odd}, \\ R[b\ell] \oplus \bigoplus_{i=1}^{b/2-1} R[b + 2i(\ell-1)] \otimes (R \oplus R[1]), & b \text{ even}. \end{cases}$$

Note that the singular cohomology $\widetilde{H}^*(\widetilde{S}^\ell(S^b), R)$ can be read off from this formula, and we get exactly the result obtained in 1957 by Nakaoka in [13]; cf. [9, 4.2].

Corollary 4.8. *When $R = \mathbb{Z}/\ell$ and $b \geq 1$, we have*

$$S^\ell_{tr}(R[b]) \cong \begin{cases} \bigoplus_{i=1}^{(b-1)/2} R[b + 2i(\ell-1)] \otimes (R \oplus R[1]), & b \text{ odd}, \\ R[b\ell] \oplus \bigoplus_{i=1}^{b/2-1} R[b + 2i(\ell-1)] \otimes (R \oplus R[1]), & b \text{ even}. \end{cases}$$

Thus the canonical map $S^\ell_{tr}(R[b])[1] \to S^\ell_{tr}(R[b+1])$ is a split injection for all $b > 0$, and the zero map for $b = 0$.

Proof. We proceed by induction on b, the cases $b = 1, 2$ being covered in Example 4.7.1. Suppose that the formula holds for b and consider the map δ in 4.7. If b is odd, $\delta = 0$ because $\mathrm{Hom}(R[b\ell], S^\ell_{\mathrm{tr}}(R[b]))$ is zero, being a sum of terms $\mathrm{Hom}(R[b\ell], R[c])$ with $c < b\ell$. If b is even, the same considerations show that $\mathrm{Hom}(R[b\ell], S^\ell_{\mathrm{tr}}(R[b])) = \mathrm{Hom}(R[b\ell], R[b\ell]) = \mathbb{Z}/\ell$. In this case, we see that the topological realization functor is an isomorphism on

$$\mathrm{Hom}(R[b\ell], S^\ell_{\mathrm{tr}}(R[b])) \xrightarrow{\cong} \mathrm{Hom}_{D^-(\mathbb{Z}/\ell)}(\tilde{C}_*(S^{b\ell}), \tilde{C}_*(\widetilde{S}^\ell(S^b))) \cong \mathbb{Z}/\ell.$$

(The last isomorphism is from 4.7.2.) Since δ maps to $\delta_{\mathrm{top}} = 0$, we have $\delta = 0$. Thus the triangle in 4.7 splits, and we obtain the inductive result for $S^\ell_{\mathrm{tr}}(R[b+1])$. ☐

It is instructive to associate the terms in $S^\ell_{\mathrm{tr}}(R[b])$ with the topological Steenrod operations P^i and βP^i ($i < b/2$) of [2] and [20]. Nakaoka proved in [13] that the $\{P^i(u), \beta P^i(u)\}$ form a basis of $\widetilde{H}^*(S^\ell(S^n), \mathbb{Z}/\ell)$.

Corollary 4.9. *When $R = \mathbb{Z}/\ell$ and $a > 0$, $S^\ell_{tr}(\mathbb{L}^a[b])[1] \to S^\ell_{tr}(\mathbb{L}^a[b+1])$ is a split injection for all b, and we have:*

$$S^\ell_{tr}(\mathbb{L}^a[1]) = \bigoplus\nolimits_{i=1}^{a} \{\mathbb{L}^{a+i(\ell-1)}[1] \oplus \mathbb{L}^{a+i(\ell-1)}[2]\};$$

$$S^\ell_{tr}(\mathbb{L}^a[b]) = S^\ell_{tr}(\mathbb{L}^a[1])[b-1] \oplus \bigoplus\nolimits_{i=1}^{k} \{\mathbb{L}^{a\ell}[2i\ell+1] \oplus \mathbb{L}^{a\ell}[2i\ell+2]\}, \quad b = 2k+1;$$

$$S^\ell_{tr}(\mathbb{L}^a[b]) = S^\ell_{tr}(\mathbb{L}^a[b-1])[1] \oplus \mathbb{L}^{a\ell}[b\ell], \quad b \geq 2 \textit{ even.}$$

Proof. Set $T = \mathbb{L}^a[b]$ for $a > 0$. We will assume the result is true for T and prove that it is true for $T[1]$ using the triangles in 4.7(b). When b is odd, any map $T^{\otimes \ell}[\ell - 1] \to S^\ell_{\mathrm{tr}}(T)[1]$ is zero for weight reasons, and hence the second triangle in 4.7 splits. This reduces us to the case of even $b \geq 0$.

When b is even, we claim that the map δ is zero in the first sequence of 4.7(b), so the sequence splits. The initial case $S^\ell_{\mathrm{tr}}(\mathbb{L}^a[1])$ as well as the inductive case $S^\ell_{\mathrm{tr}}(\mathbb{L}^a[b+1])$ will then follow from Theorem 4.5. Using the topological realization functor with $n = 2a + b$, this claim follows from the fact that δ maps to $\delta_{\mathrm{top}} = 0$ under

$$\mathrm{Hom}(\mathbb{L}^{a\ell}[b\ell], S^\ell_{\mathrm{tr}}(\mathbb{L}^a[b])) \xrightarrow{\cong} \mathrm{Hom}_{D^-(\mathbb{Z}/\ell)}(\tilde{C}_*(S^{n\ell}), \tilde{C}_*(\widetilde{S}^\ell(S^n))) \cong \mathbb{Z}/\ell.$$

As in the proof of 4.8, the triangle splits, yielding the formulas by induction. ☐

Here is the main theorem 3.70 of [30], which was originally stated in [25, 3.4.1].

Theorem 4.10. *When $R = \mathbb{Z}/\ell$, $S^\infty_{tr}(\mathbb{L}^n) = R_{tr}(K_n)$ is a proper Tate motive. There are only finitely many summands $\mathbb{L}^a[b_i]$ of any fixed weight a.*

Proof. Combine 3.7, 4.1 and 4.9 with idempotent completion. ☐

Remark 4.10.1. It also follows from the formulas in 4.9 that the terms of smallest weight in $S^\infty_{\mathrm{tr}}(\mathbb{L}^a)$ are the $\mathbb{L}^{ai}$, $\mathbb{L}^{a+(\ell-1)}$ and $\mathbb{L}^{a+(\ell-1)}[1]$.

Theorem 4.11. *(Pure Künneth formula) Suppose that X and Y are pointed simplicial normal schemes such that $R_{tr}(Y)$ is a direct sum of motives $R(q_\alpha)[p_\alpha]$, and that for each q there are only finitely many α with $q_\alpha = q$. Then the Künneth homomorphism is an isomorphism:*

$$H^{*,*}(X, R) \otimes_{H^{*,*}(k,R)} H^{*,*}(Y, R) \to H^{*,*}(X \times Y, R).$$

It induces $\widetilde{H}^{*,*}(X, R) \otimes \widetilde{H}^{*,*}(Y, R) \cong \widetilde{H}^{*,*}(X \wedge Y, R)$.

Proof. By assumption, we have $R_{\mathrm{tr}}(X \times Y) \cong \oplus R_{\mathrm{tr}}(X)(q_\alpha)[p_\alpha]$ and hence

$$H^{n,i}(X \times Y, R) \cong \prod \mathrm{Hom}_{\mathbf{DM}}(R_{\mathrm{tr}}(X)(q_\alpha)[p_\alpha], R(i)[n]).$$

The terms with $q_\alpha > i$ are summands of $H^{*,0}(X \times \mathbb{A}^{q_\alpha - i}/(\mathbb{A}^{q_\alpha - i} - 0), R)$ by the Cancellation Theorem [11, 16.25], and they vanish by [26, 3.5]. This leaves the finitely many terms with $q_\alpha \leq i$ which, by Cancellation, are:

$$H^{n-p_\alpha, i-q_\alpha}(X, R).$$

The case $X = \mathrm{Spec}(k)$ shows that $H^{*,*}(Y, R)$ is a free graded $H^{*,*}(k, R)$-module with generators in bidegrees (p_α, q_α), and the result follows. ☐

Corollary 4.12. *If $1/m! \in R$, then $H^{*,*}(S^m(Y), R) \cong Sym^m H^{*,*}(Y, R)$ for every Y such that $R_{tr}(Y)$ is a sum of Tate motives.*

Proof. By 4.11, the Künneth map $H^{*,*}(Y, R) \otimes \cdots \otimes H^{*,*}(Y, R) \xrightarrow{\sim} H^{*,*}(Y^m, R)$ is an isomorphism of free $H^{*,*}(k, R)$-modules. The symmetric group acts on both sides, and the Künneth map is equivariant, so the symmetric parts are equal. The symmetric part of $H^{\otimes m}$ is $\mathrm{Sym}^m(H)$, whence the result. ☐

The following result replaces the unproven “Lemma 2.3” in [MC/l].

Proposition 4.13. *The Künneth homomorphisms for $R = \mathbb{Z}/\ell$ and $p > 0$,*

$$H^{*,*}(K_n, \mathbb{Z}/\ell) \otimes_{H^{*,*}(k)} \cdots \otimes_{H^{*,*}(k)} H^{*,*}(K_n, \mathbb{Z}/\ell) \to H^{*,*}(K_n^p, \mathbb{Z}/\ell),$$

are isomorphisms. Hence $\widetilde{H}^{*,*}(K_n, \mathbb{Z}/\ell) \otimes \cdots \otimes \widetilde{H}^{*,*}(K_n, \mathbb{Z}/\ell) \cong \widetilde{H}^{*,*}(K_n^{\wedge p}, \mathbb{Z}/\ell)$.

Proof. By Theorem 4.10, $R_{\mathrm{tr}}(K_n)$ is a proper Tate motive, so 4.11 applies. ☐

Remark. This proposition was originally announced (in 1996) for K_n/ℓ in [25, 3.15].

5. Scalar weight

The goal of this section is to prove Theorem 5.2, which is our replacement for Lemma 2.2 of [MC/l]. It is based upon the notion of the *scalar weight* of an integral-to-modular cohomology operation.

The monoid $(\mathbb{Z}, \times)$ acts (via multiplication by scalars) on any abelian group and more generally on any presheaf F of simplicial abelian groups (such as $\mathbb{Z}(q)[p]$) and hence on its underlying presheaf of simplicial sets uF. In particular, this monoid acts on $K_n = u\mathbb{L}^n$ and $BK_n = u\mathbb{L}^n[1]$. The induced action on motivic cohomology groups such as $H^{2n,n}(X, \mathbb{Z}) = [X_+, K_n]$ is just multiplication. We will be interested in the action on the cohomology operations represented by the groups $H^{p,q}(K_n, R)$ and $H^{p,q}(BK_n, R)$. Note that if $\alpha \in H^{2n,n}(X, \mathbb{Z})$ then $(\phi \cdot m)(\alpha) = \phi(m\alpha)$.

Definition 5.1. An element $\phi \in H^{*,*}(K_n, \mathbb{Z}/\ell)$ has *scalar weight* s $(0 \leq s < \ell - 1)$ if $\phi \cdot m = m^s \phi$ for all integers m. Such a ϕ has $\phi \cdot \ell = 0$, and $\phi \cdot m = \phi \cdot m'$ if $m \equiv m' \pmod{\ell}$. Hence $\mathbb{Z}/\ell^\times$ acts on the R-submodule of elements ϕ which have scalar weight s; if $\alpha \in H^{2n,n}(X, \mathbb{Z})$ then $(\phi \cdot m)(\alpha) = m^s \phi(\alpha)$.

Example 5.1.1. A cohomology operation $\phi : H^{0,0}(-\mathbb{Z}) \to H^{p,q}(-, R)$ is represented by a sequence (ϕ_j) in R; see Example 0.1. The cohomology operation $\phi \cdot m$ is represented by the sequence (ϕ_{mj}). Only the finitely many ϕ of the form $\phi_m = am^s$ have scalar weight.

Example 5.1.2. Any cohomology operation $H^{2,1}(-\mathbb{Z}) \to H^{p,q}(-,R)$ is represented by a homogeneous polynomial $\sum a_i t^i$ of bidegree (p,q) by Corollary 3.10. The operation $\phi(x) = ax^i$, corresponding to the monomial at^i, has scalar weight i modulo $(\ell-1)$: $(\phi\cdot m)(x) = \phi(mx) = a(mx)^i = m^i\phi(x)$.

Example 5.1.3. Any stable cohomology operation, such as P^I or β, is additive by [26, 2.9], so it has scalar weight one. If ϕ_i has scalar weight s_i then the monomial $x \mapsto \phi_1(x)\cdots\phi_m(x)$ has scalar weight $\sum s_i$ modulo $(\ell-1)$.

Example 5.1.4. The cohomology operation $\phi_V : H^{2a+1,b}(-,R) \to H^{2a\ell+1,b\ell}(-,R)$ constructed in [MC/l, §3] has scalar weight one, because it satisfies $(\phi\cdot m) = m^\ell\phi$ by [MC/l, 3.5]. (See Corollary 8.2 below.)

Recall from 3.9 that cohomology operations on $H^{2n,n}(-,\mathbb{Z})$ are sums of operations corresponding to elements of $\mathrm{Hom}(S^m_{\mathrm{tr}}(\mathbb{L}^n), \mathbb{Z}/\ell(*)[*])$, arising from the decomposition $R_{\mathrm{tr}}(K_n) \cong \oplus S^m_{\mathrm{tr}}(\mathbb{L}^n)$ in (3.5). Steenrod refers to these as elements of *rank* m in [19].

Theorem 5.2. *The cohomology operations* $H^{2n,n}(-,\mathbb{Z}) \to H^{*,*}(-,\mathbb{Z}/\ell)$ *corresponding to elements of* $\mathrm{Hom}(S^m_{tr}(\mathbb{L}^n), \mathbb{Z}/\ell(*)[*])$ *have scalar weight* $m \mod (\ell-1)$.

The proof of Theorem 5.2 will occupy the rest of this section. We first dispose of the cases $m \le \ell$.

Example 5.2.1. The cohomology operation on $H^{2n,n}(-,\mathbb{Z})$ corresponding to

$$a \in H^{p-2n,q-n}(k,R) = \mathrm{Hom}(S^1_{\mathrm{tr}}(\mathbb{L}^n), R(q)[p])$$

is $x \mapsto ax$. This is additive, and so has scalar weight one. Similarly, if $1 < s < \ell$ then it follows from Example 3.8 and 4.7(a) that the cohomology operation corresponding to $a \in H^{p,q}(k,R) = \mathrm{Hom}(S^s_{\mathrm{tr}}(\mathbb{L}^n), R(q+ns)[p+2ns])$ is $x \mapsto ax^s$; these have scalar weight s.

Proposition 5.3. *The cohomology operations* $\phi : H^{2n,n}(-,\mathbb{Z}) \to H^{p,q}(-,\mathbb{Z}/\ell)$ *corresponding to elements of* $\mathrm{Hom}(S^\ell_{tr}(\mathbb{L}^n), \mathbb{Z}/\ell(q)[p])$ *are additive. Hence they have scalar weight one.*

Proof. Set $T = \mathbb{Z}/\ell(q)[p]$. By 4.9, the map $S^\ell_{\mathrm{tr}}(\mathbb{L}^n)[1] \to S^\ell_{\mathrm{tr}}(\mathbb{L}^n[1])$ is a split injection, so $\mathrm{Hom}(S^\ell_{\mathrm{tr}}(\mathbb{L}^n[1]), T[1]) \to \mathrm{Hom}(S^\ell_{\mathrm{tr}}(\mathbb{L}^n), T)$ is onto. Hence ϕ lifts to a cohomology operation $\phi_1 : H^{2n+1,n}(-,\mathbb{Z}) \to H^{p+1,q}(-,\mathbb{Z}/\ell)$, in the sense that the suspension $\Sigma\phi(x)$ is $\phi_1(\Sigma x)$. If $x, y \in H^{2n,n}(X,\mathbb{Z})$ then by [26, 2.9], the cohomology operation ϕ_1 is additive on $H^{2n+1,n}(\Sigma X,\mathbb{Z})$, so:

$$\Sigma\phi(x+y) = \phi_1(\Sigma(x+y) = \phi_1(\Sigma x + \Sigma y)) = \phi_1(\Sigma x) + \phi_1(\Sigma y) = \Sigma\phi(x) + \Sigma\phi(y).$$

Since the suspension Σ is an isomorphism of groups, we are done. □

Continuing the proof of Theorem 5.2, we next show that operations coming from $S^m_{\mathrm{tr}}(\mathbb{L}^n)$ may be factored using the ℓ-adic expansion of m. By Example 5.1.3, this reduces the proof of Theorem 5.2 to $m = a\ell^\nu$.

Lemma 5.4. *Write* $m = m_0 + m_1\ell + \cdots + m_r\ell^r$ *with* $0 \le m_i < \ell$. *Every cohomology operation* ϕ *corresponding to an element of* $\mathrm{Hom}(S^m_{tr}(\mathbb{L}^n), \mathbb{Z}/\ell(*)[*])$ *is a sum of operations* $x \mapsto \phi_0(x)\phi_1(x)\cdots\phi_r(x)$, *where the* $\phi_i \in \mathrm{Hom}(S^{m_i\ell^i}_{tr}(\mathbb{L}^n), \mathbb{Z}/\ell(*)[*])$.

Proof. By 4.1, $S^m_{\mathrm{tr}}(\mathbb{L}^n)$ is a summand of $S^G_{\mathrm{tr}}(\mathbb{L}^n)$, where G is given in (4.0a). Hence ϕ is induced from a map $S^G_{\mathrm{tr}}(\mathbb{L}^n) \to R(*)[*]$. By 4.11 and (4.0b),

$$\mathrm{Hom}(S^G_{\mathrm{tr}}(\mathbb{L}^n), \mathbb{Z}/\ell(*)[*]) \cong \otimes_{i=0}^r \mathrm{Hom}(S^{m_i \ell^i}_{\mathrm{tr}}(\mathbb{L}^n), \mathbb{Z}/\ell(*)[*]),$$

so ϕ is induced by a sum of terms $\phi_0 \otimes \phi_1 \otimes \cdots \otimes \phi_r$. □

We now establish the case $m = \ell^\nu$ by induction on ν, the case $\nu = 1$ being 5.3.

Proposition 5.5. *Any cohomology operation coming from* $\mathrm{Hom}(S^{\ell^\nu}_{tr}(\mathbb{L}^n), R(*)[*])$ *has scalar weight one.*

Proof. Recall from 4.2 that $S^{\ell^\nu}_{\mathrm{tr}}(\mathbb{L}^n)$ is a direct summand of $S^\ell_{\mathrm{tr}}(S^{\ell^{\nu-1}}_{\mathrm{tr}}(\mathbb{L}^n))$. Thus it suffices to treat cohomology operations of the form $S^{\ell^\nu}_{\mathrm{tr}}(\mathbb{L}^n) \to S^\ell_{\mathrm{tr}}(S^{\ell^{\nu-1}}_{\mathrm{tr}}(\mathbb{L}^n)) \to T$. Write $S^{\ell^{\nu-1}}_{\mathrm{tr}}(\mathbb{L}^n)$ as a sum of $\mathbb{L}^a[b]$. Then $S^\ell_{\mathrm{tr}} S^{\ell^{\nu-1}}_{\mathrm{tr}}(\mathbb{L}^n)$ is a sum of $S^\ell_{\mathrm{tr}}(\mathbb{L}^a[b])$, which are additive by Lemma 5.6 below, and terms of the form

$$S^{r_1}_{\mathrm{tr}}(\mathbb{L}^{a_{i_1}}[b_{i_1}]) \otimes \cdots \otimes S^{r_k}_{\mathrm{tr}}(\mathbb{L}^{a_{i_k}}[b_{i_k}]), \quad \sum r_i = \ell.$$

These latter terms correspond to cohomology operations which are sums of monomials $\phi_1(x) \cdots \phi_k(x)$ which have scalar weight $\sum r_i = \ell \equiv 1 \pmod{\ell - 1}$ by our inductive hypothesis. □

Lemma 5.6. *Let* $\mathbb{L}^a[b]$ *be a summand of* $S^{\ell^{\nu-1}}_{tr}(\mathbb{L}^n)$. *Then the motivic cohomology operations on* $H^{2n,n}(-,\mathbb{Z})$ *corresponding to elements of* $\mathrm{Hom}(S^\ell_{tr}(\mathbb{L}^a[b]), \mathbb{Z}/\ell(*)[*])$ *are additive. Hence they have scalar weight one.*

Proof. By 4.7, the map from $S^\ell_{\mathrm{tr}}(S^{\ell^{\nu-1}}_{\mathrm{tr}}(\mathbb{L}^n))[1]$ to $S^\ell_{\mathrm{tr}}(S^{\ell^{\nu-1}}_{\mathrm{tr}}(\mathbb{L}^n)[1])$ restricts to a map $S^\ell_{\mathrm{tr}}(\mathbb{L}^a[b])[1] \to S^\ell_{\mathrm{tr}}(\mathbb{L}^a[b+1])$. We now argue as in the proof of Proposition 5.3. By 4.9, the map $\mathrm{Hom}(S^\ell_{\mathrm{tr}}(\mathbb{L}^a[b+1]), T[1]) \to \mathrm{Hom}(S^\ell_{\mathrm{tr}}(\mathbb{L}^a[b]), T)$ is onto. Hence ϕ lifts to a cohomology operation $\phi_1 : H^{2n+1,n}(-,\mathbb{Z}) \to H^{p+1,q}(-,\mathbb{Z}/\ell)$. But then ϕ is additive by [26, 2.9]. □

The proof of Theorem 5.2 is completed by the next result.

Corollary 5.7. *If* $a < \ell$, *then every operation coming from* $\mathrm{Hom}(S^{a\ell^\nu}_{tr}(\mathbb{L}^n), R(*)[*])$ *has scalar weight* a.

Proof. By 4.1, every such operation has the form $S^{a\ell^\nu}_{\mathrm{tr}}(\mathbb{L}^n) \to S^a_{\mathrm{tr}}(S^{\ell^\nu}_{\mathrm{tr}}(\mathbb{L}^n)) \to T$. By 4.12, every element of $\mathrm{Hom}(S^a_{\mathrm{tr}}(S^{\ell^\nu}_{\mathrm{tr}}(\mathbb{L}^n)), T)$ is a sum of monomials $\phi_1 \cdots \phi_a$ where the ϕ_i belong to $\mathrm{Hom}(S^{\ell^\nu}_{\mathrm{tr}}(\mathbb{L}^n), T)$. By 5.5 and Example 5.1.3, these monomials have scalar weight a. □

6. Uniqueness of βP^n

The goal of this section is to prove Theorem 6.6, which is our replacement for Theorem 2.1 of [MC/l]. Our exposition follows §2 of [MC/l], except that the unproven (and possibly false) Lemma 2.2 is replaced by the contents of the previous section, and the equations (2.6), (2.7) and (2.8) of [MC/l] are replaced by inequalities when $m \geq \ell$ in the following result.

Theorem 6.1. *If* $R(q)[p]$ *is a Tate summand of* $S^m_{tr}(\mathbb{L}^n)$ *and* $m \equiv s \mod (\ell - 1)$ *for* $m \geq 1$ *and* $0 \leq s \leq \ell - 2$, *then:*

(1) $q \geq ns$, *with equality iff* $m < \ell$;
(2) $q \geq n(\ell - 1)$ *if* $s = 0$, *with equality iff* $m = \ell - 1$;
(3) $p \geq 2q \geq 2n$.

Proof. Recall from Theorem 4.10 that $S^m_{\rm tr}(\mathbb{L}^n)$ is a proper Tate motive, so each summand $R(q)[p]$ has the form $\mathbb{L}^q[b]$ for $b \geq 0$, with $p = 2q + b$. As $m \geq 1$, (1) and (2) imply (3). If $m < \ell$ then $S^m_{\rm tr}(\mathbb{L}^n) = \mathbb{L}^{mn}$ and $q = mn$ by 4.7(a). This yields the 'if' part of (1) and (2). To prove the 'only if' parts of (1) and (2), suppose that $m \geq \ell$ and write $m = \sum m_i \ell^i$, noting that $\sum m_i > m_0$, $\sum m_i \equiv m \mod (\ell - 1)$. We also have $q \geq (\sum m_i)n + (\ell - 1)$ by 4.1,4.6 and (4.0b). Since $\sum m_i \geq s$, we have $q > ns$. If $s = 0$ then $\sum m_i \geq \ell - 1$ and we have $q \geq (n + 1)(\ell - 1)$. $\square$

We now turn to the cohomology of $K_n \wedge \cdots \wedge K_n$ in scalar weight one. The following presentation is entirely due to Voevodsky and is taken from [MC/l, §2].

Lemma 6.2. *The scalar weight one part of* $\widetilde{H}^{p,q}(K_n^{\wedge r}, \mathbb{Z}/\ell)$ *vanishes if* $q < n\ell$ *and* $r > 1$, *and also if* $q = n\ell$ *and* $p < 2n\ell$.

Proof. ([MC/l, 2.7 and 2.8]) By 4.13, 4.10 and 5.2 it suffices to consider the monomials $x_1 \otimes \cdots \otimes x_r$ where the x_i are in $\mathrm{Hom}(S^{m_i}(\mathbb{L}^n), R(q_i)[p_i])$, with $\sum p_i = p$, $\sum q_i = q$ and $\sum m_i \equiv 1 \mod (\ell - 1)$. In the case $q < n\ell$ and $r \geq 2$ we have $m_i \not\equiv 0$ by 6.1(2) and we must have $\sum m_i \geq \ell$, which is excluded by 6.1(1) as $q \geq n \sum m_i$. When $q = n\ell$ and $p < 2n\ell$, the vanishing comes from 6.1(3). $\square$

We now analyze the motivic cohomology $\widetilde{H}^{2n\ell+2,n\ell}(BK_n, \mathbb{Z}/\ell)$. Recall that $BK_n = u(\mathbb{L}^n[1])$, which is weak equivalent to the simplicial classifying space $[r] \mapsto K_n^r$. The standard spectral sequence for the cohomology of a simplicial space with coefficients $\mathbb{Z}/\ell(n\ell)$ is

$$E_1^{r,s} = \widetilde{H}^{s,n\ell}(K_n^{\wedge r}, \mathbb{Z}/\ell) \Rightarrow \widetilde{H}^{r+s,n\ell}(BK_n, \mathbb{Z}/\ell). \tag{6.3}$$

The spectral sequence is bounded and converges for $n > 0$ by [MC/l, 2.6], because $E_1^{r,s} = 0$ for $r > \ell$. (This is because $n \geq 1$, and $K_n^{\wedge r}$ is nr-fold T-connected by 4.11.) Using Lemma 6.2, the relevant part of the spectral sequence looks like this:

$$\begin{array}{llcccc}
 & 0 & & & & \\
 & 0 & \widetilde{H}^{2n\ell+1,n\ell}(K) & \to & & \\
s = 2n\ell & 0 & \widetilde{H}^{2n\ell,n\ell}(K) & \to & \widetilde{H}^{2n\ell,n\ell}(K \wedge K) & \to \widetilde{H}^{2n\ell,n\ell}(K \wedge K \wedge K) \\
s < 2n\ell & 0 & 0 & & \text{(nothing in scalar weight one)} &
\end{array}$$

The E_1 page of the spectral sequence converging to $\widetilde{H}^{*,n\ell}(BK_n, \mathbb{Z}/\ell)$

Recall from [26, 3.7] that $H^{2n,n}(K_n, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$ on the fundamental class α.

Lemma 6.4. *For* $r \geq 2$, *the scalar weight one subgroup* Γ_r *of* $\widetilde{H}^{2n\ell,n\ell}(K_n^{\wedge r}, \mathbb{Z}/\ell)$ *is the free* $H^{*,*}(k, \mathbb{Z}/\ell)$*-module generated by the monomials of the form* $\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_r}$, *where* $\sum i_r = \ell$ *and each* $i_j > 0$.

Proof. This is [MC/l, 2.9]. The given monomials are linearly independent and have scalar weight one by the Künneth formula 4.13. The Künneth formula also shows that Γ_r is generated by terms $x_1 \otimes \cdots \otimes x_r$ where

$$x_j \in \mathrm{Hom}(S^{m_j}(\mathbb{L}^n), R(q_j)[p_j]),$$

for m_j, q_j and p_j satisfying $\sum p_j = 2n\ell$, $\sum q_j = n\ell$ and $\sum m_j \equiv 1 \mod (\ell - 1)$.

We claim that $m_j < \ell$ and $q_j = nm_j$ for each j. Now if $i < \ell$ then by 1.10, $S^i\mathbb{L}^n \cong \mathbb{L}^{ni}$ and $\mathrm{Hom}(S^i\mathbb{L}^n, R(ni)[p])$ vanishes unless $p = 2ni$ when it is R. Moreover, we know from 3.8 that α^i is a generator of $\mathrm{Hom}(S^i\mathbb{L}^n, R(ni)[2ni]) \cong R$. It will follow that each x_j is a scalar multiple of α^{m_j} and hence $x_1 \otimes \cdots \otimes x_r$ is a scalar multiple of one of our given monomials, proving the lemma.

Suppose first that $m_j \not\equiv 0 \mod (\ell-1)$ for all j. By 6.1(1), $\ell = q/n \geq \sum s_j$ with equality iff $m_j = s_j$ and $q_j = nm_j$ for all j. Since $r \geq 2$, $\sum s_j \geq 2$. As $s_j \equiv 1$ this forces $\sum s_j = \ell$ and hence $\sum m_j = \ell$ and $q_j = nm_j$ for all j. Thus we are reduced to the case in which some x_j has scalar weight 0.

Since $q_i \geq n$, the inequality 6.1(2) shows that at most one x_j can have scalar weight zero, and this can happen only if $r = 2$. If $r = 2$ and $m_1 \equiv 0$ then $m_2 \equiv 1$. Then by 6.1(1,2) we must have $m_1 = \ell - 1$, $q_1 = n(\ell-1)$, $m_2 = 1$ and $q_2 = n$, as claimed. The case $m_2 \equiv 0$ then $m_1 \equiv 1$ is treated identically. $\square$

Example 6.5. $\gamma = \alpha^{\ell-1} \wedge \alpha + \cdots + \alpha \wedge \alpha^{\ell-1}$ is an element of $\Gamma_2 \subset \widetilde{H}^{2n\ell,n\ell}(K_n \wedge K_n, \mathbb{Z}/\ell)$. A calculation shows that γ is a cycle in E_1; formally this follows from $\partial(\alpha^\ell) = \ell\gamma = (\alpha \otimes 1 + 1 \otimes \alpha)^\ell$.

Theorem 6.6. *Let $\phi : H^{2n+1,n}(-,\mathbb{Z}) \to H^{2n\ell+2,n\ell}(-,\mathbb{Z}/\ell)$ be a cohomology operation such that for all $x \in H^{2n+1,n}(X,\mathbb{Z})$:*

1. *$\phi(mx) = m\phi(x)$ for all $m \in \mathbb{Z}$;*
2. *If $x = \Sigma y$ for $y \in H^{2n,n}(X,\mathbb{Z})$ then $\phi(x) = 0$.*

Then ϕ is a multiple of βP^n.

Proof. (Voevodsky) We regard ϕ as an element of $\widetilde{H}^{2n\ell+2,n\ell}(BK_n,\mathbb{Z}/\ell)$. Condition (1) says that ϕ has scalar weight one. Condition (2) says that ϕ, like βP^n, is in the kernel of the map

$$\widetilde{H}^{2n\ell+2,n\ell}(BK_n,\mathbb{Z}/\ell) \to \widetilde{H}^{2n\ell+2,n\ell}(\Sigma K_n,\mathbb{Z}/\ell) = \widetilde{H}^{2n\ell+1,n\ell}(K_n,\mathbb{Z}/\ell)$$

defined by the inclusion of ΣK_n in BK_n as the part of simplicial degree one. That is, ϕ and βP^n both lie in the kernel of the edge map in the spectral sequence, and belong to the subgroup $E_2^{2,2n\ell}$ of $\widetilde{H}^{2n\ell+2,n\ell}(BK_n,\mathbb{Z}/\ell)$.

Voevodsky makes the following observation (at the end of §2 in [MC/l]). By a formal calculation due to Lazard in [8, 12.1], the kernel of $E_1^{2,2n\ell} \to E_1^{3,2n\ell}$ is $\mathbb{Z}/\ell$ in scalar weight one, generated by the cycle γ displayed in 6.5. Since βP^n is nonzero by [2] and/or [26, 11.5], it follows that every element in the kernel of the edge map must be a multiple of βP^n. $\square$

Remark 6.6.1. In topology, βP^n is the mod-ℓ reduction of an integral cohomology operation $H^{2n+1}(-,\mathbb{Z}) \to H^{2n\ell+2}(-,\mathbb{Z})$; see [2, Thm 5]. We will see in 8.4 that Voevodsky's operation ϕ_V provides such a lift for βP^n on $H^{2n+1,n}(-,\mathbb{Z}_{(\ell)})$.

7. $\mathfrak{X}$-DUALITY

In this short section, we introduce the notion of $\mathfrak{X}$-duality. No originality is claimed; the material is a translation of the duality in [29] for embedded schemes.

For simplicity, if M is a geometric motive we write eM for the motive $R_{\mathrm{tr}}(\mathfrak{X}) \otimes M$ and let $\mathbf{DM}^e_{\mathrm{gm}}$ denote the triangulated subcategory of $\mathbf{DM}^{\mathrm{eff}}$ generated by the objects eM with M in $\mathbf{DM}_{\mathrm{gm}}$. It is a tensor triangulated subcategory, because e is idempotent: $e(eM) \cong eM$ because $R_{\mathrm{tr}}(\mathfrak{X}) \otimes R_{\mathrm{tr}}(\mathfrak{X}) \simeq R_{\mathrm{tr}}(\mathfrak{X})$.

For any geometric motive M, the functor $eN \mapsto e\,\underline{R\,\mathrm{Hom}}(M,eN)$ is right adjoint to $eL \mapsto eL \otimes M \cong eL \otimes eM$; see [11, 14.12]. It follows that whenever $eM \cong eM'$ we have $e\,\underline{R\,\mathrm{Hom}}(M,eN) \cong e\,\underline{R\,\mathrm{Hom}}(M',eN)$, and that the object eM of $\mathbf{DM}^e_{\mathrm{gm}}$ determines a well defined functor $\underline{R\,\mathrm{Hom}}_e(eM,-)$ sending eN to $e\,\underline{R\,\mathrm{Hom}}(M,eN)$.

Applying e to the triangle $\mathfrak{X} \to R \to \Sigma\mathfrak{X}$, we see that $e(\Sigma\mathfrak{X}) \cong 0$.

Lemma 7.1. *For effective motives L, M, N with M geometric, the map $eN \to N$ induces isomorphisms* $\mathrm{Hom}(eL, eN) \cong \mathrm{Hom}(eL, N)$ *and*

$$\mathrm{Hom}(eL, e\,\underline{R\,\mathrm{Hom}}(M, eN)) \cong \mathrm{Hom}(eL, e\,\underline{R\,\mathrm{Hom}}(M, N)).$$

Thus $e\,\underline{R\,\mathrm{Hom}}(M, eN) \cong e\,\underline{R\,\mathrm{Hom}}(M, N)$. *If N is geometric this is in* $\mathbf{DM}^e_{gm}$.

Proof. For the first assertion, it suffices to show that every $f \in \mathrm{Hom}(eL, \Sigma\mathfrak{X} \otimes N)$ is zero. This follows from the observation above that $e(\Sigma\mathfrak{X}) = 0$ and the commutative diagram

$$\begin{tikzcd}
e(eL) \arrow[r, "e(f)"] \arrow[d, "\cong"'] & e(\Sigma\mathfrak{X} \otimes N) = 0 \arrow[d] \\
eL \arrow[r, "f"] & \Sigma\mathfrak{X} \otimes N.
\end{tikzcd}$$

In particular, the natural map from $\mathrm{Hom}(eL \otimes M, eN) \cong \mathrm{Hom}(eL, \underline{R\,\mathrm{Hom}}(M, eN))$ to $\mathrm{Hom}(eL \otimes M, N) \cong \mathrm{Hom}(eL, \underline{R\,\mathrm{Hom}}(M, N))$ is an isomorphism. The third assertion follows from this in a standard way, taking L to be $\underline{R\,\mathrm{Hom}}(M, eN)$ and $\underline{R\,\mathrm{Hom}}(M, N)$. The final assertion follows from the observation in [11, 20.3] that $\underline{R\,\mathrm{Hom}}(M, N)$ is a geometric motive. □

If $M = M(Y)$ is the motive of a smooth Y and $d = \dim(Y)$ we define

$$(eM)^\dagger = \underline{R\,\mathrm{Hom}}_{\,e}(eM, e\mathbb{L}^d) \otimes \mathbb{L}^{-d} = e\,\underline{R\,\mathrm{Hom}}(M, e\mathbb{L}^d) \otimes \mathbb{L}^{-d}.$$

By Lemma 7.1, this is isomorphic to $e(M^*)$. We may now mimick the development of [11, §20] to prove the following proposition.

Proposition 7.2. *The tensor category* $\mathbf{DM}^e_{gm}$ *is rigid. More precisely:*

1. *For every M in* $\mathbf{DM}_{gm}$, $(eM)^\dagger \cong e(M^*)$, *and hence* $(eM)^\dagger$ *is in* $\mathbf{DM}^e_{gm}$;
2. $(-)^\dagger$ *extends to a contravariant triangulated endo-functor on* $\mathbf{DM}^e_{gm}$;
3. *There is a natural isomorphism* $M \cong (M^\dagger)^\dagger$ *in* $\mathbf{DM}^e_{gm}$;
4. *There are natural isomorphisms (for L, M, N in* $\mathbf{DM}^e_{gm}$*)*
$$\mathrm{Hom}(L \otimes M, N) \cong \mathrm{Hom}(L, M^\dagger \otimes N);$$
5. *The internal Hom functor on* $\mathbf{DM}^e_{gm}$ *is* $\underline{\mathrm{Hom}}_e(M, N) = M^\dagger \otimes N$.

The $\mathfrak{X}$-dual $f^\dagger$ of a morphism $e(f) : e(M) \to e(N)$ in $\mathbf{DM}^e_{\mathrm{gm}}$ is $e(f^*)$. In particular, if $e(M) \cong M$ then $e(M^*) \cong M^*$ and there is a commutative diagram

$$\begin{tikzcd}
\mathfrak{X} = e(R) \arrow[r, "y^\dagger"] \arrow[d] & e(M^*) = \mathfrak{X} \otimes M^* \arrow[d, "\cong"] \\
R \arrow[r, "y^*"] & M^*.
\end{tikzcd} \tag{7.3}$$

Example 7.4. Suppose that $A \to eM \xrightarrow{\mu} eN \to$ is a triangle in $\mathbf{DM}^e_{\mathrm{gm}}$ with M, N geometric motives. Since μ may not be in the image of $\mathrm{Hom}(M, N)$, A may not have the form eL for any geometric motive L, yet $A^\dagger$ exists in $\mathbf{DM}^e_{\mathrm{gm}}$ by Proposition 7.2, and fits into a triangle in $\mathbf{DM}^e_{\mathrm{gm}}$:

$$eN^* \xrightarrow{\mu^\dagger} eM^* \to A^\dagger \to eN^*[1].$$

8. The Rost motive

We now consider the motivic operation $\phi_V : H^{2p+1,q}(-,R) \to H^{2p\ell+2,q\ell}(-,R)$, constructed by Voevodsky in [MC/l, 3.1–3.2] for any coefficient ring R containing $1/(\ell-1)!$ (we use $i = \ell - 1$).

Proposition 8.1. *If $\gamma \in H^{2r,s}(X,R)$ and $\sigma \in H^{2p+1,q}(X,R)$ then*

$$\phi_V(\gamma\sigma) = \gamma^\ell \phi_V(\sigma).$$

Proof. This is just Lemma 3.4 of [MC/l], where γ and σ are interpreted as maps $R \to R(s)[2r]$ and $R \to R(q)[2p+1]$ in the triangulated category $\mathbf{DM}(X,R)$ for a smooth simplicial X. □

Corollary 8.2. *(a) For any $x \in H^{2p+1,q}(X,R)$ and $m \in R$, $\phi_V(mx) = m^\ell \phi_V(x)$.*
(b) If $x = \Sigma y$ for $y \in H^{2p,q}(X,R)$ then $\phi_V(x) = 0$.

Proof. ([MC/l, 3.5 and 3.6]) The first assertion is just the case $\gamma = m$, and the second is just the case $\sigma \in H^{1,0}(S^1,R) \subset H^{1,0}(X \times S^1,R)$ with the observation that $\phi_V(\sigma) = 0$ because $H^{2,0}(S^1,R) = 0$. (See Example 0.2.) □

Example 8.2.1. Suppose that $n = 0$. When $R = \mathbb{Z}_{(\ell)}$, then $\phi_V = 0$ because $S^1 = BK_0$ and $H^{2,0}(S^1,R) = 0$. When $R = \mathbb{Z}/\ell$, ϕ_V is the Bockstein $\beta : H^{1,0}(X,\mathbb{Z}/\ell) \to H^{2,0}(X,\mathbb{Z}/\ell)$. (This was shown by Voevodsky in [MC/l, 3.7].) It is well known that if $L = K(\mathbb{Z}/\ell,1)$ is the Lens space then $\beta(\tau) \neq 0$, where $\tau \in H^{1,0}(L,\mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$.

Example 8.2.2. The following argument, implicit in [MC/l, 3.7], is taken from [20]. If $X = (\mathbb{P}^N)^n \times L$ and $x \in H^{2,1}(\mathbb{P}^N,\mathbb{Z}/\ell)$ is the generator, then $x_1 \otimes \cdots x_n \otimes \tau \in H^{2n+1,n}(X,\mathbb{Z}/\ell)$ satisfies

$$\phi_V(x_1 \otimes \cdots x_n \otimes \tau) = x_1^\ell \cdots x_n^\ell \beta(\tau)$$

by 8.1, and this is nonzero if $N \geq \ell$ by 8.2.1.

Let $\bar\phi_V$ denote the mod-ℓ reduction of ϕ_V, considered as an operation from $H^{2p+1,q}(-,\mathbb{Z})$ to $H^{2p\ell+2,q\ell}(-,\mathbb{Z}/\ell)$. Example 8.2.1 shows that $\bar\phi_V = 0$ when $n = 0$. Thus the argument of [MC/l, 3.7], using Example 8.2.2, does not apply to show that $\bar\phi_V \neq 0$. We substitute the following argument.

Proposition 8.3. *For any $n \geq 1$, $\bar\phi_V$ is nonzero on $H^{2n+1,n}(-,\mathbb{Z})$. That is, there exists an X and an $x \in H^{2n+1,n}(X,\mathbb{Z})$ so that $\bar\phi_V(x) \neq 0$ in $H^{2n\ell,n\ell}(X,\mathbb{Z}/\ell)$.*

Proof. It suffices to consider the case $n = 1$, by the trick of Example 8.2.2. Let L be the Lens space, and $\tau \in H^{1,0}(L,\mathbb{Z}/\ell)$ as in Example 8.2.1. We saw in 4.3 that $u \in H^{1,1}(B_{\mathrm{gm}}\mu_\ell,\mathbb{Z}/\ell)$ has $v = \beta(u)$. Set $X = L \times B_{\mathrm{gm}}\mu_\ell$, and consider the element $x = \tilde\beta(u\tau)$ of $H^{3,1}(X,\mathbb{Z})$; the mod-ℓ reduction of x is $\beta(u\tau) = v\tau - u\beta(\tau)$. Invoking 8.1, we have

$$\bar\phi_V(x) = \phi_V(v\tau) - \phi_V(u\beta(\tau)) = v^\ell \cdot \beta(\tau) - v \cdot \beta(\tau)^\ell,$$

and this is nonzero by the Künneth formula of [26]; see 4.3 or 4.11. □

Corollary 8.4. *The cohomology operations $\bar\phi_V$ and βP^n are non-zero multiples of each other.*

Proof. By 8.3, $\bar\phi_V$ is a nonzero element of the vector space $H^{2n\ell+2,n\ell}(K_n,\mathbb{Z}/\ell)$. By 8.2 and Theorem 6.6, $\bar\phi_V$ is a multiple of βP^n. □

Corollary 8.4 is our replacement for Theorem 3.8 in [MC/l]. We can now follow the presentation given by Voevodsky in [MC/l, §5].

Lemma 8.5. *There are maps* $\mathbb{Z}_{tr}(X) \xrightarrow{\lambda} S^{\ell-1}(A)$ *such that the inclusion* $X \xrightarrow{\iota} \mathfrak{X}$ *factors in* **DM** *as*

$$\mathbb{Z}_{tr}(X) \xrightarrow{\lambda} S^{\ell-1}(A) \xrightarrow{S^{\ell-1}y} \mathfrak{X}.$$

Proof. (Voevodsky [MC/l, 5.11]) Applying $\mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(X), -)$ to the triangle (1.9) yields the exact sequence

$$\mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(X), A) \xrightarrow{y} \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(X), \mathbb{Z}_{\mathrm{tr}}(\mathfrak{X})) \xrightarrow{z} \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(X), \mathbb{L}^b[1]) = 0.$$

(The group on the right vanishes since it equals $H^{2b+1,b}(X, \mathbb{Z}) = 0$.) Hence ι factors as $y\lambda_1$ for some $\lambda_1 : \mathbb{Z}_{\mathrm{tr}}(X) \to A$. Inductively, we use the second triangle of Lemma 1.12 to get

$$\mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(X), S^i A) \xrightarrow{u} \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(X), S^{i-1}(A)) \xrightarrow{r} \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(X), \mathfrak{X} \otimes \mathbb{L}^{bi}[1]) = 0.$$

Again, the group on the right is $H^{2bi+1,bi}(X, \mathbb{Z}) = 0$, so we see that there are maps $\lambda_i : \mathbb{Z}_{\mathrm{tr}}(X) \to S^i(A)$ for $i < \ell$ such that $\lambda_{i-1} = u\lambda_i$. By the construction of u, $yu^i = S^i y : S^i(A) \to \mathfrak{X}$. □

Recall that $\mathrm{Hom}(\mathbb{L}^d, \mathbb{Z}_{\mathrm{tr}}(X)) \cong H^0(X, \mathbb{Z})$ by Duality, so there is a fundamental class $\tau : \mathbb{L}^d \to \mathbb{Z}_{\mathrm{tr}}(X)$. Since $X \times \mathfrak{X} \simeq X$ we have $\mathbb{Z}_{\mathrm{tr}}(X) \otimes \mathbb{Z}_{\mathrm{tr}}(\mathfrak{X}) \cong \mathbb{Z}_{\mathrm{tr}}(X)$. Thus we may view τ as a map from $\mathbb{Z}_{\mathrm{tr}}(\mathfrak{X}) \otimes \mathbb{L}^d$ to $\mathbb{Z}_{\mathrm{tr}}(X)$.

Proposition 8.6. *The composition* $\mathbb{Z}_{tr}(\mathfrak{X}) \otimes \mathbb{L}^d \xrightarrow{\tau} \mathbb{Z}_{tr}(X) \xrightarrow{\lambda} S^{\ell-1}(A)$ *is not divisible by* ℓ.

Proof. (Voevodsky, [MC/l, 5.12]) By Lemma 1.8, $\alpha = Q_{n-1}(\mu)$ is a nonzero element of $H^{b\ell+2,b\ell}(\mathfrak{X}, \mathbb{Z}/\ell)$. Since $Q_i^2 = 0$, $Q_{n-1}(\alpha) = 0$. By the definition of ϕ_V in terms of the map s of 1.12, the restriction of ϕ_V to $S^{\ell-1}(A)$ is zero. By 8.4, βP^b also vanishes there. Since the Q_i anticommute we have $Q_i(\mu) = 0$ for $i \le n-2$. By the definition of Q_{n-1} we have

$$\alpha = Q_{n-1}(\mu) = Q_{n-2}(P^{\ell^{n-2}}\mu) = \cdots = \beta P^b(\mu),$$

and we have seen that $(S^{\ell-1}y)^*(\alpha) = 0$ By [MC/l, 4.4], applied to $\alpha \neq 0$ and Lemma 8.5, the mod-ℓ reduction of the map $\mathbb{Z}_{\mathrm{tr}}(\mathfrak{X}) \otimes \mathbb{L}^d \to S^{\ell-1}(A)$ is nonzero. □

Because $\mu : \mathfrak{X} \to \mathfrak{X} \otimes \mathbb{L}^b[1]$ is a map between Tate objects in $\mathbf{DM}^e_{\mathrm{gm}}$, it is self-dual ($\mu = \mu^\dagger \otimes \mathbb{L}^b$); see Example 7.4. It follows that $A \cong A^\dagger \otimes \mathbb{L}^b$. Since $S^i(M^\dagger) \cong (S^i M)^\dagger$ for every M we also have $S^i(A) \cong S^i(A)^\dagger \otimes \mathbb{L}^{bi}$. Cf. [MC/l, 5.7]. We write λ^D for the dual map

$$\lambda^D : S^{\ell-1}(A) \cong S^{\ell-1}(A)^\dagger \otimes \mathbb{L}^d \xrightarrow{\lambda^\dagger} \mathbb{Z}_{\mathrm{tr}}(X)^* \otimes \mathbb{L}^d \cong \mathbb{Z}_{\mathrm{tr}}(X).$$

Theorem 8.7. *The composition* $\lambda \circ \lambda^D$ *is an isomorphism on the symmetric Rost motive* $M = S^{\ell-1}(A)$ *(with coefficients* $\mathbb{Z}_{(\ell)}$ *or* $\mathbb{Z}/\ell$*), and there is an integer* $c \not\equiv 0$ (mod ℓ) *so that the following diagram commutes:*

$$\begin{tikzcd}
M \arrow[r, "\lambda\circ\lambda^D"] \arrow[d, "S^{\ell-1}y"'] & M \arrow[d, "S^{\ell-1}y"] \\
R_{tr}(\mathfrak{X}) \arrow[r, "c"] & R_{tr}(\mathfrak{X}).
\end{tikzcd}$$

In particular, M is a direct summand of $R_{tr}(X)$.

Proof. This is proven by Voevodsky in [MC/l, 5.15] □

Corollary 8.8. *When $R = \mathbb{Z}_{(\ell)}$, the maps λ and λ^D make $M = S^{\ell-1}(A)$ into a direct summand of $R_{tr}(X)$, and the following composition is an isomorphism.*

$$M \cong M^* \otimes \mathbb{L}^d \xrightarrow{\lambda^*} R_{tr}(X)^* \otimes \mathbb{L}^d \cong R_{tr}(X) \xrightarrow{\lambda} M.$$

Hence M is a Rost motive for $\underline{a}$.

Indeed, this is just a restatement of Theorem 8.7 in the form of axioms 1.3(a,b). Since axiom 1.3(c) holds by 1.12, M is a Rost motive for $\underline{a}$.

Corollary 8.8 implies that Rost motives exist for all $\underline{a}$. As proven by Voevodsky, and codified in Theorem 1.4, this implies that the norm residue homomorphisms $K_n^M(k)/\ell \longrightarrow H^n_{\text{ét}}(k, \mu_\ell^{\otimes n})$ are isomorphisms. This completes the proof, sketched in [MC/l], that the Bloch-Kato conjecture is true.

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Institute for Advanced Study, Princeton, NJ

Dept. of Mathematics, Rutgers University, New Brunswick, NJ 08901, USA
E-mail address: `weibel@math.rutgers.edu`