

On the interior motive of certain Shimura varieties: the case of Hilbert–Blumenthal varieties

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June 23, 2009

Abstract

Applying the main results of the author’s previous work [W2, W3, W4], we construct a Hecke-equivariant Chow motive whose realizations equal interior cohomology of Hilbert–Blumenthal varieties with non-constant coefficients.

Keywords: weight structures, Artin–Tate motives, relative motives, boundary motive, interior motive, Hilbert–Blumenthal varieties.

Math. Subj. Class. (2000) numbers: 14G35 (11F32, 11F41, 14C25, 14F25, 19E15, 19F27)

*Partially supported by the *Agence Nationale de la Recherche*, project no. ANR-07-BLAN-0142 “Méthodes à la Voevodsky, motifs mixtes et Géométrie d’Arakelov”.

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0 Introduction

The purpose of this paper is the construction and the analysis of the *interior motive* of Kuga–Sato families over Hilbert–Blumenthal varieties. This will be done by applying the program from [W4] to this geometrical setting.

More precisely, recall that for a smooth variety X over a perfect field k , the *boundary motive* $\partial M_{gm}(X)$ of X fits into a canonical exact triangle

$$(*) \quad \partial M_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}^c(X) \longrightarrow \partial M_{gm}(X)[1]$$

in the category $DM_{gm}^{eff}(k)$ of effective *geometrical motives*, defined and studied by Voevodsky, Suslin and Friedlander [VSF]. In order to make the construction from [W4] work, we need an idempotent endomorphism e of the exact triangle $(*)$, giving rise to a direct factor

$$\partial M_{gm}(X)^e \longrightarrow M_{gm}(X)^e \longrightarrow M_{gm}^c(X)^e \longrightarrow \partial M_{gm}(X)^e[1],$$

and assumed to satisfy the following hypothesis: the object $\partial M_{gm}(X)^e$ is *without weights* -1 and 0 with respect to the motivic *weight structure* from [Bo]. Under this hypothesis (which holds rarely for the whole of $\partial M_{gm}(X)$), the object $\partial M_{gm}(X)^e$ admits a canonical *weight filtration*

$$C_{\leq -2} \longrightarrow \partial M_{gm}(X)^e \longrightarrow C_{\geq 1} \longrightarrow C_{\leq -2}[1],$$

and a Chow motive $\mathrm{Gr}_0 M_{gm}(X)^e$ can be constructed, sitting in canonical exact triangles

$$C_{\leq -2} \longrightarrow M_{gm}(X)^e \longrightarrow \mathrm{Gr}_0 M_{gm}(X)^e \longrightarrow C_{\leq -2}[1]$$

and

$$C_{\geq 1} \longrightarrow \mathrm{Gr}_0 M_{gm}(X)^e \longrightarrow M_{gm}^c(X)^e \longrightarrow C_{\geq 1}[1].$$

Given the nature of its realizations, it is natural to call $\mathrm{Gr}_0 M_{gm}(X)^e$ the e -part of the interior motive of X . Its main properties are established in [W4, Sect. 4].

The rough organization of this article corresponds to this program. Sections 1 and 2 show how the theory of *relative Chow motives* can be employed to construct idempotents of the exact triangle (*). Sections 3–6 aim at a criterion (Theorem 6.2) allowing to verify the above hypothesis on the absence of weights -1 and 0 , when the boundary motive is *Artin–Tate*. We hope to apply both ingredients in geometrical contexts other than Hilbert–Blumenthal varieties, which explains why we chose to formulate the results from Sections 1–6 in a rather general way.

Section 7 contains the statements of our main results, Theorems 7.5 and 7.6. The base k is the field \mathbb{Q} of rational numbers, and X is the r -th power of the universal Abelian scheme over a smooth Hilbert–Blumenthal variety associated to a totally real field L of degree g . The idempotent e cuts out the direct factor of the relative Chow motive of X , on which the action of L is of type (r_1, \dots, r_g) , for $r_1 + \dots + r_g = r$. Theorem 7.5 implies in particular that in this context, the criterion from Theorem 6.2 is satisfied as soon as $r \geq 1$. Therefore, the interior motive exists in this geometrical setting. We list its principal properties, using the main results from [W4, Sect. 4]. First (Corollary 7.7), we get precise statements on the weights occurring in the motive $M_{gm}(X)^e$ and the motive with compact support $M_{gm}^c(X)^e$. Second (Corollary 7.8) the interior motive is Hecke-equivariant. This result appears particularly interesting, given that for Shimura varieties of higher dimension, Hecke-equivariant smooth compactifications are not known (and maybe not reasonable to expect) to exist. Third (Corollary 7.9), the interior motive occurs canonically as a direct factor of the (Chow) motive of any smooth compactification of X . We then discuss the special cases $g = 1$ and $g = 2$. For $g = 2$, the control of the weights from Corollary 7.7 turns out to be sufficiently precise to allow for a strengthening of the main result of [Ki] (Corollaries 7.13 and 7.14): the *special elements* in the motivic cohomology of X constructed in [loc. cit.] come indeed from motivic cohomology of a (in fact, of any) smooth compactification of X . The proofs of Theorems 7.5 and 7.6 are given in the final Section 8.

Let us now give a more detailed description of the content of this paper.

Section 1 is concerned with the functorial behavior of the exact triangle

$$(*) \quad \partial M_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}^c(X) \longrightarrow \partial M_{gm}(X)[1] .$$

Recall that the algebra of *finite correspondences* $c(X, X)$ acts on $M_{gm}(X)$ [V1]. In order to apply the results from [W4], we need an idempotent endomorphism of the whole of $(*)$. Indeed, the approach chosen in [loc. cit.] was to define a sub-algebra $c_{1,2}(X, X)$ of $c(X, X)$ (of “bi-finite correspondences”), and to show that it acts on $(*)$. Two problems occur naturally: first, one needs a source for cycles in $c_{1,2}(X, X)$; second, one needs a criterion to ensure that the action on $(*)$ of a given such cycle is idempotent. Given the application we have in mind, we are thus led to a close analysis of the behavior of the exact triangle $(*)$ under morphisms of relative Chow motives. Fix a base scheme S , which is smooth over k . Theorem 1.2 establishes the existence of a functor from the category of relative Chow motives over S to the category of exact triangles in $DM_{gm}^{eff}(k)$. On objects, it is given by mapping a proper, smooth S -scheme X to the exact triangle

$$\partial M_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}^c(X) \longrightarrow \partial M_{gm}(X)[1] .$$

We should mention that as far as the $M_{gm}(X)$ -component is concerned, the functoriality statement from Theorem 1.2 is just a special feature of results by Déglise, Cisinski–Déglise [D  2, CD  ] and Levine [Le3], at least after restriction to the sub-category generated by smooth projective schemes over S (see Remarks 1.3 and 1.13 for details). However, the application of the results from [loc. cit.] to the functor ∂M_{gm} is not obvious. This is one of the reasons why we chose to follow an alternative approach. It is based on a relative version of *moving cycles* [W2, Thm. 6.14]. This also explains why we are forced to suppose the base field k to admit a strict version of resolution of singularities. Theorem 1.5 and Corollary 1.15 then analyze the behaviour of the functor from Theorem 1.2 under change of the base S . Another reason for us to choose a cycle theoretic approach was that it becomes then easier to keep track of the correspondences on $X \times_k X$ commuting with our constructions. Our main application (Example 1.16) thus concerns correspondences “of Hecke type” yielding endomorphisms of the exact triangle $(*)$.

In Section 2, we apply these principles to Abelian schemes. More precisely, the main result of [DeM] on the Chow–K  nneth decomposition of the relative motive of an Abelian scheme A over S (recalled in Theorem 2.1) yields canonical projectors in the relative Chow group. They are easily shown to be represented by correspondences in $c_{1,2}(A, A)$ (Proposition 2.4). Given our analysis from Section 1, it follows that they act idempotently on the exact triangle

$$\partial M_{gm}(A) \longrightarrow M_{gm}(A) \longrightarrow M_{gm}^c(A) \longrightarrow \partial M_{gm}(A)[1] .$$

In Sections 3–5, we exhibit the basic structural properties of the triangulated category of Artin–Tate motives over a perfect base field k . The defini-

tion of this category will be recalled, and a number of variants will be defined in Section 3. Roughly speaking, the properties we shall be interested in, then fall into two classes. First (Section 4), we apply the main results of [Bo] to Artin–Tate motives. More precisely, we show (Theorem 4.5 (a)) that the weight structure on the category of geometrical motives of [loc. cit.] induces a weight structure on the triangulated category of Artin–Tate motives. We give a very explicit description of the *heart* of the latter (Theorem 4.5 (b), (c)), showing in particular that it is Abelian semi-simple. We also give a description of objects with two adjacent weights (Corollary 4.9), which will turn out to be useful for our analysis of the motivic cohomology of Hilbert–Blumenthal surfaces. Let us note that the results of Section 4 are valid over any perfect base field.

Second (Section 5), we generalize the main result from [Le1] from Tate motives to Artin–Tate motives, when the base field is algebraic over \mathbb{Q} . More precisely, we show (Theorem 5.1) that under this hypothesis, there is a non-degenerate *t-structure* on the triangulated category of Artin–Tate motives. The strategy of proof is identical to the one used by Levine. Our main interest lies then in the simultaneous application of both points of view: that of weight structures and that of *t-structures*. Still assuming that k is algebraic over \mathbb{Q} , we give a characterization (Theorem 5.8) of the weight structure on the triangulated category of Artin–Tate motives in terms of the *t-structure*. Specializing further to the case of number fields, we get a powerful result (Corollary 5.10), allowing to identify the weight structure *via* the Hodge theoretic or ℓ -adic *realization*.

This result is then used in Section 6 to deduce the already mentioned criterion on the absence of certain weights in the boundary motive (Theorem 6.2). Its simplified form states that when the boundary motive is Artin–Tate, then the absence of weights can be read off from the Hodge structure or the Galois action on the *boundary cohomology* (Corollary 6.4). It is this criterion that we shall verify for Hilbert–Blumenthal varieties.

Section 7 contains the statements of our main results, which have already been listed above. In the case of “non-parallel type”, i.e., the integers r_i used to construct the idempotent e are not all equal to each other, Theorem 7.6 states that the e -part of the boundary motive vanishes. In particular, the interior motive then coincides with the e -part of the motive of the (open) Kuga–Sato variety. This can be seen as a motivic explanation of [Ft, Rem. I.4.8]: “If f is a modular form, but not a cusp form, then $r_1 = \dots = r_g$.”

The final Section 8 is devoted to the verification of the criterion from Corollary 6.4 in the setting of Hilbert–Blumenthal varieties. First, we need to show that in this case, the boundary motive is indeed Artin–Tate (Theorem 8.2). This is done using a smooth *toroidal compactification*. We use

co-localization for the boundary motive [W2], in order to reduce to showing the statement for the contribution of any of the strata. The latter was identified in the general context of *mixed Shimura varieties* [W3]. For Kuga–Sato families over Hilbert–Blumenthal varieties, [loc. cit.] shows in particular that these contributions are indeed all Artin–Tate. We are thus reduced to the identification of the boundary cohomology. The resulting formula is most certainly known to the experts (see e.g. [BrL, Ha, Blt]). In the presence of a “non-parallel type”, it actually implies the vanishing of the boundary cohomology. In the remaining case, we employ the main result from [BuW], to identify the weights in the boundary cohomology, thereby completing the verification of the criterion from Corollary 6.4.

We should warn the reader that our constructions work *a priori* with \mathbb{Q} -coefficients. This seems to be necessary for at least the following reasons. First, the *triangulated category of Artin motives* is not known to admit a *t*-structure; by contrast, such a structure becomes obvious after tensoring with \mathbb{Q} (see Section 3). Second, as pointed out in [Le1], the existence of the *t*-structure on the *triangulated category of Tate motives* necessitates (and is in fact equivalent to) the validity of the *Beilinson–Soulé vanishing conjecture*; but this vanishing is only known (for algebraic base fields) after tensoring with \mathbb{Q} . Another problem, possibly related to the preceding two is that the realizations on the category of Artin–Tate motives are not known to be conservative before passage to \mathbb{Q} -coefficients. Next, the motivic decomposition of Abelian schemes of relative dimension greater than one (see Section 2) necessitates the inversion of at least one prime, and is only known to be canonical after $\otimes \mathbb{Q}$. Finally, our computations of the boundary cohomology of Hilbert–Blumenthal varieties (see Section 8) are valid only after tensoring with \mathbb{Q} . In fact, unless one deals with modular curves, very little seems to be known about the primes dividing the torsion of boundary cohomology with integer coefficients (see [Gh, Sect. 3.4]).

Part of this work was done while I was enjoying a *modulation de service pour les porteurs de projets de recherche*, granted by the *Université Paris 13*, and during a stay at the *Universität Zürich*. I am grateful to both institutions. I also wish to thank J. Ayoub, D. Blasius, F. Déglise, M. Dimitrov, P. Jørgensen, B. Kahn, F. Lecomte, M. Levine, V. Maillot, R. Pink and J. Tilouine for useful discussions and comments.

Notation and conventions: Throughout the article, k denotes a fixed perfect base field. We denote by Sch/k the category of separated schemes of finite type over k , and by $Sm/k \subset Sch/k$ the full sub-category of objects which are smooth over k . When we assume k to admit resolution of singularities, then it will be in the sense of [FIV, Def. 3.4]: (i) for any $X \in Sch/k$, there exists an abstract blow-up $Y \rightarrow X$ [FIV, Def. 3.1] whose source Y is in Sm/k , (ii) for any $X, Y \in Sm/k$, and any abstract blow-up $q : Y \rightarrow X$,

there exists a sequence of blow-ups $p : X_n \rightarrow \dots \rightarrow X_1 = X$ with smooth centers, such that p factors through q . We say that k admits strict resolution of singularities, if in (i), for any given dense open subset U of the smooth locus of X , the blow-up $q : Y \rightarrow X$ can be chosen to be an isomorphism above U , and such that arbitrary intersections of the irreducible components of the complement Z of U in Y are smooth (e.g., $Z \subset Y$ a normal crossing divisor with smooth irreducible components).

As far as motives are concerned, the notation of this paper is that of [W2, W3, W4], which in turn follows that of [V1]. We refer to [W2, Sect. 1] for a concise review of this notation, and of the definition of the triangulated categories $DM_{gm}^{eff}(k)$ and $DM_{gm}(k)$ of (effective) geometrical motives over k . Let F be a commutative \mathbb{Q} -algebra. The notation $DM_{gm}^{eff}(k)_F$ and $DM_{gm}(k)_F$ stands for the F -linear analogues of these triangulated categories defined in [An, Sect. 16.2.4 and Sect. 17.1.3]. Similarly, let us denote by $CHM^{eff}(k)$ and $CHM(k)$ the categories opposite to the categories of (effective) Chow motives, and by $CHM^{eff}(k)_F$ and $CHM(k)_F$ the pseudo-Abelian completion of the category $CHM^{eff}(k) \otimes_{\mathbb{Z}} F$ and $CHM(k) \otimes_{\mathbb{Z}} F$, respectively. Using [V2, Cor. 2] ([V1, Cor. 4.2.6] if k admits resolution of singularities), we canonically identify $CHM^{eff}(k)_F$ and $CHM(k)_F$ with a full additive sub-category of $DM_{gm}^{eff}(k)_F$ and $DM_{gm}(k)_F$, respectively.

1 Relative motives and functoriality of the boundary motive

In this and the next section, the base field k is assumed to admit strict resolution of singularities. Recall that for $X \in Sch/k$, the *boundary motive* $\partial M_{gm}(X)$ of X [W2, Def. 2.1] fits into a canonical exact triangle

$$(*) \quad \partial M_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}^c(X) \longrightarrow \partial M_{gm}(X)[1]$$

in $DM_{gm}^{eff}(k)$. If we assume X to be smooth, then the algebra of *finite correspondences* $c(X, X)$ acts on $M_{gm}(X)$ [V1, p. 190]. In order to apply the results from [W4], we need an idempotent endomorphism of the whole exact triangle (*). One of the aims of this section is to show that the theory of *relative motives* provides a source of such idempotents. This result is a special feature of an analysis of the functorial behavior of the exact triangle (*) under morphisms of relative motives (Theorems 1.2 and 1.5, Corollary 1.15). The main application (Example 1.16) concerns endomorphisms of (*) “of Hecke type”.

Let us fix a base scheme $S \in Sm/k$. Recall that by definition, objects of Sm/k are separated over k . Thus, for any two schemes X and Y over S , the

natural morphism

$$X \times_S Y \longrightarrow X \times_k Y$$

is a closed immersion. Therefore, cycles on $X \times_S Y$ can and will be considered as cycles on $X \times_k Y$. Denote by Sm/S the category of separated smooth schemes of finite type over S , by $PropSm/S \subset Sm/S$ the full sub-category of objects which are proper and smooth over S , and by $ProjSm/S \subset Sm/S$ the full sub-category of projective, smooth S -schemes.

Definition 1.1. Let $X, Y \in Sm/S$. Denote by $c_S(X, Y)$ the subgroup of $c(X, Y)$ of correspondences whose support is contained in $X \times_S Y$.

The group $c_S(X, Y)$ is at the base of the theory of (*effective*) *geometrical motives over S* , as defined and developed (for arbitrary regular Noetherian bases S) in [D 1, D 2]. Note that any cycle \mathfrak{Z} in $c_S(X, Y)$ gives rise to a morphism from $M_{gm}(X)$ to $M_{gm}(Y)$, which we shall denote by $M_{gm}(\mathfrak{Z})$. Recall from [DeM, Sect. 1.3, 1.6] the definition of the categories of (*effective*) *Chow motives over S* ; note that the approach of [loc. cit.] does not necessitate passage to \mathbb{Q} -coefficients, and that one may choose to perform the construction using schemes in $PropSm/S$ instead of just schemes in $ProjSm/S$. Denote by $CHM^{eff}(S)$ and $CHM(S)$ the respective opposites of these categories. Note that for $X, Y \in PropSm/S$ and $\mathfrak{Z} \in c_S(X, Y)$, the class of \mathfrak{Z} in the Chow group $CH^*(X \times_S Y)$ of cycles modulo rational equivalence lies in the right degree, and therefore defines a morphism from the relative Chow motive $h(X/S)$ of X to the relative Chow motive $h(Y/S)$. Our aim is to prove the following.

Theorem 1.2. (a) *There is a canonical additive covariant functor, denoted $(\partial M_{gm}, M_{gm}, M_{gm}^c) = (\partial M_{gm}, M_{gm}, M_{gm}^c)_S$, from $CHM(S)$ to the category of exact triangles in $DM_{gm}(k)$. On objects, it is characterized by the following properties:*

(a1) *for $X \in PropSm/S$, the functor $(\partial M_{gm}, M_{gm}, M_{gm}^c)$ maps $h(X/S)$ to the triangle*

$$(*)_X \quad \partial M_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}^c(X) \longrightarrow \partial M_{gm}(X)[1],$$

(a2) *the functor $(\partial M_{gm}, M_{gm}, M_{gm}^c)$ is compatible with Tate twists.*

On morphisms, the functor $(\partial M_{gm}, M_{gm}, M_{gm}^c)$ maps the class of a cycle $\mathfrak{Z} \in c_S(X, Y)$ in $CH^(X \times_S Y)$, for $X, Y \in PropSm/S$, to a morphism $(*)_X \rightarrow (*)_Y$ whose M_{gm} -component $M_{gm}(X) \rightarrow M_{gm}(Y)$ coincides with $M_{gm}(\mathfrak{Z})$.*

(b) *There is a canonical additive contravariant functor $(\partial M_{gm}, M_{gm}, M_{gm}^c)^* = (\partial M_{gm}, M_{gm}, M_{gm}^c)_S^*$ from $CHM(S)$ to the category of exact triangles in $DM_{gm}(k)$. On objects, it is characterized by the following properties:*

(b1) for an object $X \in \text{PropSm}/S$ which is of pure absolute dimension d_X , the functor $(\partial M_{gm}, M_{gm}, M_{gm}^c)^*$ maps $h(X/S)$ to the triangle

$$(*)_X^* := (*)_X(-d_X)[-2d_X],$$

(b2) the functor $(\partial M_{gm}, M_{gm}, M_{gm}^c)^*$ is anti-compatible with Tate twists.

On morphisms, the functor $(\partial M_{gm}, M_{gm}, M_{gm}^c)^*$ maps the class of a cycle $\mathfrak{Z} \in c_S(X, Y)$ in $\text{CH}^*(X \times_S Y)$, for $X, Y \in \text{PropSm}/S$ of pure absolute dimensions d_X and d_Y , respectively, to a morphism $(*)_Y^* \rightarrow (*)_X^*$ whose M_{gm}^c -component coincides with the dual of $M_{gm}(\mathfrak{Z})$.

(c) The functor $(\partial M_{gm}, M_{gm}, M_{gm}^c)^*$ is canonically identified with the composition of $(\partial M_{gm}, M_{gm}, M_{gm}^c)$ and duality in $DM_{gm}(k)$.

Note that by [V1, Thm. 4.3.7 3], the object $M_{gm}^c(X)$ is indeed dual to $M_{gm}(X)(-d_X)[-2d_X]$. Note also [V1, Cor. 4.1.6] that the functor from Theorem 1.2 (a) maps the full sub-category $CHM^{eff}(S)$ to the full sub-category [V1, Thm. 4.3.1] of exact triangles in $DM_{gm}^{eff}(k)$. Note finally that by convention, the Tate twist (n) in $CHM(S)$ corresponds to the (componentwise) operation $M \mapsto M(n)[2n]$ in $DM_{gm}(k)$. Thus, anti-compatibility of the functor $(\partial M_{gm}, M_{gm}, M_{gm}^c)^*$ with Tate twists means that for any object X of $CHM(S)$, we have

$$(\partial M_{gm}, M_{gm}, M_{gm}^c)^*(X(n)) = ((\partial M_{gm}, M_{gm}, M_{gm}^c)^*(X))(-n)[-2n].$$

Remark 1.3. As far as the M_{gm} - and M_{gm}^c -components are concerned, Theorem 1.2, or at least its restriction to the full sub-category $CHM(S)_{proj}$ of $CHM(S)$ generated by the motives of projective smooth S -schemes, is a consequence of the main results of [Dé2], especially [Dé2, Thm. 5.23], together with the existence of an adjoint pair $(La_{S, \sharp}, a_S^*)$ of exact functors [CDé, Ex. 4.12, Ex. 7.15] linking the category $DM_{gm}(S)$ of geometrical motives over S to $DM_{gm}(k)$ (here we let $a_S : S \rightarrow \mathbf{Spec} k$ denote the structure morphism of S). We should also mention that this approach would allow to avoid the hypothesis on strict resolution of singularities. However, the application of the results of [loc. cit.] to the functor ∂M_{gm} is not obvious. We are therefore forced to follow an alternative approach.

Remark 1.4. The following sheaf-theoretical phenomenon explains why one should expect a statement like Theorem 1.2. Writing $a = a_X$ for the structure morphism $X \rightarrow \mathbf{Spec} k$, for $X \in \text{Sch}/k$, there is an exact triangle of exact functors

$$(+)_X \quad a_! \longrightarrow a_* \longrightarrow a_*/a_! \longrightarrow a_![1]$$

from the derived category $D^+(X)$ of complexes of étale sheaves on X (say), bounded from below, to $D^+(\mathbf{Spec} k)$. Here, a_* denotes the derived functor of the direct image, $a_!$ is its analogue “with compact support”, and $a_*/a_!$ is a canonical choice of cone (which exists since the category of compactifications

of X is filtered). The triangle $(+)_X$ is contravariantly functorial with respect to proper morphisms. Up to a twist and a shift, it is covariantly functorial with respect to proper smooth morphisms. This shows that a suitable version of Theorem 1.2 (a) is likely to extend to the sub-category of $DM_{gm}(S)$ generated by the relative motives of schemes which are (only) proper over S .

For any proper smooth morphism $f : T \rightarrow S$ in the category Sm/k , denote by $f_{\sharp} : CHM(T) \rightarrow CHM(S)$ the canonical functor induced by $h(X/T) \mapsto h(X/S)$, for any proper smooth scheme X over T (hence, over S). For any morphism $g : U \rightarrow S$ in Sm/k , denote by $g^* : CHM(S) \rightarrow CHM(U)$ the canonical tensor functor induced by $h(Y/S) \mapsto h(Y \times_S U/U)$, for any proper smooth scheme Y over S . When g is proper and smooth, the functor g_{\sharp} is left adjoint to g^* . The following summarizes the behaviour of $(\partial M_{gm}, M_{gm}, M_{gm}^c)$ and $(\partial M_{gm}, M_{gm}, M_{gm}^c)^*$ under change of the base S .

Theorem 1.5. (a) *Let $f : T \rightarrow S$ be a proper smooth morphism in Sm/k . There are canonical isomorphisms of additive functors*

$$\alpha_{f_{\sharp}} : (\partial M_{gm}, M_{gm}, M_{gm}^c)_S \circ f_{\sharp} \xrightarrow{\sim} (\partial M_{gm}, M_{gm}, M_{gm}^c)_T$$

and

$$\alpha_{f_{\sharp}}^* : (\partial M_{gm}, M_{gm}, M_{gm}^c)_T^* \xrightarrow{\sim} (\partial M_{gm}, M_{gm}, M_{gm}^c)_S^* \circ f_{\sharp}$$

on $CHM(T)$. The formation of both $\alpha_{f_{\sharp}}$ and $\alpha_{f_{\sharp}}^*$ is compatible with composition of proper smooth morphisms in Sm/k . Under the identification of Theorem 1.2 (c), the equivalence $\alpha_{f_{\sharp}}^*$ corresponds to the dual of the equivalence $\alpha_{f_{\sharp}}$.

(b) *Let $g : U \rightarrow S$ be a proper smooth morphism in Sm/k . Then there exists a canonical transformation of additive functors*

$$\beta_{g^*, \text{id}_S} : (\partial M_{gm}, M_{gm}, M_{gm}^c)_U \circ g^* \longrightarrow (\partial M_{gm}, M_{gm}, M_{gm}^c)_S .$$

The formation of β_{g^*, id_S} is compatible with composition of proper smooth morphisms in Sm/k .

(c) *The transformations $\alpha_{f_{\sharp}}$ and β_{g^*, id_S} commute in the following sense: let $f : T \rightarrow S$ and $g : U \rightarrow S$ be proper smooth morphisms in Sm/k . Consider the cartesian diagram*

$$\begin{array}{ccc} V = T \times_S U & \xrightarrow{f'} & U \\ g' \downarrow & & \downarrow g \\ T & \xrightarrow{f} & S \end{array}$$

and the canonical identification of natural transformations

$$f'_{\sharp} \circ g'^* = g^* \circ f_{\sharp}$$

of functors from $CHM(T)$ to $CHM(U)$. Then the transformations

$$\beta_{g'^*, \text{id}_T} \circ (\alpha_{f'_{\sharp}} \circ g'^*) \quad , \quad \alpha_{f_{\sharp}} \circ (\beta_{g^*, \text{id}_S} \circ f_{\sharp})$$

of functors on $CHM(T)$

$$(\partial M_{gm}, M_{gm}, M_{gm}^c)_U \circ g^* \circ f_{\sharp} \longrightarrow (\partial M_{gm}, M_{gm}, M_{gm}^c)_T$$

coincide.

(d) Let $g : U \rightarrow S$ be a proper smooth morphism in Sm/k . Then there exists a canonical transformation of additive functors

$$\gamma_{id_S, g^*} : (\partial M_{gm}, M_{gm}, M_{gm}^c)_S^* \longrightarrow (\partial M_{gm}, M_{gm}, M_{gm}^c)_U^* \circ g^* .$$

The formation of γ_{id_S, g^*} is compatible with composition of proper smooth morphisms in Sm/k . Under the identification of Theorem 1.2 (c), the transformation γ_{id_S, g^*} corresponds to the dual of the transformation β_{g^*, id_S} .

(e) The transformations $\alpha_{f_{\sharp}}^*$ and γ_{id_S, g^*} commute in the following sense: let $f : T \rightarrow S$ and $g : U \rightarrow S$ be proper smooth morphisms in Sm/k . Consider the cartesian diagram

$$\begin{array}{ccc} V = T \times_S U & \xrightarrow{f'} & U \\ g' \downarrow & & \downarrow g \\ T & \xrightarrow{f} & S \end{array}$$

Then the transformations

$$(\gamma_{id_S, g^*} \circ f_{\sharp}) \circ \alpha_{f_{\sharp}}^* \quad , \quad (\alpha_{f'_{\sharp}}^* \circ g'^*) \circ \gamma_{id_T, g'^*}$$

of functors on $CHM(T)$

$$(\partial M_{gm}, M_{gm}, M_{gm}^c)_T^* \longrightarrow (\partial M_{gm}, M_{gm}, M_{gm}^c)_U^* \circ g^* \circ f_{\sharp}$$

coincide.

Remark 1.6. Sheaf-theoretical considerations show that parts (b)–(e) of Theorem 1.5 should hold more generally for morphisms g which are (only) proper. While this could be shown to be indeed the case, we chose to prove the statements only under the more restrictive assumption on g (the proof simplifies considerably since it is possible to make use of the functor g_{\sharp} , which only exists when g is proper and smooth).

Let us prepare the proofs of Theorems 1.2 and 1.5. They are based on the following result.

Theorem 1.7 ([W2, Thm. 6.14, Rem. 6.15]). *Let $W \in Sm/k$ be of pure dimension m , and $Z \subset W$ a closed sub-scheme such that arbitrary intersections of the irreducible components of Z are smooth. Fix $n \in \mathbb{Z}$.*

(a) *There is a canonical morphism*

$$cyc : \underline{h}^0(z_{equi}(W, m-n)_Z)(\mathbf{Spec} k) \longrightarrow \mathrm{Hom}_{DM_{gm}^{eff}(k)}(M_{gm}(W/Z), \mathbb{Z}(n)[2n]) .$$

(b) *The morphism cyc is compatible with passage from the pair $Z \subset W$ to $Z' \subset U$, for open sub-schemes U of W , and closed sub-schemes Z' of $Z \cap U$ such that arbitrary intersections of the irreducible components of Z'*

are smooth.

(c) When Z is empty, then

$$\text{cyc} : \underline{h}^0(z_{\text{equi}}(W, m-n))(\mathbf{Spec} k) \longrightarrow \text{Hom}_{DM_{gm}^{eff}(k)}(M_{gm}(W), \mathbb{Z}(n)[2n]) .$$

coincides with the morphism from [V1, Cor. 4.2.5]. In particular, it is then an isomorphism.

Some explanations are necessary. First, by definition [W2, Def. 6.13], the Nisnevich sheaf with transfers $z_{\text{equi}}(W, m-n)_Z$ associates to $T \in Sm/k$ the group of those cycles in $z_{\text{equi}}(W, m-n)(T)$ [V1, p. 228] having empty intersection with $T \times_k Z$. In particular, the group $z_{\text{equi}}(W, m-n)_Z(\mathbf{Spec} k)$ equals the group of cycles on W of dimension $m-n$, whose support is disjoint from Z . Recall then [V1, p. 207] that the group

$$\underline{h}^0(z_{\text{equi}}(W, m-n)_Z)(\mathbf{Spec} k)$$

is the quotient of $z_{\text{equi}}(W, m-n)_Z(\mathbf{Spec} k)$ by the image under the differential “pull-back via 1 minus pull-back via 0” of $z_{\text{equi}}(W, m-n)_Z(\mathbb{A}_k^1)$. Finally the object $M_{gm}(W/Z)$ denotes the relative motive associated to the immersion of Z into W [W2, Def. 6.4].

Remark 1.8. One may speculate about the validity of Theorem 1.7 for arbitrary closed sub-schemes Z of $W \in Sm/S$. While the author is optimistic about this possibility, he notes that the tools developed in [W2] to prove Theorem 1.7 require Z to satisfy our more restrictive hypotheses. It is for that reason that we are forced to suppose k to admit strict resolution of singularities.

Now note the following.

Proposition 1.9. *In the above situation, let in addition $V \subset W$ be a closed sub-scheme in Sm/S , which is disjoint from Z . Then the natural map*

$$z_{\text{equi}}(V, m-n) \longrightarrow z_{\text{equi}}(W, m-n)_Z$$

induces a morphism

$$\text{CH}_{m-n}(V) \longrightarrow \underline{h}^0(z_{\text{equi}}(W, m-n)_Z)(\mathbf{Spec} k) .$$

Corollary 1.10. *Let $W \in Sm/k$ be of pure dimension m , $V, Z \subset W$ closed sub-schemes, and $n \in \mathbb{Z}$. Suppose that arbitrary intersections of the irreducible components of Z are smooth, and that $V \cap Z = \emptyset$. Then there is a canonical morphism*

$$\text{cyc} : \text{CH}_{m-n}(V) \longrightarrow \text{Hom}_{DM_{gm}^{eff}(k)}(M_{gm}(W/Z), \mathbb{Z}(n)[2n]) .$$

Given an open immersion $j : U \hookrightarrow W$ and a closed sub-scheme Z' of the intersection $Z \cap U$ such that arbitrary intersections of the irreducible compo-

nents of Z' are smooth, the diagram

$$\begin{array}{ccc} \mathrm{CH}_{m-n}(V) & \xrightarrow{\mathrm{cyc}} & \mathrm{Hom}_{DM_{gm}^{eff}(k)}(M_{gm}(W/Z), \mathbb{Z}(n)[2n]) \\ j^* \downarrow & & \downarrow j^* \\ \mathrm{CH}_{m-n}(V \cap U) & \xrightarrow{\mathrm{cyc}} & \mathrm{Hom}_{DM_{gm}^{eff}(k)}(M_{gm}(U/Z'), \mathbb{Z}(n)[2n]) \end{array}$$

commutes.

Now fix $X, Y \in \mathrm{PropSm}/S$. Choose a compactification (over k) \overline{S} of S , and compactifications \overline{X} of X , and \overline{Y} of Y together with cartesian diagrams

$$\begin{array}{ccc} X & \hookrightarrow & \overline{X} \\ \downarrow & & \downarrow \\ S & \hookrightarrow & \overline{S} \end{array}$$

and

$$\begin{array}{ccc} Y & \hookrightarrow & \overline{Y} \\ \downarrow & & \downarrow \\ S & \hookrightarrow & \overline{S} \end{array}$$

(this is possible since X and Y are proper over S). The hypothesis on k ensures that arbitrary intersections of the irreducible components of the complements $\partial\overline{X}$ of X in \overline{X} and $\partial\overline{Y}$ of Y in \overline{Y} can be supposed to be smooth. Each of the three constituents M_{gm} , M_{gm}^c , ∂M_{gm} of the exact triangle (*) will correspond to an application of Corollary 1.10, with different choices of (W, Z) .

- (1) for M_{gm} , we define $W := X \times_k \overline{Y}$,
- (2) for M_{gm}^c , we define $W := \overline{X} \times_k Y$,
- (3) for ∂M_{gm} , we define $W := \overline{X} \times_k \overline{Y} - \partial\overline{X} \times_k \partial\overline{Y}$.

In all three cases, we put $Z := W - X \times_k Y$. That is,

- (1) $Z = X \times_k \partial\overline{Y}$,
- (2) $Z = \partial\overline{X} \times_k Y$,
- (3) $Z = X \times_k \partial\overline{Y} \cup \partial\overline{X} \times_k Y$.

We also let $V := X \times_S Y \subset X \times_k Y$ in all three cases. These choices satisfy the hypotheses of Corollary 1.10 thanks to the following.

Lemma 1.11. *The scheme $X \times_S Y$ is closed in $\overline{X} \times_k \overline{Y} - \partial\overline{X} \times_k \partial\overline{Y}$.*

Proof. Indeed, the diagram

$$\begin{array}{ccc} X \times_S Y \hookrightarrow & \overline{X} \times_k \overline{Y} - \partial \overline{X} \times_k \partial \overline{Y} \\ \downarrow & \downarrow \\ \overline{S} \hookrightarrow & \overline{S} \times_k \overline{S} \end{array} \quad \Delta$$

is cartesian.

q.e.d.

Proof of Theorem 1.2. We may clearly assume S , X and Y to be of pure absolute dimension d_S , d_X and d_Y , respectively.

Let us treat M_{gm} first. Note that by [V1, Thm. 4.3.7 3], the group of morphisms in $DM_{gm}(k)$ from $M_{gm}(X)$ to $M_{gm}(Y)$ is canonically isomorphic to

$$\mathrm{Hom}_{DM_{gm}(k)}(M_{gm}(X) \otimes M_{gm}^c(Y), \mathbb{Z}(d_Y)[2d_Y]) .$$

Localization for the motive with compact support [V1, Prop. 4.1.5] shows that $M_{gm}^c(Y) = M_{gm}(\overline{Y} / \partial \overline{Y})$. Given the definition of the tensor structure on $DM_{gm}(k)$, the above therefore equals

$$\mathrm{Hom}_{DM_{gm}(k)}(M_{gm}(X \times_k \overline{Y} / X \times_k \partial \overline{Y}), \mathbb{Z}(d_Y)[2d_Y]) .$$

By Corollary 1.10, applied to the setting (1), this group is the target of the morphism cyc_1 on $\mathrm{CH}_{d_X}(X \times_S Y) = \mathrm{CH}^{d_Y - d_S}(X \times_S Y)$. Note that on a class which comes from $\mathfrak{z} \in c_S(X, Y)$, the map cyc_1 takes indeed the value $M_{gm}(\mathfrak{z})$.

The case of M_{gm}^c is similar. First, by duality, the group of morphisms in $DM_{gm}(k)$ from $M_{gm}^c(X)$ to $M_{gm}^c(Y)$ is canonically isomorphic to

$$\mathrm{Hom}_{DM_{gm}(k)}(M_{gm}^c(X) \otimes M_{gm}(Y), \mathbb{Z}(d_Y)[2d_Y]) .$$

By localization, this group then equals

$$\mathrm{Hom}_{DM_{gm}(k)}(M_{gm}(\overline{X} \times_k Y / \partial \overline{X} \times_k Y), \mathbb{Z}(d_Y)[2d_Y]) .$$

By Corollary 1.10, applied to the setting (2), this group is the target of the morphism cyc_2 on $\mathrm{CH}^{d_Y - d_S}(X \times_S Y)$.

In order to show that for a cycle class z in $\mathrm{CH}^{d_Y - d_S}(X \times_S Y)$, the diagram

$$\begin{array}{ccc} M_{gm}(X) & \longrightarrow & M_{gm}^c(X) \\ cyc_1(z) \downarrow & & \downarrow cyc_2(z) \\ M_{gm}(Y) & \longrightarrow & M_{gm}^c(Y) \end{array}$$

commutes, we need to study the group of morphisms in $DM_{gm}(k)$ from $M_{gm}(X)$ to $M_{gm}^c(Y)$. Again by duality, it is canonically isomorphic to

$$\mathrm{Hom}_{DM_{gm}(k)}(M_{gm}(X \times_k Y), \mathbb{Z}(d_Y)[2d_Y]) .$$

The above commutativity then follows from the compatibility of cyc under restriction from $X \times_k \overline{Y}$, resp. $\overline{X} \times_k Y$, to $X \times_k Y$ (Corollary 1.10).

Now let us treat ∂M_{gm} . Note that by [W2, Thm. 6.1], the group of morphisms in $DM_{gm}(k)$ from $\partial M_{gm}(X)$ to $\partial M_{gm}(Y)$ is canonically isomorphic to

$$\mathrm{Hom}_{DM_{gm}(k)}(\partial M_{gm}(X) \otimes \partial M_{gm}(Y)[1], \mathbb{Z}(d_Y)[2d_Y]) .$$

As in [W2, pp. 650–651], one shows that $\partial M_{gm}(X) \otimes \partial M_{gm}(Y)[1]$ maps canonically to the relative motive

$$M_{gm}((\overline{X} \times_k \overline{Y} - \partial \overline{X} \times_k \partial \overline{Y}) / (\overline{X} \times_k \overline{Y} - \partial \overline{X} \times_k \partial \overline{Y} - X \times_k Y)) .$$

Hence the group of morphisms $\mathrm{Hom}_{DM_{gm}^{eff}(k)_F}(\partial M_{gm}(X), \partial M_{gm}(Y))$ receives an arrow, say α , from the group of morphisms from

$$M_{gm}((\overline{X} \times_k \overline{Y} - \partial \overline{X} \times_k \partial \overline{Y}) / (\overline{X} \times_k \overline{Y} - \partial \overline{X} \times_k \partial \overline{Y} - X \times_k Y))$$

to $\mathbb{Z}(d_Y)[2d_Y]$. By Corollary 1.10, applied to the setting (3), this group is the target of the morphism cyc_3 on $\mathrm{CH}^{d_Y-d_S}(X \times_S Y)$.

In order to show that for a cycle class z in $\mathrm{CH}^{d_Y-d_S}(X \times_S Y)$, the diagram

$$\begin{array}{ccc} M_{gm}^c(X) & \longrightarrow & \partial M_{gm}(X)[1] \\ cyc_2(z) \downarrow & & \downarrow cyc_3(z)[1] \\ M_{gm}^c(Y) & \longrightarrow & \partial M_{gm}(Y)[1] \end{array}$$

commutes, we need to study the group of morphisms in $DM_{gm}(k)$ from $M_{gm}^c(X)$ to $\partial M_{gm}(Y)[1]$. Again by [W2, Thm. 6.1], it is canonically isomorphic to

$$\mathrm{Hom}_{DM_{gm}(k)}(M_{gm}^c(X) \otimes \partial M_{gm}(Y), \mathbb{Z}(d_Y)[2d_Y]) .$$

But $M_{gm}^c(X) \otimes \partial M_{gm}(Y)$ maps canonically to $M_{gm}^c(X) \otimes M_{gm}(Y)$, which was already identified with the relative motive

$$M_{gm}(\overline{X} \times_k Y / \partial \overline{X} \times_k Y) .$$

Hence the group of morphisms $\mathrm{Hom}_{DM_{gm}(k)}(M_{gm}^c(X), \partial M_{gm}(Y)[1])$ receives an arrow, say β , from

$$\mathrm{Hom}_{DM_{gm}(k)}(M_{gm}(\overline{X} \times_k Y / \partial \overline{X} \times_k Y), \mathbb{Z}(d_Y)[2d_Y]) .$$

The desired commutativity then follows from the compatibility of cyc under restriction from $\overline{X} \times_k \overline{Y} - \partial \overline{X} \times_k \partial \overline{Y}$ to $\overline{X} \times_k Y$ (Corollary 1.10), and from the compatibility of β and the map α from above. The latter is a consequence of the compatibility of the isomorphism

$$\partial M_{gm}(Y)[1] \xrightarrow{\sim} \partial M_{gm}(Y)^*(d_Y)[2d_Y]$$

with duality $M_{gm}(Y) \cong M_{gm}^c(Y)^*(d_Y)[2d_Y]$ [W2, Thm. 6.1].

The proof of the commutativity of

$$\begin{array}{ccc} \partial M_{gm}(X) & \longrightarrow & M_{gm}(X) \\ \text{cyc}_3(z) \downarrow & & \downarrow \text{cyc}_1(z) \\ \partial M_{gm}(Y) & \longrightarrow & M_{gm}(Y) \end{array}$$

is similar.

Altogether, this proves part (a) of the statement. As for parts (b) and (c), simply compose the functor from (a) with duality in $DM_{gm}(k)$, using [V1, Thm. 4.3.7 3] and [W2, Thm. 6.1].

By [W2, Rem. 6.15], our construction is independent of the compactifications \overline{S} , \overline{X} , \overline{Y} . **q.e.d.**

Proof of Theorem 1.5. We keep the notations of the previous proof. Choose compactifications \overline{T} of T , and \overline{U} of U together with cartesian diagrams

$$\begin{array}{ccc} T & \hookrightarrow & \overline{T} \\ f \downarrow & & \downarrow \\ S & \hookrightarrow & \overline{S} \end{array}$$

and

$$\begin{array}{ccc} U & \hookrightarrow & \overline{U} \\ g \downarrow & & \downarrow \\ S & \hookrightarrow & \overline{S} \end{array}$$

(f and g are proper).

(a) Checking the definitions, the transformation $\alpha_{f_{\sharp}}$ is in fact given by the identity. Indeed, both $(\partial M_{gm}, M_{gm}, M_{gm}^c)_S \circ f_{\sharp}$ and $(\partial M_{gm}, M_{gm}, M_{gm}^c)_T$ map the object $h(X/T)$, for $X \in PropSm/T$, to the exact triangle

$$(*)_X \quad \partial M_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}^c(X) \longrightarrow \partial M_{gm}(X)[1].$$

Note that on morphisms, the functor f_{\sharp} corresponds to the push-forward

$$CH_*(X \times_T Y) \longrightarrow CH_*(X \times_S Y)$$

along the closed immersion $X \times_T Y \hookrightarrow X \times_S Y$. The latter factors the closed immersion

$$X \times_T Y \hookrightarrow \overline{X} \times_k \overline{Y} \rightarrow \partial \overline{X} \times_k \partial \overline{Y}.$$

The construction (see the preceding proof) shows then that the effects of the functors $(\partial M_{gm}, M_{gm}, M_{gm}^c)_S \circ f_{\sharp}$ and of $(\partial M_{gm}, M_{gm}, M_{gm}^c)_T$ coincide also on $CH_*(X \times_T Y)$. This shows the first half of statement (a). The second is implied formally by Theorem 1.2 (c).

(b) We first consider an auxiliary functor. The morphism g being proper and smooth, we may consider the composition $g_{\sharp} \circ g^*$ on $CHM(S)$, which on objects is given by $h(Y/S) \mapsto h(Y \times_S U/S)$, for any proper smooth scheme

Y over S . Projection onto the first component then yields a transformation of functors, namely the adjunction

$$b_{g^*, \text{id}_S} : g_{\sharp} \circ g^* \longrightarrow \text{id}_{CHM(S)} .$$

Then define β_{g^*, id_S} to be the composition of transformations

$$\beta_{g^*, \text{id}_S} := ((\partial M_{gm}, M_{gm}, M_{gm}^c)_S \circ b_{g^*, \text{id}_S}) \circ (\alpha_{g_{\sharp}} \circ g^*)^{-1} ,$$

observing the equivalence

$$\alpha_{g_{\sharp}} \circ g^* : (\partial M_{gm}, M_{gm}, M_{gm}^c)_S \circ g_{\sharp} \circ g^* \xrightarrow{\sim} (\partial M_{gm}, M_{gm}, M_{gm}^c)_U \circ g^*$$

from part (a). We leave it to the reader to check the compatibility of this construction with composition of proper smooth morphisms in Sm/k .

(c) Similarly, this commutativity statement is left as an exercise.

(d), (e) Given Theorem 1.2 (c), these statements follow formally from (b) and (c), respectively. **q.e.d.**

For $X, Y \in PropSm/S$, denote by $\bar{c}_S(X, Y)$ the quotient of $c_S(X, Y)$ by the group of cycles \mathfrak{Z} satisfying

$$M_{gm}(\mathfrak{Z}) = 0 \quad , \quad M_{gm}^c(\mathfrak{Z}) = 0 \quad , \quad \partial M_{gm}(\mathfrak{Z}) = 0 .$$

Note that composition of correspondences induces a well-defined composition on \bar{c}_S . In particular, for any $X \in PropSm/S$, the group $\bar{c}_S(X, X)$ carries the structure of an algebra.

Corollary 1.12. *Let X and Y be in $PropSm/S$. Then the projection*

$$c_S(X, Y) \longrightarrow \bar{c}_S(X, Y)$$

factors through the image of $c_S(X, Y)$ in $CH^(X \times_S Y)$. In other words, two cycles $\mathfrak{Z}_1, \mathfrak{Z}_2 \in c_S(X, Y)$ induce the same morphisms $M_{gm}(\mathfrak{Z}_i)$, resp. $M_{gm}^c(\mathfrak{Z}_i)$, resp. $\partial M_{gm}(\mathfrak{Z}_i)$, if they are rationally equivalent (on $X \times_S Y$).*

Remark 1.13. (a) Let $X, Y \in Sm/S$. As shown in [Le3, Lemma 5.18], the map

$$c_S(X, Y) \longrightarrow CH_{d_X}(X \times_S Y)$$

is surjective, whenever Y is projective, and X of pure absolute dimension d_X . Therefore, by Corollary 1.12, the group $\bar{c}_S(X, Y)$ is canonically a quotient of $CH_{d_X}(X \times_S Y)$ if $X \in PropSm/S$ and $Y \in ProjSm/S$.

(b) The observation from (a) fits in the functorial picture sketched in Remark 1.3. Indeed, [D 2, Thm. 5.23] implies that the restriction of the functor M_{gm} ,

$$M_{gm} : CHM(S)_{proj} \longrightarrow DM_{gm}(k)$$

factors canonically through a fully faithful embedding

$$CHM(S)_{proj} \hookrightarrow DM_{gm}(S) .$$

(c) In [Le3, Prop. 5.19], an embedding result analogous to (b) is proven for a dg-version of $DM_{gm}(S)$, denoted $SmMot(S)$ in [loc. cit.], from which the embedding (b) can be deduced [Le3, Cor. 7.13].

(d) Since [D 2, Thm. 5.23] is true for any regular Noetherian base scheme S , the embedding

$$CHM(S)_{proj} \hookrightarrow DM_{gm}(S)$$

holds more generally for any such S .

(e) For $S \in Sm/k$, [D 2, Thm. 5.23] can be employed to show that the image of $CHM(S)_{proj}$ in $DM_{gm}(S)$ is negative, i.e.,

$$\mathrm{Hom}_{DM_{gm}(S)}(M_1, M_2[i]) = 0$$

for any two relative Chow motives $M_1, M_2 \in CHM(S)_{proj}$, and any integer $i > 0$.

(f) When $S = \mathrm{Spec} k$, results (b) and (e) are contained in [V1, Cor. 4.2.6].

Remark 1.14. Fix a non-negative integer d , and consider the full subcategory $CHM(S)_d$ of $CHM(S)$ of Chow motives generated by the Tate twists of $h(X/S)$, for $X \in PropSm/S$ of pure absolute dimension d . The construction of the duality isomorphisms [V1, Thm. 4.3.7 3], [W2, Thm. 6.1] shows that the identification

$$(\partial M_{gm}, M_{gm}, M_{gm}^c)^* = (\partial M_{gm}, M_{gm}, M_{gm}^c)(-d)[-2d]$$

of the restriction of the functors from Theorem 1.2 to $CHM(S)_d$ admits an alternative description, when S is of pure absolute dimension, say s : on $CHM(S)_d$, the functor $(\partial M_{gm}, M_{gm}, M_{gm}^c)^*$ equals then composition of duality in the category $CHM(S)_d$ with $(\partial M_{gm}, M_{gm}, M_{gm}^c)$, followed by the functor $M \mapsto M(-s)[-2s]$. Note that on morphisms, duality in $CHM(S)_d$ corresponds to the transposition $\mathrm{CH}_*(X \times_S Y) \rightarrow \mathrm{CH}_*(Y \times_S X)$.

This observation allows to deduce the following statements from Theorem 1.5.

Corollary 1.15. (a) *Let $g : U \rightarrow S$ be a proper smooth morphism in Sm/k of pure relative dimension d_g . Then there exists a canonical transformation of additive functors*

$$\delta_{\mathrm{id}_S, g^*} : (\partial M_{gm}, M_{gm}, M_{gm}^c)_S \longrightarrow (\partial M_{gm}, M_{gm}, M_{gm}^c)_U(-d_g)[-2d_g] \circ g^* .$$

The formation of $\delta_{\mathrm{id}_S, g^}$ is compatible with composition of proper smooth morphisms in Sm/k of pure relative dimension.*

(b) *Let $g : U \rightarrow S$ be a finite  tale morphism in Sm/k of constant (fibrewise) degree u . Then the endomorphism*

$$\beta_{g^*, \mathrm{id}_S} \circ \delta_{\mathrm{id}_S, g^*}$$

of the functor $(\partial M_{gm}, M_{gm}, M_{gm}^c)_S$ equals multiplication by u .

Proof. (a) We may assume S to be of pure absolute dimension, say s . Consider the transformation

$$\gamma_{\text{id}_S, g^*} : (\partial M_{gm}, M_{gm}, M_{gm}^c)_S^* \longrightarrow (\partial M_{gm}, M_{gm}, M_{gm}^c)_U^* \circ g^*$$

from Theorem 1.5 (d). Composition with duality \mathbb{D}_S in $CHM(S)$ gives

$$\gamma_{\text{id}_S, g^*} \circ \mathbb{D}_S : (\partial M_{gm}, M_{gm}, M_{gm}^c)_S^* \circ \mathbb{D}_S \longrightarrow (\partial M_{gm}, M_{gm}, M_{gm}^c)_U^* \circ g^* \circ \mathbb{D}_S .$$

Now observe the formula

$$\mathbb{D}_U \circ g^* = g^* \circ \mathbb{D}_S$$

($\mathbb{D}_U :=$ duality in $CHM(U)$). Define $\delta_{\text{id}_S, g^*}$ as the composition of $\gamma_{\text{id}_S, g^*} \circ \mathbb{D}_S$ and $M \mapsto M(s)[2s]$, observing that source and target of $\delta_{\text{id}_S, g^*}$ are identified with $(\partial M_{gm}, M_{gm}, M_{gm}^c)_S$ and $(\partial M_{gm}, M_{gm}, M_{gm}^c)_U(-d_g)[-2d_g] \circ g^*$, respectively.

(b) The morphism g being finite and étale, we have

$$\mathbb{D}_S \circ g_{\sharp} = g_{\sharp} \circ \mathbb{D}_U .$$

This shows that g_{\sharp} is also right adjoint to g^* . Checking the definitions, the composition $\beta_{g^*, \text{id}_S} \circ \delta_{\text{id}_S, g^*}$ equals up to twist and shift the composition of the two adjunctions

$$\xi : \text{id}_{CHM(S)} \longrightarrow g_{\sharp} \circ g^* \longrightarrow \text{id}_{CHM(S)} ,$$

preceded by duality, and followed by $(\partial M_{gm}, M_{gm}, M_{gm}^c)_S^*$. These functors being additive, it suffices to show that ξ equals multiplication by u . But this identity on morphisms of relative Chow motives is classical. **q.e.d.**

The main results of this section have obvious F -linear versions, for any commutative \mathbb{Q} -algebra F . Let us now describe how our analysis of the functor $(\partial M_{gm}, M_{gm}, M_{gm}^c)$ will be used in the sequel.

Example 1.16. Let $g_1, g_2 : U \rightarrow S$ be two finite étale morphisms in Sm/k . Fix an object $X \in PropSm/S$, an idempotent e on $h(X/S)$ (possibly belonging to $\text{CH}^*(X \times_S X) \otimes_{\mathbb{Z}} F$, for some commutative \mathbb{Q} -algebra F), and a morphism

$$\varphi : g_1^*(h(X/S)^e) \longrightarrow g_2^*(h(X/S)^e)$$

in $CHM(U)$ (or $CHM(U)_F$).

(a) Let us define an endomorphism of $(\partial M_{gm}, M_{gm}, M_{gm}^c)(h(X/S)^e)$ “of Hecke type”, denoted $\varphi(g_1, g_2)$, by composing

$$\delta_{\text{id}_S, g_1^*} : (\partial M_{gm}, M_{gm}, M_{gm}^c)(h(X/S)^e) \longrightarrow (\partial M_{gm}, M_{gm}, M_{gm}^c)(g_1^*(h(X/S)^e))$$

first with $(\partial M_{gm}, M_{gm}, M_{gm}^c) \circ \varphi$, and then with

$$\beta_{g_2^*, \text{id}_S} : (\partial M_{gm}, M_{gm}, M_{gm}^c)(g_2^*(h(X/S)^e)) \rightarrow (\partial M_{gm}, M_{gm}, M_{gm}^c)(h(X/S)^e) .$$

(b) Note that unless $g_1 = g_2$, the endomorphism $\varphi(g_1, g_2)$ is in general *not* the image of an endomorphism on the relative Chow motive $h(X/S)^e$ under

the functor $(\partial M_{gm}, M_{gm}, M_{gm}^c)$.

(c) If φ is an isomorphism, with inverse ψ , then using the construction from (a), the endomorphism $\psi(g_2, g_1)$ on $(\partial M_{gm}, M_{gm}, M_{gm}^c)(h(X/S)^e)$ can be defined. If X is of pure absolute dimension d_X , then $\psi(g_2, g_1)$ equals the dual of $\varphi(g_1, g_2)$, twisted by d_X and shifted by $2d_X$, under the identification

$$(\partial M_{gm}, M_{gm}, M_{gm}^c)^*(h(X/S)) = (\partial M_{gm}, M_{gm}, M_{gm}^c)(h(X/S))(-d_X)[-2d_X]$$

from Theorem 1.2 (b1). We leave the details of the verification to the reader.

(d) In practice, the morphism $\varphi : g_1^*(h(X/S)^e) \rightarrow g_2^*(h(X/S)^e)$ will be obtained from a morphism of relative Chow motives over U

$$\varphi : h(X \times_{S, g_1} U/U) = g_1^*(h(X/S)) \longrightarrow g_2^*(h(X/S)) = h(X \times_{S, g_2} U/U)$$

satisfying the equation

$$\varphi \circ g_1^*(e) = g_2^*(e) \circ \varphi$$

in $\text{CH}^*((X \times_{S, g_1} U) \times_U (X \times_{S, g_2} U))$ (or $\text{CH}^*((X \times_{S, g_1} U) \times_U (X \times_{S, g_2} U)) \otimes_{\mathbb{Z}} F$). In that case, $\varphi(g_1, g_2)$ can be seen as an endomorphism of the whole of $(\partial M_{gm}, M_{gm}, M_{gm}^c)(h(X/S))$ commuting with e .

(e) In the setting of (d), assume that the morphism

$$\varphi : h(X \times_{S, g_1} U/U) \longrightarrow h(X \times_{S, g_2} U/U)$$

is represented by the cycle \mathfrak{Z} in

$$c_U(X \times_{S, g_1} U, X \times_{S, g_2} U)$$

(or in $c_U(X \times_{S, g_1} U, X \times_{S, g_2} U) \otimes_{\mathbb{Z}} F$). Checking the definitions, one sees that the M_{gm} -component of $\varphi(g_1, g_2)$ is then represented by the image of \mathfrak{Z} under the direct image

$$(g_1 \times_k g_2)_* : c_U(X \times_{S, g_1} U, X \times_{S, g_2} U) \longrightarrow c(X, X)$$

(or under $(g_1 \times_k g_2)_* \otimes F$).

2 Motives associated to Abelian schemes

Fix a field k admitting strict resolution of singularities, and a base $S \in \text{Sm}/S$. In this section, we combine the main result from [DeM] with the theory developed in Section 1. Recall the following.

Theorem 2.1 ([DeM, Thm. 3.1, Prop. 3.3]). *(a) Let A/S be an Abelian scheme of relative dimension g . Then there is a unique decomposition of the class of the diagonal $(\Delta) \in \text{CH}^g(A \times_S A) \otimes_{\mathbb{Z}} \mathbb{Q}$,*

$$(\Delta) = \sum_{i=0}^{2g} p_{A,i}$$

such that $p_{A,i} \circ (\Gamma_{[n]_A}) = n^i \cdot p_{A,i}$ for all i , and all integers n . The $p_{A,i}$ are mutually orthogonal idempotents, and $(\Gamma_{[n]_A}) \circ p_{A,i} = n^i \cdot p_{A,i}$ for all i .

(b) For any morphism $f : A \rightarrow B$ of Abelian schemes over S , and any i ,

$$p_{B,i} \circ (\Gamma_f) = (\Gamma_f) \circ p_{A,i} \in \mathrm{CH}^*(A \times_S B) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In other words, the decomposition in (a) is covariantly functorial in A .

(c) For any isogeny $g : B \rightarrow A$ of Abelian schemes over S , and any i ,

$$p_{B,i} \circ ({}^t\Gamma_g) = ({}^t\Gamma_g) \circ p_{A,i} \in \mathrm{CH}^*(A \times_S B) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In other words, the decomposition in (a) is contravariantly functorial under isogenies.

We use the notation Γ_h for the graph of a morphism h of S -schemes, $[n]_A$ for the multiplication by n on the Abelian scheme A , $(\mathfrak{3})$ for the class of a cycle $\mathfrak{3}$, and ${}^t\mathfrak{3}$ for its transposition. Let

$$h(A/S) = \bigoplus_i h_i(A/S)$$

be the decomposition of the relative motive of A corresponding to the decomposition $(\Delta) = \sum_i p_{A,i}$. Thus, on the term $h_i(A/S)$, the cycle class $(\Gamma_{[n]_A})$ acts *via* multiplication by n^i .

Now recall the exact triangle

$$(*)_A \quad \partial M_{gm}(A) \longrightarrow M_{gm}(A) \longrightarrow M_{gm}^c(A) \longrightarrow \partial M_{gm}(A)[1].$$

By Theorem 1.2 (a), the cycle classes $p_{A,i}$ induce endomorphisms of $(*)_A$, when considered as an exact triangle in $DM_{gm}^{eff}(k)_{\mathbb{Q}}$.

Theorem 2.2. (a) Let A/S be an Abelian scheme of relative dimension g . For $0 \leq i \leq 2g$, denote by $M_{gm}(A)_i$, $M_{gm}^c(A)_i$ and $\partial M_{gm}(A)_i$ the images of the idempotent $p_{A,i}$ on $M_{gm}(A)$, $M_{gm}^c(A)$ and $\partial M_{gm}(A)$, respectively, considered as objects of the category $DM_{gm}^{eff}(k)_{\mathbb{Q}}$. Then for any i , the triangle

$$(*)_{A,i} \quad \partial M_{gm}(A)_i \longrightarrow M_{gm}(A)_i \longrightarrow M_{gm}^c(A)_i \longrightarrow \partial M_{gm}(A)_i[1]$$

in $DM_{gm}^{eff}(k)_{\mathbb{Q}}$ is exact.

(b) The direct sum of the triangles $(*)_{A,i}$ yields a decomposition

$$(*)_A = \bigoplus_{i=0}^{2g} (*)_{A,i}.$$

It has the following properties:

(b1) for any integer n , the decomposition is respected by $[n]_A$.

(b2) for each i and n , the induced morphisms $[n]_{A,i}$ on the three terms of $(*)_{A,i}$ equal multiplication by n^i .

(c) As a decomposition of $(*)_A$ into some finite direct sum of exact triangles in $DM_{gm}^{eff}(k)_{\mathbb{Q}}$,

$$(*)_A = \bigoplus_i (*)_{A,i}$$

is uniquely determined by properties (b1) and (b2). More precisely, it is uniquely determined by the following properties:

(c1) for some integer $n \neq -1, 0, 1$, the decomposition is respected by $[n]_A$.

(c2) for the choice of n made in (c1) and each i , the induced morphism $[n]_{A,i}$ on the three terms of $(*)_{A,i}$ equals multiplication by n^i .

(d) The decomposition

$$(*)_A = \bigoplus_i (*)_{A,i}$$

is covariantly functorial under morphisms, and contravariantly functorial under isogenies of Abelian schemes over S .

Proof. Part (a) is a formal consequence of the fact that the $p_{A,i}$ are idempotent.

Parts (b) and (d) follow from Theorem 2.1 and the functoriality statement from Theorem 1.2 (a).

Part (c) is left to the reader. **q.e.d.**

The following seems worthwhile to note explicitly.

Corollary 2.3. *Let A/S be an Abelian scheme of relative dimension g . Then the boundary motive $\partial M_{gm}(A)$ decomposes functorially into a direct sum*

$$\partial M_{gm}(A) = \bigoplus_{i=0}^{2g} \partial M_{gm}(A)_i .$$

On $\partial M_{gm}(A)_i$, the endomorphism $[n]_A$ acts via multiplication by n^i , for any integer n , and any $0 \leq i \leq 2g$.

Here is an illustration of the surjectivity proved in [Le3, Lemma 5.18].

Proposition 2.4. *Let A/S be an Abelian scheme. The elements $p_{A,i}$ of $\mathrm{CH}^*(A \times_S A) \otimes_{\mathbb{Z}} \mathbb{Q}$ lie in the image of*

$$c_S(A, A) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{CH}^*(A \times_S A) \otimes_{\mathbb{Z}} \mathbb{Q} .$$

More precisely, for any integer $n \neq -1, 0, 1$,

$$\pi_{A,i,n} := \prod_{j \neq i} \frac{\Gamma_{[n]_A} - n^j}{n^i - n^j}$$

is a pre-image of $p_{A,i}$ in $c_S(A, A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. On each of the direct factors $h_j(A/S) \subset h(A/S)$, the projector $p_{A,i}$ acts *via* multiplication by the Kronecker symbol δ_{ij} , while $(\Gamma_{[n]_A})$ acts *via* multiplication by n^j . Therefore,

$$p_{A,i} = \prod_{j \neq i} \frac{(\Gamma_{[n]_A}) - n^j}{n^i - n^j} \in \mathrm{CH}^*(A \times_S A) \otimes_{\mathbb{Z}} \mathbb{Q},$$

for any integer $n \neq -1, 0, 1$. Therefore, the element $\pi_{A,i,n} \in c_S(A, A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is indeed a pre-image of $p_{A,i}$. **q.e.d.**

3 Artin–Tate motives

We fix a commutative \mathbb{Q} -algebra F , supposed to be semi-simple and Noetherian, in other words, a finite direct product of fields of characteristic zero. In this section, we define the F -linear triangulated category of Artin–Tate motives (Definition 3.3), along with a number of variants, indexed by certain sub-categories of the category of discrete representations of the absolute Galois group of our perfect base field k (Definition 3.6). We then start the analysis of this category, following the part of [Le1] valid without additional assumptions on our perfect base field k .

For any integer m , there is defined a Tate object $\mathbb{Z}(m)$ in $DM_{gm}(k)$, which belongs to $DM_{gm}^{eff}(k)$ if $m \geq 0$ [V1, p. 192]. We shall use the same notation when we consider $\mathbb{Z}(m)$ as an object of $DM_{gm}(k)_F$.

Definition 3.1 (cmp. [Le1, Def. 3.1]). Define the *triangulated category of Tate motives over k* as the strict full triangulated sub-category $DMT(k)_F$ of $DM_{gm}(k)_F$ generated by the $\mathbb{Z}(m)$, for $m \in \mathbb{Z}$.

Recall that by definition, a strict sub-category is closed under isomorphisms in the ambient category. It is easy to see that $DMT(k)_F$ is tensor triangulated.

Definition 3.2. Define the *triangulated category of Artin motives over k* as the pseudo-Abelian completion of the strict full triangulated sub-category $DMA(k)_F$ of $DM_{gm}^{eff}(k)_F$ generated by the motives $M_{gm}(X)$ of smooth zero-dimensional schemes X over k .

This category is again tensor triangulated.

Definition 3.3. Define the *triangulated category of Artin–Tate motives over k* as the strict full tensor triangulated sub-category $DMAT(k)_F$ of $DM_{gm}(k)_F$ generated by $DMA(k)_F$ and $DMT(k)_F$.

The following observation [V1, Remark 2 on p. 217] is central for what is to follow.

Proposition 3.4. *The triangulated category $DMA(k)_F$ of Artin motives is canonically equivalent to $D^b(MA(k)_F)$, the bounded derived category of the Abelian category $MA(k)_F$ of discrete representations of the absolute Galois group of k in finitely generated F -modules.*

More precisely, if X is smooth and zero-dimensional over k , and \bar{k} a fixed algebraic closure of k , then the absolute Galois group of k , when identified with the group of automorphisms of \bar{k} over k , acts canonically on the set of \bar{k} -valued points of X . The object of $MA(k)_F$ corresponding to $M(X)$ under the equivalence of Proposition 3.4 is nothing but the formal F -linear envelope of this set, with the induced action of the Galois group. Note that the category $MA(k)_F$ is semi-simple.

Corollary 3.5. *There is a canonical non-degenerate t -structure on the category $DMA(k)_F$. Its heart is equivalent to $MA(k)_F$.*

By contrast [V1, Remark 1 on p. 217], it is not clear how to construct a non-degenerate t -structure on the triangulated category $DMA(k)$ of zero motives (whose F -linearization equals $DMA(k)_F$).

For the rest of this section, let us identify the triangulated categories $DMA(k)_F$ and $D^b(MA(k)_F)$ via the equivalence of Proposition 3.4. Let us also fix a strict full Abelian semi-simple F -linear tensor sub-category \mathcal{A} of $MA(k)_F$, containing the category $triv$ of objects of $MA(k)_F$ on which the Galois group acts trivially.

Definition 3.6. Define DAT as the strict full tensor triangulated sub-category of $DMAT(k)_F$ generated by \mathcal{A} , and by $DMT(k)_F$.

Examples 3.7. (a) When \mathcal{A} equals $MA(k)_F$, then $DAT = DMAT(k)_F$.
(b) When \mathcal{A} equals $triv$, then $DAT = DMT(k)_F$.

Let us agree to set $\mathbb{Z}(n/2) := 0$ for odd integers n . For any object M of DAT and any integer n , let us write $M(n/2)$ for the tensor product of M and $\mathbb{Z}(n/2)$.

Following [Le1], let us first define $DAT_{[a,b]}$ as the full triangulated sub-category of DAT generated by the objects $N(m)$, for $N \in \mathcal{A}$ and $a \leq -2m \leq b$, for integers $a \leq b$ (we allow $a = -\infty$ and $b = \infty$). We denote $DAT_{[a,a]}$ by DAT_a .

Proposition 3.8. *The category DAT_a is zero for $a \in \mathbb{Z}$ odd. For $a \in \mathbb{Z}$ even, the exact functor*

$$DAT_a \longrightarrow DMA(k)_F, \quad M \longmapsto M(a/2)$$

induces an equivalence between DAT_a and the bounded derived category of \mathcal{A} (which is equal to the \mathbb{Z} -graded category $\text{Gr}_{\mathbb{Z}} \mathcal{A} = \bigoplus_{m \in \mathbb{Z}} \mathcal{A}$ over \mathcal{A}).

Proof. By construction, the functor is exact, and identifies DAT_a with the full triangulated sub-category of $DMA(k)_F$ of objects, whose cohomology lies in \mathcal{A} . Recall that we identified the categories $DMA(k)_F$ and $D^b(MA(k)_F)$. It remains to see that the obvious exact functor

$$D^b(\mathcal{A}) \longrightarrow D^b(MA(k)_F)$$

is fully faithful. But this is an immediate consequence of the fact that the Abelian categories \mathcal{A} and $MA(k)_F$ are semi-simple. **q.e.d.**

In particular, there is a canonical t -structure $(DAT_a^{\leq 0}, DAT_a^{\geq 0})$ on DAT_a : the category $DAT_a^{\leq 0}$ is the full sub-category of DAT_a generated by objects $N(-a/2)[r]$, for $N \in \mathcal{A}$ and $r \geq 0$, and $DAT_a^{\geq 0}$ is the full sub-category generated by objects $N(-a/2)[r]$, for $N \in \mathcal{A}$ and $r \leq 0$. If a is even, then the category \mathcal{A} is equivalent to the heart \mathcal{AT}_a of this canonical t -structure via the functor $N \mapsto N(-a/2)$.

Second, we construct auxiliary t -structures.

Proposition 3.9 (cmp. [Le1, Lemma 1.2]). *Let $a \leq n \leq b$. Then the pair $(DAT_{[a,n]}, DAT_{[n+1,b]})$ defines a t -structure on $DAT_{[a,b]}$.*

Proof. Imitate the proof of [Le1, Lemma 1.2]. The decisive ingredient is the following generalization of the vanishing from [Le1, Def. 1.1 i)]:

$$\mathrm{Hom}_{DAT}(N_1(m_1)[r], N_2(m_2)[s]) = 0, \quad \forall m_1 > m_2, N_1, N_2 \in \mathcal{A}, r, s \in \mathbb{Z}.$$

It holds because $\mathrm{Hom}_{DAT} = \mathrm{Hom}_{DM_{gm}(k)_F}$, and $\mathrm{Hom}_{DM_{gm}(\bullet)_F}$ satisfies descent for finite extensions L/k of the base field. Choosing an extension L splitting both N_1 and N_2 therefore allows to deduce the desired vanishing from that of

$$\mathrm{Hom}_{DM_{gm}(L)_{\mathbb{Q}}}(\mathbb{Z}(m_1)[r], \mathbb{Z}(m_2)[s]).$$

q.e.d.

Remark 3.10. (a) The above proof uses the relation of K -theory of L tensored with \mathbb{Q} , with $\mathrm{Hom}_{DM_{gm}(L)_{\mathbb{Q}}}$. This relation is established by work of Bloch [Blc1, Blc2] (see [Le2, Section II.3.6]), and will be used again in the proofs of Theorem 5.1 and Variant 5.2.

(b) Levine pointed out that the t -structures from Proposition 3.9, for varying n , can be used to show that the category DAT is pseudo-Abelian. We shall give an alternative proof of this result in Section 4, using Bondarko's theory of weight structures (Corollary 4.6).

Note that since $DAT_{[a,n]}$ and $DAT_{[n+1,b]}$ are themselves triangulated, the t -structure from Proposition 3.9 is necessarily degenerate. As in [Le1, Sect. 1], denote the truncation functors by

$$W_{\leq n} : DAT_{[a,b]} \longrightarrow DAT_{[a,n]}$$

and

$$W_{\geq n+1} : \mathcal{DAT}_{[a,b]} \longrightarrow \mathcal{DAT}_{[n+1,b]} ,$$

and note that for fixed n , they are compatible with change of a or b . Write gr_n for the composition of $W_{\leq n}$ and $W_{\geq n}$ (in either sense). The target of this functor is the category \mathcal{DAT}_n . We are now ready to set up the data necessary for the t -structure we shall actually be interested in.

Definition 3.11 (cmp. [Le1, Def. 1.4]). Fix $a \leq b$ (we allow $a = -\infty$ and $b = \infty$).

- (a) Define $\mathcal{DAT}_{[a,b]}^{\leq 0}$ as the full sub-category of $\mathcal{DAT}_{[a,b]}$ of objects M such that $\mathrm{gr}_n M \in \mathcal{DAT}_n^{\leq 0}$ for all integers n such that $a \leq n \leq b$.
- (b) Define $\mathcal{DAT}_{[a,b]}^{\geq 0}$ as the full sub-category of $\mathcal{DAT}_{[a,b]}$ of objects M such that $\mathrm{gr}_n M \in \mathcal{DAT}_n^{\geq 0}$ for all integers n such that $a \leq n \leq b$.

As we shall see (Theorem 5.1, Variant 5.2), the pair $(\mathcal{DAT}_{[a,b]}^{\leq 0}, \mathcal{DAT}_{[a,b]}^{\geq 0})$ defines a t -structure on $\mathcal{DAT}_{[a,b]}$, provided that the base field k is algebraic over \mathbb{Q} . In particular, we then get a canonical t -structure on \mathcal{DAT} . The vital point will be the validity of the Beilinson–Soulé vanishing conjecture for all finite field extensions of k .

4 The motivic weight structure

The purpose of this section is to first review Bondarko’s definition of weight structures on triangulated categories, and his result on the existence of such a weight structure on the categories $DM_{gm}(k)$ and $DM_{gm}(k)_F$ [Bo]. We will then show (Theorem 4.5 (a)) that the latter induces a weight structure on any of the triangulated categories constructed in Section 3. The explicit description of the heart of this weight structure (Theorem 4.5 (b), (c)) will turn out to be very useful. In particular, we derive a description of objects with two adjacent weights (Corollary 4.9), which will be used in our analysis of the motivic cohomology of Hilbert–Blumenthal surfaces (see Section 7).

Definition 4.1 (cmp. [Bo, Def. 1.1.1]). Let \mathcal{C} be a triangulated category. A *weight structure* on \mathcal{C} is a pair $w = (\mathcal{C}_{w \leq 0}, \mathcal{C}_{w \geq 0})$ of full sub-categories of \mathcal{C} , such that, putting

$$\mathcal{C}_{w \leq n} := \mathcal{C}_{w \leq 0}[n] \quad , \quad \mathcal{C}_{w \geq n} := \mathcal{C}_{w \geq 0}[n] \quad \forall n \in \mathbb{Z} ,$$

the following conditions are satisfied.

- (1) The categories $\mathcal{C}_{w \leq 0}$ and $\mathcal{C}_{w \geq 0}$ are Karoubi-closed (i.e., closed under retracts formed in \mathcal{C}).

(2) (Semi-invariance with respect to shifts.) We have the inclusions

$$\mathcal{C}_{w \leq 0} \subset \mathcal{C}_{w \leq 1} \quad , \quad \mathcal{C}_{w \geq 0} \supset \mathcal{C}_{w \geq 1}$$

of full sub-categories of \mathcal{C} .

(3) (Orthogonality.) For any pair of objects $M \in \mathcal{C}_{w \leq 0}$ and $N \in \mathcal{C}_{w \geq 1}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(M, N) = 0 .$$

(4) (Weight filtration.) For any object $M \in \mathcal{C}$, there exists an exact triangle

$$A \longrightarrow M \longrightarrow B \longrightarrow A[1]$$

in \mathcal{C} , such that $A \in \mathcal{C}_{w \leq 0}$ and $B \in \mathcal{C}_{w \geq 1}$.

It is easy to see that for any integer n and any object $M \in \mathcal{C}$, there is an exact triangle

$$A \longrightarrow M \longrightarrow B \longrightarrow A[1]$$

in \mathcal{C} , such that $A \in \mathcal{C}_{w \leq n}$ and $B \in \mathcal{C}_{w \geq n+1}$. By a slight generalization of the terminology introduced in condition 4.1 (4), we shall refer to any such exact triangle as a weight filtration of M .

Remark 4.2. Our convention concerning the sign of the weight is opposite to the one from [Bo, Def. 1.1.1], i.e., we exchanged the roles of $\mathcal{C}_{w \leq 0}$ and $\mathcal{C}_{w \geq 0}$.

Definition 4.3 ([Bo, Def. 1.2.1]). Let $w = (\mathcal{C}_{w \leq 0}, \mathcal{C}_{w \geq 0})$ be a weight structure on \mathcal{C} . The *heart of w* is the full additive sub-category $\mathcal{C}_{w=0}$ of \mathcal{C} whose objects lie both in $\mathcal{C}_{w \leq 0}$ and in $\mathcal{C}_{w \geq 0}$.

One of the main results of [Bo] is the following.

Theorem 4.4 ([Bo, Sect. 6]). (a) *If k is of characteristic zero, then there is a canonical weight structure on the category $DM_{gm}^{eff}(k)$. It is uniquely characterized by the requirement that its heart equal $CHM^{eff}(k)$.*

(b) *If k is of characteristic zero, then there is a canonical weight structure on the category $DM_{gm}(k)$, extending the weight structure from (a). It is uniquely characterized by the requirement that its heart equal $CHM(k)$.*

(c) *Let F be a commutative \mathbb{Q} -algebra. Analogues of statements (a) and (b) hold for the F -linearized categories $DM_{gm}^{eff}(k)_F$, $CHM^{eff}(k)_F$, $DM_{gm}(k)_F$, and $CHM(k)_F$, and for a perfect base field k of arbitrary characteristic.*

Let us refer to any of these weight structures as *motivic*. For a concise review of the main ingredients of Bondarko's proof, see [W4, Sect. 1].

Now fix a finite direct product F of fields of characteristic zero, and a full Abelian F -linear tensor sub-category \mathcal{A} of $MA(k)_F$, containing the category *triv*. Recall (Definition 3.6) that

$$DAT \subset DMAT(k)_F \subset DM_{gm}(k)_F$$

denotes the strict full tensor triangulated sub-category generated by \mathcal{A} , and by the triangulated category $DMT(k)_F$ of Tate motives. Intersecting with DAT , the motivic weight structure $(DM_{gm}(k)_{F,w \leq 0}, DM_{gm}(k)_{F,w \geq 0})$ from Theorem 4.4 (c) yields a pair

$$w := w_{\mathcal{A}} := (DAT_{w \leq 0}, DAT_{w \geq 0})$$

of full sub-categories of DAT .

Theorem 4.5. (a) *The pair w is a weight structure on DAT .*

(b) *The heart $DAT_{w=0}$ equals the intersection of DAT and $CHM(k)_F$. It generates the triangulated category DAT . It is Abelian semi-simple. Its objects are finite direct sums of objects of the form $N(m)[2m]$, for $N \in \mathcal{A}$ and $m \in \mathbb{Z}$.*

(c) *The functor from the \mathbb{Z} -graded category $\text{Gr}_{\mathbb{Z}} \mathcal{A}$ over \mathcal{A} to $DAT_{w=0}$*

$$\text{Gr}_{\mathbb{Z}} \mathcal{A} = \bigoplus_{m \in \mathbb{Z}} \mathcal{A} \longrightarrow DAT_{w=0}, \quad (N_m)_{m \in \mathbb{Z}} \longmapsto \bigoplus_{m \in \mathbb{Z}} N_m(m)[2m]$$

is an equivalence of categories.

Proof. Define \mathcal{K} as the full additive sub-category of DAT of objects, which are finite direct sums of objects of the form $N(m)[2m]$, for $N \in \mathcal{A}$ and $m \in \mathbb{Z}$. Note that \mathcal{K} generates the triangulated category DAT . All objects of \mathcal{K} are Chow motives. In particular, by orthogonality 4.1 (3) for the motivic weight structure (see [V1, Cor. 4.2.6]), \mathcal{K} is *negative*, i.e.,

$$\text{Hom}_{DAT}(M_1, M_2[i]) = \text{Hom}_{DM_{gm}(k)_F}(M_1, M_2[i]) = 0$$

for any two objects M_1, M_2 of \mathcal{K} , and any integer $i > 0$. Therefore, [Bo, Thm. 4.3.2 II 1] can be applied to ensure the existence of a weight structure v on DAT , uniquely characterized by the property of containing \mathcal{K} in its heart. Furthermore [Bo, Thm. 4.3.2 II 2], the heart $DAT_{v=0}$ of v is equal to the category \mathcal{K}' of retracts of objects of \mathcal{K} in DAT . In particular, it is contained in the heart $CHM(k)_F$ of the motivic weight structure. The existence of weight filtrations 4.1 (4) for the weight structure v then formally implies that

$$DAT_{v \leq 0} \subset DM_{gm}(k)_{F,w \leq 0},$$

and that

$$DAT_{v \geq 0} \subset DM_{gm}(k)_{F,w \geq 0}.$$

Now let $M_1 \in DAT_{w \leq 0} = DAT \cap DM_{gm}(k)_{F, w \leq 0}$. Then for any $M_2 \in DAT_{v \geq 1}$, we have

$$\mathrm{Hom}_{DAT}(M_1, M_2) = 0 ,$$

thanks to orthogonality 4.1 (3) for the motivic weight structure, and to the fact that $DAT_{v \geq 1}$ is contained in $DM_{gm}(k)_{F, w \geq 1}$. Axioms 4.1 (1) and (4) easily imply (see also [Bo, Prop. 1.3.3 2]) that $M_1 \in DAT_{v \leq 0}$. Therefore,

$$DAT_{w \leq 0} = DAT_{v \leq 0} .$$

In the same way, one proves that

$$DAT_{w \geq 0} = DAT_{v \geq 0} .$$

Altogether, the weight structure v coincides with the data $w = w_{\mathcal{A}}$. This proves part (a) of our claim. We also see that part (b) is formally implied by the following claim. (b') The category \mathcal{K} is Abelian semi-simple. (Since then \mathcal{K} will necessarily be pseudo-Abelian, hence $DAT_{w=0} = \mathcal{K}'$ coincides with \mathcal{K} .)

Now consider two objects N_1, N_2 of \mathcal{A} , two integers m_1, m_2 , and the group of morphisms

$$\mathrm{Hom}(N_1(m_1)[2m_1], N_2(m_2)[2m_2]) = \mathrm{Hom}(N_1, N_2(m_2 - m_1)[2(m_2 - m_1)])$$

in DAT . Two essentially different cases occur: if $m_1 \neq m_2$, then the group of morphisms is zero. Indeed, using descent for finite extensions of k as in the proof of Proposition 3.9, we reduce ourselves to the case $N_1 = N_2 = \mathbb{Z}$, where the desired vanishing follows from [V1, Prop. 4.2.9].

If $m_1 = m_2$, then

$$\mathrm{Hom}(N_1(m_1)[2m_1], N_2(m_2)[2m_2]) = \mathrm{Hom}(N_1, N_2)$$

can be calculated in the Abelian category \mathcal{A} .

Thus in any of the two cases, the group $\mathrm{Hom}(N_1(m_1)[2m_1], N_2(m_2)[2m_2])$ coincides with

$$\mathrm{Hom}_{\mathrm{Gr}_{\mathbb{Z}} \mathcal{A}}((N_1)_{m=m_1}, (N_2)_{m=m_2}) .$$

Therefore, the functor defined in part (c) of the claim is fully faithful. Furthermore, it induces an equivalence of categories between $\mathrm{Gr}_{\mathbb{Z}} \mathcal{A}$ and \mathcal{K} . The latter is therefore Abelian semi-simple. This shows (b'), hence part (b) of our claim. It also shows part (c). **q.e.d.**

Statement (b) of Theorem 4.5 should be considered as remarkable in that it happens rarely that the heart of a weight structure is Abelian. We refer to [Pa, Thm. 3.2], where this question is studied abstractly.

Corollary 4.6. *The category DAT is pseudo-Abelian.*

Proof. By Theorem 4.5 (b), the heart $DAT_{w=0}$ is pseudo-Abelian and generates the triangulated category DAT . Our claim thus follows from [Bo, Lemma 5.2.1]. **q.e.d.**

Here is another formal consequence of Theorem 4.5 (b).

Corollary 4.7. (a) *The inclusion of the heart $\iota_- : \mathcal{DA}T_{w=0} \hookrightarrow \mathcal{DA}T_{w \leq 0}$ admits a left adjoint*

$$\mathrm{Gr}_0 : \mathcal{DA}T_{w \leq 0} \longrightarrow \mathcal{DA}T_{w=0} .$$

For any $M \in \mathcal{DA}T_{w \leq 0}$, the adjunction morphism $M \rightarrow \mathrm{Gr}_0 M$ gives rise to a weight filtration

$$M_{\leq -1} \longrightarrow M \longrightarrow \mathrm{Gr}_0 M \longrightarrow M_{\leq -1}[1]$$

of M . The composition $\mathrm{Gr}_0 \circ \iota_-$ equals the identity on $\mathcal{DA}T_{w=0}$.

(b) *The inclusion of the heart $\iota_+ : \mathcal{DA}T_{w=0} \hookrightarrow \mathcal{DA}T_{w \geq 0}$ admits a right adjoint*

$$\mathrm{Gr}_0 : \mathcal{DA}T_{w \geq 0} \longrightarrow \mathcal{DA}T_{w=0} .$$

For any $M \in \mathcal{DA}T_{w \geq 0}$, the adjunction morphism $\mathrm{Gr}_0 M \rightarrow M$ gives rise to a weight filtration

$$\mathrm{Gr}_0 M \longrightarrow M \longrightarrow M_{\geq 1} \longrightarrow \mathrm{Gr}_0 M[1]$$

of M . The composition $\mathrm{Gr}_0 \circ \iota_+$ equals the identity on $\mathcal{DA}T_{w=0}$.

Proof. Let $M \in \mathcal{DA}T_{w \leq 0}$. First choose an exact triangle

$$M_{\leq -2} \longrightarrow M \longrightarrow M_{-1,0} \longrightarrow M_{\leq -2}[1] ,$$

with $M_{\leq -2} \in \mathcal{DA}T_{w \leq -2}$ and $M_{-1,0} \in \mathcal{DA}T_{w \geq -1} \cap \mathcal{DA}T_{w \leq 0}$. Orthogonality 4.1 (3), together with the fact that $M_{\leq -2}[1] \in \mathcal{DA}T_{w \leq -1}$ shows that the morphism $M \rightarrow M_{-1,0}$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{DA}T}(M_{-1,0}, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{DA}T}(M, N)$$

for any object N of the heart $\mathcal{DA}T_{w=0}$. Now choose an exact triangle

$$M'_{-1} \longrightarrow M_{-1,0} \longrightarrow M'_0 \xrightarrow{\alpha} M'_{-1}[1] ,$$

with $M'_{-1} \in \mathcal{DA}T_{w=-1}$ (hence $M'_{-1}[1] \in \mathcal{DA}T_{w=0}$) and $M'_0 \in \mathcal{DA}T_{w=0}$. Recall that according to Theorem 4.5 (b), $\mathcal{DA}T_{w=0}$ is Abelian semi-simple. Therefore, the morphism α has a kernel and an image, both of which admit direct complements in M'_0 and in $M'_{-1}[1]$, respectively. Choose a direct complement M_0 of $\ker \alpha$ in M'_0 , and a direct complement $M_{-1}[1]$ of $\mathrm{im} \alpha$ in $M'_{-1}[1]$ (for some $M_{-1} \in \mathcal{DA}T_{w=-1}$). *Via* the restriction of α , the object M_0 is isomorphic to the image. We thus get a commutative diagram

$$\begin{array}{ccc} M'_0 & \xrightarrow{\alpha} & M'_{-1}[1] \\ \downarrow & & \downarrow \\ \ker \alpha =: \mathrm{Gr}_0 M & \xrightarrow{0} & M_{-1}[1] \end{array}$$

in $DAT_{w=0}$ and in fact, a morphism of exact triangles

$$\begin{array}{ccccccc} M'_{-1} & \longrightarrow & M_{-1,0} & \longrightarrow & M'_0 & \xrightarrow{\alpha} & M'_{-1}[1] \\ \downarrow & & \parallel & & \downarrow & & \downarrow \\ M_{-1} & \longrightarrow & M_{-1,0} & \longrightarrow & \mathrm{Gr}_0 M & \xrightarrow{0} & M_{-1}[1] \end{array}$$

in DAT . By construction, and by orthogonality 4.1 (3), the morphism $M_{-1,0} \rightarrow \mathrm{Gr}_0 M$ induces an isomorphism

$$\mathrm{Hom}_{DAT}(\mathrm{Gr}_0 M, N) \xrightarrow{\sim} \mathrm{Hom}_{DAT}(M_{-1,0}, N)$$

for any object N of the heart $DAT_{w=0}$. Choosing a cone of the composition $M \rightarrow M_{-1,0} \rightarrow \mathrm{Gr}_0 M$, we get exact triangles

$$M_{\leq -1} \longrightarrow M \longrightarrow \mathrm{Gr}_0 M \longrightarrow M_{\leq -1}[1]$$

and

$$M_{\leq -2} \longrightarrow M_{\leq -1} \longrightarrow M_{-1} \longrightarrow M_{\leq -2}[1].$$

The second of the two exact triangles, together with stability of $DAT_{w \leq -1}$ under extensions (cmp. [Bo, Prop. 1.3.3 3]) shows that $M_{\leq -1}$ belongs to $DAT_{w \leq -1}$. Therefore, the first is a weight filtration of M . By construction, the morphism $M \rightarrow \mathrm{Gr}_0 M$ induces an isomorphism

$$\mathrm{Hom}_{DAT}(\mathrm{Gr}_0 M, N) \xrightarrow{\sim} \mathrm{Hom}_{DAT}(M, N)$$

for any object N of the heart $DAT_{w=0}$. From this property, it is easy to deduce the functorial behaviour of $\mathrm{Gr}_0 M$.

This proves part (a) of the claim; the proof of part (b) is dual. **q.e.d.**

Remark 4.8. (a) As the proof shows, Corollary 4.7 remains true in the general context of weight structures on triangulated categories, whose heart is Abelian semi-simple.

(b) This more general version of Corollary 4.7 should be compared to [W4, Prop. 2.2]. The conclusions on the existence of the adjoints Gr_0 are the same. On the one hand, Corollary 4.7 works without the additional assumption from [loc. cit.] on the absence of the adjacent weights -1 and 1 . On the other hand, the fact that the heart is Abelian semi-simple, as we have just seen, is a vital ingredient of the proof. There is another subtle difference between the two situations: In the setting of [W4, Prop. 2.2 (a)], the term $M_{\leq -2}$ also behaves functorially in M . This should not be expected to hold for the term $M_{\leq -1}$ from Corollary 4.7 (a).

The following observation will turn out to be useful (see the proof of Corollary 7.14).

Corollary 4.9. *Let $M \in DAT_{w \geq -1} \cap DAT_{w \leq 0}$. Then the adjunction morphism $M \rightarrow \mathrm{Gr}_0 M$ admits a right inverse and a kernel (in the category*

DAT). The latter is pure of weight -1 . Any choice of right inverse induces an isomorphism

$$M_{-1} \oplus \mathrm{Gr}_0 M \cong M$$

between M and the direct sum of $M_{-1} \in DAT_{w=-1}$ and of $\mathrm{Gr}_0 M$.

Proof. Either look at the proof of Corollary 4.7 (a). Or use its statement: indeed, the adjunction morphism can be extended to a weight filtration

$$M_{-1} \longrightarrow M \longrightarrow \mathrm{Gr}_0 M \xrightarrow{\alpha} M_{-1}[1]$$

of M , with some $M_{-1} \in DAT_{w=-1}$. Since Gr_0 is left adjoint to ι_- , and $M_{-1}[1] \in DAT_{w=0}$, the morphism α is necessarily zero. Therefore, the weight filtration splits. **q.e.d.**

Of course, the object M_{-1} occurring in Corollary 4.9 is just the shift by -1 of Gr_0 applied to $M[1] \in DAT_{w \geq 0}$.

Definition 4.10 ([W4, Def. 1.10]). Let $\alpha \leq \beta$ be two integers, and D one of the categories DAT or $DM_{gm}(k)_F$. An object M of D is said to be *without weights* α, \dots, β if there is an exact triangle

$$M_{\leq \alpha-1} \longrightarrow M \longrightarrow M_{\geq \beta+1} \longrightarrow M_{\leq \alpha-1}[1]$$

in D , with $M_{\leq \alpha-1} \in D_{w \leq \alpha-1}$ and $M_{\geq \beta+1} \in D_{w \geq \beta+1}$.

For the sequel, it will be important to know that the property of being without weights α, \dots, β is stable under extensions. Recall that L is said to be an extension of M by K if there is an exact triangle

$$K \longrightarrow L \longrightarrow M \longrightarrow K[1].$$

Proposition 4.11. *Let*

$$K \longrightarrow L \longrightarrow M \longrightarrow K[1]$$

be an exact triangle in DAT or in $DM_{gm}(k)_F$. Assume that K and M are both without weights α, \dots, β . Then L is without weights α, \dots, β .

Proof. According to the context, write D for the category DAT resp. $DM_{gm}(k)_F$ we are working in. Let

$$K_{\leq \alpha-1} \longrightarrow K \longrightarrow K_{\geq \beta+1} \longrightarrow K_{\leq \alpha-1}[1]$$

and

$$M_{\leq \alpha-1} \longrightarrow M \longrightarrow M_{\geq \beta+1} \longrightarrow M_{\leq \alpha-1}[1]$$

be exact triangles in D , with

$$K_{\leq \alpha-1}, M_{\leq \alpha-1} \in D_{w \leq \alpha-1}$$

and

$$K_{\geq \beta+1}, M_{\geq \beta+1} \in D_{w \geq \beta+1}.$$

By orthogonality 4.1 (3), there are no non-zero morphisms from $M_{<\alpha-1}[-1]$ to $K_{\geq\beta+1}$. By [Le1, Lemma 1.1] (with $f : Z_1 \rightarrow Z_2$ equal to the morphism $M[-1] \rightarrow K$), this implies the existence of exact triangles

$$M_{\leq\alpha-1}[-1] \longrightarrow K_{\leq\alpha-1} \longrightarrow L' \longrightarrow M_{\leq\alpha-1} ,$$

$$M_{\geq\beta+1}[-1] \longrightarrow K_{\geq\beta+1} \longrightarrow L'' \longrightarrow M_{\geq\beta+1} ,$$

and

$$L' \longrightarrow L \longrightarrow L'' \longrightarrow L'[1] .$$

The first of these triangles shows that $L' \in D_{w \leq \alpha-1}$. The second shows that $L'' \in D_{w \geq \beta+1}$. Therefore, the third shows that L is indeed without weights α, \dots, β . **q.e.d.**

Remark 4.12. As the proof shows, Proposition 4.11 remains true in the general context of weight structures on triangulated categories.

5 The case of an algebraic base field: the t -structure

In this section, we assume k to be algebraic over the field \mathbb{Q} of rational numbers. We first show that the data from Definition 3.11 define a t -structure on the triangulated category $D\mathcal{A}T$ (Theorem 5.1). This provides a generalization of the main result from [Le1] (which concerns the case of Tate motives). Our strategy of proof is identical to the one from [loc. cit.]. We then proceed (Theorem 5.8) to give a characterization of the weight structure on $D\mathcal{A}T$ in terms of this t -structure. For a number field k , Theorem 5.8 implies that the weight structure can be identified *via* the Hodge theoretic or ℓ -adic realization (Corollary 5.10).

Theorem 5.1. *The pair $(D\mathcal{A}T^{\leq 0}, D\mathcal{A}T^{\geq 0})$ from Definition 3.11 is a t -structure on $D\mathcal{A}T$. It has the following properties.*

- (a) *The t -structure is non-degenerate.*
- (b) *Its heart $\mathcal{A}T$ is generated (as a full Abelian sub-category of $D\mathcal{A}T$ stable under extensions) by the objects $N(m)$, for $N \in \mathcal{A}$ and $m \in \mathbb{Z}$.*
- (c) *Each object M of $\mathcal{A}T$ has a canonical weight filtration by sub-objects*

$$0 \subset \dots \subset W_{n-1}M \subset W_nM \subset \dots \subset M .$$

This filtration is functorial and exact in M . It is uniquely characterized by the properties of being finite (i.e., $W_n M = 0$ for n very small and $W_n M = M$ for n very large), and of admitting sub-quotients

$$\mathrm{gr}_n M := W_n M / W_{n-1} M, \quad n \in \mathbb{Z}$$

of the form $N_n(-n/2)$, for some $N_n \in \mathcal{A}$.

(d) The functor

$$\bigoplus_{m \in \mathbb{Z}} \mathrm{gr}_{2m}(m) : \mathcal{AT} \longrightarrow \mathrm{Gr}_{\mathbb{Z}} \mathcal{A}, \quad M \longmapsto ((\mathrm{gr}_{2m} M)(m))_m$$

is a faithful exact tensor functor to the \mathbb{Z} -graded category over \mathcal{A} . It thus identifies \mathcal{AT} with a tensor sub-category of $\mathrm{Gr}_{\mathbb{Z}} \mathcal{A}$.

(e) The natural maps

$$\mathrm{Ext}_{\mathcal{AT}}^p(M_1, M_2) \longrightarrow \mathrm{Hom}_{D\mathcal{AT}}(M_1, M_2[p])$$

($\mathrm{Ext}^p =$ Yoneda Ext-group of p -extensions) are isomorphisms, for all p , and all $M_1, M_2 \in \mathcal{AT}$. Both sides are zero for $p \geq 2$. In particular, the Abelian category \mathcal{AT} is of cohomological dimension one.

We thus get in particular the existence of two generating Abelian sub-categories, namely \mathcal{AT} and $D\mathcal{AT}_{w=0}$, of the same triangulated category $D\mathcal{AT}$. The first of these is of cohomological dimension one, and the second is semi-simple. In addition (Theorems 5.1 (d) and 4.5 (c)), the first is abstractly tensor equivalent to a tensor sub-category of the second.

Theorem 5.1 is the special case $(a, b) = (-\infty, \infty)$ of the following.

Variant 5.2 (cmp. [Le1, Thm. 1.4, Cor. 4.3]). *Fix $a \leq b$. Then the pair $(D\mathcal{AT}_{[a,b]}^{\leq 0}, D\mathcal{AT}_{[a,b]}^{\geq 0})$ is a t -structure on $D\mathcal{AT}_{[a,b]}$. It has the following properties.*

(a) *The t -structure is non-degenerate.*

(b) *Its heart $\mathcal{AT}_{[a,b]}$ is generated (as a full Abelian sub-category of $D\mathcal{AT}_{[a,b]}$ (or of $DM_{gm}(k)_F$) stable under extensions) by the objects $N(-n/2)$, for $N \in \mathcal{A}$ and $a \leq n \leq b$.*

(c) *Each object M of $\mathcal{AT}_{[a,b]}$ has a canonical weight filtration by sub-objects*

$$0 = W_{a-1} M \subset W_a M \subset \dots \subset W_{b-1} M \subset W_b M = M.$$

This filtration is functorial and exact in M . It is uniquely characterized by the property of admitting sub-quotients

$$W_n M / W_{n-1} M, \quad n \in \mathbb{Z}$$

of the form $N_n(-n/2)$, for some $N_n \in \mathcal{A}$. For all $n \in \mathbb{Z}$, we have

$$W_n M / W_{n-1} M = \text{gr}_n M$$

as objects of the heart \mathcal{AT}_n of $D\mathcal{AT}_n$.

(d) The functor

$$\bigoplus_{m \in \mathbb{Z}, a \leq 2m \leq b} \text{gr}_{2m}(m) : \mathcal{AT}_{[a,b]} \longrightarrow \bigoplus_{m \in \mathbb{Z}, a \leq 2m \leq b} \mathcal{A}$$

is a faithful exact tensor functor.

(e) The natural maps

$$\text{Ext}_{\mathcal{AT}_{[a,b]}}^p(M_1, M_2) \longrightarrow \text{Hom}(M_1, M_2[p])$$

($\text{Hom} = \text{morphisms in } D\mathcal{AT}_{[a,b]} \text{ (or in } DM_{gm}(k)_F)$) are isomorphisms, for all p , and all $M_1, M_2 \in \mathcal{AT}_{[a,b]}$. Both sides are zero for $p \geq 2$. In particular, the Abelian category $\mathcal{AT}_{[a,b]}$ is of cohomological dimension one.

(f) For $a' \leq a$ and $b \leq b'$, the inclusion of $D\mathcal{AT}_{[a,b]}$ into $D\mathcal{AT}_{[a',b']}$ as a full triangulated sub-category is compatible with the t -structures. That is, the t -structure on $D\mathcal{AT}_{[a,b]}$ is induced by the t -structure on $D\mathcal{AT}_{[a',b']}$.

Proof. The decisive ingredient is the following generalization of the vanishing from [Le1, Thm. 1.4]:

$$\text{Hom}_{D\mathcal{AT}_{[a,b]}}(N_1(m_1), N_2(m_2)[s]) = 0, \quad \forall m_1 < m_2, N_1, N_2 \in \mathcal{A}, s \leq 0.$$

It holds because $\text{Hom}_{D\mathcal{AT}_{[a,b]}} = \text{Hom}_{DM_{gm}(k)_F}$, and $\text{Hom}_{DM_{gm}(\bullet)_F}$ satisfies descent for finite extensions L/k of the base field. Choosing an extension L splitting both N_1 and N_2 therefore allows to deduce the desired vanishing from the Beilinson–Soulé vanishing conjecture

$$\text{Hom}_{DM_{gm}(L)_\mathbb{Q}}(\mathbb{Z}(m_1), \mathbb{Z}(m_2)[s]),$$

which by the work of Borel is known for all number fields, hence also for direct limits L of such.

We now faithfully imitate the proof of [Le1, Thm. 1.4], to get assertions (a), (b), and (d). We also get the following: the filtration $W_\bullet M$ induced by the grading $\text{gr}_\bullet M$ is functorial. By construction, the sub-quotient $\text{gr}_n M$ lies in \mathcal{AT}_n . Its unicity follows from the fact that there are no non-zero morphisms from objects of \mathcal{AT} of weights at most r to objects of weights at least $r + 1$. To prove this, use induction on the length of weight filtrations, and the vanishing

$$\text{Hom}_{D\mathcal{AT}}(N_1(m_1)[r], N_2(m_2)[s]) = 0, \quad \forall m_1 > m_2, N_1, N_2 \in \mathcal{A}, r, s \in \mathbb{Z}$$

(see the proof of Proposition 3.9). We thus get part (c) of our claim.

Part (f) follows from the definition of our t -structure, and from the compatibility of the functors gr_n under the inclusion of $D\mathcal{A}T_{[a,b]}$ into $D\mathcal{A}T_{[a',b']}$.

As for claim (e), we faithfully imitate the proof of [Le1, Cor. 4.3]. **q.e.d.**

The following result will not be needed on the sequel; we mention it for the sake of completeness.

Corollary 5.3. *The identity on $\mathcal{A}T$ extends canonically to an equivalence of triangulated categories*

$$D^b(\mathcal{A}T) \longrightarrow D\mathcal{A}T$$

between the bounded derived category of $\mathcal{A}T$ and $D\mathcal{A}T$. Its composition with the cohomology functor $D\mathcal{A}T \rightarrow \mathcal{A}T$ associated to the t -structure of Theorem 5.1 equals the canonical cohomology functor on $D^b(\mathcal{A}T)$.

Proof. Recall the definition of the category $Shv_{Nis}(SmCor(k))$ of Nisnevich sheaves with transfers [V1, Def. 3.1.1]. It is Abelian [V1, Thm. 3.1.4], and there is a canonical full triangulated embedding

$$DM_{gm}^{eff}(k) \hookrightarrow D^-(Shv_{Nis}(SmCor(k)))$$

into the derived category of complexes of Nisnevich sheaves bounded from above [V1, Thm. 3.2.6, p. 205]. Imitating the construction from [loc. cit.] using F instead of \mathbb{Z} as ring of coefficients, one shows that there is a canonical full triangulated embedding

$$DM_{gm}^{eff}(k)_F \hookrightarrow D^-(Shv_{Nis}(SmCor(k))_F),$$

where $Shv_{Nis}(SmCor(k))_F$ denotes the Abelian category of Nisnevich sheaves with transfers taking values in F -modules. We thus get a canonical embedding into $D(Shv_{Nis}(SmCor(k))_F)$ of any full triangulated category \mathcal{C} of $DM_{gm}^{eff}(k)_F$, and hence in particular for $\mathcal{C} = D\mathcal{A}T$. Our claim thus follows from [W5, Thm. 1.1 (a), (d)]: indeed, $\mathrm{Hom}_{D\mathcal{A}T}(M_1, M_2[2]) = 0$ for any two objects M_1, M_2 in $\mathcal{A}T$ (Theorem 5.1 (e)), and $\mathcal{A}T$ generates $D\mathcal{A}T$ (Theorem 5.1 (b)). **q.e.d.**

We already mentioned the special cases $\mathcal{A} = \mathrm{triv}$ and $\mathcal{A} = MA(k)_F$. A third case appears worthwhile mentioning; in fact, it is this case that will be of interest to us in later sections.

Definition 5.4. (a) Define the category $MD(k)_F$ as the full Abelian F -linear sub-category of $MA(k)_F$ of objects on which the Galois group acts *via* a commutative (finite) quotient.

(b) Define the *triangulated category of Dirichlet–Tate motives over k* as the strict full tensor triangulated sub-category $DMDT(k)_F$ of $DM_{gm}(k)_F$ generated by $MD(k)_F$ and $DMT(k)_F$.

Similarly, for any algebraic extension K of k , we could define the *triangulated category of Artin–Tate* (resp. *Dirichlet–Tate, resp...*) *motives over k*

trivializable over K by letting \mathcal{A} equal the full Abelian F -linear sub-category $MA(K/k)_F$ (resp. $MD(K/k)_F$, resp...) of $MA(k)_F$ (resp. $MD(k)_F$, resp...) of objects on which the absolute Galois group of K , when identified with a subgroup of the Galois group of k , acts trivially.

Corollary 5.5. *The conclusions of Theorem 5.1, Variant 5.2 and Corollary 5.3 hold in particular in any of the following three cases.*

- (1) $\mathcal{A} = \text{triv}$. In particular, this gives back the main result of [Le1]. The heart \mathcal{AT} equals the Abelian category $MT(k)_F$ of mixed Tate motives.
- (2) $\mathcal{A} = MD(k)_F$. In this case, the category $D\mathcal{AT}$ equals the triangulated category $DMDT(k)_F$ of Dirichlet–Tate motives. Its heart \mathcal{AT} equals the Abelian category $MDT(k)_F$ of mixed Dirichlet–Tate motives.
- (3) $\mathcal{A} = MA(k)_F$. In this case, the category $D\mathcal{AT}$ equals the triangulated category $DMAT(k)_F$ of Artin–Tate motives. Its heart \mathcal{AT} equals the Abelian category $MAT(k)_F$ of mixed Artin–Tate motives.

Remark 5.6. (a) An equivalent construction of the category $MAT(k)_F$, for $F = \mathbb{Q}$, is given in [DG, Sect. 2.17].

(b) Note that by construction, an inclusion $\mathcal{A} \subset \mathcal{B}$ of strict full Abelian semi-simple F -linear tensor sub-categories of $MA(k)_F$ containing triv induces first a strict full tensor triangulated embedding $D\mathcal{AT} \subset D\mathcal{BT}$, and then a strict full exact tensor embedding $\mathcal{AT} \subset \mathcal{BT}$. An object of $D\mathcal{BT}$ belongs to $D\mathcal{AT}$ if and only if its cohomology objects (with respect to the t -structure from Theorem 5.1) lie in \mathcal{AT} . The equivalences of Corollary 5.3 for \mathcal{A} and \mathcal{B} fit into a commutative diagram

$$\begin{array}{ccc} D^b(\mathcal{AT}) & \xrightarrow{\cong} & D\mathcal{AT} \\ \downarrow & & \downarrow \\ D^b(\mathcal{BT}) & \xrightarrow{\cong} & D\mathcal{BT} \end{array}$$

In particular, the bounded derived category $D^b(\mathcal{AT})$ is canonically identified with a full sub-category of the bounded derived category $D^b(\mathcal{BT})$.

For later use, let us introduce some terminology.

Definition 5.7. Let M be a mixed Artin–Tate motive, with weight filtration

$$0 \subset \dots \subset W_{r-1}M \subset W_rM \subset \dots \subset M .$$

Let m be an integer.

- (a) We say that M is of *weights* $\leq m$ if $W_mM = M$.
- (b) We say that M is of *weights* $\geq m$ if $W_{m-1}M = 0$.
- (c) We say that M is *pure of weight* m if it is both of *weights* $\leq m$ and of

weights $\geq m$, i.e., if $W_{m-1}M = 0$ and $W_m = M$.

(d) We say that M is *without weight m* if $W_{m-1}M = W_mM$, i.e., if the sub-quotient $W_mM/W_{m-1}M$ is trivial.

Of course, any mixed Artin–Tate motive is without weight m , whenever m is odd. Denote by

$$\tau^{\leq n}, \tau^{\geq n} : DAT \longrightarrow DAT$$

the truncation functors, and by

$$\mathcal{H}^n : DAT \longrightarrow \mathcal{A}T$$

the cohomology functors associated to the t -structure from Theorem 5.1. Here is the main result of this section.

Theorem 5.8. *Let $K \in DAT$, and $\alpha \leq \beta$.*

(a) *K lies in the heart $DAT_{w=0}$ of w if and only if the object \mathcal{H}^nK of $\mathcal{A}T$ is pure of weight n , for all $n \in \mathbb{Z}$.*

(b) *K lies in $DAT_{w \leq \alpha}$ if and only if \mathcal{H}^nK is of weights $\leq n + \alpha$, for all $n \in \mathbb{Z}$.*

(c) *K lies in $DAT_{w \geq \beta}$ if and only if \mathcal{H}^nK is of weights $\geq n + \beta$, for all $n \in \mathbb{Z}$.*

(d) *K is without weights α, \dots, β if and only if \mathcal{H}^nK is without weights $n + \alpha, \dots, n + \beta$, for all $n \in \mathbb{Z}$.*

Proof. Observe that the triangulated category DAT is generated by the heart $DAT_{w=0}$ of w (Theorem 4.5 (b)) as well as by the heart $\mathcal{A}T$ of t (Theorem 5.1 (a)). This will allow to simplify the proof.

The explicit description of objects K of $DAT_{w=0}$ from Theorem 4.5 (b) shows that the \mathcal{H}^nK are indeed pure of weight n , for all n (see Theorem 5.1 (c)). To show that any K whose cohomology objects \mathcal{H}^nK are pure of weight n , does belong to $DAT_{w=0}$, we may assume by the above that K is concentrated in one degree (with respect to the t -structure), say $K = M[d]$ for some $M \in \mathcal{A}T$ and $d \in \mathbb{Z}$. By assumption, the mixed Artin–Tate motive M is pure of weight $-d$, and hence (Theorem 5.1 (c)) of the form $N(d/2)$, for some Artin motive N belonging to \mathcal{A} . The latter is clearly a Chow motive, and hence so is its tensor product with the Chow motive $\mathbb{Z}(d/2)[d]$. Therefore, K is a Chow motive belonging to DAT . By Theorem 4.5, it is in the heart $DAT_{w=0}$. This shows part (a).

We leave it to the reader to deduce (b) and (c) from (a).

As for part (d), it is easy to see that the cohomology \mathcal{H}^nK of an object $K \in DAT$ without weights α, \dots, β is without weights $n + \alpha, \dots, n + \beta$, for all n (use (b) resp. (c) for the constituents of a suitable weight filtration of K). To prove the inverse implication, we use induction on the number of integers n such that $\mathcal{H}^nK \neq 0$. If this number equals one, then $K = M[d]$ for some $M \in \mathcal{A}T$ and $d \in \mathbb{Z}$. By assumption, the mixed Artin–Tate motive

M is without weights $-d + \alpha, \dots, -d + \beta$. By Definition 5.7 (d), its weight filtration thus satisfies the relation

$$W_{-d+\alpha-1}M = W_{-d+\alpha}M = \dots = W_{-d+\beta}M .$$

The sequence

$$0 \longrightarrow W_{-d+\alpha-1}M \longrightarrow M \longrightarrow M/W_{-d+\beta}M \longrightarrow 0$$

is therefore exact in \mathcal{AT} . It gives rise to an exact triangle

$$(W_{-d+\alpha-1}M)[d] \longrightarrow K \longrightarrow (M/W_{-d+\beta}M)[d] \longrightarrow (W_{-d+\alpha-1}M)[d+1]$$

in $D\mathcal{AT}$. By parts (b) and (c),

$$(W_{-d+\alpha-1}M)[d] \in D\mathcal{AT}_{w \leq \alpha-1} ,$$

and

$$(M/W_{-d+\beta}M)[d] \in D\mathcal{AT}_{w \geq \beta+1} .$$

Therefore, the object K is indeed without weights α, \dots, β .

By Proposition 4.11, the property of being without weights α, \dots, β is stable under extensions in $D\mathcal{AT}$. This allows to perform the induction step.

q.e.d.

To conclude this section, let us now consider realizations ([Hu, Sect. 2.3 and Corrigendum]; see [DG, Sect. 1.5] for a simplification of this approach). We assume from now on that k is a number field, and concentrate on two realizations (the statement from Corollary 5.10 below then formally generalizes to any of the other realizations “with weights” considered in [Hu]):

(i) the Hodge theoretic realization

$$R_\sigma : DM_{gm}(k)_F \longrightarrow D$$

associated to a fixed embedding σ of the number field k into the field \mathbb{C} of complex numbers. Here, D is the bounded derived category of mixed graded-polarizable \mathbb{Q} -Hodge structures [Be2, Def. 3.9, Lemma 3.11], tensored with F ,

(ii) the ℓ -adic realization

$$R_\ell : DM_{gm}(k)_F \longrightarrow D$$

for a prime ℓ . Here, D is the bounded “derived category” of constructible \mathbb{Q}_ℓ -sheaves on $\mathbf{Spec} k$ [E, Sect. 6], tensored with F .

Choose and fix one of these two, denote it by R , recall that it is a contravariant tensor functor, and use the same letter for its restriction to the sub-category $D\mathcal{AT}$ of $DM_{gm}(k)_F$. The category $D\mathcal{AT}$ is equipped with a t -structure. The same is true for D ; write H^n for the cohomology functors. It is easy to see that R is t -exact (since it maps \mathcal{AT} to the heart of D). In particular, it induces an exact contravariant functor R_0 from the heart \mathcal{AT}

of DAT to the heart of D , which we shall denote by \mathcal{B} . As for the weight structure on DAT , note that R_0 maps the pure Tate motive $\mathbb{Z}(m)$ to the pure Hodge structure $\mathbb{Q}(-m)$ (when $R = R_\sigma$) and to the pure \mathbb{Q}_ℓ -sheaf $\mathbb{Q}_\ell(-m)$ (when $R = R_\ell$), respectively [Hu, Thm. 2.3.3].

Proposition 5.9. *Assume k to be a number field.*

(a) *The realization*

$$R : DAT \longrightarrow D$$

is conservative. In other words, an object K of DAT is zero if and only if its image $R(K)$ under R is.

(b) *The induced functor*

$$R_0 : \mathcal{AT} \longrightarrow \mathcal{B}$$

is conservative.

(c) *The functor R_0 respects and detects weights up to inversion of the sign. More precisely, an object M of \mathcal{AT} is pure of weight n if and only if $R_0(M)$ is pure of weight $-n$.*

Note that there is a notion of purity and mixedness for objects of \mathcal{B} .

Proof of Proposition 5.9. Let $K \in DAT$. Given the t -exactness and contravariance of R , we have the formula

$$H^n R(K) = R_0(\mathcal{H}^{-n}K)$$

for all n . By Theorem 5.1 (a), the t -structure on DAT is non-degenerate. Hence (b) implies (a).

Recall that by Theorem 5.1 (c), there is a unique finite weight filtration on any object of \mathcal{AT} . Also, the functor R_0 is exact. Hence (b) is implied by conservativity of the restriction of R_0 to the sub-category of objects of \mathcal{AT} , which are pure of some weight. But this property is clearly implied by (c) (since the zero object of \mathcal{B} is pure of any weight).

Let $M \in \mathcal{AT}$. As before, we may assume that M is pure of some weight, say n . Again by Theorem 5.1 (c), M is of the form $N(-n/2)$, for some Artin-Tate motive N . Thus, $R_0(M) \cong R_0(N)(n/2)$ is pure of weight $-n$. It is zero if and only if $R_0(N)$ is, which is the case if and only if N is. **q.e.d.**

Corollary 5.10. *Assume k to be a number field. Then the realization R respects and detects the weight structure. More precisely, let $K \in DAT$, and $\alpha \leq \beta$.*

(a) *K lies in the heart $DAT_{w=0}$ of w if and only if the n -th cohomology object $H^n R(K) \in \mathcal{B}$ of $R(K)$ is pure of weight n , for all $n \in \mathbb{Z}$.*

(b) *K lies in $DAT_{w \leq \alpha}$ if and only if $H^n R(K)$ is of weights $\geq n - \alpha$, for all $n \in \mathbb{Z}$.*

(c) *K lies in $DAT_{w \geq \beta}$ if and only if $H^n R(K)$ is of weights $\leq n - \beta$, for all*

$n \in \mathbb{Z}$.

(d) K is without weights α, \dots, β if and only if $H^n R(K)$ is without weights $n - \beta, \dots, n - \alpha$, for all $n \in \mathbb{Z}$.

Proof. Recall that $H^n R(K) = R_0(\mathcal{H}^{-n}K)$. The claim thus follows from Theorem 5.8 and Proposition 5.9. **q.e.d.**

Remark 5.11. As the proof shows, the analogues of parts (a) and (b) of Proposition 5.9 continue to hold for any of the realizations (including those “without weights”) considered in [Hu]. This is true in particular for

- (iii) the Betti realization, i.e., the composition of the Hodge theoretic realization R_σ with the forgetful functor to the bounded derived category of F -modules of finite type,
- (iv) the topological ℓ -adic realization, i.e., the composition of the ℓ -adic realization R_ℓ with the forgetful functor to the bounded derived category of $F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ -modules of finite type [E, Thm. 7.2 i)].

Remark 5.12. (a) For the Hodge theoretic realization

$$R = R_\sigma : DM_{gm}(k)_F \longrightarrow D$$

($D =$ the bounded derived category of mixed graded-polarizable \mathbb{Q} -Hodge structures, tensored with F), it is possible to give a more conceptual interpretation of respect of the weight structure. In fact, there is a canonical weight structure $w_{\mathfrak{H}}$ on D , characterized by the property of admitting as heart the full sub-category \mathcal{K} of classes of complexes K of Hodge structures, whose n -th cohomology object $H^n R(K)$ is pure of weight n , for all $n \in \mathbb{Z}$. In order to prove this claim, apply [Bo, Thm. 4.3.2 II 1 and 2], observing that (1) \mathcal{K} generates the triangulated category D , (2) \mathcal{K} is negative:

$$\mathrm{Hom}_D(M_1, M_2[i]) = 0$$

for any two objects M_1, M_2 of \mathcal{K} , and any integer $i > 0$ (it is here that the polarizability assumption enters), and (3) any retract of an object of \mathcal{K} in D belongs already to \mathcal{K} .

To say that $R : DM_{gm}(k)_F \rightarrow D$ respects the weight structure means then that R respects the pairs of sub-categories $(DM_{gm}(k)_{F,w \leq 0}, DM_{gm}(k)_{F,w \geq 0})$ and $(D_{w \leq 0}, D_{w \geq 0})$:

$$R(DM_{gm}(k)_{F,w \leq 0}) \subset D_{w \geq 0} \quad , \quad R(DM_{gm}(k)_{F,w \geq 0}) \subset D_{w \leq 0}$$

(recall that R is contravariant). Given that $DM_{gm}(k)_F$ is generated by its heart, this requirement is equivalent to saying that R respects the hearts, i.e., that it maps $CHM(k)_F$ to \mathcal{K} — which is a true statement, since the Hodge structure on the n -th Betti cohomology of a proper smooth variety is indeed pure of weight n , for all $n \in \mathbb{Z}$. This observation implies immediately

the “only if” part of the statements of Corollary 5.10.

(b) By contrast, for the ℓ -adic realization

$$R = R_\ell : DM_{gm}(k)_F \longrightarrow D$$

($D =$ the bounded “derived category” of constructible \mathbb{Q}_ℓ -sheaves on $\mathbf{Spec} k$, tensored with F), there is no such interpretation, since there is no reasonable weight structure on D . Indeed, according to [J, Rem. 6.8.4 i)], for any odd integer $m \in \mathbb{Z}$,

$$\mathrm{Hom}_D(\mathbb{Q}_\ell(0), \mathbb{Q}_\ell(m)[1]) \neq 0 .$$

This is true in particular when m is negative, i.e., $\mathbb{Q}_\ell(m)[1]$ is pure of strictly positive weight $-2m + 1$. Therefore, orthogonality 4.1 (3) is violated.

6 A criterion on the existence of the interior motive

From now on, our base field k is assumed to be a number field. Our results on Artin–Tate motives imply a criterion (Theorem 6.2) on the absence of certain weights in the boundary motive. It allows to identify sufficient conditions (Corollaries 6.3 and 6.4) for [W4, Assumption 4.2] to hold. The results of [loc. cit.] can therefore be applied. In particular, it is then possible to construct the *interior motive*.

Fix $X \in Sm/k$, and consider the exact triangle

$$(*) \quad \partial M_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}^c(X) \longrightarrow \partial M_{gm}(X)[1]$$

in $DM_{gm}^{eff}(k)$. Recall from [W4, Def. 4.1 (a)] that $c(X, X)$ contains a canonical sub-algebra $c_{1,2}(X, X)$ (of “bi-finite correspondences”) acting on $(*)$. Denote by $\bar{c}_{1,2}(X, X)$ the quotient of $c_{1,2}(X, X)$ by the kernel of this action. Fix a finite direct product F of fields of characteristic zero, and an idempotent e in $\bar{c}_{1,2}(X, X) \otimes_{\mathbb{Z}} F$. Denote by $M_{gm}(X)^e$, $M_{gm}^c(X)^e$ and $\partial M_{gm}(X)^e$ the images of e on $M_{gm}(X)$, $M_{gm}^c(X)$ and $\partial M_{gm}(X)$, respectively, considered as objects of the category $DM_{gm}^{eff}(k)_F$. Recall the following assumption.

Assumption 6.1 ([W4, Asp. 4.2]). The object $\partial M_{gm}(X)^e$ is without weights -1 and 0 .

In order to apply the results from [W4, Sect. 4], one needs to verify Assumption 6.1.

Theorem 6.2. *Let $\alpha \leq \beta$ be two integers, and R one of the two realizations considered in Section 4 (Hodge theoretic or ℓ -adic). Assume that $\partial M_{gm}(X)^e$ is a successive extension of objects M of $DM_{gm}^{eff}(k)_F$, each satisfying one of the following properties.*

(i) M is without weights α, \dots, β .

(ii) M lies in the triangulated sub-category $DMAT(k)_F$ of $DM_{gm}(k)_F$ of Artin–Tate motives over k , and the cohomology object $H^n R(K)$ of its image $R(M)$ under R is without weights $n - \beta, \dots, n - \alpha$, for all $n \in \mathbb{Z}$.

Then $\partial M_{gm}(X)^e$ is without weights α, \dots, β .

Proof. Apply Proposition 4.11 and Corollary 5.10 (d). **q.e.d.**

Corollary 6.3. *If the hypotheses of Theorem 6.2 are met with $\alpha \leq -1$ and $\beta \geq 0$, then Assumption 6.1 holds.*

As far as the remaining part of this article is concerned, we shall be dealing with a situation in which the whole of $\partial M_{gm}(X)^e$ satisfies property (ii) from Theorem 6.2. It will be worthwhile to spell out that property.

Corollary 6.4. *The conclusion of Theorem 6.2 holds in particular if $\partial M_{gm}(X)^e$ lies in $DMAT(k)_F$, and if the e -part of the boundary cohomology of X*

$$(\partial H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F)^e$$

(in the Hodge theoretic setting) resp.

$$(\partial H^n(X_{\bar{k}}, \mathbb{Q}_{\ell}) \otimes_{\mathbb{Q}} F)^e$$

(in the ℓ -adic setting) is without weights $n - \beta, \dots, n - \alpha$, for all $n \in \mathbb{Z}$. If this latter condition is fulfilled with $\alpha \leq -1$ and $\beta \geq 0$, then Assumption 6.1 holds.

Recall that boundary cohomology of X is defined *via* a compactification $j : X \hookrightarrow \bar{X}$: writing $i : \partial \bar{X} \hookrightarrow \bar{X}$ for the complementary immersion, one defines $\partial H^n(\bullet)$ as cohomology of $\partial \bar{X}$ with coefficients in $i^* Rj_*(\bullet)$. Thanks to proper base change, this definition is independent of the choice of j , as is the long exact cohomology sequence

$$\dots \longrightarrow H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F \longrightarrow \partial H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F \longrightarrow$$

$$H_c^{n+1}(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F \longrightarrow H^{n+1}(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F \longrightarrow \dots$$

(in the Hodge theoretic setting) resp.

$$\dots \longrightarrow H^n(X_{\bar{k}}, \mathbb{Q}_{\ell}) \otimes_{\mathbb{Q}} F \longrightarrow \partial H^n(X_{\bar{k}}, \mathbb{Q}_{\ell}) \otimes_{\mathbb{Q}} F \longrightarrow$$

$$H_c^{n+1}(X_{\bar{k}}, \mathbb{Q}_{\ell}) \otimes_{\mathbb{Q}} F \longrightarrow H^{n+1}(X_{\bar{k}}, \mathbb{Q}_{\ell}) \otimes_{\mathbb{Q}} F \longrightarrow \dots$$

(in the ℓ -adic setting).

Note also that the algebra $\bar{c}_{1,2}(X, X)$ acts contravariantly on the boundary cohomology $\partial H^n(X(\mathbb{C}), \mathbb{Q})$ resp. $\partial H^n(X_{\bar{k}}, \mathbb{Q}_{\ell})$.

Corollary 6.4 results from Theorem 6.2 and the following.

Proposition 6.5. *Fix $X \in \text{Sm}/k$, a \mathbb{Q} -algebra F , and e as before. Then $H^n R(\partial M_{gm}(X)^e)$ is isomorphic to $(\partial H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F)^e$ (in the Hodge theoretic setting) resp. $(\partial H^n(X_{\bar{k}}, \mathbb{Q}_{\ell}) \otimes_{\mathbb{Q}} F)^e$ (in the ℓ -adic setting), for all n .*

Proof. It suffices to show that the image under R of the canonical morphism

$$\iota : M_{gm}(X) \longrightarrow M_{gm}^c(X)$$

can be $c_{1,2}(X, X)$ -equivariantly identified with the canonical morphism

$$R\Gamma_c(X) \longrightarrow R\Gamma(X)$$

in the target D of R of classes of complexes $R\Gamma_c(X)$ and $R\Gamma(X)$ computing cohomology with resp. without support. Indeed, the exact triangle $(*)$ will then show that the e -part $\partial M_{gm}(X)^e$ of the boundary motive is mapped to a cone of $R\Gamma_c(X)^e \rightarrow R\Gamma(X)^e$. It will therefore be isomorphic to the class of a complex computing the e -part of boundary cohomology.

We may assume that X is of pure dimension d . First note that for any fixed smooth compactification $j : X \hookrightarrow \bar{X}$, the morphism ι is the composition of the canonical morphism

$$M_{gm}(j) : M_{gm}(X) \longrightarrow M_{gm}(\bar{X}) ,$$

of the inverse of the duality isomorphism,

$$M_{gm}(\bar{X}) \xrightarrow{\sim} M_{gm}(\bar{X})^*(d)[2d]$$

[V1, Thm. 4.3.7 3], of the dual of $M_{gm}(j)$,

$$M_{gm}(j)^*(d)[2d] : M_{gm}(\bar{X})^*(d)[2d] \longrightarrow M_{gm}(X)^*(d)[2d] ,$$

and of the duality isomorphism

$$M_{gm}(X)^*(d)[2d] \xrightarrow{\sim} M_{gm}^c(X)$$

[V1, Thm. 4.3.7 3]. Now recall that R is compatible with the tensor structures [Hu, Cor. 2.3.5, Cor. 2.3.4], and sends the Tate motive $\mathbb{Z}(1)$ to $\mathbb{Q}(-1)$ resp. $\mathbb{Q}_{\ell}(-1)$. It follows that R is compatible with duality. Furthermore, R sends $M_{gm}(f)$ to $f^* : R\Gamma(Z) \rightarrow R\Gamma(Y)$, for any morphism $f : Y \rightarrow Z$ of smooth k -schemes [DG, pp. 6–7]. Thus, R sends ι to the composition of the duality isomorphism

$$R\Gamma_c(X) \xrightarrow{\sim} R\Gamma(X)^*(-d)[-2d] ,$$

the dual of j^* , the inverse of the duality isomorphism,

$$R\Gamma(\bar{X})^*(-d)[-2d] \xrightarrow{\sim} R\Gamma(\bar{X}) ,$$

and j^* . But this composition equals the canonical morphism

$$R\Gamma_c(X) \longrightarrow R\Gamma(X) .$$

It remains to show that the above identification is compatible with the action of $c_{1,2}(X, X)$. Let \mathfrak{Z} be a cycle on $X \times_k X$ belonging to $c_{1,2}(X, X)$, and denote by ${}^t\mathfrak{Z}$ its transpose. Both \mathfrak{Z} and ${}^t\mathfrak{Z}$ are finite over both components of $X \times X$. This is true in particular for the first component. Therefore, both induce endomorphisms of $M_{gm}(X)$. Similarly, they induce endomorphisms of $M_{gm}^c(X)$. Now the definition of the duality isomorphism

$$M_{gm}(X)^*(d)[2d] \xrightarrow{\sim} M_{gm}^c(X)$$

[V1, proof of Thm. 4.3.7 3] implies that under this isomorphism, the endomorphism \mathfrak{Z} of $M_{gm}^c(X)$ corresponds to the endomorphism ${}^t\mathfrak{Z}^*(d)[2d]$ of $M_{gm}(X)^*(d)[2d]$. We thus identify the commutative diagram

$$\begin{array}{ccc} M_{gm}(X) & \xrightarrow{\iota} & M_{gm}^c(X) \\ \mathfrak{Z} \downarrow & & \downarrow \mathfrak{Z} \\ M_{gm}(X) & \xrightarrow{\iota} & M_{gm}^c(X) \end{array}$$

with

$$\begin{array}{ccc} M_{gm}(X) & \xrightarrow{\iota} & M_{gm}(X)^*(d)[2d] \\ \mathfrak{Z} \downarrow & & \downarrow {}^t\mathfrak{Z}^*(d)[2d] \\ M_{gm}(X) & \xrightarrow{\iota} & M_{gm}(X)^*(d)[2d] \end{array}$$

By [DG, pp. 6–7], R sends \mathfrak{Y} to $\mathfrak{Y}^* : R\Gamma(W) \rightarrow R\Gamma(V)$, for any finite correspondence \mathfrak{Y} on the product $V \times_k W$ of two smooth k -schemes. It follows that R sends the latter commutative diagram to the commutative diagram

$$\begin{array}{ccc} R\Gamma(X) & \longleftarrow & R\Gamma(X)^*(-d)[-2d] \\ \mathfrak{Z}^* \uparrow & & \uparrow ({}^t\mathfrak{Z}^*)^*(-d)[-2d] \\ R\Gamma(X) & \longleftarrow & R\Gamma(X)^*(-d)[-2d] \end{array}$$

Now the endomorphism $({}^t\mathfrak{Z}^*)^*(-d)[-2d]$ of $R\Gamma(X)^*(-d)[-2d]$ corresponds to the endomorphism \mathfrak{Z}^* of $R\Gamma_c(X)$. But this means precisely that our identification of the image under R of $M_{gm}(X) \rightarrow M_{gm}^c(X)$ with the canonical morphism $R\Gamma_c(X) \rightarrow R\Gamma(X)$ is $c_{1,2}(X, X)$ -equivariant. **q.e.d.**

Remark 6.6. In the Hodge theoretic setting, the isomorphism of mixed Hodge structures

$$H^n R(\partial M_{gm}(X)) \cong (\partial H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F)$$

from Proposition 6.5 implies an isomorphism of F -modules. For quasi-projective X , this latter statement should be compared to [Ay, Lemme 3.11].

Remark 6.7. (a) Consider the situation from Section 1, that is, fix a base scheme $S \in Sm/k$. When X lies not only in Sm/k , but in $PropSm/S$, then Theorem 1.2 provides another source of endomorphisms of the exact

triangle $(*)$, namely the relative Chow group $\mathrm{CH}^g(X \times_S X)$ (assuming X to be of pure relative dimension g over S). In this setting, the algebra $\bar{c}_{1,2}(X, X)$ could be replaced by $\mathrm{CH}^g(X \times_S X)$, and the results from this section, as well as those from [W4, Sect. 4] formally carry over.

(b) In the sequel, we shall keep the approach *via* $\bar{c}_{1,2}(X, X)$, because the endomorphisms “of Hecke type” to be considered will not in general come from $\mathrm{CH}^g(X \times_S X)$ (Example 1.16 (b)), but still lie in the centralizer (in $\bar{c}_{1,2}(X, X)$) of the idempotents we shall be interested in (see Corollary 7.8). We shall consider the subgroup

$$\mathrm{CH}^g(X \times_S X)_{1,2} \subset \mathrm{CH}^g(X \times_S X)$$

defined as the image of the canonical map

$$c_S(X, X) \cap c_{1,2}(X, X) \longrightarrow \mathrm{CH}^g(X \times_S X).$$

By Corollary 1.12, the map

$$c_S(X, X) \cap c_{1,2}(X, X) \longrightarrow \bar{c}_{1,2}(X, X)$$

factors through $\mathrm{CH}^g(X \times_S X)_{1,2}$. In particular, a class in $\mathrm{CH}^g(X \times_S X)_{1,2} \otimes_{\mathbb{Z}} F$ yields an element in $\bar{c}_{1,2}(X, X) \otimes_{\mathbb{Z}} F$, and the latter is idempotent if the class in $\mathrm{CH}^g(X \times_S X)_{1,2} \otimes_{\mathbb{Z}} F$ is.

7 Statement of the main results

In order to state our main results (Theorems 7.5, 7.6), let us introduce the geometrical situation we are going to consider from now on. The base k is the field \mathbb{Q} of rational numbers, X is the r -th power of the universal Abelian scheme over a Hilbert–Blumenthal variety of dimension g , and e is associated to modular forms of weight $(r_1 + 2, \dots, r_g + 2)$, for $r_1 + \dots + r_g = r$ (see below for the precise definition). Theorems 7.5 and 7.6 imply in particular that in this context, Assumption 6.1 is satisfied as soon as $r \geq 1$: indeed, $\partial M_{gm}(X)$ lies in $DMAT(\mathbb{Q})_{\mathbb{Q}}$, and the e -part of the n -th boundary cohomology group of X is without weights $n - (r - 1), \dots, n + r$, for all $n \in \mathbb{Z}$. We then list the main consequences of this result (Corollaries 7.7, 7.8, 7.9, 7.13, 7.14), applying the theory developed in [W4, Sect. 4]. The proofs of Theorems 7.5 and 7.6 will be given in Section 8, where we shall also establish more precise statements on the nature of $\partial M_{gm}(A^r)^e$ (Theorem 8.6, Corollary 8.7), which will however not be needed in the present article.

Fix a totally real number field L of degree g . Let I_L denote the set of real embeddings of L . Denote by $\mathrm{Res}_{L/\mathbb{Q}}$ the Weil restriction from schemes over L to schemes over \mathbb{Q} . The functor $\mathrm{Res}_{L/\mathbb{Q}}$ is right adjoint to the base change $Z \mapsto Z_L := Z \times_{\mathbb{Q}} L$. Hence we have in particular a functorial adjunction morphism $Z \rightarrow \mathrm{Res}_{L/\mathbb{Q}} Z_L$ for any scheme Z over \mathbb{Q} . For any scheme Y over

L , and any subfield F of \mathbb{C} containing the images $\sigma(L)$ for all $\sigma \in I_L$, there is a canonical isomorphism

$$(\mathrm{Res}_{L/\mathbb{Q}} Y) \times_{\mathbb{Q}} F \xrightarrow{\sim} \prod_{\sigma \in I_L} Y \times_{L, \sigma} F$$

induced by the isomorphism

$$L \otimes_{\mathbb{Q}} R \xrightarrow{\sim} \prod_{\sigma \in I_L} R, \quad l \otimes r \longmapsto (\sigma(l) \cdot r)_{\sigma}$$

for any F -algebra R . The composition of the adjunction (base changed by F) and this isomorphism is simply the diagonal

$$X \times_{\mathbb{Q}} F \longrightarrow \prod_{\sigma \in I_L} X \times_{\mathbb{Q}} F.$$

By functoriality, the determinant induces a morphism of group schemes over \mathbb{Q}

$$\det : \mathrm{Res}_{L/\mathbb{Q}} \mathrm{GL}_{2,L} \longrightarrow \mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_{m,L}.$$

Definition 7.1 ([Ra, Sect. 1.27]). The group scheme G over \mathbb{Q} is defined as the fibre product

$$G := \mathbb{G}_m \times_{\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_{m,L}, \det} \mathrm{Res}_{L/\mathbb{Q}} \mathrm{GL}_{2,L}.$$

In particular, we have

$$G(\mathbb{Q}) = \{M \in \mathrm{GL}_2(L), \det(M) \in \mathbb{Q}^* \subset L^*\}.$$

Under the above isomorphism

$$(\mathrm{Res}_{L/\mathbb{Q}} \mathrm{GL}_{2,L}) \times_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \prod_{\sigma \in I_L} \mathrm{GL}_{2,\mathbb{R}},$$

we can identify

$$G(\mathbb{R}) \cong \left\{ (M_{\sigma})_{\sigma \in I_L} \in \prod_{\sigma \in I_L} \mathrm{GL}_2(\mathbb{R}), \det(M_{\sigma}) = \det(M_{\eta}) \forall \sigma, \eta \in I_L \right\}.$$

In particular, we see that $G(\mathbb{R})$ has two connected components, according to the sign of the determinant. Under these identifications, the inclusion of $G(\mathbb{Q})$ into $G(\mathbb{R})$ maps $M \in \mathrm{GL}_2(L)$ to the g -tuple

$$(\sigma(M))_{\sigma \in I_L} \in \prod_{\sigma \in I_L} \mathrm{GL}_2(\mathbb{R}).$$

Definition 7.2. (a) The analytic space \mathcal{H} is defined as

$$\mathcal{H} := \left\{ (\tau_{\sigma})_{\sigma \in I_L} \in \prod_{\sigma \in I_L} (\mathbb{C} - \mathbb{R}), \mathrm{sign}(\mathrm{im} \tau_{\sigma}) = \mathrm{sign}(\mathrm{im} \tau_{\eta}) \forall \sigma, \eta \in I_L \right\}.$$

(b) The action of $G(\mathbb{R})$ on \mathcal{H} is given by the usual componentwise action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{C} - \mathbb{R}$, and the above identification of $G(\mathbb{R})$ with a subgroup of $\prod_{\sigma \in I_L} \mathrm{GL}_2(\mathbb{R})$.

Given that the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{C} - \mathbb{R}$ is transitive and trivial on $\mathbb{G}_m(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{R})$, it is easy to see that $G(\mathbb{R})$ acts transitively on \mathcal{H} . Observe that this action is by analytical automorphisms. In fact, (G, \mathcal{H}) are *pure Shimura data* [Pi, Def. 2.1]. Their *reflex field* [Pi, Sect. 11.1] equals \mathbb{Q} . The center $Z(G)$ of G equals

$$\mathbb{G}_m \times_{\mathrm{Res}_{L/\mathbb{Q}}} \mathbb{G}_{m,L}, x \mapsto x^2 \mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_{m,L},$$

hence its neutral connected component is isogeneous to \mathbb{G}_m . In particular, the Shimura data (G, \mathcal{H}) satisfy condition (+) from [W3, Sect. 5].

Let us now fix additional data: (A) an open compact subgroup K of $G(\mathbb{A}_f)$ which is neat [Pi, Sect. 0.6], (B) a subfield F of \mathbb{C} containing the images of all embeddings $\sigma \in I_L$, (C) an integer $r \geq 0$, together with a partition

$$\underline{r} : \quad r = \sum_{\sigma \in I_L} r_\sigma$$

with g integers $r_\sigma \geq 0$. Equivalently, we may see $\underline{r} = \sum_{\sigma} r_\sigma \cdot \sigma$ as an element of the free Abelian group $\mathbb{Z}[I_L]$ on I_L .

These data (A)–(C) are used as follows (cmp. [Ki, Sect. 2.2, 2.3] for the case $g = 2$). The *Shimura variety* $S := S^K(G, \mathcal{H})$ is an object of Sm/\mathbb{Q} . This is the *Hilbert–Blumenthal variety* of level K associated to L . It is of dimension g , and admits an interpretation as modular space of Abelian varieties of dimension g with additional structures, among which a real multiplication by a sub-algebra of L which is of rank g over \mathbb{Z} , hence of finite index in the ring of integers \mathcal{O}_L (and which depends on K). In particular, there is a universal family A of Abelian varieties over S . Thus, the absolute dimension of A over \mathbb{Q} is $2g$, it is an object of Sm/\mathbb{Q} , and thanks to the modular interpretation there is a canonical ring monomorphism

$$L \hookrightarrow \mathrm{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

from L to the endomorphisms of A over S , tensored with \mathbb{Q} . The decomposition of the relative motive

$$h(A/S) = \bigoplus_i h_i(A/S)$$

from Theorem 2.1 being functorial, there is a map

$$\mathrm{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{End}_{\mathrm{CHM}(S)}(h_i(A/S)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for all $0 \leq i \leq 2g$. For $i = 1$, this map is an isomorphism of \mathbb{Q} -vector spaces [Ki, Prop. 2.2.1]. Hence, we get a ring monomorphism

$$L \hookrightarrow \text{End}_{CHM(S)}(h_1(A/S)) \otimes_{\mathbb{Z}} \mathbb{Q} .$$

Its tensor product with F gives

$$L \otimes_{\mathbb{Q}} F \hookrightarrow \text{End}_{CHM(S)}(h_1(A/S)) \otimes_{\mathbb{Z}} F .$$

The field F containing the images of all $\sigma \in I_L$, we get canonically

$$L \otimes_{\mathbb{Q}} F \xrightarrow{\sim} \prod_{\sigma \in I_L} F , \quad l \otimes f \mapsto (\sigma(l) \cdot f)_{\sigma} .$$

In particular, there are canonical idempotents e_{σ} in $L \otimes_{\mathbb{Q}} F$, indexed by I_L : by definition, e_{σ} is the projection to the copy of F corresponding to σ . Let us use the same symbol e_{σ} for its image in

$$\text{End}_{CHM(S)}(h_1(A/S)) \otimes_{\mathbb{Z}} F \subset \text{CH}^g(A \times_S A) \otimes_{\mathbb{Z}} F .$$

From our construction, the relation

$$(\Delta) = \sum_{\sigma \in I_L} e_{\sigma} \in \text{CH}^g(A \times_S A) \otimes_{\mathbb{Z}} F$$

is obvious. It induces a decomposition

$$h_1(A/S) = \bigoplus_{\sigma \in I_L} h_1(A/S)^{e_{\sigma}}$$

in $CHM(S)_F$, where $h_1(A/S)^{e_{\sigma}}$ denotes the image of the projector e_{σ} on $h_1(A/S)$.

Let us now use the partition $\underline{r} = \sum_{\sigma} r_{\sigma} \cdot \sigma \in \mathbb{Z}[I_L]$.

Definition 7.3. Define the relative Chow motive ${}^{\underline{r}}\mathcal{V} \in CHM(S)_F$ as

$${}^{\underline{r}}\mathcal{V} := \bigotimes_{\sigma \in I_L} \text{Sym}^{r_{\sigma}} h_1(A/S)^{e_{\sigma}} .$$

The tensor product is in $CHM(S)_F$, and the symmetric powers are formed with the usual convention concerning the (twist of) the natural action of the symmetric group on a power of A over S (see e.g. [Ki, p. 72]). Thus, ${}^{\underline{r}}\mathcal{V}$ is a direct factor of the relative motive $h(A^r/S)$, where A^r denotes the r -fold fibre product of A over S . That is, it is associated to an idempotent

$$e_{\underline{r}} \in \text{CH}^{rg}(A^r \times_S A^r) \otimes_{\mathbb{Z}} F .$$

Recall (Remark 6.7 (b)) that $\text{CH}^{rg}(A^r \times_S A^r)_{1,2}$ is defined as the image of

$$c_S(A^r, A^r) \cap c_{1,2}(A^r, A^r) \longrightarrow \text{CH}^{rg}(A^r \times_S A^r) .$$

Lemma 7.4. *The idempotent $e_{\underline{r}}$ lies in*

$$\text{CH}^{rg}(A^r \times_S A^r)_{1,2} \otimes_{\mathbb{Z}} F \subset \text{CH}^{rg}(A^r \times_S A^r) \otimes_{\mathbb{Z}} F .$$

Proof. We already know from the formula in Proposition 2.4 that

$$\pi_{A,1,n} = \prod_{j \neq 1} \frac{\Gamma_{[n]_A} - n^j}{n - n^j}$$

is a pre-image of $p_{A,1}$ in $c_S(A, A) \otimes_{\mathbb{Z}} \mathbb{Q}$, for any $n \neq -1, 0, 1$. It visibly lies in the intersection $(c_S(A^r, A^r) \cap c_{1,2}(A^r, A^r)) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Similarly, e_σ is seen to be the image of the composition of $\pi_{A,1,n}$ and

$$\prod_{\tau \neq \sigma} \frac{\Gamma_{\alpha(l)} - \tau(l)}{\sigma(l) - \tau(l)},$$

for any l generating L over \mathbb{Q} , and such that $\alpha(l)$ is a genuine endomorphism of A . The graph $\Gamma_{\alpha(l)}$ is a cycle in $A \times_S A$ which maps isomorphically to the first component of $A \times_S A$. Over the second component, it is necessarily finite: indeed, the element l is invertible in L , hence $\alpha(l)$ is invertible in $\text{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Altogether, this proves that $e_\sigma \in \text{CH}^g(A \times_S A) \otimes_{\mathbb{Z}} F$ comes from

$$(c_S(A, A) \cap c_{1,2}(A, A)) \otimes_{\mathbb{Z}} F.$$

The same is then true for the external product of the e_σ corresponding to the direct factor

$$\bigotimes_{\sigma \in I_L} (h_1(A/S)^{e_\sigma})^{\otimes r_\sigma}$$

of $h(A^r/S)$.

In order to get a pre-image of the idempotent e_r , it suffices to take a suitable average over the action of a suitable finite group (a product of symmetric groups). **q.e.d.**

According to Remark 6.7 (b), the idempotent e_r thus maps to an idempotent in $\bar{c}_{1,2}(A^r, A^r) \otimes_{\mathbb{Z}} F$. It will be denoted by the same symbol e_r . Theorem 1.2 (a) tells us that the relative Chow motive

$${}^r\mathcal{V} = h(A^r/S)^{e_r}$$

gives rise to an exact triangle

$$\partial M_{gm}(A^r)^{e_r} \longrightarrow M_{gm}(A^r)^{e_r} \longrightarrow M_{gm}^c(A^r)^{e_r} \longrightarrow \partial M_{gm}(A^r)^{e_r}[1]$$

in $DM_{gm}^{eff}(k)_F$. Here are our main results.

Theorem 7.5. *The boundary motive $\partial M_{gm}(A^r)$ lies in the triangulated sub-category $DMDT(\mathbb{Q})_{\mathbb{Q}}$ of $DM_{gm}(\mathbb{Q})_{\mathbb{Q}}$ of Dirichlet–Tate motives over \mathbb{Q} . Its direct factor $\partial M_{gm}(A^r)^{e_r}$ is without weights*

$$-r, -(r-1), \dots, r-1.$$

In particular, Assumption 6.1 holds for $\partial M_{gm}(A^r)^{e_r}$ whenever $r \geq 1$.

Theorem 7.6. *Assume that there are $\tau, \sigma \in I_L$ such that $r_\tau \neq r_\sigma$ (hence $g \geq 2$ and $r \geq 1$). Then $\partial M_{gm}(A^r)^{e_\tau} = 0$, and $M_{gm}(A^r)^{e_\tau} \cong M_{gm}^c(A^r)^{e_\tau}$ are effective Chow motives.*

Theorems 7.5 and 7.6 will be proved in Section 8. Let us give their main corollaries, assuming that $r \geq 1$. First, we fix a weight filtration

$$C_{\leq -(r+1)} \longrightarrow \partial M_{gm}(A^r)^{e_\tau} \longrightarrow C_{\geq r} \longrightarrow C_{\leq -(r+1)}[1]$$

avoiding weights $-r, \dots, r-1$. Thus (Corollary 4.6), $C_{\leq -(r+1)}$ and $C_{\geq r}$ are Dirichlet–Tate motives over \mathbb{Q} of weights $\leq -(r+1)$ and $\geq r$, respectively. Furthermore, $C_{\leq -(r+1)} = 0 = C_{\geq r}$ under the hypothesis of Theorem 7.6.

Corollary 7.7 ([W4, Thm. 4.3]). *Assume $r \geq 1$.*

(a) *The motive $M_{gm}(A^r)^{e_\tau}$ is without weights $-r, \dots, -1$, and the motive $M_{gm}^c(A^r)^{e_\tau}$ is without weights $1, \dots, r$. The Chow motives $\mathrm{Gr}_0 M_{gm}(A^r)^{e_\tau}$ and $\mathrm{Gr}_0 M_{gm}^c(A^r)^{e_\tau}$ [W4, Prop. 2.2] are defined, and they carry a natural action of*

$$GCen_{\bar{c}_{1,2}(A^r, A^r)}(e_\tau) := \{z \in \bar{c}_{1,2}(A^r, A^r) \otimes_{\mathbb{Z}} F, ze_\tau = e_\tau ze_\tau\}.$$

(b) *There are canonical exact triangles*

$$C_{\leq -(r+1)} \longrightarrow M_{gm}(A^r)^{e_\tau} \xrightarrow{\pi_0} \mathrm{Gr}_0 M_{gm}(A^r)^{e_\tau} \longrightarrow C_{\leq -(r+1)}[1]$$

and

$$C_{\geq r} \longrightarrow \mathrm{Gr}_0 M_{gm}^c(A^r)^{e_\tau} \xrightarrow{i_0} M_{gm}^c(A^r)^{e_\tau} \longrightarrow C_{\geq r}[1],$$

which are stable under the natural action of $GCen_{\bar{c}_{1,2}(A^r, A^r)}(e_\tau)$.

(c) *There is a canonical isomorphism $\mathrm{Gr}_0 M_{gm}(A^r)^{e_\tau} \xrightarrow{\sim} \mathrm{Gr}_0 M_{gm}^c(A^r)^{e_\tau}$ in $CHM^{eff}(k)_F$. As a morphism, it is uniquely determined by the property of making the diagram*

$$\begin{array}{ccc} M_{gm}(A^r)^{e_\tau} & \xrightarrow{u} & M_{gm}^c(A^r)^{e_\tau} \\ \pi_0 \downarrow & & \uparrow i_0 \\ \mathrm{Gr}_0 M_{gm}(A^r)^{e_\tau} & \longrightarrow & \mathrm{Gr}_0 M_{gm}^c(A^r)^{e_\tau} \end{array}$$

commute; in particular, it is $GCen_{\bar{c}_{1,2}(A^r, A^r)}(e_\tau)$ -equivariant.

(d) *Let $N \in CHM(k)_F$ be a Chow motive. Then π_0 and i_0 induce isomorphisms*

$$\mathrm{Hom}_{CHM(k)_F}(\mathrm{Gr}_0 M_{gm}(A^r)^{e_\tau}, N) \xrightarrow{\sim} \mathrm{Hom}_{DM_{gm}(k)_F}(M_{gm}(A^r)^{e_\tau}, N)$$

and

$$\mathrm{Hom}_{CHM(k)_F}(N, \mathrm{Gr}_0 M_{gm}^c(A^r)^{e_\tau}) \xrightarrow{\sim} \mathrm{Hom}_{DM_{gm}(k)_F}(N, M_{gm}^c(A^r)^{e_\tau}).$$

(e) *Let $M_{gm}(A^r)^{e_\tau} \rightarrow N \rightarrow M_{gm}^c(A^r)^{e_\tau}$ be a factorization of u through a Chow motive $N \in CHM(k)_F$. Then $\mathrm{Gr}_0 M_{gm}(A^r)^{e_\tau} = \mathrm{Gr}_0 M_{gm}^c(A^r)^{e_\tau}$ is canonically a direct factor of N , with a canonical direct complement.*

Henceforth, we identify $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\mathcal{L}}}$ and $\mathrm{Gr}_0 M_{gm}^c(A^r)^{e_{\mathcal{L}}}$ via the canonical isomorphism of Corollary 7.7 (c). Note that under the hypothesis of Theorem 7.6, we have

$$M_{gm}(A^r)^{e_{\mathcal{L}}} = \mathrm{Gr}_0 M_{gm}(A^r)^{e_{\mathcal{L}}} \quad \text{and} \quad \mathrm{Gr}_0 M_{gm}^c(A^r)^{e_{\mathcal{L}}} = M_{gm}^c(A^r)^{e_{\mathcal{L}}},$$

and the isomorphism of Corollary 7.7 (c) coincides with that of Theorem 7.6. The equivariance statements from Corollary 7.7 (a)–(c) apply in particular to cycles coming from the *Hecke algebra* associated to the Shimura variety S . More precisely, we have the following statement.

Corollary 7.8. *Assume $r \geq 1$. Then $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\mathcal{L}}}$ carries a natural action of the Hecke algebra $R(K, G(\mathbb{A}_f))$ associated to the neat open compact subgroup K of $G(\mathbb{A}_f)$. More precisely, any $x \in G(\mathbb{A}_f)$ defines a cycle denoted KxK in $c_{1,2}(A^r, A^r)$, whose class in $\bar{c}_{1,2}(A^r, A^r)$ belongs to the centralizer*

$$\mathrm{Cen}_{\bar{c}_{1,2}(A^r, A^r)}(e_{\mathcal{L}}) := \{z \in \bar{c}_{1,2}(A^r, A^r) \otimes_{\mathbb{Z}} F, z e_{\mathcal{L}} = e_{\mathcal{L}} z\} \subset \mathrm{GCen}_{\bar{c}_{1,2}(A^r, A^r)}(e_{\mathcal{L}})$$

of $e_{\mathcal{L}}$.

Proof. Fix $x \in G(\mathbb{A}_f)$. Recall that our base scheme S equals the Hilbert–Blumenthal variety $S^K(G, \mathcal{H})$. It is the target of two finite étale morphisms $g_1, g_2 : U \rightarrow S$, where U denotes the Hilbert–Blumenthal variety $S^{K \cap x^{-1}Kx}(G, \mathcal{H})$. Using the notation of [Pi, Sect. 3.4], the morphism g_1 equals $[\cdot 1]$, and the morphism g_2 equals $[\cdot x^{-1}]$. Note that by the very definition of the \mathbb{Q} -rational structure of S and U (e.g. [Pi, Def. 11.5]), both g_1 and g_2 are indeed defined over \mathbb{Q} .

Recall that A is the universal Abelian scheme over S ; denote by A_1, A_2 its base changes to U via g_1 and g_2 , respectively. To the data K and x , the following are canonically associated: a third Abelian scheme B over U admitting real multiplication, and isogenies $f_1 : B \rightarrow A_1$ and $f_2 : B \rightarrow A_2$ compatible with the real multiplications. By definition, the cycle KxK is then equal to the direct image under $g_1 \times_k g_2$ of the composition

$$\Gamma_{f_2^r} \circ {}^t \Gamma_{f_1^r} \in c_U(A_1^r, A_2^r) \cap c_{1,2}(A_1^r, A_2^r)$$

(= “pull-back via f_1^r followed by push-out via f_2^r ”).

In order to show that the class of KxK in $\bar{c}_{1,2}(A^r, A^r)$ commutes with $e_{\mathcal{L}}$, note first that $\varphi := \Gamma_{f_2^r} \circ {}^t \Gamma_{f_1^r}$ defines a morphism of relative Chow motives over U ,

$$\varphi : h(A_1^r/U) = g_1^*(h(A^r/S)) \longrightarrow g_2^*(h(A^r/S)) = h(A_2^r/U).$$

Then, since both f_1 and f_2 are isogenies, this morphism is compatible with the external products of the idempotents $p_{A_i,1}$ (Theorem 2.1 (b) and (c)). Since f_1 and f_2 also respect the real multiplication, the morphism φ is also compatible with the cycle classes $g_i^*(e_{\mathcal{L}}) = e_{\mathcal{L},i} \in \mathrm{CH}^{rg}(A_i^r \times_U A_i^r) \otimes_{\mathbb{Z}} F$, i.e., we have the relation

$$\varphi \circ g_1^*(e_{\mathcal{L}}) = g_2^*(e_{\mathcal{L}}) \circ \varphi$$

of morphisms of relative Chow motives over U . We are thus in the situation of Example 1.16 (d). Now observe that using the notation from Example 1.16, the effect of the cycle KxK on the exact triangle

$$\partial M_{gm}(A^r) \longrightarrow M_{gm}(A^r) \xrightarrow{u} M_{gm}^c(A^r) \longrightarrow \partial M_{gm}(A^r)[1]$$

coincides with $\varphi(g_1, g_2)$ (cmp. Example 1.16 (e)). Thanks to the relation $\varphi \circ g_1^*(e_{\underline{r}}) = g_2^*(e_{\underline{r}}) \circ \varphi$, we thus get that the class of KxK in $\bar{c}_{1,2}(A^r, A^r)$ belongs indeed to $Cen_{\bar{c}_{1,2}(A^r, A^r)}(e_{\underline{r}})$. **q.e.d.**

Corollary 7.9 ([W4, Cor. 4.6]). *Assume $r \geq 1$, and let \widetilde{A}^r be any smooth compactification of A^r . Then $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\underline{r}}}$ is canonically a direct factor of the Chow motive $M_{gm}(\widetilde{A}^r)$, with a canonical direct complement.*

Furthermore, [W4, Thm. 4.7, Thm. 4.8] on the Hodge theoretic and ℓ -adic realizations [Hu, Cor. 2.3.5, Cor. 2.3.4 and Corrigendum] apply, and tell us in particular that $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\underline{r}}}$ is mapped to the part of interior cohomology of A^r fixed by $e_{\underline{r}}$. In particular, the L -function of the Chow motive $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\underline{r}}}$ is computed *via* (the $e_{\underline{r}}$ -part of) interior cohomology of A^r .

Definition 7.10 ([W4, Def. 4.9]). Let $r \geq 1$. We call $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\underline{r}}}$ the $e_{\underline{r}}$ -part of the interior motive of A^r .

Remark 7.11. By [W4, Thm. 4.14], control of the reduction of *some* compactification of A^r implies control of certain properties of the ℓ -adic realization of $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\underline{r}}}$. To the best of the author's knowledge, the sharpest result known about reduction of compactifications of A^r is [DiT, Thm. 6.4]. It concerns the case when $K \subset G(\mathbb{A}_f)$ is of type Γ_1 , and states that there exist then smooth compactifications of A^r having good reduction at each prime number p dividing neither the level N of K nor the absolute discriminant d of L . [W4, Thm. 4.14] then yields the following conclusions: (a) for all primes p not dividing Nd , the p -adic realization of $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\underline{r}}}$ is crystalline, (b) if furthermore $p \neq \ell$, then the ℓ -adic realization of $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\underline{r}}}$ is unramified. Note that given the identification of the ℓ -adic realization of $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\underline{r}}}$ with intersection cohomology, conclusions (a) and (b) are already contained in [DiT, Sect. 7].

Remark 7.12. If all r_σ are strictly positive (hence $r \geq g$) then Saper's vanishing theorem on (ordinary) cohomology [Sp, Thm. 5] implies that the realizations of $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\underline{r}}}$ are concentrated in the single cohomological degree $r + g$. In particular, we expect the following relation to the Chow–Künneth decompositions constructed in [GHM, Thm. 2.4]. The base change from \mathbb{Q} to \mathbb{C} of $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\underline{r}}}$ should map monomorphically to the $(r + g)$ -th Chow–Künneth component of the motive (over \mathbb{C}) of any toroidal compactification of A^r .

Let us discuss special cases. First, for $g = 1$, we have $L = \mathbb{Q}$, S is a smooth modular curve, and A the universal family of elliptic curves over S . As coefficient field (B), we may choose $F = \mathbb{Q}$. The partition (C) amounts to fixing an integer $r \geq 0$, and ${}^z\mathcal{V} = \mathrm{Sym}^r h_1(A/S)$. In this setting, the formal implications of Theorem 7.5 are discussed in [W4, Rem. 4.17]; note that our idempotent e_r coincides with the idempotent denoted e in [loc. cit.]. Indeed, the additional action of torsion entering the definition of e is known (and easily shown) to be trivial on the relative motive $h(A/S)$. In particular [W4, Rem. 4.17 (b)], we get an alternative construction of the Grothendieck motive $M(f)$ associated to a normalized newform f of weight $r + 2$ [Sch] as a direct factor of the Grothendieck motive underlying $\mathrm{Gr}_0 M_{gm}(A^r)^{e_r}$.

Now let $g = 2$. Here, we have $[L : \mathbb{Q}] = 2$, S is a smooth Hilbert–Blumenthal surface, and A the universal family of Abelian surfaces over S . As coefficient field (B), we may choose F equal to the image of L under any of its two real embeddings $\tau, \sigma \in I_L$. The partition (C) amounts to fixing two integers $r_\tau, r_\sigma \geq 0$, whose sum is denoted r . Then

$${}^z\mathcal{V} = \mathrm{Sym}^{r_\tau} h_1(A/S)^{e_\tau} \otimes \mathrm{Sym}^{r_\sigma} h_1(A/S)^{e_\sigma} .$$

For any object M of $DM_{gm}^{eff}(\mathbb{Q})_F$, define motivic cohomology

$$H_{\mathcal{M}}^p(M, F(q)) := \mathrm{Hom}_{DM_{gm}^{eff}(\mathbb{Q})_F}(M, \mathbb{Z}(q)[p]) .$$

When $M = M_{gm}(Y)$ for a scheme $Y \in Sm/\mathbb{Q}$, this gives motivic cohomology $H_{\mathcal{M}}^p(Y, \mathbb{Z}(q))$ of Y , tensored with F . Now observe that the relative Chow motive ${}^z\mathcal{V}$ coincides with the object denoted $\mathcal{V}_K^{r_\tau, r_\sigma}$ in [Ki, Def. 2.3.1]. From now on, assume that $r_\tau \geq r_\sigma \geq 1$ (hence $r \geq 2$). The main result of [loc. cit.] gives the construction of a sub-space

$$\mathcal{K}(r_\tau, r_\sigma, n) \subset H_{\mathcal{M}}^{r+3}({}^z\mathcal{V}, F(n)) = \mathrm{Hom}_{DM_{gm}^{eff}(\mathbb{Q})_F}(M_{gm}(A^r)^{e_r}, \mathbb{Z}(n)[r+3])$$

for all integers n between $r_\tau + 2$ and $r + 2 = r_\tau + r_\sigma + 2$ [Ki, Thm. 5.2.4], and establishes a weak version of Beilinson’s conjecture for Asai L -functions [Ki, Thm. 5.2.4 (b)]. As already mentioned in [Ki, p. 62, Rem. 5.2.5 (a)], one of the shortcomings of this result is that $\mathcal{K}(r_\tau, r_\sigma, n)$ is not shown to come from motivic cohomology of a smooth compactification of A^r . It is reasonable to expect this to be true; one of the indications being [Ki, Thm. 5.2.4 (a)] that the Hodge theoretic realization of $\mathcal{K}(r_\tau, r_\sigma, n)$ lands in

$$H_{1, \mathfrak{S}}^{r+3}({}^z\mathcal{V}_{/\mathbb{R}}, \mathbb{R}(n)) \subset H_{\mathfrak{S}}^{r+3}({}^z\mathcal{V}_{/\mathbb{R}}, \mathbb{R}(n)) ,$$

which by definition [Ki, Def. (2.4.1)] is the sub-space of absolute Hodge cohomology of ${}^z\mathcal{V}_{/\mathbb{R}}$ given by the image of absolute Hodge cohomology of any smooth compactification of A^r . Our main results allow to give a significantly more precise statement.

Corollary 7.13. *Assume that $r_\tau \geq r_\sigma + 1 (\geq 2)$ or that $r_\tau + 2 \leq n \leq r + 1$. Then the map on the level of motivic cohomology induced by the morphism*

$$\pi_0 : M_{gm}(A^r)^{e_r} \rightarrow \mathrm{Gr}_0 M_{gm}(A^r)^{e_r},$$

$$\pi_0^* : \mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(\mathrm{Gr}_0 M_{gm}(A^r)^{e_r}, \mathbb{Z}(n)[r+3]) \longrightarrow H_{\mathcal{M}}^{r+3}(x\mathcal{V}, F(n))$$

is an isomorphism.

In particular, under the hypotheses of the corollary, $\mathcal{K}(r_\tau, r_\sigma, n)$ can be considered as a sub-space of $\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(\mathrm{Gr}_0 M_{gm}(A^r)^{e_r}, \mathbb{Z}(n)[r+3])$, and hence (Corollary 7.9) of motivic cohomology of any smooth compactification of A^r .

Corollary 7.14. *Assume that $r_\tau = r_\sigma (\geq 1)$, hence $r = 2r_\tau$. Then the image of the map on the level of motivic cohomology induced by π_0 ,*

$$\pi_0^*(\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(\mathrm{Gr}_0 M_{gm}(A^r)^{e_r}, \mathbb{Z}(r+2)[r+3])) \subset H_{\mathcal{M}}^{r+3}(x\mathcal{V}, F(r+2))$$

contains the sub-space $\mathcal{K}(r_\tau, r_\sigma, r+2)$.

In particular, under the hypotheses of the corollary, $\mathcal{K}(r_\tau, r_\sigma, r+2)$ comes from a sub-space of $\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(\mathrm{Gr}_0 M_{gm}(A^r)^{e_r}, \mathbb{Z}(r+2)[r+3])$.

Remark 7.15. This settles completely the problem mentioned in [Ki, Rem. 5.2.5 (a)]. At least two other points remain, in order to get a proof of Beilinson's full conjecture: first [Ki, Rem. 5.2.5 (c)], the elements in $\mathcal{K}(r_\tau, r_\sigma, n)$ should be *integral* (with respect to suitable models over $\mathbf{Spec} \mathbb{Z}$), second, the space of integral elements in motivic cohomology should be equal to $\mathcal{K}(r_\tau, r_\sigma, n)$. We have nothing to say about these two points.

Proof of Corollaries 7.13 and 7.14. Recall the exact triangle

$$C_{\leq -(r+1)} \longrightarrow M_{gm}(A^r)^{e_r} \xrightarrow{\pi_0} \mathrm{Gr}_0 M_{gm}(A^r)^{e_r} \longrightarrow C_{\leq -(r+1)}[1]$$

from Corollary 7.7 (b). Here, $C_{\leq -(r+1)}$ is a Dirichlet–Tate motive over \mathbb{Q} of weights $\leq -(r+1)$. Theorem 7.6 tells us that

$$C_{\leq -(r+1)} = 0$$

when $r_\tau \geq r_\sigma + 1$. In this case, the morphism π_0 is therefore itself an isomorphism, and thus induces an isomorphism on the level of motivic cohomology.

In general, the kernel of

$$\pi_0^* : \mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(\mathrm{Gr}_0 M_{gm}(A^r)^{e_r}, \mathbb{Z}(n)[r+3]) \longrightarrow H_{\mathcal{M}}^{r+3}(x\mathcal{V}, F(n))$$

is a quotient of

$$\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(C_{\leq -(r+1)}[1], \mathbb{Z}(n)[r+3]) = \mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(C_{\leq -(r+1)}, \mathbb{Z}(n)[r+2])$$

and the co-kernel a sub-space of

$$\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(C_{\leq -(r+1)}, \mathbb{Z}(n)[r+3]).$$

Now $\mathbb{Z}(n)$ is pure of weight $-2n$. When $n \leq r+1$, then the weights of $\mathbb{Z}(n)[r+2]$ and of $\mathbb{Z}(n)[r+3]$ are at least equal to $-2(r+1)+r+2 = -r$. Since $C_{\leq -(r+1)}$ is of weights at most $-(r+1)$, orthogonality 4.1 (3) implies that both

$\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(C_{\leq-(r+1)}, \mathbb{Z}(n)[r+2])$ and $\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(C_{\leq-(r+1)}, \mathbb{Z}(n)[r+3])$ are zero. In this case, the map π_0^* is therefore again an isomorphism.

In the sequel, let us therefore assume that $r_\tau = r_\sigma$, and that $n = r + 2$. As above, the co-kernel injects into

$$\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(C_{\leq-(r+1)}, \mathbb{Z}(n)[r+3]) .$$

Let us first show that on this latter space, the map induced by the Hodge theoretic realization is injective. Observe that the motive $\mathbb{Z}(n)[r+3]$ is pure of weight $-(r+1)$. This is the highest weight possibly occurring in $C_{\leq-(r+1)}$. Shifting by $r+1$ therefore reduces us to show the following: for any two Dirichlet–Tate motives M and N , with $M \in DMDT(\mathbb{Q})_{F,w \leq 0}$ and $N \in DMDT(\mathbb{Q})_{F,w=0}$, the map induced on

$$\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(M, N)$$

by the Hodge theoretic realization is injective. Choose an exact triangle

$$M_{\leq -2} \longrightarrow M \longrightarrow M_{-1,0} \longrightarrow M_{\leq -2}[1] ,$$

with $M_{\leq -2} \in DMDT(\mathbb{Q})_{F,w \leq -2}$ and

$$M_{-1,0} \in DMDT(\mathbb{Q})_{F,w \geq -1} \cap DMDT(\mathbb{Q})_{F,w \leq 0} .$$

By Corollary 4.9, the object $M_{-1,0}$ is a direct sum of two Dirichlet–Tate motives $M_{-1} \oplus \mathrm{Gr}_0 M$, the first being pure of weight -1 and the second pure of weight 0 . Orthogonality 4.1 (3) formally implies that the two morphisms $M \rightarrow M_{-1,0}$ and $M_{-1,0} \rightarrow \mathrm{Gr}_0 M$ induce isomorphisms

$$\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(M_{-1,0}, N) \xrightarrow{\sim} \mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(M, N)$$

and

$$\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(\mathrm{Gr}_0 M, N) \xrightarrow{\sim} \mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(M_{-1,0}, N) .$$

The Hodge theoretic realization R maps our data to an exact triangle

$$R(M_{\leq -2}) \longleftarrow R(M) \longleftarrow R(M_{-1,0}) \longleftarrow R(M_{\leq -2})[-1]$$

in the bounded derived category D of mixed graded-polarizable Hodge structures (recall that R is contravariant), and a direct sum decomposition

$$R(M_{-1}) \oplus R(\mathrm{Gr}_0 M) = R(M_{-1,0}) .$$

The functor R respects the weight structure in the sense of Remark 5.12 (a). This means that $R(M_{\leq -2})$ has weights at least 2, that $R(M_{-1})$ is pure of weight 1 and that $R(\mathrm{Gr}_0 M)$ and $R(N)$ are pure of weight 0. As above, orthogonality 4.1 (3) (for the category D) yields formally that

$$\mathrm{Hom}_D(K, R(M_{-1,0})) \xrightarrow{\sim} \mathrm{Hom}_D(K, R(M))$$

and

$$\mathrm{Hom}_D(K, R(\mathrm{Gr}_0 M)) \xrightarrow{\sim} \mathrm{Hom}_D(K, R(M_{-1,0}))$$

for any object K of the heart $D_{w=0}$, hence in particular for $K = R(N)$. Altogether, we are thus reduced to showing injectivity of

$$R : \mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(M, N) \longrightarrow \mathrm{Hom}_D(R(N), R(M))$$

under the additional assumption that M belongs to the heart $DMDT(\mathbb{Q})_{F,w=0}$, too. In other words, we must show faithfulness of the restriction of R to the heart of $DMDT(\mathbb{Q})_F$. But this follows easily from the explicit description of $DMDT(\mathbb{Q})_{F,w=0}$ given in Theorem 4.5 (c), and the (obvious) faithfulness of R on the category $MD(k)_F$ from Definition 5.4 (a).

In order to finish the proof, it remains to show that the image of the space $\mathcal{K}(r_\tau, r_\sigma, n)$ in

$$\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(C_{\leq -(r+1)}, \mathbb{Z}(n)[r+3])$$

is mapped to zero under R . Choose a smooth compactification \widetilde{A}^r of A^r . By Corollary 7.7 (d), the morphism $j : M_{gm}(A^r)^{e_x} \rightarrow M_{gm}(A^r) \rightarrow M_{gm}(\widetilde{A}^r)$ factors through $\mathrm{Gr}_0 M_{gm}(A^r)^{e_x}$. It follows that the exact triangle

$$C_{\leq -(r+1)} \longrightarrow M_{gm}(A^r)^{e_x} \longrightarrow \mathrm{Gr}_0 M_{gm}(A^r)^{e_x} \longrightarrow C_{\leq -(r+1)}[1]$$

maps to an exact triangle of the form

$$C' \xrightarrow{i} M_{gm}(A^r)^{e_x} \xrightarrow{j} M_{gm}(\widetilde{A}^r) \longrightarrow C'[1].$$

By [Ki, Thm. 5.2.4 (a)], the sub-space

$$\mathcal{K}(r_\tau, r_\sigma, n) \subset \mathrm{Hom}_{DM_{gm}^{eff}(\mathbb{Q})_F}(M_{gm}(A^r)^{e_x}, \mathbb{Z}(n)[r+3])$$

vanishes under the composition of

$$i^* : \mathrm{Hom}_{DM_{gm}^{eff}(\mathbb{Q})_F}(M_{gm}(A^r)^{e_x}, \mathbb{Z}(n)[r+3]) \longrightarrow \mathrm{Hom}_{DM_{gm}^{eff}(\mathbb{Q})_F}(C', \mathbb{Z}(n)[r+3])$$

and of R . *A fortiori*, its image in

$$\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(C_{\leq -(r+1)}, \mathbb{Z}(n)[r+3])$$

vanishes under R . **q.e.d.**

Remark 7.16. The picture for $r_\tau = r_\sigma$ and $n = r+2$ remains incomplete: it is clearly desirable to identify a *canonical* pre-image of $\mathcal{K}(r_\tau, r_\sigma, r+2)$ in

$$\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_F}(\mathrm{Gr}_0 M_{gm}(A^r)^{e_x}, \mathbb{Z}(r+2)[r+3]).$$

In order to achieve this, one needs to go into the construction of the elements from [Ki]. The vital ingredient is Beilinson's *Eisenstein symbol* [Be1], which needs to be re-interpreted in the context of the category $DM_{gm}(\mathbb{Q})_F$. We plan to treat this elsewhere.

Let us get back to the general case of arbitrary $g \geq 1$. By Corollary 7.8, the Chow motive $\mathrm{Gr}_0 M_{gm}(A^r)^{e_x}$ carries a natural action of the Hecke algebra $R(K, G(\mathbb{A}_f))$. By functoriality, the same is true for the Grothendieck motive underlying $\mathrm{Gr}_0 M_{gm}(A^r)^{e_x}$. On the latter the action of $R(K, G(\mathbb{A}_f))$ can be

studied *via* realizations. Recall that the realizations of $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\underline{r}}}$ equal the $e_{\underline{r}}$ -part of interior cohomology of A^r .

Remark 7.17. In particular, cycles in $R(K, G(\mathbb{A}_f))$ acting semi-simply on interior cohomology of A^r act semi-simply on the Grothendieck motive underlying $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\underline{r}}}$. The latter therefore decomposes further according to the different (finitely many) eigenvalues of the action of such cycles. For $g = 1$, this is the observation allowing for the construction of the Grothendieck motive $M(f)$ associated to a normalized newform f of weight $r + 2$ (see above). The same construction works in principle for arbitrary g . However, as observed in [BlsR, p. 56], it does *not* give rise to the Grothendieck motive $M(f)$ associated to a Hilbert modular form f . One of the results of [BrL] on the ℓ -adic realization of $M(f)$ (see [Ta, p. 280] for this formulation) suggests that the eigenpart of the Grothendieck motive underlying $\mathrm{Gr}_0 M_{gm}(A^r)^{e_{\underline{r}}}$ equals in fact the “tensor induced” motive of $M(f)$.

Let us finish this section with the following amusing observation.

Corollary 7.18. *Write $M_{gm,1}(A)$ for the image of the relative Chow motive $h_1(A/S)$ under the functor M_{gm} from Theorem 1.2 (a). If $g \geq 2$, then $M_{gm,1}(A)$ is an effective Chow motive.*

Proof. Indeed, by definition,

$$M_{gm,1}(A) = \bigoplus_{\underline{r}} M_{gm}(A)^{e_{\underline{r}}},$$

where the direct sum extends over all partitions

$$\underline{r} : \quad r = \sum_{\sigma \in I_L} r_{\sigma}$$

satisfying $r_{\sigma} \geq 0$ and $r = 1$. In particular, if $g \geq 2$, then for any partition \underline{r} occurring in this direct sum, there are $\tau, \sigma \in I_L$ such that $r_{\tau} = 0$ and $r_{\sigma} = 1$. We may thus apply Theorem 7.6 **q.e.d.**

8 Proof of Theorems 7.5 and 7.6

We keep the notation of the preceding section. In order to prove Theorems 7.5 and 7.6, our idea is to apply the criterion from Section 6. Let us start by fixing some notation.

Definition 8.1. Denote by V the standard two-dimensional representation of $\mathrm{GL}_{2,L}$ over L .

Thus, $\mathrm{Res}_{L/\mathbb{Q}} V$ is a $2g$ -dimensional representation of $\mathrm{Res}_{L/\mathbb{Q}} \mathrm{GL}_{2,L}$, and therefore, of G .

Theorem 8.2. *For any integer $r \geq 0$, the boundary motive $\partial M_{gm}(A^r)$ lies in the triangulated sub-category $\mathrm{DMDT}(\mathbb{Q})_{\mathbb{Q}}$ of $\mathrm{DM}_{gm}(\mathbb{Q})_{\mathbb{Q}}$ of Dirichlet–Tate motives over \mathbb{Q} .*

Proof. The variety A^r is a *mixed Shimura variety* over $S = S^K(G, \mathcal{H})$. More precisely, the representation $\mathrm{Res}_{L/\mathbb{Q}} V$ of G is easily seen to be of Hodge type $\{(-1, 0), (0, -1)\}$ in the sense of [Pi, Sect. 2.16]. The same statement is then true for the r -th power $\mathrm{Res}_{L/\mathbb{Q}} V^r$ of $\mathrm{Res}_{L/\mathbb{Q}} V$. By [Pi, Prop. 2.17], this allows for the construction of the *unipotent extension* (P^r, \mathfrak{X}^r) of (G, \mathcal{H}) by $\mathrm{Res}_{L/\mathbb{Q}} V^r$. The reader wishing an explicit description of (P^r, \mathfrak{X}^r) is referred to [Ki, Sect. 1.1], where the case $g = 2$ is treated. The description from [loc. cit.] generalizes easily to arbitrary g .

The pair (P^r, \mathfrak{X}^r) constitute *mixed Shimura data* [Pi, Def. 2.1]. By construction, they come endowed with a morphism $\pi^r : (P^r, \mathfrak{X}^r) \rightarrow (G, \mathcal{H})$ of Shimura data, identifying (G, \mathcal{H}) with the pure Shimura data underlying (P^r, \mathfrak{X}^r) . In particular, (P^r, \mathfrak{X}^r) also satisfy condition (+) from [W3, Sect. 5].

Now there is an open compact neat subgroup K^r of $P^r(\mathbb{A}_f)$, whose image under π^r equals K , and such that A^r is identified with the *mixed Shimura variety* $S^{K^r}(P^r, \mathfrak{X}^r)$ [Pi, Sect. 3.22, Thm. 11.18 and 11.16]. Furthermore, the morphism π^r of Shimura data induces a morphism $S^{K^r}(P^r, \mathfrak{X}^r) \rightarrow S^K(G, \mathcal{H})$, which is identified with the structure morphism of A^r .

In order to obtain control on the boundary motive of A^r , we fix a smooth *toroidal compactification* \widetilde{A}^r . It is associated to a K^r -*admissible complete smooth cone decomposition* \mathfrak{S} , i.e., a collection of subsets of

$$\mathcal{C}(P^r, \mathfrak{X}^r) \times P^r(\mathbb{A}_f)$$

satisfying the axioms of [Pi, Sect. 6.4]. Here, $\mathcal{C}(P^r, \mathfrak{X}^r)$ denotes the *conical complex* associated to (P^r, \mathfrak{X}^r) [Pi, Sect. 4.24].

We refer to [Pi, 9.27, 9.28] for criteria sufficient to guarantee the existence of the associated compactification $\widetilde{A}^r := S^{K^r}(P^r, \mathfrak{X}^r, \mathfrak{S})$. It comes equipped with a natural (finite) stratification into locally closed strata. The unique open stratum is A^r . Any stratum \widetilde{A}_σ^r different from the generic one is associated to a *rational boundary component* (P_1, \mathfrak{X}_1) of (P^r, \mathfrak{X}^r) [Pi, Sect. 4.11] which is *proper*, i.e., unequal to (P^r, \mathfrak{X}^r) .

First, co-localization for the boundary motive [W2, Cor. 3.5] tells us that $\partial M_{gm}(A^r)$ is a successive extension of (shifts of) objects of the form

$$M_{gm}(\widetilde{A}_\sigma^r, i_\sigma^! j_! \mathbb{Z}) .$$

Here, j denotes the open immersion of A^r into \widetilde{A}^r , i_σ runs through the immersions of the strata \widetilde{A}_σ^r different from A^r into \widetilde{A}^r , and $M_{gm}(\widetilde{A}_\sigma^r, i_\sigma^! j_! \mathbb{Z})$ is the motive of \widetilde{A}_σ^r with coefficients in $i_\sigma^! j_! \mathbb{Z}$ defined in [W2, Def. 3.1].

Next, by [W3, Thm. 6.1], there is an isomorphism

$$M_{gm}(\widetilde{A}_\sigma^r, i_\sigma^! j_! \mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}(\sigma), M_{gm}(S^{K_1}(P_1, \mathfrak{X}_1)))[\dim \sigma] .$$

Recall [W3, p. 971] that the *group of orientations* $\mathbb{Z}(\sigma)$ is (non-canonically) isomorphic to \mathbb{Z} , hence

$$\mathrm{Hom}(\mathbb{Z}(\sigma), M_{gm}(S^{K_1}(P_1, \mathfrak{X}_1))) \cong M_{gm}(S^{K_1}(P_1, \mathfrak{X}_1)) .$$

$S^{K_1}(P_1, \mathfrak{X}_1)$ is a Shimura variety associated to the data (P_1, \mathfrak{X}_1) and an open compact neat subgroup K_1 of $P_1(\mathbb{A}_f)$. In order to show our claim, we are thus reduced to showing that $M_{gm}(S^{K_1})$ is an object of $DMDT(\mathbb{Q})_{\mathbb{Q}}$, for any Shimura variety $S^{K_1} = S^{K_1}(P_1, \mathfrak{X}_1)$ associated to a proper rational boundary component (P_1, \mathfrak{X}_1) of (P^r, \mathfrak{X}^r) , and any open compact neat subgroup K_1 of $P_1(\mathbb{A}_f)$.

Given that P^r is a unipotent extension of G , the pure Shimura data underlying (P_1, \mathfrak{X}_1) coincides with the pure Shimura data underlying some proper rational boundary component (G_1, \mathcal{H}_1) of (G, \mathcal{H}) . By definition [Pi, Sect. 4.11], the group G_1 is associated to an *admissible \mathbb{Q} -parabolic subgroup* Q of G [Pi, Def. 4.5]. It is not difficult to see that the inverse image under the immersion of G into $\mathrm{Res}_{L/\mathbb{Q}} \mathrm{GL}_{2,L}$ induces a bijection on the sets of \mathbb{Q} -parabolic subgroups. Under this bijection, the group Q corresponds necessarily to a group of the form $\mathrm{Res}_{L/\mathbb{Q}} B$, for some Borel subgroup B of $\mathrm{GL}_{2,L}$. Equivalently, Q is the stabilizer in G of a subspace of $\mathrm{Res}_{L/\mathbb{Q}} V$ of the form $\mathrm{Res}_{L/\mathbb{Q}} V'$, for a one-dimensional L -subspace V' of V . A computation analogous to the one from [Pi, Ex. 4.25] for $g = 1$ shows that the pure Shimura data underlying (G_1, \mathcal{H}_1) equal the data $(\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$ from [Pi, Ex. 2.8]. Altogether, we see that (P_1, \mathfrak{X}_1) is a unipotent extension of $(\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$.

We are ready to conclude. As follows directly from the definition of the canonical model (cmp. [Pi, Sect. 11.3, 11.4]), the pure Shimura variety $S^{\pi^r(K_1)}(\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$ underlying S^{K_1} equals the spectrum of a cyclotomic field C over \mathbb{Q} . By [Pi, Prop. 11.14], the variety S^{K_1} is isomorphic to a power of the multiplicative group over C . In particular, its motive lies in $DMDT(\mathbb{Q})_{\mathbb{Q}}$.
q.e.d.

Remark 8.3. As the proof shows, Theorem 8.2 admits a version “before tensoring with \mathbb{Q} ”. That is, the boundary motive $\partial M_{gm}(A^r)$ lies in the triangulated sub-category $DMDT(\mathbb{Q})$ of $DM_{gm}(\mathbb{Q})$ generated by Tate twists and motives $M_{gm}(\mathbf{Spec} k)$, for Abelian finite field extensions k of \mathbb{Q} .

In order to apply the results from Section 6, we need to analyze the Hodge structure on the e_x -part of the boundary cohomology of A^r ,

$$(\partial H^n(A^r(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F)^{e_x} ,$$

for all integers n . Recall that there is a canonical isomorphism

$$(\mathrm{Res}_{L/\mathbb{Q}} V) \otimes_{\mathbb{Q}} F \xrightarrow{\sim} \bigoplus_{\sigma \in I_L} V_\sigma ,$$

where we set $V_\sigma := V \otimes_{L,\sigma} F$. In fact, this is an isomorphism of representations of G over F .

Definition 8.4. Denote by $V_{\underline{r}}$ the representation

$$V_{\underline{r}} := \bigotimes_{\sigma \in I_L} \text{Sym}^{r_\sigma} V_\sigma^\vee .$$

of G over F .

The tensor product is over F , and V_σ^\vee is the contragredient representation of V_σ . Recall (e.g. [W1, Thm. 2.2]) the definition of the *canonical construction* functor μ from the category of finite-dimensional algebraic representations of G to the category of admissible graded-polarizable variations of Hodge structure on S .

Proposition 8.5. *There is a canonical isomorphism of Hodge structures*

$$(\partial H^n(A^r(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F)^{e_{\underline{r}}} \xrightarrow{\sim} \partial H^{n-r}(S(\mathbb{C}), \mu(V_{\underline{r}}))$$

for all integers n .

Proof. The central observation is that the functor μ maps $\text{Res}_{L/\mathbb{Q}} V^\vee$ to the first higher direct image of $\mathbb{Q}(0)$ under the structure morphism of the Abelian variety A [W1, remark following Lemma 2.5]. The rest of the argument is purely formal; cmp. [Ki, proof of Prop. 2.3.3] for the case $g = 2$. **q.e.d.**

Thus, we need to control $\partial H^{n-r}(S(\mathbb{C}), \mu(V_{\underline{r}}))$. As the reader may expect, we use the *Baily–Borel compactification* S^* of S . The complement of S consists of finitely many cusps; the boundary cohomology of $S(\mathbb{C})$ therefore coincides with the direct sum over the cusps of the *degeneration* of the coefficients to the boundary of S^* .

Proof of Theorem 7.6. By assumption, there are $\tau, \sigma \in I_L$ such that $r_\tau \neq r_\sigma$. This implies [BrL, Lemme 2.2.8] that the boundary cohomology of $\mu(V_{\underline{r}})$ vanishes. Proposition 6.5 tells us that the Hodge theoretic realization of $\partial M_{gm}(A^r)^{e_{\underline{r}}}$ is zero. Given that $\partial M_{gm}(A^r)^{e_{\underline{r}}}$ is Artin–Tate (Theorem 8.2 and Corollary 4.6), this means that it is itself zero (Proposition 5.9 (a)). **q.e.d.**

Proof of Theorem 7.5. The first claim is Theorem 8.2. Given Theorem 7.6, we may assume that all r_σ are equal,

$$r_\sigma = s \geq 0 \quad \forall \sigma \in I_L ,$$

say. Thus,

$$V_{\underline{r}} = \bigotimes_{\sigma \in I_L} \text{Sym}^s V_\sigma^\vee$$

and $r = g \cdot s$. Note that this representation descends to \mathbb{Q} , and that it occurs as a direct factor of the representation $\mathrm{Sym}^r \mathrm{Res}_{L/\mathbb{Q}} V^\vee$. Fix a cusp y of $S^*(\mathbb{C})$, and denote by j the open immersion of S into S^* . We need to compute the weights occurring in

$$R^{n-r} j_* (\mu(V_{\underline{r}}))_y .$$

First [BrL, bottom of p. 386], the cup product

$$R^0 j_* (\mu(V_{\underline{r}}))_y \otimes_{\mathbb{Q}} R^{n-r} j_* (\mathbb{Q}(0))_y \longrightarrow R^{n-r} j_* (\mu(V_{\underline{r}}))_y$$

is an isomorphism in degrees $0 \leq n - r \leq g - 1$. Next [BrL, Thm. 1.3.4, Cor. 1.3.7], in the same range of indices, the map induced by the cup product

$$\Lambda^{n-r} R^1 j_* (\mathbb{Q}(0))_y \longrightarrow R^{n-r} j_* (\mathbb{Q}(0))_y$$

is an isomorphism. We shall show:

- (1) $R^0 j_* (\mu(V_{\underline{r}}))_y \cong \mathbb{Q}(0)$ as Hodge structures,
- (2) $R^1 j_* (\mathbb{Q}(0))_y \cong \mathbb{Q}(0)^{g-1}$ as Hodge structures, when $g \geq 2$.

Admitting these claims for the moment, we see from (1) and (2) that the Hodge structure $R^{n-r} j_* (\mu(V_{\underline{r}}))_y$ is pure of weight 0 for $0 \leq n - r \leq g - 1$, i.e., $r \leq n \leq r + g - 1$. In particular, $R^{n-r} j_* (\mu(V_{\underline{r}}))_y$ is without weights $n - (r - 1), \dots, n + r$ whenever $n \leq r + g - 1$. To deal with the complementary range of indices $n \geq r + g$, recall that the Hodge structures

$$R^m j_* (\mu(V_{\underline{r}}))_y \quad \text{and} \quad R^{2g-1-m} j_* (\mu(V_{\underline{r}}))_y(r + g)$$

are dual to each other, for all integers m . Indeed, the representation $V_{\underline{r}}$ is pure of weight r in the sense of [Pi, Sect. 1.11], therefore [Pi, Prop. 1.12] it can be G -equivariantly identified with its own contragredient, twisted by $\mathbb{Q}(-r)$, i.e.,

$$\mu(V_{\underline{r}}) \cong \mu(V_{\underline{r}})^\vee(-r)$$

as variations of Hodge structures. Given the definition of the Verdier dual \mathbb{D} in the category of algebraic Hodge modules [Sp, Prop. 2.6], we have

$$\mu(V_{\underline{r}}) \cong \mathbb{D}_S(\mu(V_{\underline{r}}))(-r + g)[-2g] .$$

Denoting by i the immersion of y into S^* , we have

$$i^* \circ Rj_* \circ \mathbb{D}_S = \mathbb{D}_y \circ i^! \circ j_! ;$$

furthermore, $i^! \circ j_! = i^* \circ Rj_*[1]$ [Sp, formulae (4.3.5) and (4.4.1)]. This implies that

$$i^* Rj_* \mu(V_{\underline{r}}) \cong \mathbb{D}_y i^* Rj_* \mu(V_{\underline{r}})(-r + g)[-2g + 1] .$$

Therefore, we see that $R^{n-r} j_* (\mu(V_{\underline{r}}))_y$ is pure of weight $2(r + g)$ whenever $g \leq n - r \leq 2g - 1$, i.e., $r + g \leq n \leq r + 2g - 1$. In particular, $R^{n-r} j_* (\mu(V_{\underline{r}}))_y$

is without weights $n - (r - 1), \dots, n + r$ in any case. Given Theorem 8.2, we thus have verified the hypotheses of Corollary 6.4.

It remains to show claims (1) and (2). We shall use the main result from [BuW] on degeneration in the Baily–Borel compactification of variations in the image of μ . The cusp y belongs to one of the strata associated to a rational boundary component (P_1, \mathfrak{X}_1) of (G, \mathcal{H}) , where P_1 is contained as a normal subgroup in one of the admissible \mathbb{Q} -parabolic subgroups Q of G [Pi, Sect. 4.11]. The latter being the stabilizer in G of a subspace of $\text{Res}_{L/\mathbb{Q}} V$ of the form $\text{Res}_{L/\mathbb{Q}} V'$, for a one-dimensional L -subspace V' of V , we see that the situation is conjugate under an element of $G(\mathbb{Q})$ to the one associated to the standard Borel subgroup. Since claims (1) and (2) are invariant under isomorphisms, we may therefore assume that we work in this setting.

It is identical to the one considered in [Blt, proof of Prop. 3.2] (with the same notation). Applying [BuW, Thm. 2.9], we see that

$$R^0 j_* (\mu(V_r))_y \cong H^0(\bar{H}_C, H^0(W_1, \text{Res}_Q^G V_r)) ,$$

while

$$R^1 j_* (\mathbb{Q}(0))_y \cong H^0(\bar{H}_C, H^1(W_1, \mathbb{Q}(0))) \oplus H^1(\bar{H}_C, H^0(W_1, \mathbb{Q}(0))) .$$

Here, W_1 denotes the unipotent radical of Q , and \bar{H}_C is free Abelian of rank $g - 1$ (cmp. [Blt, Sect. 3.2]). Note that there is a shift by the codimension of y in S^* (which equals g) in the formula of [BuW, Thm. 2.9], due to the normalization of the inclusion of the category of variations into the derived category of algebraic Hodge modules used in [loc. cit.]. In order to evaluate the first of the above expressions, one proceeds dually to [Blt, proof of Prop. 3.2], to show:

(3) $H^0(W_1, \text{Res}_Q^G V_r)$ is one-dimensional,

(4) the actions of \bar{H}_C and of P_1/W_1 on $H^0(W_1, \text{Res}_Q^G V_r)$ are both trivial.

Given that the action of P_1/W_1 determines the Hodge structure, this shows (1). As for (2), note first that

$$H^0(\bar{H}_C, H^1(W_1, \mathbb{Q}(0))) = 0$$

when $g \geq 2$ [BrL, bottom of p. 386]. Hence

$$R^1 j_* (\mathbb{Q}(0))_y \cong H^1(\bar{H}_C, H^0(W_1, \mathbb{Q}(0)))$$

in this case. Given that $H^0(W_1, \mathbb{Q}(0)) = \mathbb{Q}(0)$, and that the action of \bar{H}_C on $H^0(W_1, \mathbb{Q}(0))$ is trivial (use claim (4) for $s = 0$), we have indeed

$$R^1 j_* (\mathbb{Q}(0))_y \cong \text{Hom}(\bar{H}_C, \mathbb{Q})(0) .$$

q.e.d.

The above proof actually yields a statement on the Dirichlet–Tate motive $\partial M_{gm}(A^r)^{e_z}$, which is much more precise than Theorem 7.5.

Theorem 8.6. *Assume that $r_\sigma = s \geq 0 \forall \sigma \in I_L$, i.e., that*

$$V_{\underline{x}} = \bigotimes_{\sigma \in I_L} \text{Sym}^s V_\sigma^\vee .$$

(a) *The Dirichlet–Tate motive $\partial M_{gm}(A^r)^{e_{\underline{x}}}$ lies in*

$$DMDT(\mathbb{Q})_{\mathbb{Q},[-2(r+g),0]}$$

in the notation of Section 3.

(b) *There is an exact triangle*

$$M_{-2(r+g)} \longrightarrow \partial M_{gm}(A^r)^{e_{\underline{x}}} \longrightarrow M_0 \longrightarrow M_{-2(r+g)}[1] ,$$

such that $M_{-2(r+g)} \in DMDT(\mathbb{Q})_{\mathbb{Q},-2(r+g)}$ and $M_0 \in DMDT(\mathbb{Q})_{\mathbb{Q},0}$. The triangle is unique up to unique isomorphism.

(c) *Denoting by R the Hodge theoretic realization, and by S^∞ the complement of S in S^* , there are isomorphisms*

$$R(M_{-2(r+g)}) \cong \bigoplus_{n=r+g}^{r+2g-1} R(M_{gm}(S^\infty)^{\binom{g-1}{n-(r+g)}}(r+g)[n])$$

and

$$R(M_0) \cong \bigoplus_{n=r}^{r+g-1} R(M_{gm}(S^\infty)^{\binom{g-1}{n-r}}[n])$$

(d) *When $r \geq 1$, then the weight filtration*

$$C_{\leq -(r+1)} \longrightarrow \partial M_{gm}(A^r)^{e_{\underline{x}}} \longrightarrow C_{\geq r} \longrightarrow C_{\leq -(r+1)}[1]$$

of $\partial M_{gm}(A^r)^{e_{\underline{x}}}$ coincides with the exact triangle from (b), i.e., there are unique isomorphisms

$$C_{\leq -(r+1)} \xrightarrow{\sim} M_{-2(r+g)} \quad \text{and} \quad C_{\geq r} \xrightarrow{\sim} M_0$$

in $DMDT(\mathbb{Q})_{\mathbb{Q}}$.

Proof. Use Propositions 3.9 and 3.8, together with the (trivial) fact that R is conservative on the \mathbb{Z} -graded category over $MD(\mathbb{Q})_{\mathbb{Q}}$. **q.e.d.**

For $r = 0$, the exact triangle

$$M_{-2g} \longrightarrow \partial M_{gm}(S) \longrightarrow M_0 \longrightarrow M_{-2g}[1]$$

may therefore be considered as “the” weight filtration of $\partial M_{gm}(S)$. Write

$$C_{\leq -1} := M_{-2g} \quad \text{and} \quad C_{\geq 0} := M_0 .$$

Corollary 8.7. *The weight filtration of $\partial M_{gm}(A^r)^{e_{\underline{x}}}$ is split: there is an isomorphism*

$$\partial M_{gm}(A^r)^{e_{\underline{x}}} \cong C_{\leq -(r+1)} \oplus C_{\geq r} .$$

Proof. We have to show that the morphism $C_{\geq r} \rightarrow C_{\leq -(r+1)}[1]$ occurring in the weight filtration of $\partial M_{gm}(A^r)^{e_x}$ is trivial. Parts (c) and (d) of Theorem 8.6, together with Propositions 3.8 and 3.9 imply the existence of objects $N_n \in MD(\mathbb{Q})_{\mathbb{Q}}$, for $r \leq n \leq r + 2g - 1$, such that

$$C_{\leq -(r+1)} \cong \bigoplus_{n=r+g}^{r+2g-1} N_n(r+g)[n]$$

and

$$C_{\geq r} \cong \bigoplus_{m=r}^{r+g-1} N_m[m].$$

Thus, the group $\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_{\mathbb{Q}}}(C_{\geq r}, C_{\leq -(r+1)}[1])$ is identified with the direct sum

$$\bigoplus_{m \leq r+g-1, n \geq r+g} \mathrm{Hom}_{DM_{gm}(\mathbb{Q})_{\mathbb{Q}}}(N_m[m], N_n(r+g)[n+1]).$$

But for any pair $n \geq m+1$, the group $\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_{\mathbb{Q}}}(N_m[m], N_n(r+g)[n+1])$ equals $\mathrm{Hom}_{DM_{gm}(\mathbb{Q})_{\mathbb{Q}}}(N_m, N_n(r+g)[n-m+1])$, and is thus zero (Theorem 5.1 (e)). **q.e.d.**

Thus, the object $\partial M_{gm}(A^r)^{e_x}$ is isomorphic to a direct sum of the form

$$\bigoplus_{n=r}^{r+g-1} N_n[n] \oplus \bigoplus_{n=r+g}^{r+2g-1} N_n(r+g)[n],$$

for Dirichlet motives N_n over \mathbb{Q} . Applying the realization, we see that the complex computing boundary cohomology is isomorphic to the direct sum of its cohomology objects. This observation should be compared to [BuW, last statement of Thm. 2.6].

Remark 8.8. One should expect isomorphisms

$$C_{\leq -(r+1)} \cong \bigoplus_{n=r+g}^{r+2g-1} M_{gm}(S^{\infty})^{\binom{g-1}{n-(r+g)}}(r+g)[n]$$

and

$$C_{\geq r} \cong \bigoplus_{n=r}^{r+g-1} M_{gm}(S^{\infty})^{\binom{g-1}{n-r}}[n]$$

to exist already in $DMDT(\mathbb{Q})_{\mathbb{Q}}$. For $g = 1$, this is indeed the case [W4, Thm. 3.3 (c), Cor. 3.4 (c), Rem. 3.5 (b)].

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