THE n-MOTIVIC t-STRUCTURES FOR $n = 0$, 1 AND 2

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Abstract. For a field k and an integer $n \in \{0, 1, 2\}$, we construct a t-structure $\left({}^n\mathcal{T}^{\mathcal{M}}_{\geq 0}(k), {}^n\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)\right)$ on Voevodsky's triangulated category of motives $\mathbf{DM}_{\text{eff}}(k)$, which we call the *$n$-motivic* t-structure. When $n = 0$, this is simply the usual homotopy t-structure, but for $n \in \{1, 2\}$, these are new t-structures. We will show that the category of Deligne's 1-motives can be embedded as a full subcategory in the heart of the 1-motivic t-structure. By a rather straightforward analogy, we are led to specify a class of objects in the heart of the 2-motivic t-structure which we call *mixed 2-motives*. We will also check that these objects form an abelian category.

Keywords: Voevodsky's motives; mixed motives; motivic t-structures.

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Introduction. A major open problem in the theory of motives is the construction of a *motivic* t-structure on Voevodsky's triangulated category $\mathbf{DM}_{\text{eff}}(k)$ whose heart would be the awaited abelian category of mixed motives. This motivic t-structure should be very different from the existing *homotopy* t-structure which is an outcome of the construction of $\mathbf{DM}_{\text{eff}}(k)$ and the study of homotopy invariant presheaves with transfers [7]. However, one can speculate about the existence of a sequence of *n-motivic* t-structures $\left({}^n\mathcal{T}^{\mathcal{M}}_{\geq 0}(k), {}^n\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)\right)$ on $\mathbf{DM}_{\text{eff}}(k)$, which interpolate between the homotopy t-structure and the motivic t-structure. More precisely, we expect the following to hold.

(1) $\left({}^0\mathcal{T}^{\mathcal{M}}_{\geq 0}(k), {}^0\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)\right)$ is the homotopy t-structure.
(2) ${}^n\mathcal{T}^{\mathcal{M}}_{\geq 0}(k) \subset {}^{n+i}\mathcal{T}^{\mathcal{M}}_{\geq 0}(k)$ and ${}^n\mathcal{T}^{\mathcal{M}}_{\leq 0}(k) \supset {}^{n+i}\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)$ for $i \geq 0$.
(3) Denote ${}^\infty\mathcal{T}^{\mathcal{M}}_{\leq 0}(k) = \bigcap_{n\in\mathbb{N}} {}^n\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)$ and ${}^\infty\mathcal{T}^{\mathcal{M}}_{\geq 0}(k) \subset \mathbf{DM}_{\text{eff}}(k)$ the full subcategory of $P \in \mathbf{DM}_{\text{eff}}(k)$ such that

$$\hom_{\mathbf{DM}_{\text{eff}}(k)}(P, N[-1]) = 0$$

for all $N \in {}^\infty\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)$. Then $\left({}^\infty\mathcal{T}^{\mathcal{M}}_{\geq 0}(k), {}^\infty\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)\right)$ is the motivic t-structure.
(4) ${}^n\mathcal{T}^{\mathcal{M}}_{\geq 0}(k) \bigcap {}^{n+i}\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)$ is independent of $i \geq 1$ and is the abelian category of *mixed n-motives*.

The last property above, justifies our terminology. For us, a mixed n-motive is an object of the heart of the motivic t-structure which is also in the smallest triangulated subcategory of $\mathbf{DM}_{\text{eff}}(k)$ stable under small sums and containing the motives of smooth varieties of dimension at most n. In particular, contrary to the usual, we allow non-geometric (or non-constructible) objects.

In this paper, we propose a definition of the n-motivic t-structure for $n \in \{0, 1, 2\}$. Our construction relies on [2]. We will see that objects in ${}^0\mathcal{T}^{\mathcal{M}}_{\leq 0}(k) \cap {}^1\mathcal{T}^{\mathcal{M}}_{\leq 1}(k)$ are (possibly non-compact) 0-motives. We also see that objects in ${}^0\mathcal{T}^{\mathcal{M}}_{\leq 1}(k) \cap {}^1\mathcal{T}^{\mathcal{M}}_{\leq 2}(k)$ are (possibly non-compact) Deligne's 1-motives. Finally, we specify a class of objects in the heart of the 2-motivic t-structure which we call mixed 2-motives. We check that the category of mixed 2-motives is abelian.

Acknowledgment. I wish to thank Luca Barbieri-Viale and Bruno Kahn for their encouragement and interest in this work. An incomplete definition of mixed 2-motives, very similar to the one proposed in this paper, appeared to me sometimes ago. However, I never dared to take it seriously. Especially, the problem of showing that these mixed 2-motives form an abelian category seemed at a first sight out of reach. After I shared my definition with Luca, he kept telling me that I should try to do something with it, till one evening (during the "Algebraic K-Theory and Motivic Cohomology" workshop in Oberwolfach (June 28th - July 4th, 2009)), I decided to follow his advise. I then realized that some progress was possible, and this paper was conceived.

1. Preliminaries

1.1. **Notation and general facts.**

If not otherwise stated, we work with rational coefficients. In particular, our sheaves take values in the category of $\mathbb{Q}$-vectorspaces and we think of an isogeny of semi-abelian group-schemes as an invertible morphism (this will be made more precise later).

Fix a ground field k and denote by Sm/k the category of smooth k-schemes. Given two smooth k-schemes X and Y, we denote by $\mathrm{Cor}(X, Y)$ the group of *finite* *correspondences*, i.e., the $\mathbb{Z}$-module freely generated by closed and integral subschemes $Z \subset X \times_k Y$ which are finite and surjective on X. There is an additive category $\mathrm{Cor}(k)$ whose objects are smooth k-schemes and whose morphisms are finite correspondences (see [7, Lect. 1]). The graph of a morphism yields an inclusion $Sm/k \hookrightarrow \mathrm{Cor}(k)$.

A *presheaf with transfers* (on Sm/k) is an additive contravariant functor F from $\mathrm{Cor}(k)$ to the category of abelian groups. F is called a *Nisnevich* (resp. an *étale*) *sheaf with transfers* if its restriction to Sm/k is a sheaf for the Nisnevich (resp. the étale) topology. If not otherwise stated, a presheaf with transfers F is assumed to be uniquely divisible, i.e., takes values in the category of $\mathbb{Q}$-vectorspaces. Under this assumption, the restriction of F to Sm/k is a Nisnevich sheaf if and only if it is an étale sheaf. Thus, there will be no ambiguity in saying: F is a *sheaf with tranfers*. We denote $Shv_{tr}(k)$ the abelian category of sheaves with transfers on Sm/k. There is an embedding $\mathbb{Q}_{tr}(-) : \mathrm{Cor}(k) \hookrightarrow Shv_{tr}(k)$ which takes a smooth k-scheme X to the sheaf with transfers $\mathbb{Q}_{tr}(X) = \mathrm{Cor}(-, X) \otimes \mathbb{Q}$ represented by X.

We denote by $\mathbf{K}(Shv_{tr}(k))$ the category of complexes of sheaves with transfers endowed with its *injective model structure* (i.e., $\mathbf{W} = \{\text{quasi-isomorphisms}\}$ and $\mathbf{Cof} = \{\text{monomorphisms}\}$). The homotopy category of this model structure is the derived category $\mathbf{D}(Shv_{tr}(k))$. Following Voevodsky [7], we define $\mathbf{DM}_{\mathrm{eff}}(k)$ to be the homotopy category of the Bousfield localization of $\mathbf{K}(Shv_{tr}(k))$ with respect to the class of arrows $\mathbb{Q}_{tr}(\mathbb{A}^1_X)[n] \to \mathbb{Q}_{tr}(X)[n]$ for $X \in Sm/k$ and $n \in \mathbb{Z}$. Given a smooth k-scheme X, we denote by $\mathrm{M}(X)$ the complex $\mathbb{Q}_{tr}(X)[0]$ considered as an object of $\mathbf{DM}_{\mathrm{eff}}(k)$. This is the motive of X. From the general theory of Bousfield localizations (see [5]), we may identify (up to an equivalence) $\mathbf{DM}_{\mathrm{eff}}(k)$ with the triangulated subcategory of $\mathbf{D}(Shv_{tr}(k))$ whose objects are the $\mathbb{A}^1$-*local* complexes. Recall that $K \in \mathbf{K}(Shv_{tr}(k))$ is $\mathbb{A}^1$-local if the natural homomorphism

$$\mathrm{H}^n(X, K) \longrightarrow \mathrm{H}^n(\mathbb{A}^1_X, X)$$

is invertible for all $n \in \mathbb{Z}$ and $X \in Sm/k$. (Here, $\mathrm{H}^n(-, K)$ stands for the Nisnevich (or equivalently the étale) hypercohomology with values in K.) A central result of Voevodsky [7, Th. 24.1] asserts that the previous condition holds if and only if the homology sheaves $\mathrm{H}_i(K)$ are homotopy invariant for all $i \in \mathbb{Z}$. In particular, this implies that the canonical t-structure on $\mathbf{D}(Shv_{tr}(k))$ restricts to a t-structure on $\mathbf{DM}_{\mathrm{eff}}(k)$. This is the so-colled *homotopy t-structure* whose heart is identified with the category $\mathbf{HI}(k)$ of homotopy invariant sheaves with transfers.

N.B. — In this paper, we will use the expression *$\mathcal{H}$-sheaf* as a shorthand for: "homotopy invariant sheaf with transfers".

1.2. **Some recollection from** [2].

For $n \in \mathbb{N}$, we denote $Sm/k_{\leq n}$ the full subcategory of Sm/k whose objects are the smooth k-schemes of dimension less or equal to n. Similarly, we denote $\mathrm{Cor}(k_{\leq n})$ the full-subcategory of $\mathrm{Cor}(k)$ having the same objects as $Sm/k_{\leq n}$. A presheaf with transfers on $Sm/k_{\leq n}$ is an additive contravariant functor F from $\mathrm{Cor}(k_{\leq n})$ to the category of $\mathbb{Q}$-vectorspaces. F is a sheaf with transfers if its restriction to $Sm/k_{\leq n}$ is a sheaf for the Nisnevich topology (or equivalently, for the étale topology). We denote by $Shv_{tr}(k_{\leq n})$ the category of étale sheaves with transfers on $Sm/k_{\leq n}$. There is an adjunction [2, Lem. 1.1.12]:

$$Shv_{tr}(k_{\leq n}) \underset{\sigma_{n*}}{\overset{\sigma_n^*}{\rightleftarrows}} Shv_{tr}(k). \qquad (1)$$

Definition 1.1 — *An $\mathcal{H}$-sheaf $F \in \mathbf{HI}(k)$ is* n-presented *if the obvious morphism*

$$h_0\sigma_n^*\sigma_{n*}F \to F$$

is an isomorphism. (Here, h_0 is the left adjoint to the inclusion $\mathbf{HI}(k) \hookrightarrow Shv_{tr}(k)$.)

Remark 1.2 — In [2] (see Def. 1.1.20 of *loc. cit.*) n-presented $\mathcal{H}$-sheaves where called "n-motivic sheaves". In this paper, we use a different terminology because of an eventual conflict with the notion of *$(n,\mathcal{H})$-sheaf* which will be introduced later (for $n \in \{0, 1, 2\}$).

Let $\mathbf{HI}_{\leq n}(k)$ denotes the full-subcategory of $\mathbf{HI}(k)$ whose objects are the n-presented $\mathcal{H}$-sheaves. This is an abelian category, and the inclusion $\mathbf{HI}_{\leq n}(k) \hookrightarrow \mathbf{HI}(k)$ is right exact. It is conjectured that this inclusion is also left exact (see [2, Cor. 1.4.5] for a conjectural proof). This conjecture is known to hold for $n = 0$, 1 due to the following result (see [2, Cor. 1.2.5 and Prop. 1.3.11]).

Proposition 1.3 — *For $n = 0$, 1, the inclusions $\mathrm{H}_{\leq n}(k) \hookrightarrow Shv_{tr}(k)$ admit left adjoints denoted respectively by*

$$\pi_0 : \ Shv_{tr}(k) \longrightarrow \mathbf{HI}_{\leq 0}(k) \qquad \textit{and} \qquad \mathrm{Alb} : \ Shv_{tr}(k) \longrightarrow \mathbf{HI}_{\leq 1}(k). \qquad (2)$$

Definition 1.4 — *An $\mathcal{H}$-sheaf $F \in \mathbf{HI}(k)$ is called* 0-connected *if $\pi_0(F) = 0$. It is called* 1-connected *if $\mathrm{Alb}(F) = 0$.*

By [2, Prop. 2.3.2 and Th. 2.4.1], the functors π_0 and Alb from (2) can be left derived, yielding functors

$$\mathrm{L}\pi_0 : \ \mathbf{D}(Shv_{tr}(k)) \longrightarrow \mathbf{D}(\mathbf{HI}_{\leq 0}(k)) \quad \text{and} \quad \mathrm{LAlb} : \ \mathbf{D}(Shv_{tr}(k)) \longrightarrow \mathbf{D}(\mathbf{HI}_{\leq 1}(k)).$$

The above functors pass to the Bousfield localization, yielding functors

$$\mathrm{L}\pi_0 : \ \mathbf{DM}_{\mathrm{eff}}(k) \longrightarrow \mathbf{D}(\mathbf{HI}_{\leq 0}(k)) \quad \text{and} \quad \mathrm{LAlb} : \ \mathbf{DM}_{\mathrm{eff}}(k) \longrightarrow \mathbf{D}(\mathbf{HI}_{\leq 1}(k))$$

which are left adjoints to the obvious functors

$$\mathbf{D}(\mathbf{HI}_{\leq 0}(k)) \longrightarrow \mathbf{DM}_{\mathrm{eff}}(k) \quad \text{and} \quad \mathbf{D}(\mathbf{HI}_{\leq 1}(k)) \longrightarrow \mathbf{DM}_{\mathrm{eff}}(k).$$

These are fully faithful embedding with essential images $\mathbf{DM}_{\leq 0}(k)$ and $\mathbf{DM}_{\leq 1}(k)$ respectively. Recall that $\mathbf{DM}_{\leq n}(k)$ (with $n \in \mathbb{N}$) is the smallest triangulated subcategory of $\mathbf{DM}_{\mathrm{eff}}(k)$ stable under small sums and containing $\mathrm{M}(X)$ with $X \in Sm/k$ of dimension at most n. It follows that the obvious inclusions $\mathbf{DM}_{\leq 0}(k) \hookrightarrow \mathbf{DM}_{\mathrm{eff}}(k)$ and $\mathbf{DM}_{\leq 1}(k) \hookrightarrow \mathbf{DM}_{\mathrm{eff}}(k)$ have left adjoints, which we also denote as follows:

$$\mathrm{L}\pi_0 : \ \mathbf{DM}_{\mathrm{eff}}(k) \longrightarrow \mathbf{DM}_{\leq 0}(k) \quad \text{and} \quad \mathrm{LAlb} : \ \mathbf{DM}_{\mathrm{eff}}(k) \longrightarrow \mathbf{DM}_{\leq 1}(k).$$

1.3. **Generating t-structures.**

Let $\mathcal{T}$ be a triangulated category. Recall from [3] that a t-structure on $\mathcal{T}$ is a couple of full subcategories $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ satisfying three simple axioms. Contrary to *loc. cit.*, we will use the homological convention for t-structures. One passes back and forth between the homological and cohomological conventions via the usual rule: $\mathcal{T}_{\geq n} = \mathcal{T}^{\leq -n}$ and $\mathcal{T}_{\leq n} = \mathcal{T}^{\geq -n}$.

In this paragraph we recall the technique of generating t-structures which is described in [1, §2.1.3]. Let $\underline{G}$ a class of objects in $\mathcal{T}$.

Definition 1.5 — *(Compare with* [1, Déf. 2.1.68]*.)*

(a) *An object $N \in \mathcal{T}$ is* $\underline{G}$-negative *if for every $A \in \underline{G}$ and $n \in \mathbb{N}$, we have*

$$\hom_{\mathcal{T}}(A[n+1], N) = 0.$$

We denote $\mathcal{T}^{\underline{G}}_{\leq 0}$ the full subcategory of $\underline{G}$-negative objects and set $\mathcal{T}^{\underline{G}}_{\leq d} = \mathcal{T}^{\underline{G}}_{\leq 0}[d]$ for $d \in \mathbb{Z}$.

(b) *An object $P \in \mathcal{T}$ is* $\underline{G}$-positive *if for every $N \in \mathcal{T}^{\underline{G}}_{\leq -1}$, we have*

$$\hom_{\mathcal{T}}(P, N) = 0.$$

We denote $\mathcal{T}^{\underline{G}}_{\geq 0}$ the full subcategory of $\underline{G}$-positive objects and set $\mathcal{T}^{\underline{G}}_{\geq d} = \mathcal{T}^{\underline{G}}_{\geq 0}[d]$ for $d \in \mathbb{Z}$.

Recall that an object E of $\mathcal{T}$ is said to be an extension of E' and E'' if there exists a distinguished triangle in $\mathcal{T}$:

$$E' \longrightarrow E \longrightarrow E'' \longrightarrow E'[1].$$

We record the following fact (see [1, Prop. 2.1.70]).

Proposition 1.6 — *Assume that $\mathcal{T}$ has small sums and that $\underline{G}$ is essentially small (i.e., the isomorphism classes of objects in $\underline{G}$ form a set) and consists of compact objects. Then $(\mathcal{T}^{\underline{G}}_{\geq 0}, \mathcal{T}^{\underline{G}}_{\leq 0})$ is a t-structure on $\mathcal{T}$. Moreover, $\mathcal{T}^{\underline{G}}_{\geq 0}$ is the smallest full subcategory of $\mathcal{T}$ containing $\underline{G}$, and stable under small sums, suspensions and extensions.*

The t-structure $(\mathcal{T}^{\underline{G}}_{\geq 0}, \mathcal{T}^{\underline{G}}_{<0})$ is said to be *generated* by $\underline{G}$. Clearly $\underline{G} \subset \mathcal{T}^{\underline{G}}_{\geq 0}$, and $(\mathcal{T}^{\underline{G}}_{\geq 0}, \mathcal{T}^{\underline{G}}_{\leq 0})$ is the universal t-structure with this property in the following sense (see [1, Lem. 2.1.78]).

Lemma 1.7 — *Keep the hypothesis in* Proposition 1.6*. Let $\mathcal{S}$ be a triangulated category endowed with a t-structure $(\mathcal{S}_{\geq 0}, \mathcal{S}_{\leq 0})$. Let $F : \mathcal{T} \to \mathcal{S}$ be a triangulated functor. We assume that F commutes with small sums and that $F(\underline{G}) \subset \mathcal{S}_{\geq 0}$. Then F is t-positive, i.e., takes an object in $\mathcal{T}^{\underline{G}}_{\geq 0}$ to an object in $\mathcal{S}_{\geq 0}$.*

2. Perverting t-structures

2.1. **The abstract construction.**

In this paragraph we present a simple way to construct new t-structures out of olds. This will be applied in the next section. We begin by describing the abstract setting.

Let $\mathcal{T}$ be a triangulated category endowed with a t-structure $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$. For $n \in \mathbb{Z}$, we denote by $\tau_{\geq n}$ and $\tau_{\leq n}$ the truncation functors with respect to this t-structures.

Thus, we have a canonical distinguished triangle

$$\tau_{\geq n}(A) \longrightarrow A \longrightarrow \tau_{\leq n-1}(A) \longrightarrow \tau_{\geq n}(A)[1]$$

for every $A \in \mathcal{T}$. We also set $\mathrm{H}_n(A) = \tau_{\geq n} \circ \tau_{\leq n}(A)[-n]$. This is an object of the heart $\mathcal{H}_{\mathcal{T}} = \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0}$.

Let $\mathcal{A} \subset \mathcal{H}_{\mathcal{T}}$ be a full subcategory. We assume the following.

Hypothesis 2.1 —

(i) $\mathcal{A}$ is a thick abelian subcategory of $\mathcal{H}_{\mathcal{T}}$, i.e., stable under extensions, subobjets and quotients.
(ii) The inclusion $\mathcal{A} \hookrightarrow \mathcal{H}_{\mathcal{T}}$ admits a left adjoint $\mathrm{F} : \mathcal{H}_{\mathcal{T}} \to \mathcal{A}$.
(iii) Let

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

be a short exact sequence in $\mathcal{H}_{\mathcal{T}}$. If $A'' \in \mathcal{A}$, then $\mathrm{F}(A') \to \mathrm{F}(A)$ is a monomorphism.

Remark 2.2 —

(a) It follows from (i) that the inclusion $\mathcal{A} \hookrightarrow \mathcal{H}_{\mathcal{T}}$ is an exact functor. As it is also a full embedding, the unit of the adjunction $\phi_A : A \to \mathrm{F}(A)$ is the universal morphism from $A \in \mathcal{H}_{\mathcal{T}}$ to an object in $\mathcal{A}$. In particular, when $A \in \mathcal{A}$, ϕ_A is invertible. For general $A \in \mathcal{H}_{\mathcal{T}}$, we claim that ϕ_A is surjective. Indeed, $im(\phi_A)$ is a sub-object of $\mathrm{F}(A)$. Hence by (i), it is in $\mathcal{A}$. Applying the universal property to $A \to im(\phi_A)$, we get a retraction $\mathrm{F}(A) \to im(\phi_A)$. As the composition $\mathrm{F}(A) \to im(\phi_A) \hookrightarrow \mathrm{F}(A)$ is the identity, $im(\phi_A) \to \mathrm{F}(A)$ is an isomorphism.

(b) Being a left adjoint, the functor F is right exact. Under the conditions of (iii), we thus have a short exact sequence in $\mathcal{A}$:

$$0 \longrightarrow \mathrm{F}(A') \longrightarrow \mathrm{F}(A) \longrightarrow A'' \longrightarrow 0. \tag{3}$$

(Here we use that $A'' \simeq \mathrm{F}(A'')$.) For $A \in \mathcal{H}_{\mathcal{T}}$, we set

$$\mathrm{G}(A) = ker\{A \to \mathrm{F}(A)\}. \tag{4}$$

Thus, we have a canonical exact sequence in $\mathcal{H}_{\mathcal{T}}$:

$$0 \longrightarrow \mathrm{G}(A) \longrightarrow A \longrightarrow \mathrm{F}(A) \longrightarrow 0. \tag{5}$$

It follows from the exact sequence (3) that $\mathrm{F}(\mathrm{G}(A)) = 0$.

Definition 2.3 — *An object $A \in \mathcal{H}_{\mathcal{T}}$ is said to be* F-connected *if* $\mathrm{F}(A) = 0$. *Equivalently, any morphism from A to an object of $\mathcal{A}$ is zero.*

We have seen that for any $A \in \mathcal{H}_{\mathcal{T}}$, $\mathrm{G}(A)$ is F-connected. Moreover, this is the largest F-connected subobject of A. Indeed, let $a : B \to A$ be a morphism in $\mathcal{H}_{\mathcal{T}}$ from an F-connected object B. Then the composition $\phi_A \circ a : B \to \mathrm{F}(A)$ is zero and hence, a factors through $\mathrm{G}(A)$. This also prove that G is the right adjoint to the inclusion of the full subcategory of F-connected objects in $\mathcal{H}_{\mathcal{T}}$.

We now come the main construction of this paragraph.

Proposition 2.4 — *We define a t-structure $({}^{\backprime}\mathcal{T}_{\geq 0}, {}^{\backprime}\mathcal{T}_{\leq 0})$ on $\mathcal{T}$ as follows:*

- *${}^{\backprime}\mathcal{T}_{\geq 0}$ is the full subcategory of $P \in \mathcal{T}$ such that $\mathrm{H}_i(P) = 0$ for $i < -1$ and $\mathrm{H}_{-1}(P)$ is* F*-connected.*

- *${}^{\text{`}}\mathcal{T}_{\leq 0}$ is the full subcategory of $N \in \mathcal{T}$ such that* $\mathrm{H}_i(N) = 0$ *for* $i > 0$ *and* $\mathrm{H}_0(N) \in \mathcal{A}$.

Proof. As usual, we set ${}^{\text{`}}\mathcal{T}_{\geq n} = {}^{\text{`}}\mathcal{T}_{\geq 0}[n]$ and ${}^{\text{`}}\mathcal{T}_{\leq n} = {}^{\text{`}}\mathcal{T}_{\leq 0}[n]$ for $n \in \mathbb{Z}$. We clearly have ${}^{\text{`}}\mathcal{T}_{\geq 1} \subset {}^{\text{`}}\mathcal{T}_{\geq 0}$ and ${}^{\text{`}}\mathcal{T}_{\leq 1} \supset {}^{\text{`}}\mathcal{T}_{\leq 0}$.

Let $P \in {}^{\text{`}}\mathcal{T}_{\geq 0}$ and $N \in {}^{\text{`}}\mathcal{T}_{\leq -1}$. Then $P \in \mathcal{T}_{\geq -1}$ and $N \in \mathcal{T}_{\leq -1}$. Thus, we have

$$\hom_{\mathcal{T}}(P, N) \simeq \hom_{\mathcal{T}}(\tau_{\leq -1}(P), N) \simeq \hom_{\mathcal{T}}(\tau_{\leq -1}(P), \tau_{\geq -1}(N))$$
$$\simeq \hom_{\mathcal{H}_{\mathcal{T}}}(\mathrm{H}_{-1}(P), \mathrm{H}_{-1}(N)).$$

As $\mathrm{H}_{-1}(P)$ is F-connected and $\mathrm{H}_{-1}(N) \in \mathcal{A}$, every morphism from $\mathrm{H}_{-1}(P)$ to $\mathrm{H}_{-1}(N)$ is zero. This shows that $\hom_{\mathcal{T}}(P, N) = 0$.

To end the proof, we still need to check axiom (iii) of [3, Déf. 1.3.1]. Let $A \in \mathcal{T}$. There is a distinguished triangle

$$P_0 \xrightarrow{u_0} A \xrightarrow{v_0} N_0 \longrightarrow P_0[1]$$

where $P_0 \in \mathcal{T}_{\geq -1}$ and $N_0 \in \mathcal{T}_{\leq -2}$. Consider the composition

$$t: \ P_0 \longrightarrow \mathrm{H}_{-1}(P_0)[-1] \longrightarrow \mathrm{F}(\mathrm{H}_{-1}(P_0))[-1],$$

and form a distinguished triangle

$$P \xrightarrow{s} P_0 \xrightarrow{t} \mathrm{F}(\mathrm{H}_{-1}(P_0))[-1] \longrightarrow P[1].$$

Let $u = u_0 \circ s : P \to A$. Clearly $\mathrm{H}_i(P) = 0$ for $i < -1$, and we have an isomorphism $\mathrm{H}_i(u) : \mathrm{H}_i(P) \xrightarrow{\sim} \mathrm{H}_i(A)$ for $i > -1$. Moreover, there is a short exact sequence

$$0 \longrightarrow \mathrm{H}_{-1}(P) \xrightarrow{\mathrm{H}_{-1}(u)} \mathrm{H}_{-1}(A) \longrightarrow \mathrm{F}(\mathrm{H}_{-1}(A)) \longrightarrow 0.$$

In particular $\mathrm{H}_{-1}(P)$ is F-connected. It follows that $P \in {}^{\text{`}}\mathcal{T}_{\geq 0}$.

Now, form a distinguished triangle

$$P \xrightarrow{u} A \xrightarrow{v} N \longrightarrow P[+1].$$

Then $\mathrm{H}_i(N) = 0$ for $i \leq 0$ and

$$\mathrm{H}_{-1}(N) \simeq coker\{\mathrm{H}_{-1}(P) \to \mathrm{H}_{-1}(A)\} \simeq \mathrm{F}(\mathrm{H}_{-1}(A)).$$

This shows that $N \in {}^{\text{`}}\mathcal{T}_{\leq -1}$. The proposition is proven. □

Definition 2.5 — *Keep the above notation and assumption. The t-structure $({}^{\text{`}}\mathcal{T}_{\geq 0}, {}^{\text{`}}\mathcal{T}_{\leq 0})$ is called the* $\mathcal{A}$-perverted *t-structure.*

Remark 2.6 — We denote ${}^{\text{`}}\mathcal{H}_{\mathcal{T}} = {}^{\text{`}}\mathcal{T}_{\geq 0} \cap {}^{\text{`}}\mathcal{T}_{\leq 0}$ the heart of the $\mathcal{A}$-perverted $t$-structure. Clearly, and object $A \in \mathcal{T}$ is in ${}^{\text{`}}\mathcal{H}_{\mathcal{T}}$ if and only if it satisfies the following properties:

(1) $\mathrm{H}_i(A) = 0$ for $i \notin \{0, -1\}$;
(2) $\mathrm{H}_0(A) \in \mathcal{A}$;
(3) $\mathrm{H}_{-1}(A)$ is F-connected.

From this, it follows immediately that $\mathcal{A} = \mathcal{H}_{\mathcal{T}} \cap {}^{\text{`}}\mathcal{H}_{\mathcal{T}}$.

2.2. **The case of a generated t-structure.**

We keep the notation and assumption of §2.1. Suppose that the t-structure $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ is generated by an essentially small class $\underline{G}$ of compact objects in $\mathcal{T}$. Assume that for every $A \in \underline{G}$, we can find a distinguished triangle

$$A^{\perp} \longrightarrow A \longrightarrow A^{F} \longrightarrow A^{\perp}[1]$$

such that A^F is compact and t-positive, $\mathrm{H}_0(A^F) \in \mathcal{A}$, and the obvious morphism $\mathrm{F}(\mathrm{H}_0(A)) \to \mathrm{H}_0(A^F)$ is invertible. Let $\underline{G}^{\perp}[-1] = \{A^{\perp}[-1] \,|\, A \in \underline{G}\}$. We choose the above triangles so that $\underline{G}^{\perp}[-1]$ is again essentially small (this is clearly possible). Remark also that $\underline{G}^{\perp}[-1]$ consists of compact objects.

Proposition 2.7 — *The t-structure $({}^{`}\mathcal{T}_{\geq 0}, {}^{`}\mathcal{T}_{\leq 0})$ is generated by the essentially small class of compact objects ${}^{`}\underline{G} = \underline{G} \bigcup \underline{G}^{\perp}[-1]$.*

Proof. Denote by $({}^{`}\mathcal{T}'_{\geq 0}, {}^{`}\mathcal{T}'_{\leq 0})$ the t-structure on $\mathcal{T}$ generated by ${}^{`}\underline{G}$. It suffices to check that ${}^{`}\mathcal{T}'_{\geq 0} \subset {}^{`}\mathcal{T}_{\geq 0}$ and ${}^{`}\mathcal{T}'_{\leq 0} \subset {}^{`}\mathcal{T}_{\leq 0}$. It is easy to see that ${}^{`}\underline{G} \subset {}^{`}\mathcal{T}_{\geq 0}$. We thus have ${}^{`}\mathcal{T}'_{\geq 0} \subset {}^{`}\mathcal{T}_{\geq 0}$ by Lemma 1.7. To check the second inclusion, we fix $N \in {}^{`}\mathcal{T}'_{\leq 0}$. As $\underline{G} \subset {}^{`}\underline{G}$, we have ${}^{`}\mathcal{T}'_{\leq 0} \subset \mathcal{T}_{\leq 0}$ and thus $\mathrm{H}_i(N) = 0$ for $i > 0$. It remains to show that $\mathrm{H}_0(N) \in \mathcal{A}$.

Let $A \in \underline{G}$. Clearly, $A^{\perp}$ is t-positive. It follows that

$$\hom_{\mathcal{T}}(A^{\perp}, N) \simeq \hom_{\mathcal{H}_{\mathcal{T}}}(\mathrm{H}_0(A^{\perp}), \mathrm{H}_0(N)).$$

On the other hand, $\hom_{\mathcal{T}}(A^{\perp}, N) = 0$ by the definition of the class of ${}^{`}\underline{G}$-negative objects. Thus we get $\hom_{\mathcal{H}_{\mathcal{T}}}(\mathrm{H}_0(A^{\perp}), \mathrm{H}_0(N)) = 0$.

From the statement of the proposition, we have an exact sequence

$$\mathrm{H}_0(A^{\perp}) \longrightarrow \mathrm{H}_0(A) \longrightarrow \mathrm{F}(\mathrm{H}_0(A)) \longrightarrow 0,$$

and hence a surjective morphism $\mathrm{H}_0(A^{\perp}) \twoheadrightarrow \mathrm{G}(\mathrm{H}_0(A))$. We deduce from this an inclusion

$$\hom_{\mathcal{H}_{\mathcal{T}}}(\mathrm{G}(\mathrm{H}_0(A)), \mathrm{H}_0(N)) \hookrightarrow \hom_{\mathcal{H}_{\mathcal{T}}}(\mathrm{H}_0(A^{\perp}), \mathrm{H}_0(N)).$$

This shows that

$$\hom_{\mathcal{H}_{\mathcal{T}}}(\mathrm{G}(\mathrm{H}_0(A)), \mathrm{H}_0(N)) = 0. \tag{6}$$

Now, the abelian category $\mathcal{H}_{\mathcal{T}}$ is generated by $\mathrm{H}_0(A)$ for $A \in \underline{G}$. Thus, we may find a family $(A_i)_{i \in I} \in \underline{G}^I$ and a surjective morphism $\bigoplus_{i \in I} \mathrm{H}_0(A_i) \twoheadrightarrow \mathrm{H}_0(N)$. Consider the induced morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i\in I} \mathrm{G}(\mathrm{H}_0(A_i)) & \longrightarrow & \bigoplus_{i\in I} \mathrm{H}_0(A_i) & \longrightarrow & \bigoplus_{i\in I} \mathrm{F}(\mathrm{H}_0(A_i)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow\!\!\downarrow & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & \mathrm{G}(\mathrm{H}_0(N)) & \longrightarrow & \mathrm{H}_0(N) & \longrightarrow & \mathrm{F}(\mathrm{H}_0(N)) & \longrightarrow & 0.
\end{array}$$

As $\mathrm{G}(\mathrm{H}_0(N)) \to \mathrm{H}_0(N)$ is injective, we deduce from (6) that $\alpha = 0$. By the Snake Lemma, we have a surjective morphism $ker(\beta) \twoheadrightarrow \mathrm{G}(\mathrm{H}_0(N))$. As $ker(\beta) \in \mathcal{A}$, we get from Hypothesis 2.1, (i) that $\mathrm{G}(\mathrm{H}_0(N)) \in \mathcal{A}$. This implies that $\mathrm{G}(\mathrm{H}_0(N)) = 0$. Indeed, the identity morphism of an F-connected object which is in $\mathcal{A}$ is necessarily zero. The proposition is proven. □

2.3. Perverting subcategories of the heart.

We keep the notation and assumption of §2.1. Let $\mathcal{B} \subset \mathcal{H}_{\mathcal{T}}$ be a full subcategory satisfying to the following conditions.

Hypothesis 2.8 —

(i) $\mathcal{B}$ contains $\mathcal{A}$. Moreover, if $B \in \mathcal{B}$, then $\mathrm{G}(B)$ is also in $\mathcal{B}$.
(ii) The category $\mathcal{B}$ is abelian and the inclusion $\mathcal{B} \hookrightarrow \mathcal{H}_{\mathcal{T}}$ admits a right adjoint $\mathrm{Q} : \mathcal{H}_{\mathcal{T}} \to \mathcal{B}$.

It follows from (ii) that the inclusion $\mathcal{B} \hookrightarrow \mathcal{H}_{\mathcal{T}}$ is right exact. Given a morphism $b : B \to C$ in $\mathcal{B}$, its cokernel taken in $\mathcal{B}$ coincides with its cokernel taken in $\mathcal{H}_{\mathcal{T}}$. It will be denoted by $coker(b)$. This is a priori not the case for kernels. We will reserve the notation $ker(b)$ for the kernel taken in $\mathcal{H}_{\mathcal{T}}$ and denote $ker_{\mathcal{B}}(b)$ the kernel taken in $\mathcal{B}$. We have a canonical isomorphism $ker_{\mathcal{B}}(b) \simeq \mathrm{Q}(ker(b))$.

Definition 2.9 — *Let* $`\mathcal{B} \subset \mathcal{T}$ *be the full subcategory whose objects are the* $A \in \mathcal{T}$ *such that:*

(1) $\mathrm{H}_i(A) = 0$ *for* $i \neq \{0, -1\}$*;*
(2) $\mathrm{H}_0(A) \in \mathcal{A}$*;*
(3) $\mathrm{H}_{-1}(A) \in \mathcal{B}$ *and is* F*-connected.*

We call $`\mathcal{B}$ *the* $\mathcal{A}$-perverted *subcategory associated to* $\mathcal{B}$*. Clearly,* $`\mathcal{B}$ *is contained in* $`\mathcal{H}_{\mathcal{T}}$*.*

Lemma 2.10 — *The inclusion* $`\mathcal{B} \hookrightarrow `\mathcal{H}_{\mathcal{T}}$ *has a right adjoint* $`\mathrm{Q} : `\mathcal{H}_{\mathcal{T}} \to `\mathcal{B}$*. Moreover, for* $A \in `\mathcal{H}_{\mathcal{T}}$ *the counit of the adjunction* $`\mathrm{Q}(A) \to A$ *induces isomorphisms* $\mathrm{H}_0(`\mathrm{Q}(A)) \simeq \mathrm{H}_0(A)$ *and* $\mathrm{H}_{-1}(`\mathrm{Q}(A)) \simeq \mathrm{G} \circ \mathrm{Q}(\mathrm{H}_{-1}(A))$*.*

Proof. It suffices to construct for every $A \in `\mathcal{H}_{\mathcal{T}}$ a universal morphism $`\mathrm{Q}(A) \to A$ from an object $`\mathrm{Q}(A) \in `\mathcal{B}$. We have a functorial distinguished triangle

$$\mathrm{H}_0(A) \longrightarrow A \longrightarrow \mathrm{H}_{-1}(A)[-1] \longrightarrow \mathrm{H}_0(A)[1].$$

As both $\mathrm{H}_0(A)$ and $\mathrm{H}_{-1}(A)[-1]$ are in $`\mathcal{H}_{\mathcal{T}}$, this determines a functorial short exact sequence

$$0 \longrightarrow \mathrm{H}_0(A) \longrightarrow A \longrightarrow \mathrm{H}_{-1}(A)[-1] \longrightarrow 0.$$

Consider $\mathrm{G} \circ \mathrm{Q}(\mathrm{H}_{-1}(A))[-1]$. This is an object of $`\mathcal{H}_{\mathcal{T}}$. We define

$$`\mathrm{Q}(A) = A \times_{\mathrm{H}_{-1}(A)[-1]} \big(\mathrm{G} \circ \mathrm{Q}(\mathrm{H}_{-1}(A))[-1]\big), \tag{7}$$

the fiber product being taken in the abelian category $`\mathcal{H}_{\mathcal{T}}$. We thus have a cartesian square in $`\mathcal{H}_{\mathcal{T}}$:

$$\begin{array}{ccc} `\mathrm{Q}(A) & \longrightarrow & \mathrm{G} \circ \mathrm{Q}(\mathrm{H}_{-1}(A))[-1] \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{H}_{-1}(A)[-1]. \end{array}$$

From the construction, $\mathrm{H}_0(`\mathrm{Q}(A)) \simeq \mathrm{H}_0(A)$ and $\mathrm{H}_{-1}(`\mathrm{Q}(A)) \simeq \mathrm{G} \circ \mathrm{Q}(\mathrm{H}_{-1}(A))$. In particular, $`\mathrm{Q}(A) \in `\mathcal{B}$ as it follows from Hypothesis 2.8, (i).

We claim that $`\mathrm{Q}(A) \to A$ is the universal morphism from an object of $`\mathcal{B}$. Indeed, let B be an object of $`\mathcal{B}$. With $`A = `\mathrm{Q}(A)$, we have a commutative diagram of abelian groups

$$\begin{array}{ccccccccc}
0 \to & \hom_{\mathcal{T}}(B, \mathrm{H}_0(`A)) & \to & \hom_{\mathcal{T}}(B, `A) & \to & \hom_{\mathcal{T}}(B, \mathrm{H}_{-1}(`A)[-1]) & \to & \hom_{\mathcal{T}}(B, \mathrm{H}_0(`A)[1]) \\
& \downarrow\sim & & \downarrow & & \downarrow & & \downarrow\sim \\
0 \to & \hom_{\mathcal{T}}(B, \mathrm{H}_0(A)) & \longrightarrow & \hom_{\mathcal{T}}(B, A) & \longrightarrow & \hom_{\mathcal{T}}(B, \mathrm{H}_{-1}(A)[-1]) & \longrightarrow & \hom_{\mathcal{T}}(B, \mathrm{H}_0(A)[1])
\end{array}$$

with exact rows. By the Five Lemma, we are reduced to show that third vertical homomorphism is bijective. The latter can be identified with

$$\hom_{\mathcal{H}_{\mathcal{T}}}(\mathrm{H}_{-1}(B), \mathrm{G} \circ \mathrm{Q}(\mathrm{H}_{-1}(A))) \longrightarrow \hom_{\mathcal{H}_{\mathcal{T}}}(\mathrm{H}_{-1}(B), \mathrm{H}_{-1}(A)).$$

This is a bijection as $\mathrm{H}_{-1}(B)$ is F-connected and in $\mathcal{B}$. □

Proposition 2.11 — *Keep the above notation and assumptions. The category $`\mathcal{B}$ is abelian.*

Proof. We split the proof in two parts.

Part A: Let $a : A \to B$ be a morphism in $`\mathcal{B}$. Here we prove that $coker(a)$, taken in $`\mathcal{H}_{\mathcal{T}}$ is an object of $`\mathcal{B}$. Denote $N = ker(a)$, $C = im(a)$ and $D = coker(a)$, all taken in $`\mathcal{H}_{\mathcal{T}}$. From the two short exact sequences:

$$0 \longrightarrow N \longrightarrow A \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow C \longrightarrow B \longrightarrow D \longrightarrow 0$$

we deduce two exact sequences in $\mathcal{H}_{\mathcal{T}}$:

$$\mathrm{H}_{-1}(N) \longrightarrow \mathrm{H}_{-1}(A) \longrightarrow \mathrm{H}_{-1}(C) \longrightarrow 0$$

$$\mathrm{H}_{-1}(C) \longrightarrow \mathrm{H}_{-1}(B) \longrightarrow \mathrm{H}_{-1}(D) \longrightarrow 0$$

which can be put together to get another exact sequence:

$$\mathrm{H}_{-1}(A) \longrightarrow \mathrm{H}_{-1}(B) \longrightarrow \mathrm{H}_{-1}(D) \longrightarrow 0.$$

Now, as $\mathrm{H}_{-1}(A)$ and $\mathrm{H}_{-1}(B)$ are in $\mathcal{B}$, and the inclusion $\mathcal{B} \hookrightarrow \mathcal{H}_{\mathcal{T}}$ is right exact, we deduce that $\mathrm{H}_{-1}(D) \in \mathcal{B}$. This shows that $coker(a) \in `\mathcal{B}$.

Part B: The previous part shows that cokernels exist in $`\mathcal{B}$ and can be computed in $\mathcal{H}_{\mathcal{T}}$. On the other hand, by Lemma 2.10, the kernels also exist in $`\mathcal{B}$. Indeed, if $a : A \to B$ is a morphism in $`\mathcal{B}$, then its kernel in $`\mathcal{B}$ is $ker_{`\mathcal{B}}(a) = `\mathrm{Q}(ker(a))$. It remains to show that images and coimages coincide in $`\mathcal{B}$.

Fix a morphism $a : A \to B$ in $`\mathcal{B}$. $`\mathcal{H}_{\mathcal{T}}$ being an abelian category, the canonical morphism

$$coker\{ker(a) \to A\} \longrightarrow ker\{B \to coker(a)\}$$

is invertible. Applying $`\mathrm{Q}$, we get an isomorphism

$$`\mathrm{Q}\big(coker\{ker(a) \to A\}\big) \xrightarrow{\sim} `\mathrm{Q}\big(ker\{B \to coker(a)\}\big).$$

$`\mathrm{Q}$ being a left adjoint, it commutes with cokernels. It follows that:

$$`\mathrm{Q}\big(coker\{ker(a) \to A\}\big) \simeq coker\{`\mathrm{Q}(ker(a)) \to `\mathrm{Q}(A)\} \simeq coker\{ker_{`\mathcal{B}}(a) \to A\}.$$

Thus the obvious morphism

$$coker\{ker_{`\mathcal{B}}(a) \to A\} \longrightarrow ker_{`\mathcal{B}}\{B \to coker(a)\}$$

is invertible. This finishes the proof of the proposition. □

3. The n-motivic t-structure for $n = 0$ and 1

3.1. The 0-motivic t-structure.

We bring in the notation from §1.1. From the introduction, we are led to make the following definition.

Definition 3.1 — *The* 0-motivic t-structure $\left({}^0\mathcal{T}^{\mathcal{M}}_{\geq 0}(k), {}^0\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)\right)$ *on* $\mathbf{DM}_{\text{eff}}(k)$ *is the usual homotopy t-structure. An object in* ${}^0\mathcal{T}^{\mathcal{M}}_{\geq 0}(k)$ *will be called* ${}^0t^{\mathcal{M}}$-positive. *An object in* ${}^0\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)$ *will be called* ${}^0t^{\mathcal{M}}$-negative. *For $n \in \mathbb{Z}$, we denote* ${}^0\tau^{\mathcal{M}}_{\geq n}$ *and* ${}^0\tau^{\mathcal{M}}_{\leq n}$ *the truncation functors, and* ${}^0\mathrm{H}^{\mathcal{M}}_n(-) = {}^0\tau^{\mathcal{M}}_{\geq n} \circ {}^0\tau^{\mathcal{M}}_{\leq n}(-)[-n]$. *We also denote* ${}^0\mathcal{H}^{\mathcal{M}}(k) = {}^0\mathcal{T}^{\mathcal{M}}_{\geq 0}(k) \bigcap {}^0\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)$, *the heart of 0-motivic t-structure. An object of* ${}^0\mathcal{H}^{\mathcal{M}}(k)$ *is a called a* $(0, \mathcal{H})$-sheaf.

Strictly speaking, the category ${}^0\mathcal{H}^{\mathcal{M}}(k)$ is equivalent (and not isomorphic) to the category $\mathbf{HI}(k)$ of $\mathcal{H}$-sheaves (i.e., homotopy invariant sheaves with transfers). This equivalence takes a $(0, \mathcal{H})$-sheaf to its zero homology $\mathcal{H}$-sheaf. However, it is safe enough to identify both categories, and we will often do this.

Remark 3.2 — In the sequel, we will keep using the notation $\tau_{\leq n}$, $\tau_{\geq n}$, H_n and $\mathbf{HI}(k)$ relative to the homotopy t-structure. In fact, the only reason we introduced the new terminology in Definition 3.1, is to stress the analogy between the 0-motivic, 1-motivic and 2-motivic t-structures.

Proposition 3.3 — *The 0-motivic t-structure on* $\mathbf{DM}_{\text{eff}}(k)$ *is generated by the essentially small class* $\{\mathrm{M}(X) \,|\, X \in Sm/k\}$.

Proof. This is well-known (see [8]). For the sake of completeness, we provide an argument. For $K \in \mathbf{K}(Shv_{tr}(k))$, the following two conditions are equivalent:

(i) The homology sheaves $\mathrm{H}_i(K)$ are zero for $i < 0$;
(ii) The (Nisnevich) hyper-cohomology groups $\mathrm{H}^{-i}(X, K)$ are zero for $i > 0$ and $X \in Sm/k$.

If moreover we assume that K is $\mathbb{A}^1$-local, the second condition can be rewritten as follows:

(ii′) The groups $\hom_{\mathbf{DM}_{\text{eff}}(k)}(\mathrm{M}(X)[i], K)$ are zero for $i > 0$ and $X \in Sm/k$.

In other words, K is $\{\mathrm{M}(X) \,|\, X \in Sm/k\}$-negative. This proves the proposition. □

Definition 3.4 — *We denote by* $\mathbf{M}_0(k) \subset {}^0\mathcal{H}^{\mathcal{M}}(k)$ *the full subcategory whose objects are the 0-presented $\mathcal{H}$-sheaves which we will also call* mixed 0-motives *(or simply* 0-motive*).*

In [2], the category $\mathbf{M}_0(k)$ is denoted by $\mathbf{HI}_{\leq 0}(k)$ and their objects were called 0-motivic sheaves. It is the heart of the restriction of the homotopy t-structure on $\mathbf{DM}_{\leq 0}(k)$. By [2, Lem. 1.2.2], $\mathbf{M}_0(k)$ is canonically equivalent to the category $Shv_{tr}(k_{\leq 0})$. The latter is equivalent to the category of $\mathbb{Q}$-linear representations V of the absolute Galois group $\mathrm{Gal}(k^s/k)$ of k such that the stabilizer of each element of V is open (i.e., of finite index). This justifies our terminology.

3.2. The 1-motivic t-structure.

The subcategory $\mathbf{M}_0(k) \subset {}^0\mathcal{H}^{\mathcal{M}}(k)$ satisfies Hypothesis 2.1. Indeed, (i) and (ii) are contained in [2, Cor. 1.2.5 and Prop. 1.2.7]. To check (iii), we use [2, Cor. 2.3.3].

It asserts that the left adjoint $\pi_0 : {}^0\mathcal{H}^{\mathcal{M}} \to \mathbf{M}_1(k)$ is induced on the hearts by a t-positive triangulated functor $\mathrm{L}\pi_0 : \mathbf{DM}_{\text{eff}}(k) \to \mathbf{DM}_{\leq 0}(k)$. Given an exact sequence of $\mathcal{H}$-sheaves

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we deduce a distinguished triangle

$$\mathrm{L}\pi_0(M') \longrightarrow \mathrm{L}\pi_0(M) \longrightarrow \mathrm{L}\pi_0(M'') \longrightarrow \mathrm{L}\pi_0(M')[1],$$

and thus an exact sequence of 0-motives

$$\mathrm{H}_1(\mathrm{L}\pi_0(M'')) \longrightarrow \pi_0(M') \longrightarrow \pi_0(M).$$

Now, assume that M'' is a 0-motive. Then $M'' \simeq \mathrm{L}\pi_0(M'')$ and $\mathrm{H}_1(\mathrm{L}\pi_0(M'')) = 0$. This proves that $\pi_0(M') \to \pi_0(M)$ is injective.

We are now in position to apply the construction from §2.1.

Definition 3.5 — *The* 1-motivic t-structure $\left({}^1\mathcal{T}^{\mathcal{M}}_{\geq 0}(k), {}^1\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)\right)$ *on* $\mathbf{DM}_{\text{eff}}(k)$ *is the* $\mathbf{M}_0(k)$*-perverted t-structure associated to the* 0*-motivic t-structure. An object in* ${}^1\mathcal{T}^{\mathcal{M}}_{\geq 0}(k)$ *will be called* ${}^1t^{\mathcal{M}}$-positive. *An object in* ${}^1\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)$ *will be called* ${}^1t^{\mathcal{M}}$-negative. *For* $n \in \mathbb{Z}$*, we denote* ${}^1\tau^{\mathcal{M}}_{\geq n}$ *and* ${}^1\tau^{\mathcal{M}}_{\leq n}$ *the truncation functors and* ${}^1\mathrm{H}^{\mathcal{M}}_n(-) = {}^1\tau^{\mathcal{M}}_{\geq n} \circ {}^1\tau^{\mathcal{M}}_{\leq n}(-)[-n]$*. We also denote* ${}^1\mathcal{H}^{\mathcal{M}}(k) = {}^1\mathcal{T}^{\mathcal{M}}_{\geq 0}(k) \bigcap {}^1\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)$ *the heart of* 1*-motivic t-structure. An object of* ${}^1\mathcal{H}^{\mathcal{M}}(k)$ *is a called a* $(1, \mathcal{H})$-sheaf.

Remark 3.6 — From the construction, we have the following description of the 1-motivic t-structure.

1. An object $P \in \mathbf{DM}_{\text{eff}}(k)$ is ${}^1t^{\mathcal{M}}$-positive if and only if it satisfies:
   - (a) $\mathrm{H}_n(P) = 0$ for $n < -1$;
   - (b) $\mathrm{H}_{-1}(P)$ is a 0-connected $\mathcal{H}$-sheaf.
2. An objet $N \in \mathbf{DM}_{\text{eff}}(k)$ is ${}^1t^{\mathcal{M}}$-negative if and only if it satisfies:
   - (a) $\mathrm{H}_n(N) = 0$ for $n > 0$;
   - (b) $\mathrm{H}_0(N)$ is a 0-motive.
3. An object $M \in \mathbf{DM}_{\text{eff}}(k)$ is a $(1, \mathcal{H})$-sheaf if and only if it satisfies:
   - (a) $\mathrm{H}_i(M) = 0$ for $i \notin \{0, -1\}$;
   - (b) $\mathrm{H}_0(M)$ is a 0-motive;
   - (c) $\mathrm{H}_{-1}(M)$ is a 0-connected $\mathcal{H}$-sheaf.

For $X \in Sm/k$ we choose a distinguished triangle

$$\mathrm{M}_{\geq 1}(X) \longrightarrow \mathrm{M}(X) \longrightarrow \mathrm{L}\pi_0(\mathrm{M}(X)) \longrightarrow \mathrm{M}_{\geq 1}(X)[1].$$

From the construction in [2, §2.3], we have $\mathrm{L}\pi_0(\mathrm{M}(X)) \simeq \mathrm{M}(\pi_0(X))$ where $\pi_0(X)$ is the étale k-scheme of connected components of X. It follows that $\mathrm{M}_{\geq 1}(X)$ is isomorphic in $\mathbf{DM}_{\text{eff}}(k)$ to $ker\{\mathbb{Q}_{tr}(X) \to \mathbb{Q}_{tr}(\pi_0(X))\}[0]$. The following result is a direct consequence of Proposition 2.7.

Proposition 3.7 — *The* 1*-motivic t-structure on* $\mathbf{DM}_{\text{eff}}(k)$ *is generated by the essentially small class* $\{\mathrm{M}(X),\ \mathrm{M}_{\geq 1}(X)[-1] \,|\, X \in Sm/k\}$.

3.3. Mixed 1-motives.

Definition 3.8 — *An object* $M \in \mathbf{DM}_{\text{eff}}(k)$ *is called a* mixed 1-motive *if it satisfies the following conditions:*

1. $\mathrm{H}_i(M) = 0$ *for* $i \notin \{0, -1\}$*;*
2. $\mathrm{H}_0(M)$ *is a* 0*-motivic sheaf;*

(3) $\mathrm{H}_{-1}(M)$ *is a* 0*-connected* 1*-presented* $\mathcal{H}$*-sheaf.*

The full subcategory of mixed 1*-motives will be denoted by* $\mathbf{M}_1(k)$*.*

Clearly, $\mathbf{M}_1(k)$ is the $\mathbf{M}_0(k)$-perverted subcategory associated to $\mathbf{HI}_{\leq 1}(k) \subset \mathbf{HI}(k) \simeq {}^0\mathcal{H}^{\mathcal{M}}(k)$. In particular, it is abelian. In fact, we have more as the following result shows.

Proposition 3.9 — $\mathbf{M}_1(k)$ *is a thick abelian subcategory of* ${}^1\mathcal{H}^{\mathcal{M}}(k)$*, i.e., stable under extensions, subobjects and quotients.*

Proof. Indeed, consider a short exact sequence of $(1, \mathcal{H})$-sheaves:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0. \tag{8}$$

It suffices to show that M is a mixed 1-motive if and only if M' and M'' are mixed 1-motives. In other words, we need to show that the $\mathcal{H}$-sheaf $\mathrm{H}_{-1}(M)$ is 1-presented if and only if the $\mathcal{H}$-sheaves $\mathrm{H}_{-1}(M')$ and $\mathrm{H}_{-1}(M'')$ are 1-presented.

From (8), we get an exact sequence of $\mathcal{H}$-sheaves:

$$0 \longrightarrow im\{\mathrm{H}_0(M) \to \mathrm{H}_0(M'')\} \longrightarrow \mathrm{H}_{-1}(M') \longrightarrow \mathrm{H}_{-1}(M) \longrightarrow \mathrm{H}_{-1}(M'') \longrightarrow 0.$$

The $\mathcal{H}$-sheaf $im\{\mathrm{H}_0(M) \to \mathrm{H}_0(M'')\}$ is 0-presented and hence 1-presented. The lemma follows now as $\mathbf{HI}_{\leq 1}(k)$ is a thick abelian subcategory of $\mathbf{HI}(k)$ (see [2, Cor. 1.3.5]). □

Lemma 3.10 — *Let M be a mixed* 1*-motive. Then, M decomposes into a direct sum*

$$M \simeq \mathrm{H}_0(M)[0] \bigoplus \mathrm{H}_{-1}(M)[-1].$$

Proof. We have a distinguished triangle in $\mathbf{DM}_{\mathrm{eff}}(k)$:

$$\mathrm{H}_0(M) \longrightarrow M \longrightarrow \mathrm{H}_{-1}(M)[-1] \xrightarrow{\epsilon} \mathrm{H}_0(M)[1]. \tag{9}$$

We need to show that ϵ is zero. By [2, Th. 2.4.1, (i)], $\mathbf{HI}_{\leq 1}(k)$ is contained in $\mathbf{DM}_{\leq 1}(k)$, and hence $M \in \mathbf{DM}_{\leq 1}(k)$. Also, by [2, Th. 2.4.1, (i)], we have an equivalence of categories $\mathbf{D}(\mathbf{HI}_{\leq 1}(k)) \simeq \mathbf{DM}_{\leq 1}(k)$. Thus, we may view (9) as a triangle in the derived category $\mathbf{D}(\mathbf{HI}_{\leq 1}(k))$ and ϵ as a element of:

$$\hom_{\mathbf{D}(\mathbf{HI}_{\leq 1}(k))}\big(\mathrm{H}_{-1}(M)[-1], \mathrm{H}_0(M)[1]\big) \simeq \mathrm{ext}^2_{\mathbf{HI}_{\leq 1}(k)}\big(\mathrm{H}_{-1}(M), \mathrm{H}_0(M)\big).$$

On the other hand, the cohomological dimension of $\mathbf{HI}_{\leq 1}(k)$ is 1 by [2, Prop. 2.4.10]. This shows that $\mathrm{ext}^2_{\mathbf{HI}_{\leq 1}(k)}\big(\mathrm{H}_{-1}(M), \mathrm{H}_0(M)\big) = 0$, and hence $\epsilon = 0$. □

In the reminder of this paragraph, we describe the link between our notion of mixed 1-motives and the classical notion of Deligne's 1-motives. We do this in order to justify our terminology. However, this material will not be used elsewhere in the paper and can safely be skipped by the reader.

Recall (*cf.* [4]) that a Deligne 1-motive is a morphism of group-schemes $[L \xrightarrow{u} G]$ with L a lattice (i.e., locally for the étale topology isomorphic to $\mathbb{Z}^r$) and G a semi-abelian variety. We denote by $\mathcal{M}_1(k)$ the category of 1-motives. Given two 1-motives $M_1 = [L_1 \xrightarrow{u_1} G_1]$ and $M_2 = [L_2 \xrightarrow{u_2} G_2]$, we have

$$\hom_{\mathcal{M}_1}(M_1, M_2) = \{(a : L_1 \to L_2,\, b : G_1 \to G_2) \,|\, b \circ u_1 = u_2 \circ a\} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

(Where a and b above are morphisms of group-schemes.) There is a functor

$$\mathrm{T} : \mathcal{M}_1(k) \to \mathbf{DM}_{\mathrm{eff}}(k)$$

which takes a 1-motive $[L \to G]$ to the complex

$$[\cdots \to 0 \to L \otimes \mathbb{Q} \to G \otimes \mathbb{Q} \to 0 \to \cdots]$$

where L and G are identified with the sheaves they represents and $L \otimes \mathbb{Q}$ is placed in degree 0.

Proposition 3.11 — *The functor* T *induces an exact full embedding of* $\mathcal{M}_1(k)$ *into* $\mathbf{M}_1(k)$.

Proof. This is a special case of the main result in [9]. For the sake of completeness, we give a proof. Clearly, the image of T is contained in $\mathbf{M}_1(k)$. Let $M_i = [L_i \to G_i]$ (for $i \in \{1, 2\}$) be two Deligne 1-motives. We need to show that

$$\hom_{\mathcal{M}_1}(M_1, M_2) \longrightarrow \hom_{\mathbf{M}_1(k)}(\mathrm{T}(M_1), \mathrm{T}(M_2)) \tag{10}$$

is a bijection. We can decompose M_i as follows

$$M_i \simeq L'_i \bigoplus [L''_i \to G_i]$$

where L'_i and L''_i are sub-lattices of L_i such that:

- $L'_i \cap L''_i = 0$ and $L'_i + L''_i$ is of finite index in L_i;
- $L''_i \to G_i$ is injective.

We are then reduced to check that (10) is bijective in the following cases:

(a) G_i is zero for $i \in \{1, 2\}$;
(b) G_1 is zero and $L_2 \hookrightarrow G_2$ is injective;
(c) $L_1 \hookrightarrow G_1$ is injective and $G_2 = 0$;
(d) $L_i \hookrightarrow G_i$ is injective for $i \in \{1, 2\}$.

Case (a) is easy. In case (b), both sides of (10) are zero. In case (c), both sides of (10) are canonically isomorphic to $\hom(L_1, L_2) \otimes \mathbb{Q}$. Finally, in case (d), both sides of (10) are given by the sub-vectorspace of $e \in \hom(G_1, G_2) \otimes \mathbb{Q}$ such that $e(L_1 \otimes \mathbb{Q}) \subset e(L_2 \otimes \mathbb{Q})$. □

Remark 3.12 — Using [2, Th. 1.3.10] and Lemma 3.10, it is possible to show that $\mathbf{M}_1(k)$ is equivalent to the category of ind-objects in $\mathcal{M}_1(k)$. We leave the details to the reader.

3.4. n-Presented $(1, \mathcal{H})$-sheaves.

Definition 3.13 — *Let $n \geq 0$ be an integer. A $(1, \mathcal{H})$-sheaf M is* n-presented *if the $\mathcal{H}$-sheaf $\mathrm{H}_{-1}(M)$ is n-presented. We denote by ${}^1\mathcal{H}^{\mathcal{M}}_{\leq n}(k) \subset {}^1\mathcal{H}^{\mathcal{M}}(k)$ the full subcategory of n-presented $(1, \mathcal{H})$-sheaves.*

Remark 3.14 — A $(1, \mathcal{H})$-sheaf M is 0-presented if and only if $\mathrm{H}_{-1}(M) = 0$. It follows that ${}^1\mathcal{H}^{\mathcal{M}}_{\leq 0}(k) = \mathbf{M}_0(k)$. Also, 1-presented $(1, \mathcal{H})$-sheaves are exactly the mixed 1-motives, i.e, ${}^1\mathcal{H}^{\mathcal{M}}_{\leq 1}(k) = \mathbf{M}_1(k)$.

Clearly, ${}^1\mathcal{H}^{\mathcal{M}}_{\leq n}(k)$ is the $\mathbf{M}_0(k)$-perverted subcategory of ${}^1\mathcal{H}^{\mathcal{M}}(k)$ associated to the full subcategory ${}^0\mathcal{H}^{\mathcal{M}}_{\leq n}(k) = \mathbf{HI}_{\leq n}(k)$ of ${}^0\mathcal{H}^{\mathcal{M}}(k)$. It is easy to check Hypothesis 2.8 for the 0-motivic t-structure with $\mathcal{A} = \mathbf{M}_0(k)$ and $\mathcal{B} = \mathbf{HI}_{\leq n}(k)$. Indeed, the inclusion $\mathbf{HI}_{\leq n}(k) \hookrightarrow \mathbf{HI}(k)$ admits a right adjoint given by $\mathrm{Q}_n = h_0\sigma_n^*\sigma_{n*}$. Moreover, a short exact sequence of $\mathcal{H}$-sheaves

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

such that F'' is 0-presented has a splitting. This clearly implies that $\mathrm{Q}_n(F') \to \mathrm{Q}_n(F)$ is injective. From Lemma 2.10 and Proposition 2.11, we deduce the following result.

Corollary 3.15 — *${}^1\mathcal{H}^{\mathcal{M}}_{\leq n}(k)$ is an abelian category and there is a functor*

$$ {}^1\mathrm{Q}_n : {}^1\mathcal{H}^{\mathcal{M}}(k) \longrightarrow {}^1\mathcal{H}^{\mathcal{M}}_{\leq n}(k),$$

which is a right adjoint to the obvious inclusion.

4. The 2-motivic t-structure

4.1. The construction.

By Proposition 3.9, $\mathbf{M}_1(k) \subset {}^1\mathcal{H}^{\mathcal{M}}(k)$ is a thick abelian subcategory. Thus the first condition in Hypothesis 2.1 is satisfied. We will see in a moment that the two other conditions are satisfied as well. First, we note the following result which is of independent interest.

Proposition 4.1 — *The* 1*-motivic t-structure restricts to a t-structure on $\mathbf{DM}_{\leq 1}(k)$ whose heart is $\mathbf{M}_1(k)$.*

Proof. We know that the homotopy t-structure restricts to a t-structure on $\mathbf{DM}_{\leq 1}(k)$ whose heart is $\mathbf{HI}_{\leq 1}(k)$. The subcategory $\mathbf{M}_0(k) \subset \mathbf{HI}_{\leq 1}(k)$ satisfies the conditions in Hypothesis 2.1. Thus, we may consider the $\mathbf{M}_0(k)$-perverted t-structure on $\mathbf{DM}_{\leq 1}(k)$ associated to the homotopy t-structure. By a straightforward inspection, we see that the inclusion $\mathbf{DM}_{\leq 1}(k) \hookrightarrow \mathbf{DM}_{\mathrm{eff}}(k)$ is exact with respect to the $\mathbf{M}_0(k)$-perverted t-structures. This proves the proposition. □

Definition 4.2 — *The restriction of the* 1*-motivic t-structure to $\mathbf{DM}_{\leq 1}(k)$ is also called the* 1-motivic *t-structure.*

Lemma 4.3 — *The functor* $\mathrm{LAlb} : \mathbf{DM}_{\mathrm{eff}}(k) \to \mathbf{DM}_{\leq 1}(k)$ *is ${}^1t^{\mathcal{M}}$-positive (i.e., t-positive with respect to the* 1*-motivic t-structures).*

Proof. This is clear as LAlb is the left adjoint to the inclusion $\mathbf{DM}_{\leq 1}(k) \hookrightarrow \mathbf{DM}_{\mathrm{eff}}(k)$ which is ${}^1t^{\mathcal{M}}$-exact. □

We are now in position to check Hypothesis 2.1, (ii) and (iii) for $\mathbf{M}_1(k) \subset {}^1\mathcal{H}^{\mathcal{M}}(k)$.

Lemma 4.4 — *The inclusion $\mathbf{M}_1(k) \hookrightarrow {}^1\mathcal{H}^{\mathcal{M}}(k)$ has a left adjoint*

$$\mathrm{Alb}^{\mathcal{M}} : {}^1\mathcal{H}^{\mathcal{M}}(k) \longrightarrow \mathbf{M}_1(k).$$

Moreover, given an exact sequence of $(1, \mathcal{H})$-sheaves

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

with M'' a mixed 1*-motive, the morphism $\mathrm{Alb}^{\mathcal{M}}(M') \to \mathrm{Alb}^{\mathcal{M}}(M)$ is injective.*

Proof. Given a $(1,\mathcal{H})$-sheaf M, we set

$$\mathrm{Alb}^{\mathcal{M}}(M) = {}^1\mathrm{H}_0^{\mathcal{M}}(\mathrm{LAlb}(M)).$$

That this is a left adjoint to the obvious inclusion, follows immediately from Lemma 4.3. To prove the second part, we use the distinguished triangle

$$\mathrm{LAlb}(M') \longrightarrow \mathrm{LAlb}(M) \longrightarrow \mathrm{LAlb}(M'') \longrightarrow \mathrm{LAlb}(M')[1].$$

We deduce an exact sequence of mixed 1-motives

$${}^1\mathrm{H}_1^{\mathcal{M}}(\mathrm{LAlb}(M'')) \longrightarrow {}^1\mathrm{H}_0^{\mathcal{M}}(\mathrm{LAlb}(M')) \longrightarrow {}^1\mathrm{H}_0^{\mathcal{M}}(\mathrm{LAlb}(M)).$$

Now, if M'' is a mixed 1-motive, $M'' \simeq \mathrm{LAlb}(M'')$ and thus ${}^1\mathrm{H}_1^{\mathcal{M}}(\mathrm{LAlb}(M'')) = 0$. This finishes the proof of the lemma. □

Definition 4.5 — *The* 2-motivic *t-structure* $\left({}^2\mathcal{T}^{\mathcal{M}}_{\geq 0}(k), {}^2\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)\right)$ *on* $\mathbf{DM}_{\mathrm{eff}}(k)$ *is the* $\mathbf{M}_1(k)$*-perverted t-structure associated to the* 1*-motivic t-structure. An object in* ${}^2\mathcal{T}^{\mathcal{M}}_{\geq 0}(k)$ *will be called* ${}^2t^{\mathcal{M}}$-positive. *An object in* ${}^2\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)$ *will be called* ${}^2t^{\mathcal{M}}$-negative. *For* $n \in \mathbb{Z}$*, we denote* ${}^2\tau^{\mathcal{M}}_{\geq n}$ *and* ${}^2\tau^{\mathcal{M}}_{\leq n}$ *the truncation functors and* ${}^2\mathrm{H}^{\mathcal{M}}_n(-) = {}^2\tau^{\mathcal{M}}_{\geq n} \circ {}^2\tau^{\mathcal{M}}_{\leq n}(-)[-n]$*. We also denote* ${}^2\mathcal{H}^{\mathcal{M}}(k) = {}^2\mathcal{T}^{\mathcal{M}}_{\geq 0}(k) \bigcap {}^2\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)$ *the heart of* 2*-motivic t-structure. An object of* ${}^2\mathcal{H}^{\mathcal{M}}(k)$ *is a called a* $(2,\mathcal{H})$-sheaf.

Remark 4.6 — We will say that a $(1,\mathcal{H})$-sheaf M is 1-*connected* if $\mathrm{Alb}^{\mathcal{M}}(M) = 0$. From the construction, we have the following.

1. An object $P \in \mathbf{DM}_{\mathrm{eff}}(k)$ is ${}^2t^{\mathcal{M}}$-positive if and only if it satisfies:
   - (a) ${}^1\mathrm{H}^{\mathcal{M}}_n(P) = 0$ for $n < -1$;
   - (b) ${}^1\mathrm{H}^{\mathcal{M}}_{-1}(P)$ is a 1-connected $(1,\mathcal{H})$-sheaf.
2. An objet $N \in \mathbf{DM}_{\mathrm{eff}}(k)$ is ${}^2t^{\mathcal{M}}$-negative if and only if it satisfies:
   - (a) ${}^1\mathrm{H}^{\mathcal{M}}_n(N) = 0$ for $n > 0$;
   - (b) ${}^1\mathrm{H}^{\mathcal{M}}_0(N)$ is a mixed 1-motive.
3. An object $M \in \mathbf{DM}_{\mathrm{eff}}(k)$ is a $(2,\mathcal{H})$-sheaf if and only if it satisfies:
   - (a) ${}^1\mathrm{H}^{\mathcal{M}}_i(M) = 0$ for $i \notin \{0,-1\}$;
   - (b) ${}^1\mathrm{H}^{\mathcal{M}}_0(M)$ is a 0-motive;
   - (c) ${}^1\mathrm{H}^{\mathcal{M}}_{-1}(M)$ is a 0-connected $\mathcal{H}$-sheaf.

In the next paragraph, we will give equivalent formulations of the above conditions in terms of the homotopy t-structure.

For $X \in Sm/k$ we choose a distinguished triangle

$$\mathrm{M}_{\geq 2}(X) \longrightarrow \mathrm{M}(X) \longrightarrow \mathrm{LAlb}(\mathrm{M}(X)) \longrightarrow \mathrm{M}_{\geq 2}(X)[1].$$

We have the following result.

Proposition 4.7 — *The* 2*-motivic t-structure on* $\mathbf{DM}_{\mathrm{eff}}(k)$ *is generated by the essentially small class* $\{\mathrm{M}(X),\ \mathrm{M}_{\geq 1}(X)[-1],\ \mathrm{M}_{\geq 2}(X)[-2] \mid X \in Sm/k\}$.

Proof. For $X \in Sm/k$, we choose a distinguished triangle

$$\mathrm{M}'_{\geq 2}(X) \longrightarrow \mathrm{M}_{\geq 1}(X) \longrightarrow \mathrm{LAlb}(\mathrm{M}_{\geq 1}(X)) \longrightarrow \mathrm{M}'_{\geq 2}(X)[1].$$

A direct application of Propositions 2.7 and 3.7 yields the following generating class for the 2-motivic t-structure:

$$\{\mathrm{M}(X),\ \mathrm{M}_{\geq 1}(X)[-1],\ \mathrm{M}_{\geq 2}(X)[-1],\ \mathrm{M}'_{\geq 2}(X)[-2] \mid X \in Sm/k\}.$$

The proposition would follow if we prove that $\mathrm{M}'_{\geq 2}(X) \simeq \mathrm{M}_{\geq 2}(X)$. There is a morphism of distinguished triangles

$$\begin{array}{ccccccc} \mathrm{M}'_{\geq 2}(X) & \longrightarrow & \mathrm{M}_{\geq 1}(X) & \longrightarrow & \mathrm{LAlb}(\mathrm{M}_{\geq 1}(X)) & \longrightarrow & \mathrm{M}'_{\geq 2}(X)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{M}_{\geq 2}(X) & \longrightarrow & \mathrm{M}(X) & \longrightarrow & \mathrm{LAlb}(\mathrm{M}(X)) & \longrightarrow & \mathrm{M}_{\geq 2}(X)[1]. \end{array}$$

By the Verdier's octahedral axiom (and the fact that LAlb is a triangulated functor), it suffices to show that

$$Cone\{\mathrm{M}_{\geq 1}(X) \to \mathrm{M}(X)\} \simeq \mathrm{LAlb}\big(Cone\{\mathrm{M}_{\geq 1}(X) \to \mathrm{M}(X)\}\big).$$

This is indeed the case as $Cone\{\mathrm{M}_{\geq 1}(X) \to \mathrm{M}(X)\} \simeq \mathrm{L}\pi_0(\mathrm{M}(X)) \in \mathbf{DM}_{\leq 0}(k) \subset \mathbf{DM}_{\leq 1}(k)$. The proposition is proven. □

4.2. **A more explicit description.**

The description of the 2-motivic t-structure in term of the 1-motivic t-structure given in Remark 4.6 is rather abstract and not intuitive. Here we give a more down-to-earth description which only uses the homotopy t-structure.

Proposition 4.8 —

(1) *An object $P \in \mathbf{DM}_{\text{eff}}(k)$ is ${}^2t^{\mathcal{M}}$-positive if and only if it satisfies:*
  - (a′) $\mathrm{H}_i(P) = 0$ *for $i < -2$;*
  - (b′) $\mathrm{H}_{-2}(P)$ *is a 1-connected $\mathcal{H}$-sheaf;*
  - (c′) *For every 0-motive L,* $\hom_{\mathbf{DM}_{\text{eff}}(k)}(P, L[-1]) = 0$.

(2) *An object N in $\mathbf{DM}_{\text{eff}}(k)$ is ${}^2t^{\mathcal{M}}$-negative if and only if it satisfies:*
  - (a′) $\mathrm{H}_i(N) = 0$ *for $i > 0$;*
  - (b′) $\mathrm{H}_0(N)$ *is a 0-presented $\mathcal{H}$-sheaf;*
  - (c′) $\mathrm{H}_{-1}(N)$ *is a 1-presented $\mathcal{H}$-sheaf.*

Proof. We will compare the conditions of the statement with those in Remark 4.6. We split the proof into four parts.

Part A: Let $P \in \mathbf{DM}_{\text{eff}}(k)$ be a ${}^2t^{\mathcal{M}}$-positive object. We will show that P satisfies conditions (1a′), (1b′) and (1c′).

We have a chain of inclusions ${}^2\mathcal{T}^{\mathcal{M}}_{\geq 0}(k) \subset {}^1\mathcal{T}^{\mathcal{M}}_{\geq -1}(k) \subset {}^0\mathcal{T}^{\mathcal{M}}_{\geq -2}(k)$. It follows that $\mathrm{H}_i(P) = 0$ for $i < -2$. This is condition (1a′) of the statement.

By Remark 4.6, (1b), the $(1, \mathcal{H})$-sheaf ${}^1\mathrm{H}^{\mathcal{M}}_{-1}(P)$ is 1-connected. As

$${}^1\mathrm{H}^{\mathcal{M}}_{-1}(\mathrm{LAlb}(P)) \simeq \mathrm{Alb}^{\mathcal{M}}({}^1\mathrm{H}^{\mathcal{M}}_{-1}(P)) = 0,$$

we deduce that $\mathrm{LAlb}(P)$ is ${}^1t^{\mathcal{M}}$-positive. This implies that $\mathrm{LAlb}(P)[1]$ is t-positif. It follows that

$$\mathrm{Alb}(\mathrm{H}_{-2}(P)) \simeq \mathrm{H}_{-2}(\mathrm{LAlb}(P)) = 0.$$

Hence $\mathrm{H}_{-2}(P)$ is a 1-connected $\mathcal{H}$-sheaf. This is condition (1b′) of the statement.

Condition (1c′) of the statement is clear as P is ${}^2t^{\mathcal{M}}$-positive and $L[-1]$ is strictly ${}^2t^{\mathcal{M}}$-negative.

Part B: Let $P \in \mathbf{DM}_{\text{eff}}(k)$ satisfying conditions (1a′), (1b′) and (1c′). We will show that P is ${}^2t^{\mathcal{M}}$-positive.

The $\mathcal{H}$-sheaf $\mathrm{H}_{-2}(P)$ being 1-connected, is also 0-connected. It follows that $P \in {}^1\mathcal{T}^{\mathcal{M}}_{\geq -1}(k)$, i.e., ${}^1\mathrm{H}^{\mathcal{M}}_i(P) = 0$ for $i < -1$. It remains to show that the $(1, \mathcal{H})$-sheaf ${}^1\mathrm{H}^{\mathcal{M}}_{-1}(P)$ is 1-connected.

Consider $\mathrm{LAlb}(P)$. We have $\mathrm{H}_{-2}(\mathrm{LAlb}(P)) \simeq \mathrm{Alb}(\mathrm{H}_{-2}(P)) = 0$ by condition (1b′). It follows that $\mathrm{LAlb}(P)[1]$ is t-positive. On the other hand, we claim that $\mathrm{H}_{-1}(\mathrm{LAlb}(P))$ is a 0-connected $\mathcal{H}$-sheaf. Indeed, given any 0-motive L, we have

$$\hom_{\mathbf{HI}(k)}(\pi_0(\mathrm{H}_{-1}(\mathrm{LAlb}(P))), L) \simeq \hom_{\mathbf{HI}(k)}(\mathrm{H}_{-1}(\mathrm{LAlb}(P)), L)$$
$$\simeq \hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(\mathrm{LAlb}(P), L[-1]) \simeq \hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(P, L[-1]) = 0.$$

By Yoneda's Lemma, this shows that $\pi_0(\mathrm{H}_{-1}(\mathrm{LAlb}(P))) = 0$. Thus, we have proven that $\mathrm{LAlb}(P) \in {}^1\mathcal{T}^{\mathcal{M}}_{\geq 0}(k)$. But, we have

$$\mathrm{Alb}^{\mathcal{M}}({}^1\mathrm{H}^{\mathcal{M}}_{-1}(P)) \simeq {}^1\mathrm{H}^{\mathcal{M}}_{-1}(\mathrm{LAlb}(P)) = 0.$$

This proves that P satisfies condition (1b) of Remark 4.6.

Part C: Let $N \in \mathbf{DM}_{\mathrm{eff}}(k)$ be a ${}^2t^{\mathcal{M}}$-negative object. We will show that N satisfies conditions (2a′), (2b′) and (2c′).

We have a chain of inclusions ${}^2\mathcal{T}^{\mathcal{M}}_{\leq 0}(k) \subset {}^1\mathcal{T}^{\mathcal{M}}_{\leq 0}(k) \subset {}^0\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)$. It follows that $\mathrm{H}_i(N) = 0$ for $i > 0$. This is condition (2a′) of the statement.

We have $\mathrm{H}_0(N) \simeq \mathrm{H}_0({}^1\mathrm{H}^{\mathcal{M}}_0(N))$, and the latter is 0-presented because ${}^1\mathrm{H}^{\mathcal{M}}_0(N)$ is a mixed 1-motive by Remark 4.6, (2b). This is condition (2b′) of the statement.

We clearly have $\mathrm{H}_{-1}(N) \simeq \mathrm{H}_{-1}({}^1\tau^{\mathcal{M}}_{\geq -1}(N))$. Using the distinguished triangle

$${}^1\mathrm{H}^{\mathcal{M}}_0(N) \longrightarrow {}^1\tau^{\mathcal{M}}_{\geq -1}(N) \longrightarrow {}^1\mathrm{H}^{\mathcal{M}}_{-1}(N)[-1] \longrightarrow {}^1\mathrm{H}^{\mathcal{M}}_0(N)[1],$$

we deduce an exact sequence of $\mathcal{H}$-sheaves

$$0 \longrightarrow \mathrm{H}_{-1}({}^1\mathrm{H}^{\mathcal{M}}_0(N)) \longrightarrow \mathrm{H}_{-1}(N) \longrightarrow \mathrm{H}_0({}^1\mathrm{H}^{\mathcal{M}}_{-1}(N)). \tag{11}$$

This shows that the $\mathcal{H}$-sheaf $\mathrm{H}_{-1}(N)$ is 1-presented. This is condition (2c′) of the statement.

Part D: Let $N \in \mathbf{DM}_{\mathrm{eff}}(k)$ satisfying conditions (2a′), (2b′) and (2c′). Then N is ${}^2t^{\mathcal{M}}$-negative. Indeed, $N \in {}^1\mathcal{T}^{\mathcal{M}}_{\leq 0}(k)$. On the other hand, from (11), we deduce that ${}^1\mathrm{H}^{\mathcal{M}}_0(N)$ is a mixed 1-motive. This finishes the proof of the proposition. □

The next lemma shows that we may replace condition (1c′) of Proposition 4.8 by a more concrete condition.

Lemma 4.9 — *Let $P \in \mathbf{DM}_{\mathrm{eff}}(k)$ be an object satisfying* (1a′) *and* (1b′) *of* Proposition 4.8. *Then the following conditions are equivalent:*

- (c′) *For every* 0*-motive L,* $\hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(P, L[-1]) = 0$*,*
- (c″) *For every* 0*-motive L,* $\mathrm{ext}^1_{\mathbf{HI}(k)}(\mathrm{H}_{-2}(P), L) = 0$ *and $L[-1]$ is not a direct summand of P unless $L = 0$.*

Proof. First, assume that P satisfies condition (c′). Then clearly, $L[-1]$ cannot be a direct summand of P unless $L = 0$.

On the other hand, from the distinguished triangle

$$\tau_{\geq -1}(P)[1] \longrightarrow P[1] \longrightarrow \mathrm{H}_{-2}(P)[-1] \longrightarrow \tau_{\geq -1}(P)[2], \tag{12}$$

we deduce an exact sequence

$$\hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(\tau_{\geq -1}(P)[2], L) \longrightarrow \hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(\mathrm{H}_{-2}(P)[-1], L) \longrightarrow \hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(P[1], L).$$

Both extremal terms are zero; the left one is zero for degree reasons and the right one is zero by assumption. We are done as the middle term is isomorphic

$\mathrm{ext}^1_{\mathbf{HI}(k)}(\mathrm{H}_{-2}(P), L)$. Indeed, both groups classify $\mathcal{H}$-sheaves which are extensions of $\mathrm{H}_{-2}(P)$ by L.

Conversely, assume that P satisfies condition (c$''$). We argue by contradition. Thus, let $\alpha : P[1] \to L$ be a non-zero morphism. As L is a direct sum of indecomposable 0-motives, we may assume that L is itself indecomposable.

By the long exact sequence associated to the distinguished triangle (12), the obvious homomorphism

$$\hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(P[1], L) \longrightarrow \hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(\tau_{\geq -1}(P)[1], L)$$

is injective. (Here again, we use that $\hom_{\mathbf{DM}_{\mathrm{eff}}(k)}(\tau_{\geq -1}(P)[2], L)$ is zero for degree reasons.) In other words, the composition

$$\alpha' : \ \tau_{\geq -1}(P)[1] \longrightarrow P[1] \xrightarrow{\alpha} L$$

is also non-zero. Moreover, α' uniquely factors through $\pi_0(\mathrm{H}_{-1}(P))$ yielding a commutative diagram

$$\begin{array}{ccccc}
\tau_{\geq -1}(P)[1] & \longrightarrow & & \longrightarrow & P[1] \\
\downarrow & & \searrow^{\alpha'} & & \downarrow{\scriptstyle \alpha} \\
\mathrm{H}_{-1}(P) & \longrightarrow & \pi_0(\mathrm{H}_{-1}(P)) & \xrightarrow[\alpha'']{} & L.
\end{array}$$

As α'' is non-zero and L is indecomposable, we deduce that α'' is surjective. It follows that $\mathrm{H}_{-1}(P) \to L$ is also surjective. Hence, given a distinguished triangle

$$Q \longrightarrow \tau_{\geq -1}(P)[1] \xrightarrow{\alpha'} L \longrightarrow Q[1],$$

the object Q is t-positive and thus the morphism $L \to Q[1]$ is zero. This shows that α' has a section $\beta' : L \to \tau_{\geq -1}(P)[1]$. Clearly, the composition

$$\beta : \ L \xrightarrow{\beta'} \tau_{\geq -1}(P)[1] \longrightarrow P[1]$$

is a section to α. We have proven that L is a non-trivial direct summand of $P[1]$ which is a 0-motive. This contradicts (c$''$). □

Corollary 4.10 — *An object $M \in \mathbf{DM}_{\mathrm{eff}}(k)$ is a $(2, \mathcal{H})$-sheaf if and only if it satisfies:*

- (a$'$) $\mathrm{H}_i(M) = 0$ *for* $i \notin \{0, -1, -2\}$;
- (b$'$) $\mathrm{H}_0(M)$ *is a* 0*-presented* $\mathcal{H}$*-sheaf;*
- (c$'$) $\mathrm{H}_{-1}(M)$ *is a* 1*-presented* $\mathcal{H}$*-sheaf;*
- (d$'$) $\mathrm{H}_{-2}(M)$ *is a* 1*-connected* $\mathcal{H}$*-sheaf;*
- (e$'$) $M[1]$ *does not have any non-trivial direct summand which is a* 0*-motive;*
- (f$'$) *If L is a* 0*-presented* $\mathcal{H}$*-sheaf, then* $\mathrm{ext}^1_{\mathbf{HI}(k)}(\mathrm{H}_2(M), L) = 0$, *i.e., every extension of* $\mathrm{H}_{-2}(M)$ *by L splits.*

4.3. n-Presented $(2, \mathcal{H})$-sheaves and mixed 2-motives.

Proposition 4.11 — *Let $n \geq 1$ be an integer. Then* Hypothesis 2.8 *is satisfied for the* 1*-motivic t-structure with* $\mathcal{A} = \mathbf{M}_1(k)$ *and* $\mathcal{B} = {}^1\mathcal{H}^{\mathcal{M}}_{\leq n}(k)$.

Proof. By Corollary 3.15, we have a right adjoint ${}^1\mathrm{Q}_n : {}^1\mathcal{H}^{\mathcal{M}}(k) \to {}^1\mathcal{H}^{\mathcal{M}}_{\leq n}(k)$ to the obvious inclusion. Next, consider a short exact sequence of $(1, \mathcal{H})$-sheaves

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

where M'' is a mixed 1-motive. We need to show that ${}^1\mathrm{Q}_n(M') \to {}^1\mathrm{Q}_n(M)$ is injective. This is easily seen to be equivalent to the condition that

$$U = ker\{{}^1\mathrm{Q}_n(M) \to M''\}$$

is n-presented. Consider the exact sequence of $\mathcal{H}$-sheaves

$$0 \longrightarrow coker\{\mathrm{H}_0(M) \to \mathrm{H}_0(M'')\} \longrightarrow \mathrm{H}_{-1}(U) \longrightarrow \mathrm{H}_{-1}({}^1\mathrm{Q}_n(M)) \longrightarrow \mathrm{H}_{-1}(M'') \longrightarrow 0.$$

By Lemma 4.12 below, $ker\{\mathrm{H}_{-1}({}^1\mathrm{Q}_n(M)) \to \mathrm{H}_{-1}(M'')\}$ is n-presented. It follows from [2, Lem. 1.1.22] that $\mathrm{H}_{-1}(U)$ is also n-presented. This imples that U is an n-presented $(1, \mathcal{H})$-sheaf. □

Lemma 4.12 — *Consider a short exact sequence of $\mathcal{H}$-sheaves:*

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0.$$

Assume that F is n-presented (with $n \geq 1$) and F'' is 1-presented. Then F' is n-presented.

Proof. Clearly, we can write F as a filtered colimit as follows:

$$F = \underset{i \in I}{colim}\, F_i, \quad \text{where} \quad F_i = coker\{\alpha_i : h_0(Y_i) \to h_0(X_i)\}.$$

Above, I is a filtered ordered set, Y_i and X_i are smooth k-schemes of dimension at most n, and α_i is a morphism of sheaves. Let $F''_i = im\{F_i \to F''\}$ and $F'_i = ker\{F_i \to F''\}$. Then $F' = colim_{i \in I}\, F'_i$ (use that filtered colimits are exact) and we have short exact sequences

$$0 \longrightarrow F'_i \longrightarrow F_i \longrightarrow F''_i \longrightarrow 0. \tag{13}$$

Clearly, it suffices to show that each F'_i is n-presented. As F''_i is a subsheaf of F'', it is 1-presented. Hence, the exact sequence (13) satisfies to the conditions of the statement. In other words, we may assume that

$$F = coker\{\alpha : h_0(Y) \to h_0(X)\}$$

with X and Y of dimension at most n.

Given a smooth k-variety V, we denote by $h_0^{\geq 2}(V) = ker\{h_0(V) \to \mathrm{Alb}(V)\}$. We also set

$$F^{\geq 2} = coker\{h_0^{\geq 2}(Y) \to h_0^{\geq 2}(X)\}.$$

As F'' is 1-presented and $F^{\geq 2}$ is 1-connected, the composition $F^{\geq 2} \to F \to F''$ is zero. It follows that $F^{\geq 2} \to F$ factors through F', yielding a morphism

$$\beta : \; F^{\geq 2} \longrightarrow F'.$$

The kernel and cokernel of β are 1-presented $\mathcal{H}$-sheaves. Indeed, $ker(\beta)$ is a subquotient of $\mathrm{Alb}(Y)$ and $coker(\beta)$ is a subquotient of $\mathrm{Alb}(X)$. Using [2, Lem. 1.1.22], we are reduced to show that $F^{\geq 2}$ is n-presented. By a second application of [2, Lem. 1.1.22], we are further reduced to check that $h_0^{\geq 2}(V)$ is n-presented for V a k-smooth scheme of dimension at most n.

One can find a smooth curve $C \subset V$ such that the composition

$$h_0(C) \to h_0(V) \to \mathrm{Alb}(V)$$

is surjective. Let $E = coker\{h_0(C) \to h_0(V)\}$. This is an n-presented $\mathcal{H}$-sheaf. The morphism $\gamma : \ h_0^{\geq 2}(V) \longrightarrow E$ is clearly surjective and its kernel is a subquotient of $h_0(C)$, and hence is 1-presented. We use again [2, Lem. 1.1.22] to conclude. □

Definition 4.13 — *Let $n \geq 1$ be an integer. We denote by ${}^2\mathcal{H}^{\mathcal{M}}_{\leq n}(k) \subset {}^2\mathcal{H}^{\mathcal{M}}(k)$ the $\mathbf{M}_1(k)$-perverted subcategory associated to ${}^1\mathcal{H}^{\mathcal{M}}_{\leq n}(k) \subset {}^1\mathcal{H}^{\mathcal{M}}(k)$. A $(2, \mathcal{H})$-sheaf which is in ${}^2\mathcal{H}^{\mathcal{M}}_{\leq n}(k)$ is called* n-presented.

Clearly, a 1-presented $(2, \mathcal{H})$-sheaf is simply a mixed 1-motive, i.e., ${}^2\mathcal{H}^{\mathcal{M}}_{\leq 1}(k) = \mathbf{M}_1(k)$. By convention, a 0-presented $(2, \mathcal{H})$-sheaf is a 0-motive and we set ${}^2\mathcal{H}^{\mathcal{M}}_{\leq 0}(k) = \mathbf{M}_0(k)$.

Lemma 4.14 — *Let $n \geq 1$ be an integer. A $(2, \mathcal{H})$-sheaf M is n-presented if and only if the $\mathcal{H}$-sheaf $\mathrm{H}_{-2}(M)$ is n-presented.*

Proof. Indeed, M is n-presented if and only if the $(1, \mathcal{H})$-sheaf ${}^1\mathrm{H}^{\mathcal{M}}_{-1}(M)$ is n-presented which is equivalent to $\mathrm{H}_{-1}({}^1\mathrm{H}^{\mathcal{M}}_{-1}(M)) \simeq \mathrm{H}_{-2}(M)$ being an n-presented $\mathcal{H}$-sheaf. □

Proposition 4.15 — *${}^2\mathcal{H}^{\mathcal{M}}_{\leq n}(k)$ is an abelian category and there is a functor*

$${}^2\mathrm{Q}_n : {}^2\mathcal{H}^{\mathcal{M}}(k) \to {}^2\mathcal{H}^{\mathcal{M}}_{\leq n}(k),$$

which is a right adjoint to the obvious inclusion.

We now come to our definition of mixed 2-motives.

Definition 4.16 — *An object $M \in \mathbf{DM}_{\mathrm{eff}}(k)$ is a* mixed 2-motive *if it satisfies the following conditions:*

(a) $\mathrm{H}_i(M) = 0$ *for* $i \notin \{0, -1, -2\}$;
(b) $\mathrm{H}_0(M)$ *is a 0-presented $\mathcal{H}$-sheaf;*
(c) $\mathrm{H}_{-1}(M)$ *is a 1-presented $\mathcal{H}$-sheaf;*
(d) $\mathrm{H}_{-2}(M)$ *is a 1-connected and 2-presented $\mathcal{H}$-sheaf;*
(e) $M[1]$ *does not have any non-trivial summand which is a 0-motive;*
(f) *If L is a 0-presented $\mathcal{H}$-sheaf, then* $\mathrm{ext}^1_{\mathbf{HI}(k)}(\mathrm{H}_2(M), L) = 0$, *i.e., every extension of* $\mathrm{H}_{-2}(M)$ *by L splits.*

We denote by $\mathbf{M}_2(k)$ the full subcategory of mixed 2-motives.

Obviously, mixed 2-motives are exactly the 2-presented $(2, \mathcal{H})$-sheaves, i.e., $\mathbf{M}_2(k) = {}^2\mathcal{H}^{\mathcal{M}}_{\leq 2}(k)$. In particular, $\mathbf{M}_2(k)$ is an abelian category.

4.4. Mixed 2-motives associated to surfaces.

Ideally, we should have the following.

Conjecture 4.17 — *Let S be a k-surface (possibly singular). Then ${}^2\mathrm{H}^{\mathcal{M}}_i(\mathrm{M}(S))$ is a mixed 2-motive for all $i \in \mathbb{Z}$.*

In fact, ${}^2\mathrm{H}^{\mathcal{M}}_i(\mathrm{M}(S))$ is expected to vanish for $i \notin [\![0, 4]\!]$ and more precisely, whenever the ℓ-adic homology group $\mathrm{H}^{\text{ét}}_i(S \otimes_k k^s, \mathbb{Q}_\ell)$ vanishes (here, k^s is a separable closure of the base field k and ℓ is a prime which is invertible in k). Unfortunately, Conjecture 4.17 seems out of reach of the actual techniques. However, it is possible to attach to S a sequence ${}^{\text{t}}\mathrm{H}_i^{\mathcal{M}}(S)$ of mixed 2-motives which hopefully coincide with those in Conjecture 4.17.

For simplicity, we assume that S is smooth and, except for the next result, we consider the cases where S is affine or S is projective.

Lemma 4.18 — *Let S be a smooth surface. Then ${}^2\mathrm{H}_n^{\mathcal{M}}(\mathrm{M}(S))$ is a mixed n-motive for $n \in \{0,1,2\}$.*

Proof. Consider the distinguished triangle

$$\mathrm{M}_{\geq 1}(S) \longrightarrow \mathrm{M}(S) \longrightarrow \mathrm{M}(\pi_0(S)) \longrightarrow \mathrm{M}_{\geq 1}(S)[1].$$

Clearly, $\mathrm{M}(\pi_0(S))$ is ${}^2t^{\mathcal{M}}$-negative, whereas $\mathrm{M}_{\geq 1}(S)$ is strictly ${}^2t^{\mathcal{M}}$-positive. This shows that ${}^2\mathrm{H}_0^{\mathcal{M}}(\mathrm{M}(S)) \simeq \mathrm{M}(\pi_0(S))$ is a 0-motive.

Similarly, consider the distinguished triangle

$$\widetilde{\mathrm{M}}_{\geq 2}(S)[-1] \longrightarrow \mathrm{M}_{\geq 1}(S)[-1] \longrightarrow (\mathrm{Alb}^0(S) \otimes \mathbb{Q})[-1] \longrightarrow \widetilde{\mathrm{M}}_{\geq 2}(S),$$

with $\mathrm{Alb}^0(S)$ the connected component of the Albanese scheme of S. Clearly, $(\mathrm{Alb}^0(S) \otimes \mathbb{Q})[-1]$ is ${}^2t^{\mathcal{M}}$-negative, whereas $\widetilde{\mathrm{M}}_{\geq 2}(S)[-1]$ is strictly ${}^2t^{\mathcal{M}}$-positive. This shows that ${}^2\mathrm{H}_1^{\mathcal{M}}(\mathrm{M}(S)) \simeq (\mathrm{Alb}^0(S) \otimes \mathbb{Q})[-1]$ is a mixed 1-motive.

Finally, to prove that ${}^2\mathrm{H}_2^{\mathcal{M}}(\mathrm{M}(S))$ is a mixed 2-motives, it suffices to show that $\mathrm{H}_{-2}({}^2\mathrm{H}_2^{\mathcal{M}}(\mathrm{M}(S)))$ is 2-presented. But, the latter is given by $ker\{h_0(S) \to \mathrm{Alb}(S)\}$. We conclude using Lemma 4.12. □

Now, assume that S is affine. Then ${}^2\mathrm{H}_i^{\mathcal{M}}(\mathrm{M}(S))$ are expected to be zero for $i \notin \{0,1,2\}$. Thus, we can make the following definition.

Definition 4.19 — *For $i \in [\![0,2]\!]$, we set ${}^{\text{t}}\mathrm{H}_i^{\mathcal{M}}(S) = {}^2\mathrm{H}_i^{\mathcal{M}}(\mathrm{M}(S))$. These are the (possibly non-zero) mixed 2-motives associated to S.*

Next, assume that S is projective. It is classical that the Chow motive of S admits a Künneth decomposition (see for example [6]). As the category of Chow motives is embedded into $\mathbf{DM}_{\text{eff}}(k)$, we deduce a decomposition

$$\mathrm{M}(S) = \bigoplus_{i=0}^{4} \mathrm{M}_i(S)[i]$$

such that $\mathrm{M}_i(S)$ corresponds under the ℓ-adic realization to $\mathrm{H}_i^{\text{ét}}(S \otimes_k k^s, \mathbb{Q}_\ell)$. We know that $\mathrm{M}_0(S) = \mathrm{M}(\pi_0(S))$ and that

$$\mathrm{M}_4(S) = \underline{\mathrm{Hom}}(\mathrm{M}_0(S), \mathbb{Q}(2)) \simeq \mathrm{M}_0(S)^\vee(2),$$

where $\mathrm{M}_0(S)^\vee = \underline{\mathrm{Hom}}(\mathrm{M}_0(S), \mathbb{Q}(0))$ is the dual 0-motive to $\mathrm{M}_0(S)$. Also, $\mathrm{M}_1(S)$ is a pure 1-motive given by the complex $(\mathrm{Alb}^0(S) \otimes \mathbb{Q})[-1]$ with $\mathrm{Alb}^0(S)$ the connected component of the Albanese scheme of S. Moreover,

$$\mathrm{M}_3(S) = \underline{\mathrm{Hom}}(\mathrm{M}_1(S), \mathbb{Q}(2)) \simeq \mathrm{M}_1(S)^\vee(1),$$

with $\mathrm{M}_1(S)^\vee = \underline{\mathrm{Hom}}(\mathrm{M}_1(S), \mathbb{Q}(1))$, the Cartier dual of the 1-motive $\mathrm{M}_1(S)$.

Lemma 4.20 — $\mathrm{M}_0(S)$ *is a* 0*-motive and* $\mathrm{M}_1(S)$ *is a* 1*-motive. Moreover, for $i \in \{2,3,4\}$, ${}^2\mathrm{H}_0^{\mathcal{M}}(\mathrm{M}_i(S))$ is a mixed* 2*-motive.*

Proof. The first statement is obvious. To prove the second statement, it suffices to show that for $i \in \{2, 3, 4\}$, the $\mathcal{H}$-sheaf

$$\mathrm{H}_{-2}({}^2\mathrm{H}_0^{\mathcal{M}}(\mathrm{M}_i(S))) \simeq \mathrm{H}_{-2}(\mathrm{M}_i(S))$$

is 2-presented. By an easy inspection, we see that:

$$\mathrm{H}_{-2}(\mathrm{M}_2(S)) = ker\{h_0(S) \to \mathrm{Alb}(S)\}, \quad \mathrm{H}_{-2}(\mathrm{M}_3(S)) = \mathrm{K}_1^M \otimes \mathrm{Alb}^0(S)^\vee$$
$$\text{and} \qquad \mathrm{H}_{-2}(\mathrm{M}_4(S)) = \mathrm{K}_2^M \otimes \mathbb{Q}_{tr}(\pi_0(S))^\vee.$$

Above, K_n^M is the n-th Milnor K-theory sheaf and the tensor product is taken in $\mathbf{HI}(k)$. This proves the lemma. □

Definition 4.21 — *Under the above hypothesis, we set* ${}^{\text{`}}\mathrm{H}_i^{\mathcal{M}}(S) = {}^2\mathrm{H}_0^{\mathcal{M}}(\mathrm{M}_i(S))$ *for* $i \in [\![0, 4]\!]$*. These are the (possibly non-zero) mixed* 2*-motives attached to* S*.*

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