

Brown representability in \mathbb{A}^1 -homotopy theory

Niko Naumann and Markus Spitzweck

ABSTRACT

We prove the following result of V. Voevodsky. If S is a finite dimensional noetherian scheme such that $S = \cup_{\alpha} \text{Spec}(R_{\alpha})$ for *countable* rings R_{α} , then the stable motivic homotopy category over S satisfies Brown representability.

1. Introduction

Let S be a noetherian scheme of finite Krull dimension. F. Morel and V. Voevodsky construct the stable motivic homotopy category $\text{SH}(S)$ guided by the intuition that there is a homotopy theory of schemes in which the affine line plays the role of the unit interval in classical homotopy theory. It is important to know which of the familiar structural properties of the classical homotopy category hold for $\text{SH}(S)$. It is well known that $\text{SH}(S)$ is a compactly generated triangulated category but this is a rather weak statement: Whereas the classical homotopy category admits the sphere spectrum as a single compact generator, $\text{SH}(S)$ requires an infinite set of such generators.

A more subtle question is whether Brown representability holds for $\text{SH}(S)$, cf. [N1] for a discussion of this notion and the fact that the following main result implies Brown representability for $\text{SH}(S)$. Denote by $\text{SH}(S)^c \subseteq \text{SH}(S)$ the full subcategory of compact objects.

THEOREM 1. *If Sm/S , the category of smooth S -schemes of finite type, is countable, then so is $\text{SH}(S)^c$.*

This result is due to V. Voevodsky [V, Proposition 5.5].¹ We obtain it here from some unstable results which are of independent interest and which we now sketch, giving an outline of this paper: In Section 2 we show that, for general S , the homotopy category of pointed motivic spaces $\mathbf{H}_{\bullet}(S)$ over S admits an almost finitely generated and monoidal model. The proof of this gives us a controlled fibrant replacement functor which allows to show in Section 3 the following unstable finiteness result.

THEOREM 2. *If Sm/S is countable, $F \in \Delta^{op}\text{Pre}(\text{Sm}/S)_{\bullet}$ is sectionwise countable and $(X, x) \in \Delta^{op}\text{Pre}(\text{Sm}/S)_{\bullet}$ is of finite type, then*

$$\mathbf{H}(S)_{\bullet}((X, x), F)$$

is countable.

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¹In *loc. cit.* V. Voevodsky works with the hypothesis that there is a Zariski open cover $S = \cup_{\alpha} \text{Spec}(R_{\alpha})$ for countable rings R_{α} . Given that S is noetherian, in particular quasi-compact, it is easy to see that this hypothesis is equivalent to Sm/S being countable.

Section 4 starts by establishing [V, Theorem 5.2]:

THEOREM 3. *If $(X, x) \in \Delta^{op}\text{Pre}(Sm/S)_\bullet$ is of finite type and $E = (E_n)_{n \geq 0}$ is a \mathbb{P}^1 -spectrum, then*

$$\text{SH}(S)(\Sigma_{\mathbb{P}^1}^\infty(X, x), E) = \text{colim}_n \mathbf{H}_\bullet(S)((\mathbb{P}^1, \infty)^{\wedge n} \wedge (X, x), E_n).$$

Then we give the proof of Theorem 1.

We conclude the Introduction with a general remark concerning Brown representability for $\text{SH}(S)$. As anticipated in [V], it is much more difficult to apply than its classical counterpart. This is essentially because, given a “cohomology theory” on Sm/S (of which there are many interesting examples), it is generally difficult to extend it to $\text{SH}(S)^c$, a minimum requirement for Brown representability to apply.

However, in recent joint work with P. A. Østvær [NSØ], we constructed many new motivic (ring) spectra, using the full strength of Theorem 1, and this was our initial motivation for documenting its proof.

ACKNOWLEDGEMENTS

Theorems 1 and 3 are due to V. Voevodsky but his proofs remain unpublished. In finding the present proofs via the above unstable results, we have been guided by [Ho1, Section 4],[B] and [PPR].

2. An almost finitely generated model for motivic spaces

Let S be a noetherian scheme of finite Krull dimension. The homotopy category of pointed motivic spaces over S , denoted $\mathbf{H}_\bullet(S)$ [MV, Section 3.2], is the homotopy category of the category $\Delta^{op}\text{Pre}(Sm/S)_\bullet$ of pointed simplicial presheaves on the category Sm/S of smooth S -schemes of finite type with respect to a suitable model structure. The purpose of this Section is to show that this model structure can be chosen to be monoidal and almost finitely generated, cf. [Ho1, 4.1]. To set the stage, denote by \mathcal{M} the model category $\Delta^{op}\text{Pre}(Sm/S)_\bullet$ with the objectwise flasque model structure [I, Theorem 3.7,a)]. We will show

PROPOSITION 4. *There exists a set $\mathcal{S} \subseteq \text{Mor}(\mathcal{M})$ such that the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ of \mathcal{M} with respect to \mathcal{S} exists, is a proper, cellular, simplicial, monoidal and almost finitely generated model category and satisfies $\text{Ho}(L_{\mathcal{S}}\mathcal{M}) \simeq \mathbf{H}_\bullet(S)$.*

Proof. By [I, Section 5] and [PPR, Theorem A.3.11], for a suitable choice of \mathcal{S} to be recalled presently, $L_{\mathcal{S}}\mathcal{M}$ exists, is proper, cellular and simplicial and satisfies $\text{Ho}(L_{\mathcal{S}}\mathcal{M}) \simeq \mathbf{H}_\bullet(S)$.

By [I, Theorem 4.9], we can choose

$$\mathcal{S} = \{\text{Cyl}((U \amalg_{U \times_X V} V \rightarrow X)_+)\}_\alpha \cup \{U_+ \xrightarrow{0} (U \times_S \mathbb{A}_S^1)_+ \mid U \in Sm/S\}$$

where 0 indicates the zero section and α runs through all elementary distinguished squares [MV, Definition 3.3]

$$\alpha = \begin{array}{ccc} Z & \xrightarrow{j} & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X. \end{array}$$

We use [Ho1, Proposition 4.2] to see that $L_{\mathcal{S}}\mathcal{M}$ is almost finitely generated: \mathcal{M} itself is proper and cellular by [I, Theorem 3.7,a)] and finitely generated by inspection of the generating (trivial) cofibrations of \mathcal{M} , given in [I, Definition 3.2]. Actually, while *loc. cit.* claims “cellular”, it only proves “cofibrantly generated”, so let us quickly explain why the additional properties [Hi, Definition 12.1.1,(1)-(3)] are true, i.e. \mathcal{M} is cellular: (1) and (2) are implied by \mathcal{M} being finitely generated and (3) says that all cofibrations in \mathcal{M} are effective monomorphisms. By [I, Lemma 3.8], every cofibration of \mathcal{M} is an injective cofibration, i.e. a monomorphism, and it is easy to see that all monomorphisms in \mathcal{M} are effective.

Since \mathcal{S} consists of cofibrations with compact² domains and codomains, $L_{\mathcal{S}}\mathcal{M}$ is almost finitely generated by [Ho1, Proposition 4.2].

To see that $L_{\mathcal{S}}\mathcal{M}$ is monoidal requires some argument, cf. [I, Section 6]: First, the \mathbb{A}^1 -local *injective* model structure on $\Delta^{op}\text{Pre}(Sm/S)_{\bullet}$ is monoidal since smashing with every pointed simplicial presheaf preserves \mathbb{A}^1 -weak equivalences [Mo, page 27]. Also, \mathcal{M} itself is monoidal by [I, Proposition 3.14] (for the unpointed variant of \mathcal{M}) and [Ho2, Proposition 4.2.9] (for the passage from the unpointed case to \mathcal{M}). Now, let i and j be cofibrations in $L_{\mathcal{S}}\mathcal{M}$. Since $L_{\mathcal{S}}\mathcal{M}$ and \mathcal{M} have the same cofibrations, the push-out product $i \wedge j$ is a cofibration in $L_{\mathcal{S}}\mathcal{M}$. Assume in addition that one of i and j is acyclic, i.e. an \mathbb{A}^1 -weak equivalence. Then so is $i \wedge j$ by the above reminder on the injective structure. \square

3. Unstable results

We employ the following notions of finiteness: A set X is countable if there is an injective map $X \rightarrow \mathbb{N}$. A simplicial set X is countable if the disjoint union $\cup_{n \geq 0} X_n$ is countable. A presheaf of (pointed) simplicial sets on a category \mathcal{C} is sectionwise countable if for all $U \in \mathcal{C}$, the (pointed) simplicial set $X(U)$ is countable. A category \mathcal{C} is countable if it is equivalent to a category \mathcal{C}' such that the disjoint union $\cup_{c_1, c_2 \in \mathcal{C}'} \mathcal{C}'(c_1, c_2)$ is countable.

We will need the following application of the small object argument.

PROPOSITION 5. *Let \mathcal{C} be a category, $\Delta^{op}\text{Pre}(\mathcal{C})_{\bullet}$ the category of pointed simplicial presheaves on \mathcal{C} and $I \subseteq \text{Mor}(\Delta^{op}\text{Pre}(\mathcal{C})_{\bullet})$ a subset such that*

- i) I is countable.
- ii) For every $\alpha \in I$, the domain $d(\alpha) \in \Delta^{op}\text{Pre}(\mathcal{C})_{\bullet}$ of α is compact.
- iii) For every $\alpha \in I$ and $G \in \Delta^{op}\text{Pre}(\mathcal{C})_{\bullet}$ sectionwise countable, the set $\Delta^{op}\text{Pre}(\mathcal{C})_{\bullet}(d(\alpha), G)$ is countable.
- iv) For every $\alpha \in I$ and $U \in \mathcal{C}$, the set $\Delta^{op}\text{Pre}(\mathcal{C})_{\bullet}(U_+, c(\alpha))$ is countable, where $c(\alpha)$ denotes the codomain of α .

²To avoid confusion arising from conflicting terminology in the literature, we make precise that we call an object c of a category \mathcal{C} compact, if for all filtering colimits $\text{colim } {}_i c_i$ which exist in \mathcal{C} , the canonical map of sets $\mathcal{C}(c, \text{colim } {}_i c_i) \rightarrow \text{colim } {}_i \mathcal{C}(c, c_i)$ is bijective.

Then every map $F \rightarrow G$ in $\Delta^{op}\text{Pre}(\mathcal{C})_\bullet$ can be functorially factored into $F \xrightarrow{\iota} F' \xrightarrow{\pi} G$ such that

- a) ι is a relative I -cell complex [Hi, Definition 10.5.8, (1)].
- b) π has the right lifting property with respect to I .
- c) If $G = \bullet$ is the final object and F is sectionwise countable, then F' is sectionwise countable.

Proof. The small object argument [Hi, Proposition 10.5.16] applies by *ii*), yielding a functorial factorization satisfying *a*) and *b*).

To see *c*), we assume F sectionwise countable and run through this argument in some detail:

We construct

$$F = F_0 \rightarrow F_1 \rightarrow \dots \rightarrow F_n \rightarrow F_{n+1} \rightarrow \dots$$

by induction on $n \geq 0$ such that all F_n are sectionwise countable as follows: Consider the set \mathcal{D} of all commutative squares

$$\begin{array}{ccc} a & \longrightarrow & F_n \\ \downarrow f & & \downarrow \\ b & \longrightarrow & \bullet \end{array}$$

with $f \in I$. Then \mathcal{D} is countable by *i*), *iii*). Define F_{n+1} to be the push-out

$$\begin{array}{ccc} \coprod_{\mathcal{D}} a & \longrightarrow & F_n \\ \downarrow & & \downarrow \\ \coprod_{\mathcal{D}} b & \longrightarrow & F_{n+1}. \end{array}$$

Then F_{n+1} is sectionwise countable by *iv*).

Now let $F \xrightarrow{\iota} F' := \text{colim}_n F_n \xrightarrow{\pi} \bullet$ be the canonical maps. These satisfy *a*) trivially and *b*) by *ii*) (i.e. we do not need any longer transfinite compositions). Clearly, F' is sectionwise countable since all the F_n are. \square

When coupled with the work from Section 2, this yields our following key technical finiteness result.

PROPOSITION 6. *Let S be a noetherian scheme of finite Krull dimension such that Sm/S is countable and $F \in \Delta^{op}\text{Pre}(Sm/S)_\bullet$ sectionwise countable. Then there is a trivial cofibration*

$$F \hookrightarrow F'$$

in $L_S\mathcal{M}$ such that F' is fibrant and sectionwise countable.

Proof. We apply Proposition 5 with $\mathcal{C} := Sm/S$ and $I := \Lambda(S) \cup J$ where J is the set of generating trivial cofibrations for \mathcal{M} given in [I, Definition 3.2,1)], \mathcal{S} is as in the proof of Proposition 4 and

$$\Lambda(\mathcal{S}) := \{a \otimes \Delta^n \amalg_{a \otimes \partial \Delta^n} b \otimes \partial \Delta^n \rightarrow b \otimes \Delta^n \mid (a \rightarrow b) \in \mathcal{S}, n \geq 0\}.$$

We check the assumptions *i) – iv)* of Proposition 5 for this set I :

i) I is countable since Sm/S is.

ii) We already know that all domains of \mathcal{S} are compact, hence so are those of $\Lambda(\mathcal{S})$. Obviously the domains of J are compact, so all domains of I are compact.

iii) Assume $G \in \Delta^{op}\text{Pre}(Sm/S)_\bullet$ sectionwise countable and $\alpha \in I$. To see that $T := \Delta^{op}\text{Pre}(Sm/S)_\bullet(d(\alpha), G)$ is countable, we distinguish two cases: If $\alpha \in J$, then

$$d(\alpha) = (\cup \mathcal{U}_+ \wedge \Delta_+^n) \amalg_{\cup \mathcal{U}_+ \wedge \Lambda_{k,+}^n} (X_+ \wedge \Lambda_{k,+}^n)$$

for a finite collection $\mathcal{U} = \{U_i \rightarrow X\}_{U_i, X \in Sm/S}$ of monomorphisms. Now, T is countable since G is sectionwise countable. If $\alpha \in \Lambda(\mathcal{S})$, then $d(\alpha) = a \otimes \Delta^n \amalg_{a \otimes \partial \Delta^n} b \otimes \partial \Delta^n$ for some $(a \rightarrow b) \in \mathcal{S}$, and it suffices to see that $\Delta^{op}\text{Pre}(Sm/S)_\bullet(a, G)$ and $\Delta^{op}\text{Pre}(Sm/S)_\bullet(b, G)$ are countable. Since a and b arise from representable presheaves by taking finite colimits and tensors with finite simplicial sets, this follows again from G being sectionwise countable.

iv) Assume $\alpha \in I$ with codomain $c(\alpha)$ and $U \in Sm/S$. We need to see that $T := \Delta^{op}\text{Pre}(Sm/S)_\bullet(U_+, c(\alpha))$ is countable: For $\alpha \in J$ we have $c(\alpha) = X_+ \wedge \Delta_+^n$ for some $X \in Sm/s, n \geq 0$, hence the result since Sm/S is countable. For $\alpha \in \Lambda(\mathcal{S})$ we have $c(\alpha) = b \otimes \Delta^n$ for some $b = c(\beta), \beta \in \mathcal{S}$, hence $T = (b(U) \wedge \Delta_+^n)_0$. By construction of \mathcal{S} , b is a finite push-out of tensors of finite simplicial sets with representables, so T is countable since Sm/S is.

Applying now Proposition 5,c), we obtain maps $F \xrightarrow{\iota} F' \xrightarrow{\pi} \bullet$ in $\Delta^{op}\text{Pre}(Sm/S)_\bullet$ such that F' is sectionwise countable, ι is a relative I -cell complex and π has the right-lifting property with respect to I .

We need to check that ι (resp. π) is a trivial cofibration (resp. a fibration) in $L_S\mathcal{M}$: J is a generating set of trivial cofibrations for \mathcal{M} and since \mathcal{S} consists of cofibrations with cofibrant domain in \mathcal{M} , $\Lambda(\mathcal{S})$ consists of trivial cofibrations in $L_S\mathcal{M}$. Hence every relative I -cell complex, in particular ι , is a trivial cofibration in $L_S\mathcal{M}$. Since π has the right-lifting property with respect to J , F' is fibrant in \mathcal{M} . Since π has the right-lifting property with respect to $I = \Lambda(\mathcal{S}) \cup J$, $F' \in \mathcal{M}$ is S -local [Hi, Proposition 4.2.4]. So F' is fibrant in $L_S\mathcal{M}$ by [Hi, Proposition 3.3.16, (1)]. \square

The previous result admits the following immediate stable analogue.

PROPOSITION 7. *Let S be a noetherian scheme of finite Krull dimension such that Sm/S is countable and $E = (E_n)$ a \mathbb{P}^1 -spectrum $[J]$ such that all E_n are sectionwise countable. Then there is a level-fibrant replacement $E' = (E'_n)$ of E such that all E'_n are sectionwise countable.*

Proof. One constructs E'_n and structure maps $E'_n \wedge \mathbb{P}^1 \rightarrow E'_{n+1}$ inductively: $E'_0 := E_0^f$, where $(-)^f$ denotes the fibrant replacement provided by Proposition 6, and for all $n \geq 0$

$$E'_{n+1} := (E'_n \wedge \mathbb{P}^1 \cup_{E_n \wedge \mathbb{P}^1} E_{n+1} \wedge \mathbb{P}^1)^f.$$

The evident maps $E'_n \wedge \mathbb{P}^1 \rightarrow E'_{n+1}$ define a \mathbb{P}^1 -spectrum E' and the obvious map $E \rightarrow E'$ is a level-equivalence, E' is level fibrant and all E'_n are sectionwise countable by an inductive application of Proposition 6. \square

We recall (the simplicial variant of) the following definition from [V].

DEFINITION 8. *Let S be a noetherian scheme of finite Krull dimension.*

- i) *The category of motivic spaces of finite type over S is the smallest strictly full subcategory $\mathrm{Spc}^{ft} \subseteq \Delta^{op}\mathrm{Pre}(Sm/S)$ such that:*
- a) *$(Sm/S) \subseteq \mathrm{Spc}^{ft}$.*
 - b) *For all push-outs*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in $\Delta^{op}\mathrm{Pre}(Sm/S)$ such that $A, B, C \in \mathrm{Spc}^{ft}$ and i is a monomorphism, we have $D \in \mathrm{Spc}^{ft}$.

- ii) *We denote by $\mathrm{Spc}_\bullet^{ft} \subseteq \Delta^{op}\mathrm{Pre}(Sm/S)_\bullet$ the strictly full subcategory of objects (X, x) such that $X \in \mathrm{Spc}^{ft}$.*

We denote by map_\bullet (resp. $\underline{\mathrm{Hom}}_\bullet$) the mapping spaces (respectively internal homs) of the simplicial monoidal model category $L_S\mathcal{M}$ and their derived analogues by Rmap (resp. $\underline{\mathrm{RHom}}_\bullet$).

THEOREM 9. *Let S be a noetherian scheme of finite Krull dimension such that Sm/S is countable, $F \in \Delta^{op}\mathrm{Pre}(Sm/S)_\bullet$ sectionwise countable and $(X, x) \in \mathrm{Spc}_\bullet^{ft}$. Then, for all $n \geq 0$,*

$$\pi_n \mathrm{Rmap}_\bullet((X, x), F)$$

is countable.

Proof. We first show that the class

$$\{(X, x) \in \mathbf{H}_\bullet(S) \mid \forall n \geq 0 : \pi_n \mathrm{Rmap}_\bullet((X, x), F) \text{ is countable} \} \subseteq \mathbf{H}_\bullet(S)$$

is stable under homotopy push-outs. For this, it suffices to see that if

$$\begin{array}{ccc} X & \longrightarrow & X_1 \\ \downarrow & & \downarrow f_1 \\ X_3 & \xrightarrow{f_3} & X_2 \end{array}$$

is a homotopy pull-back of simplicial sets such that all homotopy sets of X_1, X_2 and X_3 are countable, then so are those of X . We can assume that all X_i are Kan complexes and f_1, f_3 are Kan fibrations and now use some basic results about minimal Kan complexes/fibrations [Ma, Sections 9 and 10]: There is a minimal Kan complex $Y_2 \subseteq X_2$ which is a deformation retract of X_2 and we can replace the pull-backs of f_i to Y_2 with minimal Kan fibrations $g_i : Y_i \rightarrow Y_2$ ($i = 1, 3$). Then all Y_i are minimal Kan complexes and by minimality and our assumption about their homotopy, they are countable. Then X is weakly equivalent to the countable Kan complex $Y_1 \times_{Y_2} Y_3$.

Since the identity is a simplicial (left) Quillen equivalence from $L_S\mathcal{M}$ to $\Delta^{op}\mathrm{Pre}(Sm/S)_\bullet$ with the \mathbb{A}^1 -local injective structure, every $(X, x) \in \mathrm{Spc}_\bullet^{ft}$ can be obtained from representables by finitely

many homotopy push-outs. We may thus assume that $(X, x) = (U, u)$ for some $U \in Sm/S$. For a sectionwise countable fibrant replacement F' of F in $L_S\mathcal{M}$ as in Proposition 6, we then see that

$$\mathrm{Rmap}_\bullet((U, u), F) = \mathrm{map}_\bullet((U, u), F') \subseteq \mathrm{map}(U, F') \simeq F'(U)$$

is a countable Kan complex. □

4. Stable results

Let S be a noetherian scheme of finite Krull dimension and $\mathrm{SH}(S)$ the homotopy category of \mathbb{P}^1 -spectra over S [J]. We first establish the following result [V, Theorem 5.2].

THEOREM 10. *If $(X, x) \in \mathrm{Spc}_\bullet^{ft}$ and $E = (E_n)_{n \geq 0}$ is a \mathbb{P}^1 -spectrum, then*

$$\mathrm{Rmap}_{\mathrm{SH}(S)}(\Sigma_{\mathbb{P}^1}^\infty(X, x), E) \simeq \mathrm{hocolim}_n \mathrm{Rmap}_{\mathbf{H}_\bullet(S)}(\Sigma_{\mathbb{P}^1}^n(X, x), E_n).$$

In particular,

$$\mathrm{SH}(S)(\Sigma_{\mathbb{P}^1}^\infty(X, x), E) = \mathrm{colim}_n \mathbf{H}_\bullet(S)(\Sigma_{\mathbb{P}^1}^n(X, x), E_n).$$

Proof. Using an argument very similar to the first part of the proof of Theorem 9 powered by the facts that $\Sigma_{\mathbb{P}^1}^\infty$ and $\Sigma_{\mathbb{P}^1}^n$ preserve homotopy push-outs and that filtered (homotopy) colimits commute with finite (homotopy) limits in simplicial sets, we can assume that $(X, x) = (U, u)$ for some $U \in Sm/S$.

We now check the hypothesis of [Ho1, Corollary 4.13]: $\mathcal{D} := L_S\mathcal{M}$ is (left) proper, cellular and almost finitely generated, $T := (\mathbb{P}^1, \infty) \wedge - : \mathcal{D} \rightarrow \mathcal{D}$ is left Quillen since \mathcal{D} is monoidal and the right-adjoint of T is $U = \Omega_{\mathbb{P}^1}$ and preserves filtered colimits. For every $m \geq 0$, $A := (U, u) \wedge S^m \in \mathcal{D}$ is compact and cofibrant with compact cylinder object $A \wedge \Delta_+^1$. Finally, let Y denote a level fibrant replacement of E , then

$$\pi_m \mathrm{Rmap}_{\mathrm{SH}(S)}(\Sigma_{\mathbb{P}^1}^\infty(U, u), E) = \mathrm{SH}(S)(\Sigma_{\mathbb{P}^1}^\infty(A), Y) =$$

$$\mathrm{colim}_n \mathbf{H}_\bullet(S)(\Sigma_{\mathbb{P}^1}^n(A), Y_n) = \pi_m(\mathrm{hocolim}_n \mathrm{Rmap}_{\mathbf{H}_\bullet(S)}(\Sigma_{\mathbb{P}^1}^n(U, u), E_n)).$$

Here, the second equality is [Ho1, Corollary 4.13] and we used [PPR, Theorem A.5.6] to know that $\mathrm{Ho}(Sp^{\mathbb{N}}(\mathcal{D}, T)) = \mathrm{SH}(S)$. □

Let s_- denote the shift functor for \mathbb{P}^1 -spectra, i.e. $(s_-(E))_n = E_{n+1}$. We need the following observation about its homotopical properties.

LEMMA 11. *Let S be a noetherian scheme of finite Krull dimension and $E = (E_n)$ a \mathbb{P}^1 -spectrum. Then*

$$\mathbb{L}((\mathbb{P}^1, \infty) \wedge -)(E) \simeq s_-(E) \text{ in } \mathrm{SH}(S).$$

Proof. We use somewhat freely results and notations from [Ho1, Sections 3 and 4]. We know that $\mathbb{L}((\mathbb{P}^1, \infty) \wedge -) \simeq \mathbb{R}s_-$. Construct

$$E \xrightarrow{\iota} E' \xrightarrow{j_{E'}} \Theta^\infty E',$$

where ι is a level-fibrant replacement and $j_{E'}$ and Θ^∞ are as in *loc. cit.* Then $\Theta^\infty E'$ is stably fibrant, hence

$$\mathbb{R}s_-(E) \simeq s_- \Theta^\infty E' \simeq \Theta^\infty s_- E',$$

since $s_- \circ \Theta \simeq \Theta \circ s_-$ by direct inspection. On the other hand, $s_-(\iota) : s_- E \rightarrow s_- E'$ is a level-equivalence and thus a stable equivalence, and $j_{s_- E'} : s_- E' \rightarrow \Theta^\infty s_- E'$ was shown to be a stable weak equivalence in *loc. cit.* Combining, we see that $\mathbb{R}s_-(E) \simeq s_- E$, as desired. \square

To resume our work on Brown representability, recall that

$$\mathcal{G} := \{ \Sigma^{p,q} \Sigma_{\mathbb{P}^1}^\infty U_+ \mid U \in Sm/S, p, q \in \mathbb{Z} \} \subseteq SH(S)$$

is a set of compact generators, where $\Sigma^{p+q,q} := S^p \wedge \mathbb{G}_m^q \wedge (-)$ (for $p, q \geq 0$). Denoting by $SH(S)^c \subseteq SH(S)$ the full subcategory of compact objects, we first deduce the stable analogue of Theorem 9.

THEOREM 12. *Let S be a noetherian scheme of finite Krull dimension such that Sm/S is countable, $F \in SH(S)^c$ and $E = (E_n)$ a \mathbb{P}^1 -spectrum such that for all $n \geq 0$, E_n is sectionwise countable. Then, for all $p, q \in \mathbb{Z}$,*

$$E^{p,q}(F) := SH(S)(F, \Sigma^{p,q} E)$$

is countable.

Proof. It is clear that the class

$$\{ F' \in SH(S) \mid \forall p, q \in \mathbb{Z} : E^{p,q}(F) \text{ is countable} \} \subseteq SH(S)$$

is a thick subcategory stable under $S^1 \wedge -$ and $(\mathbb{G}_m, 1) \wedge -$.

$SH(S)^c \subseteq SH(S)$ is the thick subcategory generated by \mathcal{G} [N2, Theorem 2.1.3], so we can assume that $F = \Sigma_{\mathbb{P}^1}^\infty U_+$ for some $U \in Sm/S$.

Given $p, q \in \mathbb{Z}$, choose integers $p', q', k \geq 0$ such that $(p, q) = (p' + q', q') - (2k, k)$. Then

$$E^{-p,-q}(F) = SH(S)(\Sigma_{\mathbb{P}^1}^\infty (U_+ \wedge S^{p'} \wedge \mathbb{G}_m^{q'}), \mathbb{L}((\mathbb{P}^1, \infty) \wedge -)^k(E)).$$

By Lemma 11, there is a stable equivalence $\mathbb{L}((\mathbb{P}^1, \infty) \wedge -)^k(E) \simeq (s_-)^k(E) =: E'$. Since $(X, x) := U_+ \wedge S^{p'} \wedge \mathbb{G}_m^{q'}$ is isomorphic in $H_\bullet(S)$ to a space of finite type (because $S^1 \simeq \mathbb{A}^1/(0 \sim 1)$), Theorem 10 implies that

$$E^{-p,-q}(F) = \operatorname{colim}_n H_\bullet(S)(\Sigma_{\mathbb{P}^1}^n(X, x), E'_n).$$

Since $(X, x) \in L_{\mathcal{S}}\mathcal{M}$ is cofibrant, we have

$$X_n := \Sigma_{\mathbb{P}^1}^n(X, x) = (\mathbb{P}^1, \infty)^{\wedge n} \wedge (X, x),$$

which again is isomorphic in $H_{\bullet}(S)$ to a space of finite type. Now, for every $n \geq 0$, $H_{\bullet}(S)(X_n, E'_n)$ is countable by Theorem 9, hence so is $E^{-p, -q}(F)$. \square

Finally, we can establish [V, Proposition 5.5] which makes Brown representability available in \mathbb{A}^1 -homotopy theory.

THEOREM 13. *Let S be a noetherian scheme of finite Krull dimension such that Sm/S is countable. Then the category $SH(S)^c$ is countable.*

Proof. According to the proof of [HPS, Proposition 2.3.5], $SH(S)^c$ is equivalent to an increasing union of subcategories

$$\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \dots \subseteq SH(S)^c \subseteq SH(S)$$

such that \mathcal{S}_0 is the full subcategory spanned by the set of objects \mathcal{G} and for every $n \geq 0$, if \mathcal{S}_n is countable, so is \mathcal{S}_{n+1} . It thus suffices to see that \mathcal{S}_0 is countable. Since \mathcal{G} clearly is, this means we need to show that for all $p, q \in \mathbb{Z}, U, V \in Sm/S$ the set

$$SH(S)(\Sigma_{\mathbb{P}^1}^{\infty}U_+, \Sigma^{p, q}\Sigma_{\mathbb{P}^1}^{\infty}V_+)$$

is countable, which is true by Theorem 12 because for all $n \geq 0$, $E_n := (\Sigma_{\mathbb{P}^1}^{\infty}V_+)_n = (\mathbb{P}^1, \infty)^{\wedge n} \wedge V_+$ is sectionwise countable. Indeed, for all $U \in Sm/S$, the (simplicial) set $E_n(U)$ is a quotient of

$$[(Sm/S)(U, \mathbb{P}^1)]^n \times (Sm/S)(U, V) = \mathbb{P}^1(U)^n \times V(U)$$

which is countable since the category Sm/S is. \square

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Niko Naumann niko.naumann@mathematik.uni-regensburg.de, markus.spitzweck@mathematik.uni-regensburg.de
 NWF I- Mathematik, Universität Regensburg, 93040 Regensburg

Markus Spitzweck markus.spitzweck@mathematik.uni-regensburg.de
 NWF I- Mathematik, Universität Regensburg, 93040 Regensburg