COMPUTING BOREL'S REGULATOR II

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Abstract. In [3] we described a power series formula for the Borel regulator evaluated on the odd-dimensional homology of the general linear group of a number field and, concentrating on dimension three for simplicity, described a computer algorithm which calculates the value to any chosen degree of accuracy. In this sequel we give an algorithm for the construction of the input homology classes and describe the results of one cyclotomic field computation.

Contents

1. Introduction

In [3] we described an algorithm which computes the Borel regulator homomorphism

$$b_n : H_{2n+1}(GL_N(F); \mathbb{Z}) \to \mathbb{R}$$

from the homology of the discrete group of invertible $N \times N$ matrices with entries in a number field F to the real numbers. For illustrative purposes we concentrated on dimension 3. In this paper we shall describe an algorithm, based on the free differential calculus of ([9]; see also [2],[6],[7],[8]), for constructing homology cycles in the standard resolution for $GL_N(F)$ which is the input for the algorithm of [3].

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As in [3] we concentrate on dimension 3 but in ([3] §4) we gave a power series formula from which one may give an algorithm for evaluating the Borel regulator b_n in higher odd dimensions and in §4 of this paper we describe our algorithm for constructing homology classes in $H_3(GL_N(F);\mathbb{Z})$ can be used to construct homology classes related to $H_{2n+1}(GL_N(F);\mathbb{Z})$ in higher dimensions by applying it with F replaced by the coordinate ring of the boundary of a simplex. Currently this is not practical for regulator calculations because (see the Problem posed in §4) we do not have a regulator algorithm whose input is homology classes for the general linear group of the coordinate ring of the boundary of a simplex.

2. Free Fox Differentiation

Let G be a discrete group and write $\underline{B}_n G$ for the degree n part of the bar resolution ([12],[23]). Therefore $\underline{B}_n G$ is the free left $\mathbb{Z}[G]$-module with basis consisting of n-tuples $[g_1|g_2|\ldots|g_n]$ with each $g_i \in G$ for $n \geq 1$ and $[\,]$ when $n = 0$. The differential is given by

$$\begin{aligned} d([g_1|g_2|\ldots|g_n]) &= g_1[g_2|\ldots|g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1|g_2|\ldots|g_i g_{i+1}|\ldots|g_n] \\ &\quad + (-1)^n [g_1|g_2|\ldots|g_{n-1}]. \end{aligned}$$

In [3] we used the standard resolution ([12],[23]) given in dimension n by $B_n G$ which is the free abelian group on $(n+1)$-tuples $(g_0, g_1, \ldots, g_n)$ on which $g \in G$ acts by $g(g_0, g_1, \ldots, g_n) = (gg_0, gg_1, \ldots, gg_n)$. The well-known chain isomorphisms between these resolutions [12] Chapter VI §13)

$$\psi_n : \underline{B}_n G \longrightarrow B_n G \text{ and } \phi_n : B_n G \longrightarrow \underline{B}_n G$$

are given by the formulae

$$\phi_n(1, y_1, \ldots, y_n) = [y_1|y_1^{-1}|\ldots|y_{n-1}^{-1}y_n]$$

and

$$\psi_n[x_1|\ldots|x_n] = (1, x_1, x_1x_2, \ldots, x_1x_2\ldots x_n).$$

The following material was introduced in [9], motivated by the study of knot invariants. Let F_G denote the free group on symbols s_g with $g \in G$ and denote by $\phi : F_G \to G$ the group homomorphism given by $\phi(s_g) = g$.

Definition 2.1. The free derivative

$$\partial : F_G \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} \underline{B}_2 G$$

is characterised by the properties:

(i) $\partial(e) = 0$ (e is the identity element of G) and

(ii) $\partial(us_x) = \partial(u) + 1 \otimes_{\mathbb{Z}[G]} [\phi(u)|x]$ for $u \in F_G$.

Note that (ii) implies

$$\partial(u) = \partial(us_x^{-1}s_x) = \partial(us_x^{-1}) + 1 \otimes_{\mathbb{Z}[G]} [\phi(u)x^{-1}|x]$$

which means that

$$\partial(us_x^{-1}) = \partial(u) - 1 \otimes_{\mathbb{Z}[G]} [\phi(u)x^{-1}|x].$$

Lemma 2.2.

For $1 \le i \le r$ suppose $x_i \in G$ and that $\epsilon_i = \pm 1$. Set $z_i = x_i^{-1}$ if $\epsilon_i = -1$ and $z_i = 1$ otherwise. Then

$$\partial(s_{x_1}^{\epsilon_1} s_{x_2}^{\epsilon_2} \dots s_{x_r}^{\epsilon_r}) = \sum_{i=1}^{r} \epsilon_i \otimes_{\mathbb{Z}[G]} [x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_{i-1}^{\epsilon_{i-1}} z_i | x_i]$$

which is well-defined on F_G.

Proof

By conditions (i) and (ii) of Definition 2.1 and induction on r we see that $\partial(s_{x_1}^{\epsilon_1} s_{x_2}^{\epsilon_2} \dots s_{x_r}^{\epsilon_r})$ must be given by the above formula. It remains to verify that this formula agrees with

$$\partial(s_{x_1}^{\epsilon_1} s_{x_2}^{\epsilon_2} \dots s_{x_{j-2}}^{\epsilon_{j-2}} s_{x_{j+1}}^{\epsilon_{j+1}} \dots s_{x_r}^{\epsilon_r})$$

when $x_{j-1} = x = x_j$ and $\epsilon_{j-1} + \epsilon_j = 0$. The last $r-j$ terms in the two formulae coincide. Therefore it suffices to observe that

$$\begin{aligned} &\partial(s_{x_1}^{\epsilon_1} s_{x_2}^{\epsilon_2} \dots s_{x_{j-2}}^{\epsilon_{j-2}} s_x^{\epsilon_{j-1}} s_x^{\epsilon_j}) \\ &= \partial(s_{x_1}^{\epsilon_1} s_{x_2}^{\epsilon_2} \dots s_{x_{j-2}}^{\epsilon_{j-2}}) + \epsilon_{j-1} \otimes_{\mathbb{Z}[G]} [x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_{j-2}^{\epsilon_{j-2}} z_{j-1} | x] \\ &\qquad + \epsilon_j \otimes_{\mathbb{Z}[G]} [x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x^{\epsilon_{j-1}} z_j | x] \\ &= \partial(s_{x_1}^{\epsilon_1} s_{x_2}^{\epsilon_2} \dots s_{x_{j-2}}^{\epsilon_{j-2}}) \end{aligned}$$

since the last two terms are respectively

$$(-1) \otimes_{\mathbb{Z}[G]} [x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_{j-2}^{\epsilon_{j-2}} x^{-1} | x] + 1 \otimes_{\mathbb{Z}[G]} [x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x^{\epsilon_{j-1}} x^{-1} | x] = 0$$

if $\epsilon_j = 1$ and

$$1 \otimes_{\mathbb{Z}[G]} [x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_{j-2}^{\epsilon_{j-2}} | x] + (-1) \otimes_{\mathbb{Z}[G]} [x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x x^{-1} | x] = 0$$

if $\epsilon_j = -1$.□

Definition 2.3. Given some $w \in F_G$ written in the following form:

$$w = u_1 (s_{x_1} s_{y_1} s_{x_1 y_1}^{-1})^{n_1} u_1^{-1} u_2 (s_{x_2} s_{y_2} s_{x_2 y_2}^{-1})^{n_2} u_2^{-1} \dots u_k (s_{x_k} s_{y_k} s_{x_k y_k}^{-1})^{n_k} u_k^{-1}$$

where $x_i, y_i \in G$ and $u_i \in F_G$.

Define $W(w) \in \mathbb{Z} \otimes_{\mathbb{Z}[G]} \underline{B}_3(G)$ by the formula

$$W(w) = \sum_{i=1}^{k} n_i \otimes_{\mathbb{Z}[G]} [\phi(u_i) | x_i | y_i].$$

Note that W is not a well-defined function on F_G (in contrast to the free derivative which is).

Theorem 2.4.

In Definition 2.3

$$(1 \otimes_{\mathbb{Z}[G]} d)(W(w)) = \sum_{i=1}^{k} n_i \otimes_{\mathbb{Z}[G]} [x_i | y_i] - \partial(w) \in \mathbb{Z} \otimes_{\mathbb{Z}[G]} \underline{B}_2(G).$$

Proof

First we note that

$$\phi(u_i(s_{x_i}s_{y_i}s_{x_iy_i}^{-1})^{n_i}u_i^{-1}) = \phi(u_i)\phi(s_{x_i}s_{y_i}s_{x_iy_i}^{-1})^n\phi(u_i)^{-1} = \phi(u_i)\phi(u_i)^{-1} = e$$

and, by the formula of Lemma 2.2, it is easy to see that

$$\partial(w) = \sum_{i=1}^{k} n_i\partial(u_i(s_{x_i}s_{y_i}s_{x_iy_i}^{-1})u_i^{-1}).$$

For ease of notation let us denote, for the duration of this proof, the product $h_1 \dots h_t$ (in a possibly non-abelian group) simply by $\prod_{i=1}^{t} h_1$. With this convention, let $u_i = \prod_{j=1}^{l} s_{z_j}^{m_j}$, where m_j is either 1 or -1. Then, by the formula of Lemma 2.2,

$\partial(u_i(s_{x_i}s_{y_i}s_{x_iy_i}^{-1})u_i^{-1})$

$= \sum_{j \mid m_j=1} 1 \otimes_{\mathbb{Z}[G]} [\phi(\prod_{p=1}^{j-1} z_p^{m_p})|z_j] - \sum_{j \mid m_j=-1} 1 \otimes_{\mathbb{Z}[G]} [\phi(\prod_{p=1}^{j} z_p^{m_p})|z_j]$

$+1 \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_i] + 1 \otimes_{\mathbb{Z}[G]} (\phi(u_i)x_i|y_i] - 1 \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_iy_i]$

$-\sum_{j|m_j=1} 1 \otimes_{\mathbb{Z}[G]} [\phi(\prod_{p=1}^{j-1} z_p^{m_p})|z_j] + \sum_{j|m_j=-1} 1 \otimes_{\mathbb{Z}[G]} [\phi(\prod_{p=1}^{j} z_p^{m_p})|z_j]$

$= 1 \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_i] + 1 \otimes_{\mathbb{Z}[G]} (\phi(u_i)x_i|y_i] - 1 \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_iy_i]$.

Thus we have

$\partial(w) = \sum_{i=1}^{k} (n_i \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_i] + n_i \otimes_{\mathbb{Z}[G]} (\phi(u_i)x_i|y_i] - n_i \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_iy_i])$

$= \sum_{i=1}^{k} n_i \otimes_{\mathbb{Z}[G]} [x_i|y_i]$

$-\sum_{i=1}^{k} (n_i \otimes_{\mathbb{Z}[G]} [x_i|y_i] - n_i \otimes_{\mathbb{Z}[G]} [\phi(u_i)x_i|y_i]$

$+n_i \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_iy_i] - n_i \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_i])$

$= \sum_{i=1}^{k} n_i \otimes_{\mathbb{Z}[G]} [x_i|y_i] - (1 \otimes_{\mathbb{Z}[G]} d)(W(w))$,

as required. □

Definition 2.5. Any element in $\mathrm{Ker}(\phi : F_G \longrightarrow G)$ will be called a relator.

2.6. *Constructing boundary relations*

We come now to the key step in our algorithm for the construction of 3-dimensional cycles in $\mathbb{Z} \otimes_{\mathbb{Z}[G]} \underline{B}_* G$. In our applications G will be either the general linear group or the Steinberg group of a commutative ring with a 1. The basis of the method will be to find 2-dimensional cycles which have two canonical reasons for being a boundary. By a "canonical reason" we mean precisely a relation in F_G to which we may apply the Theorem 2.4 to obtain a relation expressing the chosen 2-cycle as a boundary of a 3-chain.

The method exploits the fact that the map ∂ of Definition 2.1 and Lemma 2.2 is well-defined on F_G whereas the linearisation operation W of Definition 2.3 is not. Let us consider an example.

Suppose we have a relation in F_G of the form

$$[R,u](s_as_bs_{ab}^{-1})[R',u'](s_cs_ds_{cd}^{-1}) = [R'',u''][R''',u''']$$

where $[R,u] = RuR^{-1}u^{-1}$, R,R',R'',R''' are relators and u,u',u'',u''' are words. Any relator may be expressed as a product of terms of the form $s_xs_ys_{xy}^{-1}$. For example, if $abc = e$ then $R = s_as_bs_c$ may be written

$$s_as_bs_c = (s_as_bs_{ab}^{-1})(s_{ab}s_cs_{abc}^{-1})(s_es_es_e^{-1}).$$

Therefore both sides of the above relation may be written as words to which we may apply W of Definition 2.3.

Appplying Theorem 2.4 to each side of the relation and cancelling the common value of the free derivatives, we obtain the boundary relation

$$(1\otimes_{\mathbb{Z}[G]} d)(W([R,u](s_as_bs_{ab}^{-1})[R',u'](s_cs_ds_{cd}^{-1})) - W([R'',u''][R''',u''']))$$

$$= 1\otimes_{\mathbb{Z}[G]} [a|b] + 1\otimes_{\mathbb{Z}[G]} [c|d]$$

in $\mathbb{Z}\otimes_{\mathbb{Z}[G]} \underline{B}_2G$.

Let us consider some simple examples of short boundary relations which we shall use later.

Example 2.7. Let a,b,c be elements of G such that $[a,c] = [b,c] = e$. Consider the identity

$$e = [s_a,[s_b,s_c]][s_b,s_c][s_a,s_c][s_c,s_as_c].$$

Using the given relations one finds that this relation may be rewritten in the form

$$\begin{aligned} e \;=\; & s_a(s_bs_cs_{bc}^{-1})s_a^{-1}s_a(s_cs_bs_{bc}^{-1})^{-1}s_a^{-1}(s_as_cs_{ac}^{-1})(s_cs_as_{ac}^{-1})^{-1} \\ & (s_cs_{ab}s_{cab}^{-1})(s_{ab}s_cs_{cab}^{-1})^{-1}(s_{ab}s_cs_{ab}^{-1})(s_as_bs_{ab}^{-1})(s_{ab}s_cs_{ab}^{-1})^{-1}(s_as_bs_{ab}^{-1})^{-1}. \end{aligned}$$

Let w denote the right side of this relation then

$$\begin{aligned} W(w) \;&= 1\otimes_{\mathbb{Z}[G]} [a|b|c] - 1\otimes_{\mathbb{Z}[G]} [a|c|b)] \\ &+1\otimes_{\mathbb{Z}[G]} [e|a|c] - 1\otimes_{\mathbb{Z}[G]} [e|c|a] \\ &+1\otimes_{\mathbb{Z}[G]} [e|c|ab] - 1\otimes_{\mathbb{Z}[G]} [e|ab|c] \\ &+1\otimes_{\mathbb{Z}[G]} [c|a|b] - 1\otimes_{\mathbb{Z}[G]} [e|a|b]. \end{aligned}$$

If we write

$$\{a,b\} = 1\otimes_{\mathbb{Z}[G]} [a|b] - 1\otimes_{\mathbb{Z}[G]} [b|a]$$

the Theorem 2.4 applied to this relation yields

$$(1\otimes_{\mathbb{Z}[G]} d)(W(w)) = \{a,c\} + \{b,c\} - \{ab,c\}.$$

When $G = GL(A)$ this boundary relation is related to bilinearity of the Steinberg symbol ([14] p.64).

In fact this boundary relation simplifies by the subtraction of a 3-cycle to leave

$$(1 \otimes_{\mathbb{Z}[G]} d)(1 \otimes_{\mathbb{Z}[G]} [a|b|c] - 1 \otimes_{\mathbb{Z}[G]} [a|c|b]) + 1 \otimes_{\mathbb{Z}[G]} [c|a|b])$$

$$= \{a, c\} + \{b, c\} - \{ab, c\}.$$

Example 2.8. In particular, if $a^n = e = [a, b]$ we may add n of the relations of Example 2.7 to obtain

$$(1 \otimes_{\mathbb{Z}[G]} d)(1 \otimes_{\mathbb{Z}[G]} [e|e|b] + 1 \otimes_{\mathbb{Z}[G]} [b|e|e])$$

$$+(1 \otimes_{\mathbb{Z}[G]} d)(\textstyle\sum_{r=1}^{n-1} (1 \otimes_{\mathbb{Z}[G]} [a^r|a|b] - 1 \otimes_{\mathbb{Z}[G]} [a^r|b|a] + 1 \otimes_{\mathbb{Z}[G]} [b|a^r|a])$$

$$= n\{a, b\}.$$

Example 2.9. Let $a, b, w \in G$ satisfy the relations $waw^{-1} = b, wbw^{-1} = a, ab = ba$. In F_G one has a relation

$$[s_w, [s_a, s_b]](s_a s_b s_{ab}^{-1})(s_b s_a s_{ba}^{-1})^{-1}$$

$$= [(s_w s_a s_w^{-1} s_b^{-1}), s_b(s_w s_b s_w^{-1} s_a^{-1}) s_a s_b^{-1}](s_b s_a s_{ba}^{-1})(s_a s_b s_{ab}^{-1})^{-1}[s_a s_b s_a^{-1}, (s_w s_b s_w^{-1} s_a^{-1})].$$

Theorem 2.4 is applicable to this relation and if we denote the left and right sides of the relation by X and Y respectively we obtain

$$(1 \otimes_{\mathbb{Z}[G]} d)(W(X) - W(Y)) = 2\{a, b\}$$

in the notation of Example 2.7. As in the Example 2.7, this boundary relation simplifies by the subtraction of a 3-cycle to leave

$$(1 \otimes_{\mathbb{Z}[G]} d)(1 \otimes_{\mathbb{Z}[G]} [w|a|b] - 1 \otimes_{\mathbb{Z}[G]} [a|b|w] - 1 \otimes_{\mathbb{Z}[G]} [w|b|a]$$

$$+1 \otimes_{\mathbb{Z}[G]} [b|a|w] + 1 \otimes_{\mathbb{Z}[G]} [a|w|a] - 1 \otimes_{\mathbb{Z}[G]} [b|w|b])$$

$$= 2\{a, b\}.$$

In this example we have particularly in mind the case when $G = GL_3(A)$ and

$$a = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; b = \begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & u^{-1} \end{pmatrix}, \; w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & v \\ 0 & -v^{-1} & 0 \end{pmatrix}$$

for a suitable choice of units u, v.

Example 2.10. *A short 3-cycle from the Steinberg symbol* $\{-u, u\}$

Suppose that F is a field and that $u \in F$ satisfies $u^n = 1$. In $G = SL_3F$ let a, b be as in Example 2.9 then we have a boundary relation of the form

$$(1 \otimes_{\mathbb{Z}[SL_3F]} d)(Z_1) = 2\{a, b\}$$

and from Example 2.8 another of the form

$$(1 \otimes_{\mathbb{Z}[SL_3F]} d)(Z_2) = n\{a, b\}$$

so that

$$nZ_1 - 2Z_2 \in \mathbb{Z} \otimes_{SL_3F} \underline{B}_3 SL_3F$$

is a 3-cycle which defines a, possibly trivial, homology class

$$[nZ_1 - 2Z_2] \in H_3(SL_3F; \mathbb{Z}).$$

This homology class has been constructed from two reasons for the cycle $2n\{a, b\}$ to be a boundary by taking their difference. This procedure may be considered as taking the difference of two reasons for $2n$ times the Steinberg symbol $\{-u, u\} \in K_2(A)$ ([14] p.64) to vanish. Explicitly the connection is as follows. The Hurewicz homomorphism gives an isomorphism of the form $K_2(F) \cong H_2(SLF; \mathbb{Z}) \cong H_3(SL_3F; \mathbb{Z})$ under which $\{-u, u\}$ corresponds to the homology class of the 2-cycle $\{ac, b\}$ where

$$c = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

However, by Example 2.7, this homology class is equal to the sum of the classes of $\{a, b\}$ and of $\{c, b\}$. On the other hand, by Example 2.7 again, the homology class of $2\{c, b\}$ equals that of $\{c^2, b\} = \{e, b\}$ which is trivial, again by Example 2.7.

In fact, inspection of the matrices in the 3-chains Z_1 and Z_2 shows that the homology class $[nZ_1 - 2Z_2]$ originates in the homology of a group whose order divides $4n^2$ so that it has finite order. In particular, if F were a number field, this class would be in the kernel of the Borel regulator of [3].

3. Algorithms producing 3-cycles for $G = SL_3A$

3.1. *The strategy*

In Example 2.10 we gave an example, based on applications of the Fox free derivative, of (possibly trivial) 3-dimensional torsion homology classes derived from the reasons underlying two relations in $K_2(A)$ of the form $2\{-u, u\} = 0 = n\{-u, u\}$. Specifically the "reasons" were skew symmetry and bilinearity of the Steinberg symbol, which implies that the first identity is true for any unit u, and for bilinearity combined with the fact that $u^n = 1$ implies the second.

We would like to use the strategy behind Example 2.10 to construct explicit 3-cycles which represent torsion free homology classes in $H_3(SL_NA; \mathbb{Z})$. There are two universal relations involving Steinberg symbols $\{-u, u\} = 0$ and $\{1 - u, u\} = 0$ ([14] Lemma 9.8) of which the first holds for all units u while for the second both u and $1 - u$ must be units.

Our approach can be used either (i) in the case when $u^n = 1$ and $1 - u$ is a unit (using the reasons behind the relations $\{1 - u, u\} = 0$ and $n\{1 - u, u\} = \{1 - u, u^n\} = 0$) or (ii) when u is any unit (using the reasons behind the relations $\{-u, u\} = 0$ and $2\{-u, u\} = \{(-1)^2, u\} + 2\{u, u\} = 0$).

We shall concentrate on the second of these cases because it is applicable to a unit in any commutative ring with a 1. The method would be the same in both cases. Let StA denote the infinite Steinberg group of A ([14] §5) and let EA denote the subgroup of $SL_\infty A$ generated by elementary matrices and let $\phi : StA \longrightarrow EA$ denote the homomorphism which given by $\phi(x_{ij}^{\lambda}) = e_{ij}^{\lambda}$, the elementary matrix ([14] p.40).

In ([14] pp.71-75) one finds a proof (due to Robert Steinberg), consisting of manipulations in StA, of each of the relations $\{-u, u\} = 0$ and $\{1-u, u\} = 0$. Below we are going to describe a computer-assisted algorithm which produces from each of these group-theoretic proofs a relation in F_{StA} to which we may apply Theorem 2.4 to write the 2-cycle representing the Steinberg symbol in $H_2(E_2A; \mathbb{Z}) \cong K_2(A)$ as a boundary. As described below, this boundary relation is very long. Nonetheless it can be mapped via ϕ to the bar resolution for EA and a multiple subtracted from one of the boundary relations from Examples 2.7-2.9 as in Example 2.10 to produce a 3-cycle and hence an element of $H_3(EA; \mathbb{Z})$.

It is important to notice that up to this point our algorithm could be performed universally – that is, for the Laurent polynomial ring $A = \mathbb{Z}[u^{\pm 1}]$ – but this would only produce a torsion element represented by a very long 3-cycle! This is because

$$K_3(\mathbb{Z}[u^{\pm 1}]) \cong \mathbb{Z}/48 \oplus \mathbb{Z}/2$$

and

$$0 = K_3(\mathbb{Z}[u^{\pm 1}]) \otimes \mathbb{Q} \cong H_3(EA; \mathbb{Q}).$$

However, in the process of shortening this universal 3-cycle one uses relations between its individual terms which depend on the structure of the ring A. This is similar to the spirit of the construction of $K_2(A)$ which measures the non-universal relations between elementary matrices which come from the structure of the ring A.

3.2. *Relations in F_{StA}*

We shall now describe how to carry out the strategy of §3.1 starting with the relation $\{u, -u\} = 0$ which takes place in $K_2(A) \subset StA$. In the Steinberg group this relation is more appropriately written as $\{u, -u\} = [h_{13}^{-u}, h_{12}^{u}]^{-1} = e$. As usual, we write x_{ij}^{λ} for the generators of the Steinberg group [14]. Recall the Steinberg relations

$$\begin{aligned} x_{ij}^{u} x_{ij}^{v} (x_{ij}^{u+v})^{-1} &= e \\ x_{ij}^{u} x_{jk}^{v} (x_{ij}^{u})^{-1} (x_{jk}^{v})^{-1} (x_{ik}^{uv})^{-1} &= e, \text{ for } i \neq k \\ x_{ij}^{u} x_{kl}^{v} (x_{ij}^{u})^{-1} (x_{kl}^{v})^{-1} &= e, \text{ for } i \neq l, j \neq k. \end{aligned}$$

All the steps required to prove $\{u, -u\} = e$ are described in ([14] pp. 71-75). Here we shall consider expanded versions of the working. Recall that $h_{ij}^{u} = w_{ij}^{u} w_{ij}^{-1}$ and $w_{ij}^{u} = x_{ij}^{u} x_{ji}^{-u^{-1}} x_{ij}^{u}$. Steinberg's argument to show that $\{u, -u\} =$

$[h_{13}^{-u}, h_{12}^{u}]^{-1} = e$ is given by

$$\begin{aligned}
& [h_{13}^{-u}, h_{12}^{u}] \\
= \;& h_{13}^{-u} h_{12}^{u} (h_{13}^{-u})^{-1} (h_{12}^{u})^{-1} \\
= \;& h_{13}^{-u} (w_{12}^{u} w_{12}^{-1}) (h_{13}^{-u})^{-1} (h_{12}^{u})^{-1} \\
= \;& (h_{13}^{-u} w_{12}^{u} (h_{13}^{-u})^{-1}) (h_{13}^{-u} w_{12}^{-1} (h_{13}^{-u})^{-1}) (h_{12}^{u})^{-1} \\
= \;& w_{12}^{-u^2} w_{12}^{u} (h_{12}^{u})^{-1} \\
= \;& w_{12}^{-u^2} w_{12}^{u} (w_{12}^{u} w_{12}^{-1})^{-1} \\
= \;& w_{12}^{-u^2} w_{12}^{u} (w_{12}^{-1})^{-1} (w_{12}^{u})^{-1} \\
= \;& w_{12}^{-u^2} (w_{12}^{u} w_{12}^{-1} w_{12}^{-u})^{-1} \\
= \;& w_{12}^{-u^2} (w_{12}^{-u^2})^{-1} \\
= \;& 1
\end{aligned}$$

On the above working, we need to explain explicitly why $h_{13}^{-u} w_{12}^{\lambda} (h_{13}^{-u})^{-1} = w_{12}^{-u\lambda}$ for $\lambda = u$ or -1 and why $w_{12}^{u} w_{12}^{-1} w_{12}^{-u} = w_{12}^{-u^2}$. We do the latter first.

$$\begin{aligned}
& w_{12}^{u} w_{12}^{-1} w_{12}^{-u} \\
= \;& w_{12}^{u} x_{12}^{-1} x_{21}^{1} x_{12}^{-1} w_{12}^{-u} \\
= \;& (w_{12}^{u} x_{12}^{-1} w_{12}^{-u})(w_{12}^{u} x_{21}^{1} w_{12}^{-u})(w_{12}^{u} x_{12}^{-1} w_{12}^{-u}) \\
= \;& (x_{21}^{u^{-2}})(x_{12}^{-u^2})(x_{21}^{u^{-2}}) \\
= \;& w_{21}^{u^{-2}} \\
= \;& w_{21}^{u^{-2}} w_{21}^{u^{-2}} w_{21}^{-u^{-2}} \\
= \;& w_{21}^{u^{-2}} (x_{21}^{u^{-2}})(x_{12}^{-u^2})(x_{21}^{u^{-2}}) w_{21}^{-u^{-2}} \\
= \;& (w_{21}^{u^{-2}} x_{21}^{u^{-2}} w_{21}^{-u^{-2}})(w_{21}^{u^{-2}} x_{12}^{-u^2} w_{21}^{-u^{-2}})(w_{21}^{u^{-2}} x_{21}^{u^{-2}} w_{21}^{-u^{-2}}) \\
= \;& (x_{12}^{-u^2})(x_{21}^{u^{-2}})(x_{12}^{-u^2}) \\
= \;& w_{12}^{-u^2}
\end{aligned}$$

Note that here we used $w_{ij}^{u} x_{ij}^{\lambda} w_{ij}^{-u} = x_{ji}^{-v\lambda v}$, where $v = u^{-1}$, and $w_{ij}^{u} x_{ji}^{\lambda} w_{ij}^{-u} = x_{ij}^{-u\lambda u}$. The proof of this in ([14] Lemma 9.2 (proof)) which we shall make explicit below.

The other promised relation is given by

$$\begin{aligned}
& h_{13}^{-u} w_{12}^{\lambda} (h_{13}^{-u})^{-1} \\
= \;& (w_{13}^{-u} w_{13}^{-1})(x_{12}^{\lambda} x_{21}^{-\lambda^{-1}} x_{12}^{\lambda})(w_{13}^{1} w_{13}^{u}) \\
= \;& w_{13}^{-u} (w_{13}^{-1} x_{12}^{\lambda} w_{13}^{1})(w_{13}^{-1} x_{21}^{-\lambda^{-1}} w_{13}^{1})(w_{13}^{-1} x_{12}^{\lambda} w_{13}^{1}) w_{13}^{u} \\
= \;& w_{13}^{-u} (x_{32}^{\lambda})(x_{23}^{-\lambda^{-1}})(x_{32}^{\lambda}) w_{13}^{u} \\
= \;& (w_{13}^{-u} x_{32}^{\lambda} w_{13}^{u})(w_{13}^{-u} x_{23}^{-\lambda^{-1}} w_{13}^{u})(w_{13}^{-u} x_{32}^{\lambda} w_{13}^{u}) \\
= \;& (x_{12}^{-u\lambda})(x_{21}^{(u\lambda)^{-1}})(x_{12}^{-u\lambda}) \\
= \;& w_{12}^{-u\lambda}
\end{aligned}$$

where here we have used more identities from ([14] Lemma 9.2 (proof)). We reproduce a few of these identities, which we shall make explicit below, for arbitrary u and λ with $i \neq j \neq k$ and $v = u^{-1}$,

$$\begin{aligned}
w_{ij}^u x_{ij}^\lambda w_{ij}^{-u} &= x_{ji}^{-v\lambda v} \\
w_{ij}^u x_{ji}^\lambda w_{ij}^{-u} &= x_{ij}^{-u\lambda u} \\
w_{ij}^u x_{ik}^\lambda w_{ij}^{-u} &= x_{jk}^{-v\lambda} \\
w_{ij}^u x_{ki}^\lambda w_{ij}^{-u} &= x_{kj}^{-u\lambda} \\
w_{ij}^u x_{kj}^\lambda w_{ij}^{-u} &= x_{ki}^{v\lambda} \\
w_{ij}^u x_{jk}^\lambda w_{ij}^{-u} &= x_{ik}^{u\lambda}
\end{aligned}$$

To make these relations explicit we start from the last

$$\begin{aligned}
& w_{ij}^u x_{jk}^\lambda w_{ij}^{-u} \\
= \ & x_{ij}^u x_{ji}^{-u^{-1}} (x_{ij}^u x_{jk}^\lambda x_{ij}^{-u}) x_{ji}^{u^{-1}} x_{ij}^{-u} \\
= \ & x_{ij}^u x_{ji}^{-u^{-1}} x_{ik}^{u\lambda} (x_{jk}^\lambda x_{ji}^{u^{-1}}) x_{ij}^{-u} \\
= \ & x_{ij}^u (x_{ji}^{-u^{-1}} x_{ik}^{u\lambda} x_{ji}^{u^{-1}}) x_{jk}^\lambda x_{ij}^{-u} \\
= \ & x_{ij}^u (x_{jk}^{-\lambda} x_{ik}^{u\lambda}) x_{jk}^\lambda x_{ij}^{-u} \\
= \ & x_{ij}^u x_{ik}^{u\lambda} (x_{jk}^{-\lambda} x_{jk}^\lambda) x_{ij}^{-u} \\
= \ & x_{ij}^u x_{ik}^{u\lambda} x_{ij}^{-u} \\
= \ & x_{ik}^{u\lambda}
\end{aligned}$$

$$\begin{aligned}
& w_{ij}^u x_{kj}^\lambda w_{ij}^{-u} \\
= \ & x_{ij}^u x_{ji}^{-u^{-1}} (x_{ij}^u x_{kj}^\lambda x_{ij}^{-u}) x_{ji}^{u^{-1}} x_{ij}^{-u} \\
= \ & x_{ij}^u x_{ji}^{-u^{-1}} x_{kj}^\lambda x_{ji}^{u^{-1}} x_{ij}^{-u} \\
= \ & x_{ij}^u x_{kj}^\lambda (x_{kj}^{-\lambda} x_{ji}^{-u^{-1}} x_{kj}^\lambda x_{ji}^{u^{-1}}) x_{ij}^{-u} \\
= \ & (x_{ij}^u x_{kj}^\lambda) x_{ki}^{\lambda u^{-1}} x_{ij}^{-u} \\
= \ & x_{kj}^\lambda x_{ij}^u x_{ki}^{\lambda u^{-1}} x_{ij}^{-u} \\
= \ & x_{kj}^\lambda x_{ki}^{\lambda u^{-1}} (x_{ki}^{-\lambda u^{-1}} x_{ij}^u x_{ki}^{\lambda u^{-1}} x_{ij}^{-u}) \\
= \ & x_{kj}^\lambda x_{ki}^{\lambda u^{-1}} x_{kj}^{-\lambda} \\
= \ & x_{ki}^{\lambda u^{-1}}
\end{aligned}$$

$$
\begin{aligned}
& w_{ij}^u x_{ki}^\lambda w_{ij}^{-u} \\
= \;& x_{ij}^u x_{ji}^{-u^{-1}} x_{ij}^u x_{ki}^\lambda x_{ij}^{-u} x_{ji}^{u^{-1}} x_{ij}^{-u} \\
= \;& x_{ij}^u x_{ji}^{-u^{-1}} x_{ki}^\lambda (x_{ki}^{-\lambda} x_{ij}^u x_{ki}^\lambda x_{ij}^{-u}) x_{ji}^{u^{-1}} x_{ij}^{-u} \\
= \;& x_{ij}^u (x_{ji}^{-u^{-1}} x_{ki}^\lambda) x_{kj}^{-\lambda u} x_{ji}^{u^{-1}} x_{ij}^{-u} \\
= \;& x_{ij}^u x_{ki}^\lambda x_{ji}^{-u^{-1}} x_{kj}^{-\lambda u} x_{ji}^{u^{-1}} x_{ij}^{-u} \\
= \;& x_{ij}^u x_{ki}^\lambda x_{kj}^{-\lambda u} (x_{kj}^{\lambda u} x_{ji}^{-u^{-1}} x_{kj}^{-\lambda u} x_{ji}^{u^{-1}}) x_{ij}^{-u} \\
= \;& (x_{ij}^u x_{ki}^\lambda x_{kj}^{-\lambda u}) x_{ki}^{-\lambda} x_{ij}^{-u} \\
= \;& x_{kj}^{-\lambda u} (x_{ij}^u x_{ki}^\lambda x_{ki}^{-\lambda} x_{ij}^{-u}) \\
= \;& x_{kj}^{-\lambda u}
\end{aligned}
$$

$$
\begin{aligned}
& w_{ij}^u x_{ik}^\lambda w_{ij}^{-u} \\
= \;& x_{ij}^u x_{ji}^{-u^{-1}} (x_{ij}^u x_{ik}^\lambda x_{ij}^{-u}) x_{ji}^{u^{-1}} x_{ij}^{-u} \\
= \;& x_{ij}^u (x_{ji}^{-u^{-1}} x_{ik}^\lambda x_{ji}^{u^{-1}}) x_{ij}^{-u} \\
= \;& x_{ij}^u x_{jk}^{-u^{-1}\lambda} (x_{ik}^\lambda x_{ij}^{-u}) \\
= \;& (x_{ij}^u x_{jk}^{-u^{-1}\lambda} x_{ij}^{-u}) x_{ik}^\lambda \\
= \;& (x_{ik}^{-\lambda} x_{jk}^{-u^{-1}\lambda}) x_{ik}^\lambda \\
= \;& x_{jk}^{-u^{-1}\lambda} (x_{ik}^{-\lambda} x_{ik}^\lambda) \\
= \;& x_{jk}^{-u^{-1}\lambda}
\end{aligned}
$$

$$
\begin{aligned}
& w_{ij}^u x_{ji}^\lambda w_{ij}^{-u} \\
= \;& w_{ij}^u x_{jk}^\lambda x_{ki}^1 x_{jk}^{-\lambda} x_{ki}^{-1} w_{ij}^{-u} \\
= \;& (w_{ij}^u x_{jk}^\lambda w_{ij}^{-u})(w_{ij}^u x_{ki}^1 w_{ij}^{-u})(w_{ij}^u x_{jk}^{-\lambda} w_{ij}^{-u})(w_{ij}^u x_{ki}^{-1} w_{ij}^{-u}) \\
= \;& x_{ik}^{u\lambda} x_{kj}^{-u} x_{ik}^{-u\lambda} x_{kj}^u \\
= \;& x_{ij}^{-u\lambda u}
\end{aligned}
$$

$$
\begin{aligned}
& w_{ij}^u x_{ij}^\lambda w_{ij}^{-u} \\
= \;& w_{ij}^u x_{ik}^\lambda x_{kj}^1 x_{ik}^{-\lambda} x_{kj}^{-1} w_{ij}^{-u} \\
= \;& (w_{ij}^u x_{ik}^\lambda w_{ij}^{-u})(w_{ij}^u x_{kj}^1 w_{ij}^{-u})(w_{ij}^u x_{ik}^{-\lambda} w_{ij}^{-u})(w_{ij}^u x_{kj}^{-1} w_{ij}^{-u}) \\
= \;& x_{jk}^{-u^{-1}\lambda} x_{ki}^{u^{-1}} x_{jk}^{u^{-1}\lambda} x_{ki}^{-u^{-1}} \\
= \;& x_{jk}^{-u^{-1}\lambda u^{-1}}
\end{aligned}
$$

Hopefully we have impressed upon the reader that the calculation is tedious. Unfortunately, it only gets more tedious and we will just have to omit more and more details as we proceed.

To use the Fox free derivative we have to be in the free group F_{StA} and to use Theorem 2.4 we have to write each side of the equation into words of a certain form. Our first step is to redo all the calculations above in the free group, but now keeping track of when we used the Steinberg relations.

Warning: To ease notation, we shall use the *same* letters to represent elements in the free group – that is, we shall denote by $g \in F_G$ the element denoted by s_g in Definition 2.1.

In F_{StA} we no longer have the Steinberg relations. As a first step, we want an equation of the form

$$[h_{13}^{-u}, h_{12}^{u}] = (w_1 R_1^{\pm 1} w_1^{-1})(w_2 R_2^{\pm 1} w_2^{-1}) \cdots$$

where the w_i are words and R_i are relators of three basic types, corresponding to the three Steinberg relations. In particular, we define three important relators as follows

$$\begin{aligned} S_{ij}^{u,v} &= (x_{ij}^{u+v})^{-1} x_{ij}^{u} x_{ij}^{v} \\ T_{ijk}^{u,v} &= x_{ij}^{u} x_{jk}^{v} (x_{ij}^{u})^{-1} (x_{jk}^{v})^{-1} (x_{ik}^{uv})^{-1}, \text{ for } i \neq k \\ U_{ijkl}^{u,v} &= x_{ij}^{u} x_{kl}^{v} (x_{ij}^{u})^{-1} (x_{kl}^{v})^{-1}, \text{ for } i \neq l, j \neq k. \end{aligned}$$

As a motivation we pause to remind ourselves of our ultimate aim and to explain how we get there. When we apply Theorem 2.4 we shall create 3-chains with certain boundaries which we can control. We can arrange that the boundary on the left hand side to become $\{A, B\}$ plus the Fox free derivative and that the right hand side has boundary equal to the Fox free derivative only. And since the Fox free derivative is well defined, we can subtract the LHS 3-chain from the RHS 3-chain to give us a boundary $\{A, B\}$. To control the boundary, we consider words in the free group of the form $[R, u]$, where R is a relator and u is any word. Such a word will create a 3-chain whose boundary is equal to the Fox free derivative only. We will supply details shortly, as we explain how this plan is possible in our application. As a start, we consider

$$w_{ij}^{u} x_{jk}^{\lambda} (w_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1}$$

We know this is the identity in the Steinberg group, so this is a relator in the free group and can be thus rewritten in the form

$$\prod_i (w_i R_i^{\pm 1} w_i^{-1})$$

The idea is simple in that we follow exactly what we did before when proving the identity but now everytime we use a Steinberg relation, we have to put in an S, T or U in the free group. We will work this out as an example.

$$
\begin{aligned}
& w_{ij}^{u} x_{jk}^{\lambda} (w_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \\
= \;& x_{ij}^{u} x_{ji}^{-u^{-1}} (x_{ij}^{u} x_{jk}^{\lambda} (x_{ij}^{u})^{-1}) (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \\
= \;& x_{ij}^{u} x_{ji}^{-u^{-1}} \left(x_{ij}^{u} x_{jk}^{\lambda} (x_{ij}^{u})^{-1} (x_{jk}^{\lambda})^{-1} (x_{ik}^{u\lambda})^{-1} x_{ik}^{u\lambda} x_{jk}^{\lambda} \right) (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \\
= \;& x_{ij}^{u} x_{ji}^{-u^{-1}} T_{ijk}^{u,\lambda} x_{ik}^{u\lambda} x_{jk}^{\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \\
= \;& x_{ij}^{u} x_{ji}^{-u^{-1}} T_{ijk}^{u,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} \\
& x_{ij}^{u} x_{ji}^{-u^{-1}} x_{ik}^{u\lambda} x_{jk}^{\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \\
= \;& x_{ij}^{u} x_{ji}^{-u^{-1}} T_{ijk}^{u,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} \\
& x_{ij}^{u} x_{ji}^{-u^{-1}} x_{ik}^{u\lambda} (x_{ji}^{-u^{-1}})^{-1} x_{ji}^{-u^{-1}} x_{jk}^{\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \\
= \;& x_{ij}^{u} x_{ji}^{-u^{-1}} T_{ijk}^{u,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} \\
& x_{ij}^{u} T_{jik}^{-u^{-1},u\lambda} x_{jk}^{-\lambda} x_{ik}^{u\lambda} x_{ji}^{-u^{-1}} x_{jk}^{\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \\
= \;& x_{ij}^{u} x_{ji}^{-u^{-1}} T_{ijk}^{u,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} \\
& x_{ij}^{u} T_{jik}^{-u^{-1},u\lambda} U_{jkik}^{-\lambda,u\lambda} x_{ik}^{u\lambda} x_{jk}^{-\lambda} x_{ji}^{-u^{-1}} x_{jk}^{\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \\
= \;& \left(x_{ij}^{u} x_{ji}^{-u^{-1}} T_{ijk}^{u,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} \right) \left(x_{ij}^{u} T_{jik}^{-u^{-1},u\lambda} U_{jkik}^{-\lambda,u\lambda} (x_{ij}^{u})^{-1} \right) \\
& x_{ij}^{u} x_{ik}^{u\lambda} U_{jkji}^{-\lambda,-u^{-1}} x_{ji}^{-u^{-1}} x_{jk}^{-\lambda} x_{jk}^{\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \\
= \;& \left(x_{ij}^{u} x_{ji}^{-u^{-1}} T_{ijk}^{u,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} \right) \left(x_{ij}^{u} T_{jik}^{-u^{-1},u\lambda} U_{jkik}^{-\lambda,u\lambda} (x_{ij}^{u})^{-1} \right) \\
& x_{ij}^{u} x_{ik}^{u\lambda} U_{jkji}^{-\lambda,-u^{-1}} x_{ji}^{-u^{-1}} x_{jk}^{0} S_{jk}^{-\lambda,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \\
= \;& \left(x_{ij}^{u} x_{ji}^{-u^{-1}} T_{ijk}^{u,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} \right) \left(x_{ij}^{u} T_{jik}^{-u^{-1},u\lambda} U_{jkik}^{-\lambda,u\lambda} (x_{ij}^{u})^{-1} \right) \\
& x_{ij}^{u} x_{ik}^{u\lambda} U_{jkji}^{-\lambda,-u^{-1}} x_{ji}^{-u^{-1}} S_{jk}^{0,0} S_{jk}^{-\lambda,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \\
= \;& \left(x_{ij}^{u} x_{ji}^{-u^{-1}} T_{ijk}^{u,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} \right) \left(x_{ij}^{u} T_{jik}^{-u^{-1},u\lambda} U_{jkik}^{-\lambda,u\lambda} (x_{ij}^{u})^{-1} \right) \\
& U_{ijik}^{u,u\lambda} x_{ik}^{u\lambda} x_{ij}^{u} U_{jkji}^{-\lambda,-u^{-1}} x_{ji}^{-u^{-1}} S_{jk}^{0,0} S_{jk}^{-\lambda,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \\
= \;& \left(x_{ij}^{u} x_{ji}^{-u^{-1}} T_{ijk}^{u,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} \right) \left(x_{ij}^{u} T_{jik}^{-u^{-1},u\lambda} U_{jkik}^{-\lambda,u\lambda} (x_{ij}^{u})^{-1} \right) U_{ijik}^{u,u\lambda} \\
& \left(x_{ik}^{u\lambda} x_{ij}^{u} U_{jkji}^{-\lambda,-u^{-1}} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \right) \left(x_{ik}^{u\lambda} x_{ij}^{u} x_{ji}^{-u^{-1}} S_{jk}^{0,0} S_{jk}^{-\lambda,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \right) \\
= \;& \left(x_{ij}^{u} x_{ji}^{-u^{-1}} T_{ijk}^{u,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} \right) \left(x_{ij}^{u} T_{jik}^{-u^{-1},u\lambda} (x_{ij}^{u})^{-1} \right) \left(x_{ij}^{u} U_{jkik}^{-\lambda,u\lambda} (x_{ij}^{u})^{-1} \right) \\
& \left(U_{ijik}^{u,u\lambda} \right) \left(x_{ik}^{u\lambda} x_{ij}^{u} U_{jkji}^{-\lambda,-u^{-1}} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \right) \left(x_{ik}^{u\lambda} x_{ij}^{u} x_{ji}^{-u^{-1}} S_{jk}^{0,0} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \right) \\
& \left(x_{ik}^{u\lambda} x_{ij}^{u} x_{ji}^{-u^{-1}} S_{jk}^{-\lambda,\lambda} (x_{ji}^{-u^{-1}})^{-1} (x_{ij}^{u})^{-1} (x_{ik}^{u\lambda})^{-1} \right)
\end{aligned}
$$

The steps involved in such a calculation are repetitive and extremely well suited for a computer to do. In fact, we programmed a computer to do such a calculation, which gave us

$$[h_{13}^{-u}, h_{12}^{u}] = \prod_{i}^{m}(w_i R_i^{\pm 1} w_i^{-1})$$

with $m = 392$. Printing out the right hand side of this equation would take about 40 pages (but an electronic version of this exists). One notes that none of the relations used required us to include an extra index, so only 1, 2, and 3 were used as indices. What we do next is we replace every occurrence of x_{ij}^{λ} by $[x_{i4}^{1}, x_{4j}^{\lambda}]$. Or, just to emphasise,

$$x_{ij}^{\lambda} \to x_{i4}^{1} x_{4j}^{\lambda} (x_{i4}^{1})^{-1} (x_{4j}^{\lambda})^{-1}$$

Note that the resulting equation is still an equation because while creating the RHS we essentially put in terms, albeit many of them, of the form $x^{-1}x$ and thus it remains a free group identity. Notice, for example, that the left hand side would now have, after writing it out fully, $4 \times 2 \times 3 \times 4 = 96$ terms. Also, while this is an identity in the free group over the Steinberg group, we treat it now as a free group over E, the group of elementary matrices. We use the same notation however. In other words, by x_{ij}^{u}, we now mean $s_{\phi(x_{ij}^{u})}$. The resulting equation takes the following form

$$[w_B, w_A] = \prod_{i}^{m}(\bar{w}_i \bar{R}_i^{\pm 1} \bar{w}_i^{-1})$$

where $w_B = (x_{14}^{+1})\ (x_{43}^{-u})\ (x_{14}^{+1})^{-1}\ (x_{43}^{-u})^{-1}\ (x_{34}^{+1})\ (x_{41}^{+v})\ (x_{34}^{+1})^{-1}\ (x_{41}^{+v})^{-1}\ (x_{14}^{+1})$ $(x_{43}^{-u})\ (x_{14}^{+1})^{-1}\ (x_{43}^{-u})^{-1}\ (x_{14}^{+1})\ (x_{43}^{-1})\ (x_{14}^{+1})^{-1}\ (x_{43}^{-1})^{-1}\ (x_{34}^{+1})\ (x_{41}^{+1})\ (x_{34}^{+1})^{-1}\ (x_{41}^{+1})^{-1}$ $(x_{14}^{+1})\ (x_{43}^{-1})\ (x_{14}^{+1})^{-1}\ (x_{43}^{-1})^{-1}$

$w_A = (x_{14}^{+1})\ (x_{42}^{+u})\ (x_{14}^{+1})^{-1}\ (x_{42}^{+u})^{-1}\ (x_{24}^{+1})\ (x_{41}^{-v})\ (x_{24}^{+1})^{-1}\ (x_{41}^{-v})^{-1}\ (x_{14}^{+1})\ (x_{42}^{+u})$ $(x_{14}^{+1})^{-1}\ (x_{42}^{+u})^{-1}\ (x_{14}^{+1})\ (x_{42}^{-1})\ (x_{14}^{+1})^{-1}\ (x_{42}^{-1})^{-1}\ (x_{24}^{+1})\ (x_{41}^{+1})\ (x_{24}^{+1})^{-1}\ (x_{41}^{+1})^{-1}\ (x_{14}^{+1})$ $(x_{42}^{-1})\ (x_{14}^{+1})^{-1}\ (x_{42}^{-1})^{-1}$
where $v = u^{-1}$.

Since we are now in F_{EA} rather than F_{StA}, the relators remain relators. The point of our last substitution is that now we can, using results of ([14] pp.48-51), derive expressions for each $\bar{R}_i$ and write them into the form

$$[K_1, v_1]^{\pm 1}[K_2, v_2]^{\pm 1}[K_3, v_3]^{\pm 1} \cdots$$

where the K_i are relators and the v_i are words. The discussion of [14] explains how it is done and we repeat the steps. However, as above, we record every time a relation is used. It is again tedious and we reproduce here only the working for $U_{ijkl}^{\lambda,u}$, which after the substitution has become

$$\begin{aligned}
& [[x_{i4}^1, x_{4j}^\lambda], [x_{k4}^1, x_{4l}^u]] \\
= \; & [T_{i4j}^{1,\lambda} x_{ij}^\lambda, [x_{k4}^1, x_{4l}^u]] \\
= \; & [T_{i4j}^{1,\lambda}, x_{ij}^\lambda [x_{k4}^1, x_{4l}^u](x_{ij}^\lambda)^{-1}][x_{ij}^\lambda, [x_{k4}^1, x_{4l}^u]] \\
= \; & [T_{i4j}^{1,\lambda}, x_{ij}^\lambda [x_{k4}^1, x_{4l}^u](x_{ij}^\lambda)^{-1}] x_{ij}^\lambda [x_{k4}^1, x_{4l}^u](x_{ij}^\lambda)^{-1}[x_{k4}^1, x_{4l}^u]^{-1} \\
= \; & [T_{i4j}^{1,\lambda}, x_{ij}^\lambda [x_{k4}^1, x_{4l}^u](x_{ij}^\lambda)^{-1}][x_{ij}^\lambda x_{k4}^1 (x_{ij}^\lambda)^{-1}, x_{ij}^\lambda x_{4l}^u (x_{ij}^\lambda)^{-1}][x_{k4}^1, x_{4l}^u]^{-1} \\
= \; & [T_{i4j}^{1,\lambda}, x_{ij}^\lambda [x_{k4}^1, x_{4l}^u](x_{ij}^\lambda)^{-1}][U_{ijk4}^{\lambda,1} x_{k4}^1, U_{ij4l}^{\lambda,u} x_{4l}^u][x_{k4}^1, x_{4l}^u]^{-1} \\
= \; & [T_{i4j}^{1,\lambda}, x_{ij}^\lambda [x_{k4}^1, x_{4l}^u](x_{ij}^\lambda)^{-1}][U_{ijk4}^{\lambda,1}, x_{k4}^1 U_{ij4l}^{\lambda,u} x_{4l}^u (x_{k4}^1)^{-1}][x_{k4}^1, U_{ij4l}^{\lambda,u} x_{4l}^u][x_{k4}^1, x_{4l}^u]^{-1} \\
= \; & [T_{i4j}^{1,\lambda}, x_{ij}^\lambda [x_{k4}^1, x_{4l}^u](x_{ij}^\lambda)^{-1}][U_{ijk4}^{\lambda,1}, x_{k4}^1 U_{ij4l}^{\lambda,u} x_{4l}^u (x_{k4}^1)^{-1}] \\
& [x_{k4}^1 U_{ij4l}^{\lambda,u} (x_{k4}^1)^{-1}, x_{k4}^1 x_{4l}^u (x_{k4}^1)^{-1}(x_{4l}^u)^{-1}(x_{k4}^1)^{-1}].
\end{aligned}$$

We see that it takes only 3 square brackets to rewrite U. On the other hand, it takes 4 and 18 square brackets to rewrite S and T respectively. We note here that it was during the rewriting of T that we needed to introduce index 5. This means that in later steps we have to consider 5×5 matrices. If we could find a way to rewrite T without going to index 5, then we would only need 4 by 4 matrices and this would be much better computationally. The steps undertaken to rewrite S, T, and U crucially required ingenious identities due to Robert Steinberg ([14] pp. 71-75). For example, for T we used

$$[u, \big[v, w\big]] \; [v, w] \; [u, w] \; [w, \big[u, v\big]] \; [u, v] \; [w, v] \; [v, \big[w, u\big]] \; [w, u] \; [v, u] = 1.$$

Our next step is to substitute the $\bar{R}_i$ into this new format. In particular, we have

$$\bar{w}_i \bar{R}_i \bar{w}_i^{-1} \to \bar{w}_i \left(\prod_j^p [K_j, v_j] \right) \bar{w}_i^{-1} = \prod_j^p [\bar{w}_i K_j \bar{w}_i^{-1}, \bar{w}_i^{-1} v_j \bar{w}_i^{-1}]$$

where p is 3,4 or 18, and the K_j and v_j depend on the $\bar{R}_i$. Notice that $\bar{w}_i K_j \bar{w}_i^{-1}$ is still a relator. Again, the computer does this substitutions for us, and we end up with

$$[w_B, w_A] = \prod_j^m [R'_j, u_j]$$

where $m = 2392$, R'_j are all relators and u_j are words in the free group. This is in good form to apply the lemma. Just for motivation, take for example, for some fixed j, $R'_j = s_x s_y s_{xy}^{-1}$ – reverting to the notation of Definition 2.1 from now on. Then

$$\begin{aligned}
[R'_j, u_j] &= R'_j u_j (R'_j)^{-1} u_j^{-1} \\
&= s_x s_y s_{xy}^{-1} \left(u_j s_x s_y s_{xy}^{-1} u_j^{-1} \right)^{-1}
\end{aligned}$$

Applying Theorem 2.4, we get

$$\begin{aligned}
(1 \otimes_{\mathbb{Z}[EA]} d)(W([R'_j, u_j])) &= 1 \otimes_{\mathbb{Z}[EA]} [x|y] - 1 \otimes_{\mathbb{Z}[EA]} [x|y] - \partial([R'_j, u_j]) \\
&= -\partial([R'_j, u_j])
\end{aligned}$$

We shall now show that each R'_j can be written out in the form $\prod(s_{x_i}s_{y_i}s_{x_iy_i}^{-1})^{\pm 1}$ and thus we deduce that the 3-chain created by lifting the right hand side will have boundary equal to the Fox free derivative only. Suppose that R'_j is a word of the form $s_a s_b s_c s_d \cdots$. We can then rewrite this as $s_a s_b s_{ab}^{-1} s_{ab} s_c s_d \cdots$. We know R'_j is a relator, so $abcd \cdots = e$. Thus at the end of the word, we have $\cdots s_x s_c = \cdots s_x s_c s_{xc}^{-1} s_{xc} = \cdots (s_x s_c s_{xc}^{-1})(s_{xc} s_e s_e^{-1})$. In general, we have the following rules.

$$\begin{array}{rcl} \cdots s_a s_b \cdots & = & \cdots (s_a s_b s_{ab}^{-1}) s_{ab} \cdots \\ \cdots s_a s_b^{-1} \cdots & = & \cdots (s_{ab^{-1}} s_b s_a)^{-1} s_{ab^{-1}} \cdots \end{array}$$

In the case that the relator begins with an inverse, say $s_a^{-1} \cdots$, we rewrite this as $(s_e s_e s_e^{-1})^{-1}(s_{a^{-1}} s_a s_e^{-1})^{-1} s_{a^{-1}}$. Interestingly, this case never happens.

Finally we apply Theorem 2.4 to the right hand side of the equation. What this means is that we just create the 3-chain. We do not need to worry about the boundary or the Fox free derivative. The number of elements in the 3-chain constructed is equal to the total number of letters in all the 2392 R'_j's. We naturally got the computer to do the calculation and there are in total 11123 distinct 3-tuples using at most 3691 matrices. Let us call this 3-chain X_{RHS}.

We now focus our attention on the relatively much smaller left hand side. Being much smaller, we do the following by inspection. We have

$$[w_B, w_A] = [K_B, v_B](s_B s_A s_{AB}^{-1})(s_A s_B s_{AB}^{-1})^{-1}[K_A, v_A]^{-1}$$

where we have

$$A = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} -u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -u^{-1} \end{pmatrix},$$

$K_A = w_A s_A^{-1}$, $K_B := w_B s_B^{-1}$, $v_A := s_A s_B s_A^{-1}$ and $v_B = s_B w_A s_B^{-1}$. Note K_A and K_B are relators. We then apply Theorem 2.4 to this as well, and the resulting 3-chain has boundary equal to the Fox free derivative plus $\{B, A\}$ in the notation of Example 2.7. Let us call this 3-chain X_{LHS}.

The next step is to consider $X = X_{RHS} - X_{LHS}$. As in Examples 2.7-2.10, the Fox free derivative cancels and we deduce that X has boundary $\{A, B\}$. After some simplification (that is, by removing 3-cycles, which happen to be boundaries) X reduces to a 3-chain, still denoted by X, with 6844 distinct 3-tuples, featuring 3265 distinct matrices. This 3-chain can be further reduced using the structure of the ring A to effect some cancellations between the individual terms in X. We continue to denote the result by X.

Using Examples 2.7, 2.9 and 2.10 we can find a short 3-chain Y whose boundary is the 2-cycle is $2\{A, B\}$ and therefore obtain a homology class

$$[Y - 2X] \in H_3(EA; \mathbb{Z}).$$

This algorithm works for any unit u in A.

If instead $u^n = 1$ then we have the 3-chain Z of Example 2.8 which has boundary $n\{A, B\}$. In this case we may also create the homoology class

$$[Z - nX] \in H_3(EA; \mathbb{Z}).$$

4. The Fox free derivative and higher dimensional homology classes

In this section we shall record how one uses results of ([4], [5], [21]; see also [20], [22] and [16]) to promote the construction of 3-dimensional K-theory classes to higher dimensions.

In ([21] §3) a Karoubi-Villamayor type of K-theory is defined (with coefficients in Λ if necessary). It is denoted by $KV_*(A;\Lambda)$ where the dimension index ranges through the integers. There is a natural homomorphism

$$K_*(A;\Lambda) \longrightarrow KV_*(A;\Lambda)$$

from Quillen K-theory to Karoubi-Villamayor K-theory. When A is a regular ring this map is an isomorphism

$$K_n(A) \xrightarrow{\cong} KV_n(A)$$

for all n. These groups are zero for $n < 0$ ([4] p.137). This can be proved via the Gersten-Anderson spectral sequence ([21] p.404). When the integer (not necessarily prime) q is invertible in A then the map is an isomorphism ([21] Theorem 1.2) for all $n \geq 1$

$$K_n(A;\mathbb{Z}/q) \xrightarrow{\cong} KV_n(A;\mathbb{Z}/q).$$

For $n \geq 2$ let $\partial\Delta^n_A$ denote the coordinate ring of the boundary of the $(n+1)$-simplex over A

$$\partial\Delta^n_A = A[x_0,\ldots,x_n]/(x_0x_1\ldots x_n(1-\sum_{i=0}^{n} x_i)).$$

Then, by ([4] §§2.4, 4.8, 5.1 and 5.2; [5]), there is an isomorphism

$$KV_i(\partial\Delta^n_A) \cong KV_i(A) \oplus KV_{i+n}(A)$$

for all integers i. Therefore, if A is regular, we have in particular an isomorphism

$$KV_3(\partial\Delta^n_A) \cong K_3(A) \oplus K_{3+n}(A)$$

and if the integer q is invertible in A this simplifies to

$$K_3(\partial\Delta^n_A;\mathbb{Z}/q) \cong K_3(A;\mathbb{Z}/q) \oplus K_{3+n}(A;\mathbb{Z}/q).$$

We shall apply this discussion to the case when A is equal to a number field F or a localisation of its ring of algebraic integers $\mathcal{O}_F$. Suppose that $n \geq 2$ so that $K_{3+n}(\mathcal{O}_F) \cong K_{3+n}(F)$ is a finitely generated abelian group which is isomorphic to K_{3+n} of any localisation of $\mathcal{O}_F$. Therefore it suffices to consider the case $A = F$. We have isomorphisms of the form

$$K_3(\partial\Delta^n_F) \otimes \mathbb{Q} \cong H_3(\partial\Delta^n_F;\mathbb{Q})$$

and

$$KV_3(\partial\Delta^n_F) \otimes \mathbb{Q} \cong K_3(F) \otimes \mathbb{Q} \oplus K_{3+n}(F) \otimes \mathbb{Q}$$

and a natural homomorphism

$$K_3(\partial\Delta^n_F) \otimes \mathbb{Q} \longrightarrow KV_3(\partial\Delta^n_F) \otimes \mathbb{Q}.$$

Therefore if $x \in K_{3+n}(F)$ has infinite order it will give rise to an element of infinite order in $KV_3(\partial\Delta^n_F)$. Therefore, by applying the algorithm of §3 to a unit u of $\partial\Delta^n_F$ one may hope to produce an element of $H_3(\partial\Delta^n_F;\mathbb{Z})$ whose image in $KV_3(\partial\Delta^n_F)\otimes\mathbb{Q}$ corresponds to x.

There are two problems with this. Firstly there are probably few useful units in $\partial\Delta^n_F$. This problem may be countered by attempting to replace $\partial\Delta^n_F$ by $\partial\Delta^n_F[1/u]$ for some non-zero divisor u and attempt to construct the element of $H_3(\partial\Delta^n_F[1/u];\mathbb{Z})$ which corresponds to some small integer multiple of the image of x. Secondly, we do not currently have a formula for the Borel regulator

$$H_{3+n}(GL_N F;\mathbb{Z}) \longrightarrow \mathbb{R}$$

in terms of $H_3(\partial\Delta^n_F;\mathbb{Z})$. This raises the following problem:

Problem: Construct a homomorphism

$$H_3(\partial\Delta^n_F;\mathbb{Q}) \longrightarrow \mathbb{R}$$

which is compatible the Borel regulator on $K_{3+n}(F)\otimes\mathbb{Q}$ under the above isomorphisms.

5. Numerical results when $A = \mathbb{Q}(e^{2\pi\sqrt{-1}/3})$ in dimension 3

Let $A = \mathbb{Q}(e^{2\pi\sqrt{-1}/3})$ and set $u = e^{2\pi\sqrt{-1}/3}$ in the algorithm which is described in §3. In each of the cases (i) and (ii) (with $n = 3$) of §3.1 the algorithm gives explicit elements x_1 and x_2, respectively, in $H_3(SL_5\mathbb{Q}(e^{2\pi\sqrt{-1}/3});\mathbb{Z})$. To the image of each of these elements in $H_3(GL_5\mathbb{C};\mathbb{Z})$ we can apply the power series algorithm of [3] to compute an approximation to the value of Borel's regulator

$$b_1 : H_3(GL_5\mathbb{C};\mathbb{Z}) \longrightarrow \mathbb{R}.$$

In the power series algorithm of ([3] §4) there is a multinomial parameter $\underline{m}$. A program was written in C to do this large calculation using the parameter range $|\underline{m}| \leq 9$. We took the 3-cycle from case (i) of §3.1 but the choice is immaterial since the 3-cycle from case (ii) has the same regulator value. The program started with about 7000 distinct elements. After rewriting them in the standard resolution (rather than the bar resolution), the first step was to find the elements with a repeated matrix and remove them. We also combined elements with the same matrices in different permutations. After these simplifications, we have 3450 distinct elements to calculate. The program has been written and we can calculate the power series for $\underline{m}$ as argument. Theoretically, and practically, increasing the parameter range for $|\underline{m}|$ by one multiplies the run time by three. On a 2.4GHz computer, it took about 12 hours to compute for the range $|\underline{m}| \leq 9$. The output gave

$$-0.0375 \leq b_1(x_2) \leq -0.0125.$$

Let $R_{-1}(\mathbb{Q}(e^{2\pi\sqrt{-1}/3})) \in \mathbb{R}$ denote the value of the regulator homomorphism

$$\mathbb{Z} \cong \frac{K_3(\mathbb{Q}(e^{2\pi\sqrt{-1}/3}))}{\text{Torsion}} \stackrel{\text{Hurewicz}}{\longrightarrow} H_3(GL_\infty\mathbb{Q}(e^{2\pi\sqrt{-1}/3});\mathbb{Z}) \longrightarrow \mathbb{R}$$

on a generator (this number is the K_3-regulator in the arithmetic sense). Then the Lichtenbaum conjecture, proved for cyclotomic fields (for example, [17], [18], [19]), states that

$$\zeta^*_{\mathbb{Q}(e^{2\pi\sqrt{-1}/3})}(-1) = \pm 2^{\epsilon}\frac{R_{-1}(\mathbb{Q}(e^{2\pi\sqrt{-1}/3}))|K_2(\mathbb{Z}[e^{2\pi\sqrt{-1}/3}])|}{|\mathrm{Tors}\,K_3(\mathbb{Z}[e^{2\pi\sqrt{-1}/3}])|}$$

for some integer ϵ where $\zeta^*_{\mathbb{Q}(e^{2\pi\sqrt{-1}/3})}(-1)$ denotes the leading term at $s=-1$ of the Taylor series for the Dirichlet zeta function. In the appendix to [1], Tate shows that $K_2(\mathbb{Z}[e^{2\pi\sqrt{-1}/3}])|$ is trivial. Therefore

$$R_{-1}(\mathbb{Q}(e^{2\pi\sqrt{-1}/3})) = \pm 2^{a}\zeta^*_{\mathbb{Q}((e^{2\pi\sqrt{-1}/3})}(-1)|\mathrm{Tors}\,K_3(\mathbb{Q}(e^{2\pi\sqrt{-1}/3}))|$$

for some integer a. However the powers of 2 are determined for cyclotomic fields in Ion Rada's thesis [15] from which we find, in this example, that

$$R_{-1}(\mathbb{Q}(e^{2\pi\sqrt{-1}/3})) = \pm\zeta^*_{\mathbb{Q}(e^{2\pi\sqrt{-1}/3})}(-1).$$

However

$$\begin{aligned}\zeta^*_{\mathbb{Q}(e^{2\pi\sqrt{-1}/3})}(-1) &= -\left(\frac{3}{2^2\pi^2}\right)^{\frac{3}{2}}\zeta_{\mathbb{Q}(e^{2\pi\sqrt{-1}/3})}(2)\\ &= -0.02692216226828754283 8\end{aligned}$$

where we used PARI to calculate $\zeta_{\mathbb{Q}(e^{2\pi\sqrt{-1}/3})}(2)$. (PARI calculates this via the Dirichlet series which is calculated given the field polynomial.)

Hence we conclude that $b_1(x_2) = R_{-1}(\mathbb{Q}(e^{2\pi\sqrt{-1}/3}))$ and therefore, modulo torsion, the Hurewicz image of a generator of $\mathbb{Z} \cong \frac{K_3(\mathbb{Q}(e^{2\pi\sqrt{-1}/3}))}{\mathrm{Torsion}}$ is equal to

$$\pm x_2 \in \frac{H_3(GL_\infty\mathbb{Q}(e^{2\pi\sqrt{-1}/3});\mathbb{Z})}{\mathrm{Torsion}}.$$

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6. Appendix: Power series formula

Presented here is the power series formula written in MAPLE. At the beginning, a random 4-tuple has been included as a test case. It took about 2 hours to calculate this test tuple, as presented, to accuracy of degree 5. In comparison, the same formula in C calculated a full cycle of over 3000 4-tuples to degree 9 in 12 hours. Nevertheless, the symbolic nature of MAPLE corroborates the formula. We note, for example, the output is purely imaginary, as it should be. Also, other simple test cases like tuples with repeated matrices, or cycles with rational matrices, output 0.

```
> restart;

> with(LinearAlgebra):

> X[i] i = 0..3:

> X[0] := Matrix(3,[[1,0,0],[0,1,1/2 + sqrt(3)*I/2],[0,0,1]]):

> X[1] := Matrix(3,[[1,1/2 - sqrt(3)*I/2,0],[0,1,0],[0,0,1]]):

> X[2] := Matrix(3,[[1,0,0],[0,1,0],[0,- 1/2 - sqrt(3)*I/2,1]]):

> X[3] := Matrix(3,[[1,0,0],[0,1,0],[0,0,1]]):
```

```
> Xay := (X[0],X[1], X[2], X[3]);
```

$$Xay := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2}+\frac{1}{2}I\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \frac{1}{2}-\frac{1}{2}I\sqrt{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2}-\frac{1}{2}I\sqrt{3} & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> XX[0]:=Matrix(1..3,1..3,shape=hermitian):
> XX[1]:=Matrix(1..3,1..3,shape=hermitian):
> XX[2]:=Matrix(1..3,1..3,shape=hermitian):
> XX[3]:=Matrix(1..3,1..3,shape=hermitian):
> for i from 0 by 1 to 3 do
> XX[i]:=simplify(X[i] . HermitianTranspose(X[i])):
> XXe[i]:=Eigenvalues(XX[i], implicit):
> end do:
> XXay := (XX[0],XX[1],XX[2],XX[3]);
```

$$XXay := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & \frac{1}{2}+\frac{1}{2}I\sqrt{3} \\ 0 & \frac{1}{2}-\frac{1}{2}I\sqrt{3} & 1 \end{bmatrix}, \begin{bmatrix} 2 & \frac{1}{2}-\frac{1}{2}I\sqrt{3} & 0 \\ \frac{1}{2}+\frac{1}{2}I\sqrt{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2}+\frac{1}{2}I\sqrt{3} \\ 0 & -\frac{1}{2}-\frac{1}{2}I\sqrt{3} & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> maxE := 0:
> for i from 0 by 1 to 3 do
> for j from 1 by 1 to 3 do
> if (abs(evalf(XXe[i][j])) > maxE)
> then maxi := i; maxj:=j; maxE := abs(evalf(XXe[i][j]));
```

```
> end if;

> end do;

> end do;

> lambda := simplify(XXe[maxi][maxj]);

> evalf(lambda);
```

$$\lambda := \frac{3}{2} + \frac{\sqrt{5}}{2}$$

2.618033988

So far, we have calculated the maximum eigenvalue of the positive definite Hermitian matrices, call it λ. Now we create four other elements such that Xay = − Z1 + Z2 − Z3 + Z4. These new elements will converge under the power series formulas.

```
> XI := Matrix(3,3,shape=identity):

> Z1[0]:=X[1]/lambda:

> Z1[1]:=X[2]/lambda:

> Z1[2]:=X[3]/lambda:

> Z1[3]:=XI:

> Z1ay:=(evalf(Z1[0]),evalf(Z1[1]),evalf(Z1[2]),evalf(Z1[3])):

> Z2[0]:=X[0]/lambda:

> Z2[1]:=X[2]/lambda:

> Z2[2]:=X[3]/lambda:

> Z2[3]:=XI:

> Z2ay:=(evalf(Z2[0]),evalf(Z2[1]),evalf(Z2[2]),evalf(Z2[3])):

> Z3[0]:=X[0]/lambda:

> Z3[1]:=X[1]/lambda:

> Z3[2]:=X[3]/lambda:

> Z3[3]:=XI:

> Z3ay:=(evalf(Z3[0]),evalf(Z3[1]),evalf(Z3[2]),evalf(Z3[3])):
```

```
> Z4[0]:=X[0]/lambda:
> Z4[1]:=X[1]/lambda:
> Z4[2]:=X[2]/lambda:
> Z4[3]:=XI:
> Z4ay:=(evalf(Z4[0]),evalf(Z4[1]),evalf(Z4[2]),evalf(Z4[3])):
>
> Hamida := proc( Y0, Y1, Y2, Y3, m_start, m_end )
> description "Calculate Complex Hamida";
> local Id,U,m,n,k,L,Ltemp,i,jA,jB,l1,l2,A,B,Cmnk,Cmn,Cm,result;
> Id := Matrix(3,3,shape=identity);
> result := 0;
> U[0]:=((Y0 . HermitianTranspose(Y0)) - Id);
> U[1]:=((Y1 . HermitianTranspose(Y1))-(Y0 . HermitianTranspose(Y0)));
> U[2]:=((Y2 . HermitianTranspose(Y2))-(Y0 . HermitianTranspose(Y0)));
> #Compute Cm
> for m from m_start by 1 to m_end do
> Cm := 0;
> #Compute Cmn
> for n from 1 by 1 to m do
> Cmn := 0;
> #Compute Cmnk
> for k from 0 by 1 to (3^m-1) do
> #Convert k into base 3
> Ltemp:=0;
> for i from 1 by 1 to m do
> L(i):=(modp(k,(3^i)) - Ltemp)/(3^(i-1));
> Ltemp := Ltemp + (L(i)*3^(i-1));
> end do;
```

```
> #Create A (while counting l1,l2)
> l1:=0; l2:=0; A := Id;
> for jA from 1 by 1 to n do
> A := A . U[L(jA)];
> if (L(jA) = 1) then l1:=l1+1;
> elif (L(jA)=2) then l2:=l2+1;
> end if;
> end do;
> #Create B (while counting l1,l2)
> B := Id;
> for jB from (n+1) by 1 to m do
> B := B . U[L(jB)];
> if (L(jB) = 1) then l1:=l1+1;
> elif (L(jB)=2) then l2:=l2+1;
> end if;
> end do;
> Cmnk := (A.U[1].B.U[2]) - (A.U[2].B.U[1]);
> Cmn := Cmn + Trace(Cmnk)*factorial(l1)*factorial(l2)/factorial(l1+l2+2);
> Cmn := simplify(Cmn);
> end do;
> Cm := Cm + (((-1)^(m-1))*3*n*(Cmn)/(m+2));
> Cm := simplify(Cm);
> end do;
> result := result + Cm;
> result := simplify(result);
> end do;
> result;
> end proc;
```

```
>
> ms := 1:
> ml := 2:
> H4 := Hamida(Z4[0],Z4[1],Z4[2],Z4[3],ms,ml):
> H3 := Hamida(Z3[0],Z3[1],Z3[2],Z3[3],ms,ml):
> H2 := (Hamida(Z2[0],Z2[1],Z2[2],Z2[3],ms,ml)):
> H1 := (Hamida(Z1[0],Z1[1],Z1[2],Z1[3],ms,ml)):
> H4 := simplify(H4):
> H3 := simplify(H3):
> H2 := simplify(H2):
> H1 := simplify(H1):
> Answer := H4 - H3 + H2 - H1:
> Ans:=simplify(Answer);
```

$$Ans := \frac{-64\,I\,\sqrt{3}}{(3+\sqrt{5})^8}$$

```
> evalf[50](Ans);
```

$$-0.00019619972396514289239076477898724383468400458873754\,I$$