K-THEORY OF AZUMAYA ALGEBRAS OVER SCHEMES

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Abstract. Let X be a connected, noetherian scheme and $\mathcal{A}$ be a sheaf of Azumaya algebras on X which is a locally free $\mathcal{O}_X$-module of rank a. We show that the kernel and cokernel of $K_i(X) \to K_i(\mathcal{A})$ are torsion groups with exponent a^m for some m and any $i \geq 0$, when X is regular or X is of dimension d with an ample sheaf (in this case $m \leq d+1$). As a consequence, $K_i(X, \mathbb{Z}/m) \cong K_i(\mathcal{A}, \mathbb{Z}/m)$, for any m relatively prime to a.

An Azumaya algebra over a scheme is a sheaf of algebra which is one (étale) extension away from being a full matrix algebra. This should indicate that the functors arising from linear algebra would be similar over the Azumaya algebra and its base algebra. In [CW] it was shown that this is the case for Hochschild homology (over affine schemes). The aim of this note is to show that we have a similar result for K-functors. Indeed, we will show that, when X is of dimension d with an ample sheaf, e.g. affine or quasi-projective, or X is regular, and $\mathcal{A}$ is a sheaf of Azumaya algebras on X that is a locally free $\mathcal{O}_X$-module of rank a, then $K_i(X)$ is isomorphic to $K_i(\mathcal{A})$ up to a-torsion. In order to prove these results, we first show that the kernel and cokernel of $K_i(X) \to K_i(\mathcal{A})$ are torsion groups with exponent a^m for some m. When $\mathcal{A}$ is free over $\mathcal{O}_X$, this is a straightforward argument using Morita theory. When $\mathcal{A}$ is locally free, and X is of finite dimension with an ample sheaf, we will use an extension of Bass' result on K-theory of rings to do this. An alternative argument is given when X is regular, where we use a Mayer-Vietoris sequence to piece together local results into a global one.

Here $K_i(\mathcal{A})$ is the Quillen K-theory of the exact category of sheaves of $\mathcal{O}_X-$ locally free, left $\mathcal{A}$-modules of finite type. If $\mathcal{A}$ is non-commutative, we define a locally free, left $\mathcal{A}$ module to be a left $\mathcal{A}$ module $\mathcal{M}$ such that $\mathcal{M}$ is locally free as an $\mathcal{O}_X$ module, and we use the category $\mathcal{A}-mod$ and the corresponding exact subcategory of locally free left $\mathcal{A}$ modules to calculate $G_i(\mathcal{A})$ and $K_i(\mathcal{A})$ respectively. We follow the notation in [S] throughout this note. X and Y will always denote schemes, $\mathcal{R}$ and $\mathcal{S}$ commutative $\mathcal{O}_X$-algebras, and $\mathcal{A}$ and $\mathcal{B}$ possibly non-commutative $\mathcal{O}_X$-algebras that are locally free and of finite type as $\mathcal{O}_X$-modules. Tensor products are over $\mathcal{O}_X$ unless otherwise indicated. If X is a scheme and $\mathcal{A}$ is a not necessarily commutative sheaf of $\mathcal{O}_X$-algebras, we follow standard notation and denote the category of left $\mathcal{A}$-modules (resp. right $\mathcal{A}$-modules, $\mathcal{A}$-bimodules) by $\mathcal{A}-mod$ (resp. $mod-\mathcal{A}$, $\mathcal{A}-mod-\mathcal{A}$). If $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module, $[\mathcal{F}]$ denotes its class in $K_0(X)$.

Let $\mathcal{C}$ be a category, and let Ab be the category of abelian groups. If $F : \mathcal{C} \to Ab$ is any functor and $f : A \to B$ is a morphism in $\mathcal{C}$, define the functors $\mathrm{ZF}(B/A)$ and $\mathrm{CF}(B/A)$ by the exact sequence

$$0 \longrightarrow \mathrm{ZF}(B/A) \longrightarrow F(A) \xrightarrow{F(f)} F(B) \longrightarrow \mathrm{CF}(B/A) \longrightarrow 0 \tag{1}$$

In case $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{A}-mod$ and $\mathcal{B}-mod$ respectively and F is G_i, we shorten this to $\mathrm{ZG}_*(\mathcal{B}/\mathcal{A})$ and $\mathrm{CG}_*(\mathcal{B}/\mathcal{A})$ respectively. We adopt similar notation, $\mathrm{ZK}_*(\mathcal{B}/\mathcal{A})$ and $\mathrm{CK}_*(\mathcal{B}/\mathcal{A})$, when restricting to the corresponding exact subcategory of locally free, left modules.

If $\phi : \mathcal{A} \to \mathcal{B}$ is an $\mathcal{O}_X$-algebra homomorphism that makes $\mathcal{B}$ a sheaf of flat $\mathcal{A}$ modules, then $\mathcal{B} \in \mathcal{B}-mod-\mathcal{A}$ and $\phi^i : K_i(\mathcal{A}) \to K_i(\mathcal{B})$ denotes the functorial map defined by $\mathcal{B} \otimes_{\mathcal{A}} - : \mathcal{A}-mod \to \mathcal{B}-mod$. Note that $\mathrm{ZK}_i(\mathcal{B}/\mathcal{A})$ and $\mathrm{CK}_i(\mathcal{B}/\mathcal{A})$ become functors on the category whose objects are $\mathcal{O}_X$ algebra homomorphisms. We denote the map defined by restricting the action of $\mathcal{B}$ to an action of $\mathcal{A}$ by $res^{\mathcal{A}}_{\mathcal{B}} : \mathcal{B}-mod \to \mathcal{A}-mod$. Since this is exact it induces $\phi_i : K_i(\mathcal{B}) \to K_i(\mathcal{A})$.

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Proposition 1. *Let $\alpha : \mathcal{A} \to \mathcal{B}$, $\beta : \mathcal{B} \to \mathcal{C}$ be homomorphisms of possibly non-commutative sheaves of $\mathcal{O}_X$ algebras such that $\mathcal{B}$ is locally free when considered as a left $\mathcal{A}$ and also a right $\mathcal{A}$ module and $\mathcal{C}$ is locally free when considered as a left $\mathcal{B}$ and also as a right $\mathcal{B}$ module. Then, for any i, there is an exact sequence*

$$\begin{array}{rl} 0 \longrightarrow & \mathrm{ZK}_i\left(\mathcal{B}/\mathcal{A}\right) \longrightarrow \mathrm{ZK}_i\left(\mathcal{C}/\mathcal{A}\right) \longrightarrow \mathrm{ZK}_i\left(\mathcal{C}/\mathcal{B}\right) \\ \longrightarrow & \mathrm{CK}_i\left(\mathcal{B}/\mathcal{A}\right) \longrightarrow \mathrm{CK}_i\left(\mathcal{C}/\mathcal{A}\right) \longrightarrow \mathrm{CK}_i\left(\mathcal{C}/\mathcal{B}\right) \longrightarrow 0. \end{array}$$

Proof. This is a straghtforward diagram chase once the four term sequence (1) is broken into two short exact sequences in the diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ZK}_i(\mathcal{B}/\mathcal{A}) & \xrightarrow{i_{\mathcal{B}/\mathcal{A}}} & K_i(\mathcal{A}) & \longrightarrow & \mathrm{Im}\left(i_{\mathcal{B}/\mathcal{A}}\right) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{ZK}_i(\mathcal{C}/\mathcal{A}) & \xrightarrow{i_{\mathcal{C}/\mathcal{A}}} & K_i(\mathcal{A}) & \longrightarrow & \mathrm{Im}\left(i_{\mathcal{C}/\mathcal{A}}\right) & \longrightarrow & 0 \end{array} \tag{2}$$

and

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathrm{ZK}_i(\mathcal{C}/\mathcal{B}) & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathrm{Im}(i_{\mathcal{B}/\mathcal{A}}) & \longrightarrow & K_i(\mathcal{B}) & \longrightarrow & \mathrm{CK}_i(\mathcal{B}/\mathcal{A}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Im}(i_{\mathcal{C}/\mathcal{A}}) & \longrightarrow & K_i(\mathcal{C}) & \longrightarrow & \mathrm{CK}_i(\mathcal{C}/\mathcal{A}) & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & \mathrm{CK}_i(\mathcal{C}/\mathcal{B}) & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array} \tag{3}$$

□

In order to effectively use this result we need some information about one of the groups.

Proposition 2. *Let X be a scheme. Let $\mathcal{A}$ and $\mathcal{B}$ be possibly non-commutative sheaves of $\mathcal{O}_X$ algebras that are both locally free $\mathcal{O}_X$ modules of finite type. Let $i_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ be the homomorphism defined by $i_{\mathcal{A}}(x) = x \otimes 1$. Then $\mathrm{ZK}_i\left(\mathcal{A} \otimes \mathcal{B}/\mathcal{A}\right)$ is annihilated by $[\mathcal{B}] \in K_0\left(\mathcal{O}_X\right)$. If $\mathcal{A}$ is an Azumaya algebra, then $\mathrm{CK}_i\left(\mathcal{A} \otimes \mathcal{B}/\mathcal{B}\right)$ is annihilated by $[\mathcal{A}] \in K_0\left(\mathcal{O}_X\right)$. Moreover if $\mathcal{F}$ is a locally free sheaf of $\mathcal{O}_X$ modules, then*

$$\begin{array}{rcl} \mathrm{ZK}_i\left(\mathcal{A} \otimes End\left(\mathcal{F}\right)/\mathcal{A}\right) & \cong & \mathrm{Ker}\left(\left[\mathcal{F}^{\vee}\right] : K_i\left(\mathcal{A}\right) \to K_i\left(\mathcal{A}\right)\right) \ \textit{and} \\ \mathrm{CK}_i\left(\mathcal{A} \otimes End\left(\mathcal{F}\right)/\mathcal{A}\right) & \cong & K_i\left(\mathcal{A}\right)/\left[\mathcal{F}^{\vee}\right] \cdot K_i\left(\mathcal{A}\right). \end{array}$$

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{B} - mod & & \\ {\scriptstyle -\otimes\mathcal{B}} \uparrow & \searrow^{res^{\mathcal{A}}_{\mathcal{A}\otimes\mathcal{B}}} & \\ \mathcal{A} - mod & \xrightarrow[\mathcal{B}\otimes -]{} & \mathcal{A} - mod \end{array} \tag{4}$$

Here the horizontal map induces multiplication by $[\mathcal{B}] \in K_0\left(X\right)$ on $K_i\left(\mathcal{A}\right)$ and so $[\mathcal{B}] \cdot \mathrm{ZK}_i\left(\mathcal{A} \otimes \mathcal{B}/\mathcal{A}\right) = 0$.

If $\mathcal{A}$ is a sheaf of Azumaya algebras, then $\mathcal{A} \otimes \mathcal{A}^{op} \cong \mathrm{End}(\mathcal{A})$. The Morita Theorems [K] then show that the functor $\mathcal{A} \otimes - : \mathcal{B} - mod \to \mathcal{A}^{op} \otimes \mathcal{A} \otimes \mathcal{B} - mod$ is an equivalence of categories in the diagram:

$$\begin{array}{ccccc} \mathcal{A} \otimes \mathcal{B} - mod & \xrightarrow{\mathcal{A}^{op} \otimes -} & \mathcal{A}^{op} \otimes \mathcal{A} \otimes \mathcal{B} - mod & \xrightarrow{res^{\mathcal{A} \otimes \mathcal{B}}_{\mathcal{A}^{op} \otimes \mathcal{A} \otimes \mathcal{B}}} & \mathcal{A} \otimes \mathcal{B} - mod \\ & & \mathcal{A}\otimes- \uparrow \cong & \nearrow \mathcal{A}\otimes- & \\ & & \mathcal{B} - mod & & \end{array} \tag{5}$$

Evaluating the composite of the functors in the row on an $M \in \mathcal{A} \otimes \mathcal{B} - mod$, sends M to $\mathcal{A}^{op} \otimes M$. Consequently this induces multiplication by $[\mathcal{A}] = [\mathcal{A}^{op}] \in K_0(X)$ on $K_i(\mathcal{A} \otimes \mathcal{B})$ and this multiplication factors through $K_i(\mathcal{B}) \to K_i(\mathcal{A} \otimes \mathcal{B})$ from which the desired result follows easily.

The Morita Theorems show that $\mathcal{F} \otimes - : \mathcal{A} - mod \to \mathrm{End}(\mathcal{F}) \otimes \mathcal{A} - mod$ is an equivalence of categories that takes locally free left $\mathcal{A}$ modules to locally free, left $\mathrm{End}(\mathcal{F}) \otimes \mathcal{A}$ modules. Consequently $K_i(\mathrm{End}(\mathcal{F}) \otimes \mathcal{A})$ may be identified with $K_i(\mathcal{A})$, and the map $\phi^i : K_i(\mathcal{A}) \to K_i(\mathrm{End}(\mathcal{F}) \otimes \mathcal{A})$ becomes, with this identification,

$$\left[\mathcal{F}^\vee\right] : K_i(\mathcal{A}) \to K_i(\mathcal{A})$$

from which the assertion follows. □

Corollary 3. *Let $\mathcal{A}$ be a sheaf of Azumaya algebras on a scheme X. Then, for all $i \geq 0$,*

$$[\mathcal{A}] \cdot \mathrm{ZK}_i(\mathcal{A}/X) = 0 = [\mathcal{A}] \cdot \mathrm{CK}_i(\mathcal{A}/X).$$

In particular, if $\mathcal{A}$ is free over $\mathcal{O}_X$ of rank a then $\mathrm{ZK}_i(\mathcal{A}/\mathcal{O}_X)$ and $\mathrm{CK}_i(\mathcal{A}/\mathcal{O}_X)$ are torsion abelian groups of exponent dividing a.

While this gives us some control over the situation for a general commutative ring or scheme, it still leaves open the question of what multiplication by $[\mathcal{A}] : K_i(X) \to K_i(X)$ looks like. We would like to find a bound when $\mathcal{A}$ is not free over $\mathcal{O}_X$. The following result which provides some information should be compared to a result of Bass ([B], Corollary 16.2).

Proposition 4. *Let X be a connected, noetherian scheme of dimension d with an ample sheaf. If $\mathcal{F}$ is a locally free sheaf of rank f, then there is $Z \in K_0(X)$ such that $[\mathcal{F}] \cdot Z = f^m$ for some integer $m \leq d+1$. In particular*

$$\mathrm{Ker}\left([\mathcal{F}] : K_i(X) \to K_i(X)\right)$$

is a torsion group of exponent dividing f^m.

Proof. Recall, ([FL], V Corollary 3.10), that $I^{d+1} = 0$ if $I := \mathrm{Ker}\,[rk : K_0(X) \to \mathbb{Z}]$. Since X is connected, $\mathcal{F}$ has constant rank f and so $(f - [\mathcal{F}]) \in I$. Hence $(f - [\mathcal{F}])^m = 0$ for some $m \leq d+1$. Expanding this product and moving f^m to the other side gives the equation

$$f^m = [\mathcal{F}]\left(mf^{m-1} + \cdots - (-1)^m [\mathcal{F}]^{m-1}\right) \in K_0(X).$$

Let $Z = \left(mf^{m-1} + \cdots - (-1)^m [\mathcal{F}]^{m-1}\right)$. □

Remarks 5.

(1) If X is not noetherian and finite dimensional but has an ample sheaf, then X can be written as a limit of finite dimensional, noetherian schemes with an ample sheaf, $\mathcal{F}$ will be defined on one of them, and so the result will still hold but without an explicit bound on m.

(2) In the affine case, $X = \mathrm{Spec}(R)$, we can say a little more. Bass ([B], Proposition 15.6) has shown that if $x \in K_0(R)$ and $rk(x) \geq d$, then $x = [Q]$ for some projective module Q where $d = dim(Max(R))$ and if $[Q_1] = [Q_2] \in K_0(R)$ and $rk[Q_1] > d$, then $Q_1 \cong Q_2$. As a consequence if $rk(Z) \geq d$, e.g. $f^{m-1} \geq d$, then there is a projective R module Q such that $\mathcal{F} \otimes Q \cong R^{f^m}$.

This gives the following result.

Proposition 6. *Let X be a connected, noetherian scheme of dimension d with an ample sheaf. If $\mathcal{A}$ and $\mathcal{B}$ are sheaves of Azumaya algebras on X which are locally free of rank a and b as $\mathcal{O}_X$-modules, respectively, then, for all i, $\mathrm{ZK}_i(\mathcal{A}\otimes\mathcal{B}/\mathcal{A})$ and $\mathrm{CK}_i(\mathcal{A}\otimes\mathcal{B}/\mathcal{A})$ are torsion groups of exponent dividing b^m for some integer $m \leq d+1$. In particular, $\mathrm{ZK}_i(\mathcal{A}/X)$ and $\mathrm{CK}_i(\mathcal{A}/X)$ are torsion groups of exponent dividing a^m for some integer $m \leq d+1$.*

Proof. By Proposition 2, the groups $\mathrm{ZK}_i(\mathcal{A}\otimes\mathcal{B}/\mathcal{A})$ and $\mathrm{CK}_i(\mathcal{A}\otimes\mathcal{B}/\mathcal{A})$ are annihilated by $[\mathcal{B}] \in K_0(X)$. But by Proposition 4, there is a $Z \in K_i(X)$ such that $[\mathcal{B}]\cdot Z = b^m$ with $m \leq d+1$. The second statement is now immediate. □

The following corollary shows that the groups ZK_i and CK_i respect the "primary decomposition".

Corollary 7. *Let $\mathcal{A}$ and $\mathcal{B}$ be sheaves of Azumaya algebras on a scheme X as in Proposition 6. If $a = rank(\mathcal{A})$ is relatively prime to $b = rank(\mathcal{B})$, then $\mathrm{ZK}_i(\mathcal{A}\otimes\mathcal{B}/X) \cong \mathrm{ZK}_i(\mathcal{A}/X) \oplus \mathrm{ZK}_i(\mathcal{B}/X)$ and $\mathrm{CK}_i(\mathcal{A}\otimes\mathcal{B}/X) \cong \mathrm{CK}_i(\mathcal{A}/X) \oplus \mathrm{CK}_i(\mathcal{B}/X)$.*

Proof. In the six term exact sequence of Proposition 1 associated to $O_X \to \mathcal{A} \to \mathcal{A}\otimes\mathcal{B}$, the first and fourth groups are a^n torsion while the third and sixth groups are b^n torsion for n sufficiently large by Propositions 2 and 4. Hence $\mathrm{ZK}_i(\mathcal{A}\otimes\mathcal{B}/\mathcal{A}) \to \mathrm{CK}_i(\mathcal{A}/X)$ is zero and $\mathrm{ZK}_i(\mathcal{A}/X)$ and $\mathrm{CK}_i(\mathcal{A}/X)$ are the a primary components of $\mathrm{ZK}_i(\mathcal{A}\otimes\mathcal{B}/X)$ and $\mathrm{CK}_i(\mathcal{A}\otimes\mathcal{B}/X)$ respectively. The same argument deals with the b primary part by interchanging A and $\mathcal{B}$. □

There is an alternative argument using the Mayer-Vietoris exact sequence to piece together the local results from Corollary 3 into a global one. Although for this formulation we don't need the existence of an ample sheaf on X, we are, however, forced to restrict our attention to regular, noetherian schemes. We use the following counting lemma.

Lemma 8. *Let $A_i \xrightarrow{f_i} B_i, 1 \leq i \leq 5$, be a family of maps between exact sequences of abelian groups giving rise to a commutative diagram:*

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} & & \\
\cdots & \longrightarrow & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 & \longrightarrow & \cdots
\end{array}
\tag{6}
$$

Suppose that m_i, n_i annihilate $\mathrm{Ker}(f_i), \mathrm{Cok}(f_i)$ respectively for $i = 1,2,4,5$. Then $\mathrm{Ker}(f_3)$ and $\mathrm{Cok}(f_3)$ are annihilated by $m_2 n_1 m_4$ and $n_2 m_5 n_4$ respectively.

Proof. This is a straightforward diagram chase. □

We then apply this to our setting.

Lemma 9. *Let X be a noetherian scheme and $\mathcal{A}$ a sheaf of Azumaya algebras on X. Suppose there is a covering of X by open sets U_t such that $\mathcal{A}$ is free of rank a on each U_t. Let $V_k = \cup_{t=1}^k U_t$. Then $\mathrm{ZG}_i(\mathcal{A}|_{V_k}/V_k)$ and $\mathrm{CG}_i(\mathcal{A}|_{V_k}/V_k)$ are torsion abelian groups of exponent dividing $a^{k(k-1)+1}$ and a^{2k-1} respectively. If X is a regular, noetherian scheme, the same bounds apply to $\mathrm{ZK}_i(\mathcal{A}|_{V_k}/V_k))$ and $\mathrm{CK}_i(\mathcal{A}|_{V_k}/V_k)$.*

Proof. We use induction, the Mayer-Vietoris sequence for G-theory ([S], 5.16), and Lemma 8 to establish this result for G-theory. The argument leading up to Corollary 3 works equally well for G-theory and shows that $\mathrm{ZG}_i(\mathcal{A}|_{U_t}/U_t))$ and $\mathrm{CG}_i(\mathcal{A}|_{U_t}/U_t)$ are torsion abelian groups whose exponent divides a. Then we apply Lemma 8 to the map between Mayer-Vietoris sequences for $G_i(-)$ and $G_i(\mathcal{A}|_-)$ below. We are in the following setting

$$\begin{array}{ccccccccc}
G_{i+1}(V_k)\oplus G_{i+1}(U) & \longrightarrow & G_{i+1}(V_k\cap U) & \longrightarrow & G_i(V_{k+1}) & \longrightarrow & G_i(V_k)\oplus G_i(U) & \longrightarrow & G_i(V_k\cap U)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
G_{i+1}(\mathcal{A}|_{V_k})\oplus G_{i+1}(\mathcal{A}|_U) & \longrightarrow & G_{i+1}(\mathcal{A}|_{V_k\cap U}) & \longrightarrow & G_i\left(\mathcal{A}|_{V_{k+1}}\right) & \longrightarrow & G_i(\mathcal{A}|_{V_k})\oplus G_i(\mathcal{A}|_U) & \longrightarrow & G_i(\mathcal{A}|_{V_k\cap U})
\end{array}$$

where $U = U_{k+1}$. Multiplication by a annihilates the kernel and cokernel of the second and fifth columns since $\mathcal{A}\mid_{V_k\cap U_{k+1}}\cong \bigoplus_1^a \mathcal{O}_{V_k\cap U_{k+1}}$. Define $z(k)$ and $c(k)$ to be the integer exponents resulting from applying Lemma 8 to this diagram so that

$$a^{z(k)}\,\mathrm{ZG}_i\left(\mathcal{A}\mid_{V_k}/\mathcal{O}_{V_k}\right) = 0 = a^{c(k)}\,\mathrm{CG}_i\left(\mathcal{A}\mid_{V_k}/\mathcal{O}_{V_k}\right).$$

The resulting recursion relations become

$$z(k+1) = z(k) + c(k) + 1 \tag{7}$$

and

$$c(k+1) = c(k) + 2 \tag{8}$$

subject to the initial conditions $z(1) = 1$ and $c(1) = 1$. Thus $c(k) = 2k-1$ and substituting into the equation for $z(k+1)$ yields $z(k+1) = z(k) + 2k$. Consequently $z(k) = k(k-1)+1$.

For the second statement, when X is a regular, noetherian scheme, G-theory coincides with K-theory. □

Corollary 10. *Let X be a noetherian scheme of dimension d and $\mathcal{A}$ a sheaf of Azumaya algebras on X such that $\mathcal{A}$ is locally free of rank a. Then $\mathrm{ZG}_i(\mathcal{A}/\mathcal{O}_X)$ and $\mathrm{CG}_i(\mathcal{A}/\mathcal{O}_X)$ are torsion abelian groups of exponent dividing $a^{d(d+1)+1}$ and a^{2d+1} respectively. If X is also regular, then the same bounds apply to $\mathrm{ZK}_i(\mathcal{A}/\mathcal{O}_X)$ and $\mathrm{CK}_i(\mathcal{A}/\mathcal{O}_X)$.*

Proof. We need only to produce a covering of X by $d+1$ open sets satisfying the hypotheses of the Lemma 9. We proceed by induction. If $d = 0$, then X_{red} is a finite set of points and the result is obvious. Let $\eta_1,\dots,\eta_r$ be generic points of each irreducible component of X. Then there is an open set U_1 containing $\eta_1,\dots,\eta_r$ such that $\mathcal{A}\mid_{U_1}\cong\bigoplus_1^a\mathcal{O}_{U_1}$ as an $\mathcal{O}_{U_1}$-module. If $U_1 = X$, we are done. Otherwise $\dim(X-U_1)\le d-1$, and so $X-U_1$ can be covered by d open sets, $\{U_t\}$ with $2\le t\le d+1$, satisfying the hypothesis of the lemma. Adding U_1 to this collection finishes the induction step. □

Recall the basic property of K-theory with coefficients, that for any scheme X, an integer m and $i\ge 1$, there is a functorial exact sequence

$$0\longrightarrow K_i(X)/m\longrightarrow K_i(X,\mathbb{Z}/m)\longrightarrow {}_mK_{i-1}(X)\longrightarrow 0 \tag{9}$$

Theorem 11. *Let X be a connected, noetherian scheme and $\mathcal{A}$ a sheaf of Azumaya algebras on X such that $\mathcal{A}$ is locally free of rank a. If X has an ample sheaf or X is regular, then $K_i(X,\mathbb{Z}/m) = K_i(\mathcal{A},\mathbb{Z}/m)$, for any m relatively prime to a and $i\ge 0$.*

Proof. If $\mathcal{A}$ is free over X of rank a, it follows directly from Corollary 3 that $\mathrm{ZK}_i(\mathcal{A}/\mathcal{O}_X)$ and $\mathrm{CK}_i(\mathcal{A}/\mathcal{O}_X)$ are torsion groups of bounded exponent a (here, there is no need for any assumption on X). For a locally free Azumaya sheaf $\mathcal{A}$, if X has a finite dimension with an ample sheaf then by Proposition 6, $\mathrm{ZK}_i(\mathcal{A}/\mathcal{O}_X)$ and $\mathrm{CK}_i(\mathcal{A}/\mathcal{O}_X)$ are torsion groups with exponent dividing some power of a. If X is regular, then since X is quasi-compact, it can be covered by a finite number of open subsets U_t, such that $\mathcal{A}$ is free of rank a over U_t. Thus by Lemma 9, $\mathrm{ZK}_i(\mathcal{A}/\mathcal{O}_X)$ and $\mathrm{CK}_i(\mathcal{A}/\mathcal{O}_X)$ are torsion groups with exponent dividing some power of a.

Now tensor the exact sequence

$$0 \to \mathrm{ZK}_i(\mathcal{A}) \to K_i(X) \to K_i(\mathcal{A}) \to \mathrm{CK}_i(\mathcal{A}) \to 0,$$

with $\mathbb{Z}[1/a]$. Since $\mathrm{CK}_i(\mathcal{A}) \otimes \mathbb{Z}[1/a]$ and $\mathrm{ZK}_i(\mathcal{A}) \otimes \mathbb{Z}[1/a]$ vanish, it follows that

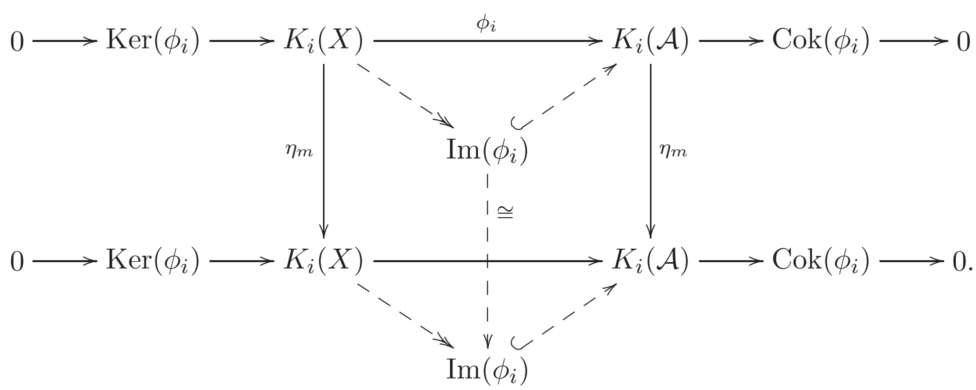
$$K_i(X) \otimes \mathbb{Z}[1/a] \cong K_i(\mathcal{A}) \otimes \mathbb{Z}[1/a]. \tag{10}$$

Now from the exact sequence (9) we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i(X) \otimes \mathbb{Z}/m & \longrightarrow & K_i(X, \mathbb{Z}/m) & \longrightarrow & {}_mK_{i-1}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_i(\mathcal{A}) \otimes \mathbb{Z}/m & \longrightarrow & K_i(\mathcal{A}, \mathbb{Z}/m) & \longrightarrow & {}_mK_{i-1}(\mathcal{A}) & \longrightarrow & 0. \end{array} \tag{11}$$

Since a and m are relatively prime, one can show that the outer vertical maps are isomorphisms, and therefore so is the middle vertical map. For example, for the left vertical map, consider the diagram below and use the snake lemma for the two short exact sequences. It follows that $K_i(X) \otimes \mathbb{Z}/m \cong \mathrm{Im}(\phi_i) \otimes \mathbb{Z}/m \cong K_i(\mathcal{A}) \otimes \mathbb{Z}/m$.

(12)



Hence $K_i(X, \mathbb{Z}/m) \cong K_i(\mathcal{A}, \mathbb{Z}/m)$. □

Remark 12. Let A be an Azumaya algebra free over its center R, of rank n^2. By Theorem 11,

$$K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n]$$

(see the first paragraph of the proof and (10)). For the case of a division algebra over a field, this was proved in Green, et al. [GHR] using the Skolem-Noether theorem, a result of Dawkins and Halperin on direct limits of division algebras along with several facts from K-theory (see also [HM]).

Corollary 13. *Let (R, m) be a Henselian local ring, and let A be an Azumaya algebra over R of rank n^2. Then*

$$K_i\left(A, \mathbb{Z}/d\right) \cong K_i\left(\overline{A}, \mathbb{Z}/d\right)$$

where $\overline{A} = A \otimes R/\mathrm{m}$, d is invertible in R and $(n, d) = 1$.

Proof. Consider the following commutative diagram where the horizontal arrows are isomorphisms by Theorem 11 and the left vertical arrow by Gabber's theorem (see Theorem 1 in [G]).

$$\begin{array}{ccc} K_i(R, \mathbb{Z}/d) & \xrightarrow{\cong} & K_i(A, \mathbb{Z}/d) \\ \cong \downarrow & & \downarrow \\ K_i(R/\mathrm{m}, \mathbb{Z}/d) & \xrightarrow{\cong} & K_i(\overline{A}, \mathbb{Z}/d) \end{array} \tag{13}$$

The result now follows. □

Here is an amusing application of our results.

Theorem 14. *Let $\mathcal{O}_v$ be a equicharacteristic dvr with residue field k and field of quotients K. Suppose that A_v is an Azumaya $\mathcal{O}_v$ algebra with $A_v \otimes_{\mathcal{O}_v} K := A$ and with residue algebra $A_v \otimes_{\mathcal{O}_v} k = A_k$. Then there is a six term exact sequence*

$$\begin{aligned} 0 &\longrightarrow \mathrm{ZK}_i\left(A_v/\mathcal{O}_v\right) \longrightarrow \mathrm{ZK}_i\left(A/K\right) \longrightarrow \mathrm{ZK}_{i-1}\left(A_k/k\right) \\ &\longrightarrow \mathrm{CK}_i\left(A_v/\mathcal{O}_v\right) \longrightarrow \mathrm{CK}_i\left(A/K\right) \longrightarrow \mathrm{CK}_{i-1}\left(A_k/k\right) \longrightarrow 0. \end{aligned}$$

for any $i > 0$.

Proof. Panin [P] has shown that the Quillen-Gersten sequence is exact for an equicharacteristic regular local ring or for an Azumaya algebra over such a ring. Thus the two middle rows in the diagram below are exact since a dvr has only one prime of height one and the theorem follows from the snake lemma.

(14)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathrm{CK}_i\left(A_v/\mathcal{O}_v\right) & \longrightarrow & \mathrm{CK}_i\left(A/K\right) & \longrightarrow & \mathrm{CK}_{i-1}\left(A_k/k\right) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K_i\left(A_v\right) & \longrightarrow & K_i\left(A\right) & \longrightarrow & K_{i-1}\left(A_k\right) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K_i\left(\mathcal{O}_v\right) & \longrightarrow & K_i\left(K\right) & \longrightarrow & K_{i-1}\left(k\right) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathrm{ZK}_i\left(A_v/\mathcal{O}_v\right) & \longrightarrow & \mathrm{ZK}_i\left(A/K\right) & \longrightarrow & \mathrm{ZK}_{i-1}\left(A_k/k\right) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

□

Corollary 15. *Let $\mathcal{O}_v$ be an equicharacteristic dvr with residue field k and field of quotients K. Let A_v be an Azumaya algebra of rank n^2 over $\mathcal{O}_v$. Then*

$$\#\left(\mathrm{CK}_1\left(A/K\right)\right) = \#\left(\mathrm{CK}_1\left(A_v/\mathcal{O}_v\right)\right)\left(\frac{n}{\mathrm{ind}\left(A_k\right)}\right).$$

In particular if A is a division algebra over K and $\mathrm{CK}_1\left(A\right) = 0$, then A_k is a division algebra.

Proof. If A is an Azumaya algebra of rank n^2 over a field L such that $A \cong M_r\left(D\right)$ for some division algebra D over L, then $D \otimes_L - : K_0\left(L\right) \to K_0\left(D\right)$ is an isomorphism. Consequently $A \otimes_L - : K_0\left(L\right) \to K_0\left(A\right)$ is an injection with cokernel of cardinality

$$r = \frac{n}{\mathrm{ind}\left(D\right)}$$

by Proposition 2. □

Remark 16. For a division algebra D over its center F of index n, thanks to the Dieudonné determinant, $\mathrm{CK}_1(D) = D^*/K^*D'$ where D^* is the multiplicative group of D and D' is its commutator subgroup. This group has a direct application in solving the open problem whether a multiplicative group of a division algebra has a maximal subgroup. Indeed, since $\mathrm{CK}_1(D)$ is torsion of bounded exponent (by Corollary 3), if it is not trivial, it has maximal subgroups and therefore D^* has (normal) maximal subgroups. Thus finding the maximal subgroups in D^* reduces to the case that $\mathrm{CK}_1(D)$ is trivial. It is a conjecture that $\mathrm{CK}_1(D)$ is trivial if and only if D is an ordinary quaternion division algebra over a Pythagorean field.

It was shown that such division algebras do have (non-normal) maximal subgroups. Thus if the above conjecture is settled positively, one concludes that the multiplicative group of a division algebra does have a maximal subgroup (see [HW] and references there).

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