ON THE FUNCTORIALITY OF THE SLICE FILTRATION

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Abstract. Let k be a field with resolution of singularities, and X a separated k-scheme of finite type with structure map g. We show that the slice filtration commutes with pullback along g. Restricting the field further to the case of characteristic zero, we are able to compute in $\mathcal{SH}_X$ the slices of homotopy invariant K-theory extending the result of Levine [Lev08], and also the zero slice of the sphere spectrum extending the result of Levine [Lev08] and Voevodsky [Voe04]. We also show that the zero slice of the sphere spectrum is a strict cofibrant ring spectrum $\mathbf{HZ}_X^{sf}$ which is stable under pullback and that all the slices have a canonical structure of strict modules over $\mathbf{HZ}_X^{sf}$. If we consider rational coefficents and assume that X is geometrically unibranch then relying on the work of Cisinski and Déglise [CD09], we get that the zero slice of the sphere spectrum is given by Voevodsky's rational motivic cohomology spectrum $\mathbf{HZ}_X \otimes \mathbb{Q}$ and that the slices have transfers. This proves several conjectures of Voevodsky [Voe02, conjectures 1, 7, 10, 11].

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1. Introduction

Let X be a Noetherian separated scheme of finite Krull dimension, and $\mathcal{M}_X$ be the category of pointed simplicial presheaves in the smooth Nisnevich site Sm_X over X equipped with the motivic model structure introduced in [PPR07, theorem A.17]. We define T_X in $\mathcal{M}_X$ as the pointed simplicial presheaf represented by $S^1 \wedge \mathbb{G}_m$, where $\mathbb{G}_m$ is the multiplicative group $\mathbb{A}^1_X - \{0\}$ pointed by 1, and S^1 denotes the simplicial circle. Let $Spt(\mathcal{M}_X)$ denote Jardine's category of symmetric T_X-spectra on $\mathcal{M}_X$ equipped with the motivic model structure defined in [PPR07, theorem A.38] and $\mathcal{SH}_X$ denote its homotopy category, which is triangulated.

For every integer $q \in \mathbb{Z}$, we consider the following family of symmetric T_X-spectra

$$C^q_{eff}(X) = \{F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \mid n, r, s \geq 0; s - n \geq q; U \in Sm_X\}$$

where F_n is the left adjoint to the n-evaluation functor

$$ev_n : Spt(\mathcal{M}_X) \to \mathcal{M}_X$$

Voevodsky [Voe02] defines the slice filtration as the following family of triangulated subcategories of $\mathcal{SH}_X$

$$\cdots \subseteq \Sigma_T^{q+1}\mathcal{SH}_X^{eff} \subseteq \Sigma_T^{q}\mathcal{SH}_X^{eff} \subseteq \Sigma_T^{q-1}\mathcal{SH}_X^{eff} \subseteq \cdots$$

where $\Sigma_T^q\mathcal{SH}_X^{eff}$ is the smallest full triangulated subcategory of $\mathcal{SH}_X$ which contains $C^q_{eff}(X)$ and is closed under arbitrary coproducts.

It follows from the work of Neeman [Nee96], [Nee01] that the inclusion

$$i_q : \Sigma_T^q\mathcal{SH}_X^{eff} \to \mathcal{SH}_X$$

has a right adjoint $r_q : \mathcal{SH}_X \to \Sigma_T^q\mathcal{SH}_X^{eff}$, and that the following functors

$$f_q : \mathcal{SH}_X \to \mathcal{SH}_X$$

$$s_q : \mathcal{SH}_X \to \mathcal{SH}_X$$

are exact, where f_q is defined as the composition $i_q \circ r_q$, and s_q is characterized by the fact that for every $E \in \mathcal{SH}_X$, we have the following distinguished triangle in $\mathcal{SH}_X$

$$f_{q+1}E \xrightarrow{\rho_q^E} f_qE \xrightarrow{\pi_q^E} s_qE \longrightarrow \Sigma_T^{1,0} f_{q+1}E$$

We will refer to f_qE as the $(q-1)$-connective cover of E, and to s_qE as the q-slice of E. It follows directly from the definition that the q-slice of E satisfies the following property:

$$\mathrm{Hom}_{\mathcal{SH}_X}(K, s_qE) = 0$$

for every symmetric T_X-spectrum K in $\Sigma_T^{q+1}\mathcal{SH}_X^{eff}$.

2. A general criterion

In this section $g : X \to Y$ will be a map of schemes, where X and Y are Noetherian, separated and of finite Krull dimension. Our goal is to introduce a general criterion which implies the compatibility between the slice filtration and pullback along g:

$$\mathbf{L}g^* : \mathcal{SH}_Y \to \mathcal{SH}_X$$

Lemma 2.1. *For every integer $q \in \mathbb{Z}$ we have that $\mathbf{L}g^*(\Sigma_T^q \mathcal{SH}_Y^{eff}) \subseteq \Sigma_T^q \mathcal{SH}_X^{eff}$, i.e. the functor $\mathbf{L}g^* : \mathcal{SH}_Y \to \mathcal{SH}_X$ respects connective objects.*

Proof. This follows directly from the fact that $g^*(T_Y) = T_X$. □

It follows immediately from lemma 2.1 that for any integer $q \in \mathbb{Z}$, we have a pair of natural transformations $\alpha_q : \mathbf{L}g^* \circ f_q \to f_q \circ \mathbf{L}g^*$, $\beta_q : \mathbf{L}g^* \circ s_q \to s_q \circ \mathbf{L}g^*$ such that for every $E \in \mathcal{SH}_Y$ the following diagram

$$\begin{array}{ccccccc}
\mathbf{L}g^*(f_{q+1}E) & \xrightarrow{\mathbf{L}g^*(\rho_q^E)} & \mathbf{L}g^*(f_qE) & \xrightarrow{\mathbf{L}g^*(\pi_q^E)} & \mathbf{L}g^*(s_qE) & \longrightarrow & \mathbf{L}g^*(\Sigma^{1,0}_{T_Y} f_{q+1}E) \\
\downarrow{\scriptstyle \alpha_{q+1}(E)} & & \downarrow{\scriptstyle \alpha_q(E)} & & {\scriptstyle \beta_q(E)}\downarrow & & {\scriptstyle \Sigma^{1,0}_{T_X}(\alpha_{q+1}(E))}\downarrow \\
f_{q+1}(\mathbf{L}g^*E) & \xrightarrow[\rho_q^{\mathbf{L}g^*E}]{} & f_q(\mathbf{L}g^*E) & \xrightarrow[\pi_q^{\mathbf{L}g^*E}]{} & s_q(\mathbf{L}g^*E) & \longrightarrow & \Sigma^{1,0}_{T_X} f_{q+1}(\mathbf{L}g^*E)
\end{array} \tag{2.1}$$

is commutative and its rows are distinguished triangles in $\mathcal{SH}_X$.

Definition 2.2. *We say that the slice filtration is* compatible with pullbacks *along g, if β_q is a natural isomorphism for every $q \in \mathbb{Z}$.*

Definition 2.3. *Let $E \in \mathcal{SH}_X$ be a symmetric T_X-spectrum and $q \in \mathbb{Z}$. We say that E is* q-orthogonal with respect to the slice filtration in $\mathcal{SH}_X$, *if one of the following equivalent conditions holds:*

(1) $f_qE = 0$.
(2) $\mathrm{Hom}_{\mathcal{SH}_X}(F, E) = 0$ *for every* $F \in \Sigma_T^q \mathcal{SH}_X^{eff}$.

Lemma 2.4. *Let $\mathcal{SH}_X^{\perp}(q)$ denote the full subcategory of $\mathcal{SH}_X$ generated by the symmetric T_X-spectra which are q-orthogonal with respect to the slice filtration in $\mathcal{SH}_X$. We have that $\mathcal{SH}_X^{\perp}(q)$ is a triangulated subcategory of $\mathcal{SH}_X$.*

Proof. It follows immediately from the fact that the functor $\mathrm{Hom}_{\mathcal{SH}_X}(A, -)$ is homological for every $A \in \mathcal{SH}_X$. □

Lemma 2.5. *Let $\mathbf{R}g_* : \mathcal{SH}_X \to \mathcal{SH}_Y$ be the right adjoint of $\mathbf{L}g^* : \mathcal{SH}_Y \to \mathcal{SH}_X$. Then the functor $\mathbf{R}g_*$ is compatible with the q-orthogonal objects with respect to the slice filtration, i.e.*

$$\mathbf{R}g_*(\mathcal{SH}_X^{\perp}(q)) \subseteq \mathcal{SH}_Y^{\perp}(q)$$

Proof. It suffices to show that for every symmetric T_X-spectrum F in $\mathcal{SH}_X$ which is q-orthogonal with respect to the slice filtration, and for every symmetric T_Y-spectrum H in $\mathcal{SH}_Y$ which is in $\Sigma_T^q \mathcal{SH}_Y^{eff}$, we have

$$\mathrm{Hom}_{\mathcal{SH}_Y}(H, \mathbf{R}g_*F) = 0$$

However, by adjointness

$$\mathrm{Hom}_{\mathcal{SH}_Y}(H, \mathbf{R}g_*F) \cong \mathrm{Hom}_{\mathcal{SH}_X}(\mathbf{L}g^*H, F)$$

on the other hand, lemma 2.1 implies that $\mathbf{L}g^*H \in \Sigma_T^q \mathcal{SH}_X^{eff}$. Hence

$$\mathrm{Hom}_{\mathcal{SH}_Y}(H, \mathbf{R}g_*F) \cong \mathrm{Hom}_{\mathcal{SH}_X}(\mathbf{L}g^*H, F) = 0$$

since F is in $\mathcal{SH}_X^{\perp}(q)$. This finishes the proof. $\square$

Lemma 2.6. *Let $E \in \mathcal{SH}_Y$ be a symmetric T_Y-spectrum and $q \in \mathbb{Z}$. If the following condition holds:*

$$\mathbf{L}g^*(s_qE) \in \mathcal{SH}_X^{\perp}(q+1) \tag{2.2}$$

then the natural maps:

$$\begin{aligned}
\alpha_{q+1}(f_qE) &: \mathbf{L}g^*(f_{q+1}f_qE) \longrightarrow f_{q+1}(\mathbf{L}g^*(f_qE)) \\
\alpha_q(f_qE) &: \mathbf{L}g^*(f_qf_qE) \longrightarrow f_q(\mathbf{L}g^*(f_qE)) \\
\beta_q(f_qE) &: \mathbf{L}g^*(s_qf_qE) \longrightarrow s_q(\mathbf{L}g^*(f_qE))
\end{aligned}$$

are all isomorphisms in $\mathcal{SH}_X$.

Proof. Consider the commutative diagram (2.1) for f_qE:

$$\begin{array}{ccccccc}
\mathbf{L}g^*(f_{q+1}f_qE) & \xrightarrow{\mathbf{L}g^*(\rho_q^{f_qE})} & \mathbf{L}g^*(f_qf_qE) & \xrightarrow{\mathbf{L}g^*(\pi_q^{f_qE})} & \mathbf{L}g^*(s_qf_qE) & \longrightarrow & \mathbf{L}g^*(\Sigma_{T_Y}^{1,0} f_{q+1}f_qE) \\
\downarrow{\scriptstyle \alpha_{q+1}(f_qE)} & & \downarrow{\scriptstyle \alpha_q(f_qE)} & & \downarrow{\scriptstyle \beta_q(f_qE)} & & \downarrow \\
f_{q+1}(\mathbf{L}g^*f_qE) & \xrightarrow[\rho_q^{\mathbf{L}g^*f_qE}]{} & f_q(\mathbf{L}g^*f_qE) & \xrightarrow[\pi_q^{\mathbf{L}g^*f_qE}]{} & s_q(\mathbf{L}g^*f_qE) & \longrightarrow & \Sigma_{T_X}^{1,0} f_{q+1}(\mathbf{L}g^*f_qE)
\end{array}$$

It follows from lemma 2.1 that $\alpha_q(f_qE)$ is an isomorphism. Using the octahedral axiom we get the following commutative diagram where all the rows and columns are distinguished triangles in $\mathcal{SH}_X$:

$$\begin{array}{ccccccc}
\mathbf{L}g^*(f_{q+1}f_qE) & \xrightarrow{\mathbf{L}g^*(\rho_q^{f_qE})} & \mathbf{L}g^*(f_qf_qE) & \xrightarrow{\mathbf{L}g^*(\pi_q^{f_qE})} & \mathbf{L}g^*(s_qf_qE) & \longrightarrow & \mathbf{L}g^*(\Sigma_{T_Y}^{1,0} f_{q+1}f_qE) \\
\downarrow{\scriptstyle \alpha_{q+1}(f_qE)} & & \downarrow{\scriptstyle \alpha_q(f_qE)} & & \downarrow{\scriptstyle \beta_q(f_qE)} & & \downarrow \\
f_{q+1}(\mathbf{L}g^*f_qE) & \xrightarrow[\rho_q^{\mathbf{L}g^*f_qE}]{} & f_q(\mathbf{L}g^*f_qE) & \xrightarrow[\pi_q^{\mathbf{L}g^*f_qE}]{} & s_q(\mathbf{L}g^*f_qE) & \longrightarrow & \Sigma_{T_X}^{1,0} f_{q+1}(\mathbf{L}g^*f_qE) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & 0 & \longrightarrow & \Sigma_{T_X}^{1,0}A & = & \Sigma_{T_X}^{1,0}A
\end{array}$$

Thus, it suffices to show that $\Sigma_{T_X}^{1,0}A \cong 0$ in $\mathcal{SH}_X$. It follows from lemma 2.1 that $\mathbf{L}g^*(f_{q+1}f_qE)$ is in $\Sigma_T^{q+1}\mathcal{SH}_X^{eff}$, and by construction $f_{q+1}(\mathbf{L}g^*f_qE)$ is also in $\Sigma_T^{q+1}\mathcal{SH}_X^{eff}$. Hence, A and $\Sigma_{T_X}^{1,0}A$ are both in $\Sigma_T^{q+1}\mathcal{SH}_X^{eff}$.

On the other hand, by hypothesis $\mathbf{L}g^*(s_qE) \cong \mathbf{L}g^*(s_qf_qE)$ is in $\mathcal{SH}_X^{\perp}(q+1)$; therefore, lemma 2.4 implies that $\Sigma_{T_X}^{1,0}A$ is in $\mathcal{SH}_X^{\perp}(q+1)$, since $s_q(\mathbf{L}g^*f_qE)$ is in $\mathcal{SH}_X^{\perp}(q+1)$ by construction.

We then have

$$\mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_{T_X}^{1,0}A, \Sigma_{T_X}^{1,0}A) = 0$$

and from this it follows at once that $\Sigma_{T_X}^{1,0}A \cong 0$ in $\mathcal{SH}_X$, as we wanted. $\square$

Theorem 2.7. *If the condition (2.2) in lemma 2.6 holds for every symmetric T_Y-spectrum in $\mathcal{SH}_Y$ and for every integer $\ell \in \mathbb{Z}$, we have that the slice filtration is compatible with pullbacks along g, i.e. we have a natural isomorphism $\beta_\ell : \mathbf{L}g^* \circ s_\ell \to s_\ell \circ \mathbf{L}g^*$ for every $\ell \in \mathbb{Z}$.*

Proof. Let E be a symmetric T_Y-spectrum in $\mathcal{SH}_Y$ and fix an integer $q \in \mathbb{Z}$. Then $E \cong hocolim_{p \leq q} f_p E$, and since $\mathbf{L}g^*$ and s_q commute with homotopy colimits we have that $\beta_q(E) : \mathbf{L}g^*(s_q E) \to s_q(\mathbf{L}g^* E)$ is given by $hocolim_{p \leq q} \beta_q(f_p E)$. Hence, it suffices to show that $\beta_q(f_p E) : \mathbf{L}g^*(s_q(f_p E)) \to s_q \mathbf{L}g^*(f_p E)$ is an isomorphism in $\mathcal{SH}_X$ for every integer $p \leq q$.

Lemma 2.6 implies that $\beta_q(f_q E)$ is an isomorphism. We now proceed by induction, and assume that $\beta_q(f_r E)$ is an isomorphism for some $r \leq q$. It only remains to show that in this situation, $\beta_q(f_{r-1} E)$ is also an isomorphism. Consider the following commutative diagram in $\mathcal{SH}_X$:

$$\begin{array}{ccc}
\mathbf{L}g^*(s_q(f_r E)) & \xrightarrow{\beta_q(f_r E)} & s_q(\mathbf{L}g^*(f_r E)) \\
{\scriptstyle \mathbf{L}g^* s_q(\rho^E_{r-1})}\downarrow & & \downarrow{\scriptstyle s_q \mathbf{L}g^*(\rho^E_{r-1})} \\
\mathbf{L}g^*(s_q(f_{r-1} E)) & \xrightarrow[\beta_q(f_{r-1} E)]{} & s_q(\mathbf{L}g^*(f_{r-1} E))
\end{array}$$

Since $r \leq q$, the left vertical map is an isomorphism and our induction hypothesis says that $\beta_q(f_r E)$ is also an isomorphism. Thus, it is enough to check that $s_q \mathbf{L}g^*(\rho^E_{r-1})$ is an isomorphism in $\mathcal{SH}_X$. However, we have the following commutative diagram in $\mathcal{SH}_X$:

$$\begin{array}{ccc}
s_q(\mathbf{L}g^*(f_r E)) & \xrightarrow{\cong} & s_q(\mathbf{L}g^*(f_r f_{r-1} E)) \\
{\scriptstyle s_q \mathbf{L}g^*(\rho^E_{r-1})}\downarrow & & \downarrow{\scriptstyle s_q(\alpha_r(f_{r-1} E))} \\
s_q(\mathbf{L}g^*(f_{r-1} E)) & \xrightarrow[\cong]{} & s_q(f_r(\mathbf{L}g^*(f_{r-1} E)))
\end{array}$$

where the rows are both canonical isomorphisms and the right vertical map is also an isomorphism by lemma 2.6. Thus, $s_q \mathbf{L}g^*(\rho^E_{r-1})$ is an isomorphism in $\mathcal{SH}_X$. This finishes the proof. □

Remark 2.8. *It is clear that theorem 2.7 holds for any exact functor*

$$F : \mathcal{SH}_Y \to \mathcal{SH}_X$$

which satisfies the following axioms:

(1) *For every $q \in \mathbb{Z}$, $F(\Sigma^q_T \mathcal{SH}^{eff}_Y) \subseteq \Sigma^q_T \mathcal{SH}^{eff}_X$.*
(2) *F commutes with homotopy colimits.*

Interesting examples are the following:

(1) *$A \wedge - : \mathcal{SH}_X \to \mathcal{SH}_X$, where A is a cofibrant symmetric T_X-spectrum in $\mathcal{SH}^{eff}_X$.*
(2) *$\mathbf{L}g_\sharp : \mathcal{SH}_X \to \mathcal{SH}_Y$, where $g : X \to Y$ is a smooth map of finite type.*

Lemma 2.9. *Assume that $g : X \to Y$ is a smooth map. We have that for every symmetric T_Y-spectrum in $\mathcal{SH}_Y$ and for every integer $\ell \in \mathbb{Z}$, the condition (2.2) in lemma 2.6 holds; and as a consequence we get that the slice filtration is compatible with pullbacks along g in the sense of definition 2.2.*

Proof. Consider a symmetric T_Y-spectrum E in $\mathcal{SH}_Y$ and fix an integer $q \in \mathbb{Z}$. By theorem 2.7 it suffices to show that $\mathbf{L}g^*(s_q E)$ is in $\mathcal{SH}_X^{\perp}(q+1)$.

Let $K = F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$, where $n, r, s \geq 0$; $s - n \geq q+1$ and $U \in Sm_X$. Since g is smooth we have that $\mathbf{L}g^* : \mathcal{SH}_Y \to \mathcal{SH}_X$ has a left adjoint $\mathbf{L}g_\sharp : \mathcal{SH}_X \to \mathcal{SH}_Y$. It is easy to see that $\mathbf{L}g_\sharp K = K$ where we look at U as a smooth scheme over Y using the map g, and $\mathbb{G}_m$ as the multiplicative group over Y (see [MV99, proposition 1.23(2)]). Therefore

$$\mathrm{Hom}_{\mathcal{SH}_X}(K, \mathbf{L}g^* s_q E) \cong \mathrm{Hom}_{\mathcal{SH}_Y}(\mathbf{L}g_\sharp K, s_q E) \cong \mathrm{Hom}_{\mathcal{SH}_Y}(K, s_q E)$$

However, it is clear that K is in $\Sigma_T^{q+1}\mathcal{SH}_Y^{eff}$ and by construction we have that $s_q E$ is in $\mathcal{SH}_Y^{\perp}(q+1)$. Thus

$$\mathrm{Hom}_{\mathcal{SH}_X}(K, \mathbf{L}g^* s_q E) \cong \mathrm{Hom}_{\mathcal{SH}_Y}(K, s_q E) = 0$$

and this finishes the proof since the family

$$C_{eff}^{q+1}(X) = \{F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \mid n, r, s \geq 0; s - n \geq q+1; U \in Sm_X\}$$

is a set of compact generators for $\Sigma_T^{q+1}\mathcal{SH}_X^{eff}$. □

3. The case of schemes defined over a field with resolution of singularities

In this section k will denote a field with resolution of singularities and X will be a separated k-scheme of finite type with structure map $g: X \to k$. Our goal is to show that the condition (2.2) of lemma 2.6 holds for every symmetric T_k-spectrum in $\mathcal{SH}_k$ and for every integer $q \in \mathbb{Z}$. Thus, by theorem 2.7 we have that in this situation there exists compatibility between the slice filtration and pullback along g in the sense of definition 2.2.

Proposition 3.1. *Let E be an arbitrary symmetric T_k-spectrum in $\mathcal{SH}_k$ and $q \in \mathbb{Z}$ an arbitrary integer. Then*

$$\mathbf{L}g^*(s_q E) \in \mathcal{SH}_X^{\perp}(q+1)$$

Proof. We will proceed by Noetherian induction. Since our base field has resolution of singularities, we have the following fibre product diagrams:

$$\begin{array}{ccc} p^{-1}Y & \xrightarrow{\tilde{i}} & W \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[i]{} & X \end{array} \qquad\qquad \begin{array}{ccc} p^{-1}U & \xrightarrow{\tilde{j}} & W \\ {\scriptstyle h}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle p} \\ U = X\backslash Y & \xrightarrow[j]{} & X \end{array}$$

where Y is a nowhere dense closed subscheme of X, p is projective, dominant and birational, W is smooth over k (with structure map $g \circ p$) and h is an isomorphism.

It follows from [Ayo07, scholium 1.4.2] that the following diagram is a distinguished triangle in $\mathcal{SH}_W$, where F denotes $\mathbf{L}(g \circ p)^*(s_q E)$

$$\tilde{j}_! \circ \tilde{j}^!(F) \xrightarrow{\epsilon_F} F \xrightarrow{\eta_F} \mathbf{R}\tilde{i}_* \circ \mathbf{L}\tilde{i}^*(F) \longrightarrow \Sigma_{T_W}^{1,0} \tilde{j}_! \circ \tilde{j}^!(F)$$

Now, lemma 2.9 implies that $F \cong \mathbf{L}(g \circ p)^*(s_q E)$ is in $\mathcal{SH}_W^{\perp}(q+1)$, since $g \circ p : W \to k$ is a smooth map. By Noetherian induction, we have that $\mathbf{L}\tilde{i}^*(F) \cong \mathbf{L}(g \circ p \circ \tilde{i})^*(s_q E)$ is in $\mathcal{SH}_{p^{-1}Y}^{\perp}(q+1)$, thus by lemma 2.5 we get that $\mathbf{R}\tilde{i}_* \circ \mathbf{L}\tilde{i}^*(F)$ is in $\mathcal{SH}_W^{\perp}(q+1)$. Therefore, it follows from lemma 2.4 that $\tilde{j}_! \circ \tilde{j}^!(F)$ is also in $\mathcal{SH}_W^{\perp}(q+1)$.

On the other hand, lemma 2.5 implies that

$$\mathbf{R}p_* \circ \tilde{j}_! \circ \tilde{j}^!(F) \cong \mathbf{R}p_* \circ \tilde{j}_! \circ \tilde{j}^! \circ \mathbf{L}p^*(\mathbf{L}g^* s_q E)$$

is in $\mathcal{SH}_X^{\perp}(q+1)$. But since p is projective, we have the following natural isomorphisms (see [Ayo07, scholium 1.4.2])

$$\begin{array}{ccc} p_! & \xrightarrow{\cong} & \mathbf{R}p_* \\ \mathbf{L}j^* \circ \mathbf{R}p_* & \xrightarrow{\cong} & \mathbf{R}h_* \circ \mathbf{L}\tilde{j}^* \\ h_! \circ \tilde{j}^! & \xrightarrow{\cong} & j^! \circ p_! \end{array} \tag{3.1}$$

hence we get the following natural isomorphisms in $\mathcal{SH}_X$

$$\begin{aligned}
(3.2)\quad & \mathbf{R}p_* \circ \tilde{j}_! \circ \tilde{j}^! \circ \mathbf{L}p^*(\mathbf{L}g^* s_q E) &\cong\ & p_! \circ \tilde{j}_! \circ \tilde{j}^! \circ \mathbf{L}p^*(\mathbf{L}g^* s_q E) \\
(3.3)\quad & &\cong\ & j_! \circ h_! \circ \tilde{j}^! \circ \mathbf{L}p^*(\mathbf{L}g^* s_q E) \\
(3.4)\quad & &\cong\ & j_! \circ j^! \circ p_! \circ \mathbf{L}p^*(\mathbf{L}g^* s_q E) \\
(3.5)\quad & &\cong\ & j_! \circ j^! \circ \mathbf{R}p_* \circ \mathbf{L}p^*(\mathbf{L}g^* s_q E) \\
(3.6)\quad & &\cong\ & \mathbf{L}j_\sharp \circ \mathbf{L}j^* \circ \mathbf{R}p_* \circ \mathbf{L}p^*(\mathbf{L}g^* s_q E) \\
(3.7)\quad & &\cong\ & \mathbf{L}j_\sharp \circ \mathbf{R}h_* \circ \mathbf{L}\tilde{j}^* \circ \mathbf{L}p^*(\mathbf{L}g^* s_q E) \\
(3.8)\quad & &\cong\ & \mathbf{L}j_\sharp \circ \mathbf{R}h_* \circ \mathbf{L}h^* \circ \mathbf{L}j^*(\mathbf{L}g^* s_q E) \\
(3.9)\quad & &\cong\ & \mathbf{L}j_\sharp \circ \mathbf{L}j^*(\mathbf{L}g^* s_q E) \\
(3.10)\quad & &\cong\ & j_! \circ j^!(\mathbf{L}g^* s_q E)
\end{aligned}$$

where (3.2), (3.4), (3.5), (3.7) follow from the natural isomorphisms mentioned in (3.1); (3.3), (3.8) follow from functoriality; (3.9) follows from the fact that h is an isomorphism and (3.6), (3.10) follow from the fact that j is an open embedding (so $j_! \circ j^!$ is naturally isomorphic to $\mathbf{L}j_\sharp \circ \mathbf{L}j^*$). Therefore, we have that $j_! \circ j^!(\mathbf{L}g^* s_q E)$ is in $\mathcal{SH}_X^\perp(q+1)$. On the other hand, by Noetherian induction we can assume that $\mathbf{L}i^*(\mathbf{L}g^* s_q E)$ is in $\mathcal{SH}_Y^\perp(q+1)$, and using lemma 2.5 we get that $\mathbf{R}i_* \circ \mathbf{L}i^*(\mathbf{L}g^* s_q E)$ is in $\mathcal{SH}_X^\perp(q+1)$.

Finally, it follows from [Ayo07, scholium 1.4.2] that the following diagram is a distinguished triangle in $\mathcal{SH}_X$,

$$j_! \circ j^!(\mathbf{L}g^* s_q E) \longrightarrow \mathbf{L}g^* s_q E \longrightarrow \mathbf{R}i_* \circ \mathbf{L}i^*(\mathbf{L}g^* s_q E) \longrightarrow \Sigma_{T_X}^{1,0} j_! \circ j^!(\mathbf{L}g^* s_q E)$$

and lemma 2.4 implies that $\mathbf{L}g^*(s_q E)$ is in $\mathcal{SH}_X^\perp(q+1)$, as we wanted. □

Theorem 3.2. *Let X be a separated k-scheme of finite type with structure map $g : X \to k$, where k has resolution of singularities. Then the slice filtration is compatible with pullbacks along g in the sense of definition 2.2.*

Proof. It follows directly from theorem 2.7 together with proposition 3.1. □

4. Applications

In this section we assume that all our schemes are of finite type over a field k of characteristic zero.

Definition 4.1. *We will denote by* $\mathbf{1}_X$, $\mathbf{KH}_X$, $\mathbf{HZ}_X$, $\mathbf{HZ}_X^{sf} \in Spt(\mathcal{M}_X)$ *respectively the sphere spectrum, the spectrum representing homotopy invariant* K*-theory, the spectrum representing motivic cohomology and* $s_0(\mathbf{1}_X)$.

The following theorem proves several conjectures of Voevodsky [Voe02, conjectures 1, 7, 10, 11].

Theorem 4.2. *Let* X *be a separated* k*-scheme of finite type with structure map* $g : X \to k$.

(1) *The zero slice of the sphere spectrum,* $\mathbf{HZ}_X^{sf}$ *is isomorphic to* $\mathbf{L}g^*(\mathbf{HZ}_k)$ *in* $\mathcal{SH}_X$.
(2) *The zero slice of the sphere spectrum,* $\mathbf{HZ}_X^{sf}$ *is a commutative ring spectrum in* $\mathcal{SH}_X$ *and a cofibrant ring spectrum in* $Spt(\mathcal{M}_X)$.
(3) *For every integer* q*, we have that* $s_q(\mathbf{KH}_X)$ *is isomorphic to* $\Sigma_{T_X}^{q,q}\mathbf{HZ}_X^{sf}$ *in* $\mathcal{SH}_X$.
(4) *If we consider rational coefficients and* X *is geometrically unibranch then* $\mathbf{HZ}_X^{sf}\otimes\mathbb{Q}$, $s_q(\mathbf{KH}_X)\otimes\mathbb{Q}$ *are respectively isomorphic in* $\mathcal{SH}_X$ *to* $\mathbf{HZ}_X\otimes\mathbb{Q}$, $\Sigma_{T_X}^{q,q}\mathbf{HZ}_X\otimes\mathbb{Q}$.

Proof. (1): It is clear that $\mathbf{1}_X \cong \mathbf{L}g^*(\mathbf{1}_k)$ in $\mathcal{SH}_X$. Therefore, by theorem 3.2 we have the following natural isomorphisms in $\mathcal{SH}_X$

$$s_0(\mathbf{1}_X) \cong s_0(\mathbf{L}g^*\mathbf{1}_k) \cong \mathbf{L}g^*(s_0\mathbf{1}_k)$$

Finally, the result follows from the work of Levine [Lev08, theorem 10.5.1] and Voevodsky [Voe04, theorem 6.6], which implies that the unit map $u : \mathbf{1}_k \to \mathbf{HZ}_k$ induces the following isomorphisms in $\mathcal{SH}_k$

$$s_0(u) : s_0\mathbf{1}_k \to s_0\mathbf{HZ}_k \cong \mathbf{HZ}_k$$

(2): We have that $\mathbf{HZ}_k$ is a commutative ring spectrum in $Spt(\mathcal{M}_X)$ (see [DRØ03, lemma 4.6]). Moreover, using [SS00, theorem 4.1(3)] together with [PPR07, theorem A.38] and [Jar00, proposition 4.19], we get a weak equivalence

$$w : \mathbf{HZ}_k^c \to \mathbf{HZ}_k$$

in $Spt(\mathcal{M}_k)$ such that $\mathbf{HZ}_k^c$ is a cofibrant ring spectrum in $Spt(\mathcal{M}_k)$. On the other hand, proposition A.47 in [PPR07] implies that

$$g^* : Spt(\mathcal{M}_k) \to Spt(\mathcal{M}_X)$$

is a strict symmetric monoidal left Quillen functor. Therefore, $g^*(\mathbf{HZ}_k^c)$ is a cofibrant ring spectrum in $Spt(\mathcal{M}_X)$ which is isomorphic to $\mathbf{L}g^*(\mathbf{HZ}_k)$ in $\mathcal{SH}_X$. Thus, the result follows from (1) above.

(3): It follows from [Voe98, section 6.2] that $\mathbf{KH}_X = \mathbf{L}g^*(\mathbf{KH}_k)$. Now, using theorem 3.2 we get the following natural isomorphisms in $\mathcal{SH}_X$

$$s_q\mathbf{KH}_X \cong s_q(\mathbf{L}g^*\mathbf{KH}_k) \cong \mathbf{L}g^*(s_q\mathbf{KH}_k)$$

Finally, the work of Levine [Lev08, theorems 6.4.2 and 9.0.3] implies that $s_q\mathbf{KH}_k$ is isomorphic in $\mathcal{SH}_k$ to $\Sigma_{T_k}^{q,q}\mathbf{HZ}_k$. Thus

$$s_q\mathbf{KH}_X \cong \mathbf{L}g^*(s_q\mathbf{KH}_k) \cong \mathbf{L}g^*(\Sigma_{T_k}^{q,q}\mathbf{HZ}_k) \cong \Sigma_{T_X}^{q,q}\mathbf{L}g^*(\mathbf{HZ}_k) \cong \Sigma_{T_X}^{q,q}\mathbf{HZ}_X^{sf}$$

as we wanted.

(4): The work of Cisinski and Déglise [CD09, corollary 15.1.6(2)] implies that under these conditons $\mathbf{L}g^*(\mathbf{HZ}_k)\otimes\mathbb{Q}$ is isomorphic to $\mathbf{HZ}_X\otimes\mathbb{Q}$ in $\mathcal{SH}_X$. Therefore, the result follows from (1) and (3) above. □

Remark 4.3. *We may consider theorem 4.2 as an extension of the computation of Levine [Lev08, theorems 6.4.2 and 9.0.3] from fields to schemes of finite type, however notice that we need to assume that our base scheme is defined over a field of characteristic zero whereas [Lev08] holds over perfect fields.*

Similarly, we may consider theorem 4.2 as an extension of the computation of Voevodsky [Voe04, theorem 6.6] and Levine [Lev08, theorem 10.5.1], but [Lev08] also holds over perfect fields whereas we need to assume that our base scheme is defined over a field of characteristic zero.

Theorem 4.4. *Let E be an arbitrary symmetric T_X-spectrum in $Spt(\mathcal{M}_X)$ and $q\in\mathbb{Z}$ an arbitrary integer.*

(1) *The q-slice of E, $s_q(E)$ has a natural structure of $\mathbf{HZ}_X^{sf}$-module in $Spt(\mathcal{M}_X)$.*
(2) *If we consider rational coefficients and X is geometrically unibranch then $s_q(E)\otimes\mathbb{Q}$ has a natural structure of $\mathbf{HZ}_X\otimes\mathbb{Q}$-module in $Spt(\mathcal{M}_X)$, in particular $s_q(E)\otimes\mathbb{Q}$ has transfers.*

Proof. This follows directly from theorem 4.2 and [Pel09, theorem 2.1], [Pel08, lemma 3.6.21(3) and theorem 3.6.20]. □

Definition 4.5. *Let $\mathbf{HZ}_X^{sf}$-mod be the category of left $\mathbf{HZ}_X^{sf}$-modules in $Spt(\mathcal{M}_X)$ equipped with the model structure induced by the adjuntion*

$$(\mathbf{HZ}_X^{sf}\wedge -, U, \varphi): Spt(\mathcal{M}_X)\to \mathbf{HZ}_X^{sf}\text{-mod}$$

i.e. a map f in $\mathbf{HZ}_X^{sf}$-mod is a fibration or a weak equivalence if and only if Uf is a fibration or a weak equivalence in $Spt(\mathcal{M}_X)$. We will denote by DM_X^{sf} the homotopy category of $\mathbf{HZ}_X^{sf}$-mod, which is triangulated.

Theorem 4.6. *The 2-functor $X\mapsto DM_X^{sf}$ has the structure of a motivic category in the sense of Cisinski and Déglise [CD09], and the adjunction*

$$(\mathbf{HZ}_X^{sf}\wedge^{\mathbf{L}} -, \mathbf{R}U, \varphi): \mathcal{SH}_X\to DM_X^{sf}$$

is a morphism of motivic categories $\mathcal{SH}\to DM$ in the category of separated k-schemes of finite type.

In particular, $X\mapsto DM_X^{sf}$ is a homotopic stable 2-functor in the sense of Ayoub and is equipped with the formalism of the six operations [Ayo07, scholium 1.4.2].

Proof. Theorem 4.2(1)-(2) implies that $X\mapsto \mathbf{HZ}_X^{sf}$ is a family of cofibrant ring spectra which is stable under pullback in the category of separated k-schemes of finite type. Hence the result follows immediately from propositions 4.2.11, 4.2.16 and corollary 2.4.9 in [CD09]. □

Let $\mathbf{H}_{\mathbb{B},X}\in Spt(\mathcal{M}_X)$ denote the Beilinson motivic cohomology spectrum introduced by Cisinski and Déglise (cf. [CD09, definition 13.1.2]). It follows in particular from Corollary 13.2.6 in [CD09] that $\mathbf{H}_{\mathbb{B},X}$ is a commutative cofibrant ring spectrum in $Spt(\mathcal{M}_X)$ which is stable under pullback in the category of separated schemes of finite type over k.

Theorem 4.7. *The Beilinson motivic cohomology spectrum* $\mathbf{H}_{\mathbb{B},X}$ *is naturally isomorphic to* $\mathbf{HZ}_X^{sf} \otimes \mathbb{Q}$ *in* $\mathcal{SH}_X$*, thus the homotopy category of* $\mathbf{H}_{\mathbb{B},X}$*-modules* $\mathrm{Ho}(\mathbf{H}_{\mathbb{B},X})$ *is equivalent to the homotopy category of left* $\mathbf{HZ}_X^{sf}$*-modules with rational coefficients.*

Hence, we get that modulo torsion $\mathrm{Ho}(\mathbf{H}_{\mathbb{B},X})$ *and* DM_X^{sf} *are equivalent.*

Proof. By theorem 4.2(1) we have that $\mathbf{HZ}_X^{sf} \otimes \mathbb{Q}$ is stable under pullback in the category of separated schemes of finite type over k, on the other hand corollary 13.2.6 in [CD09] implies in particular that $\mathbf{H}_{\mathbb{B},X}$ is also stable under pullback. Therefore, it suffices to show that $\mathbf{H}_{\mathbb{B},k}$ and $\mathbf{HZ}_k^{sf} \otimes \mathbb{Q}$ are isomorphic in $\mathcal{SH}_k$ for the base field k.

However, corollary 15.1.6(1) in [CD09] implies that $\mathbf{H}_{\mathbb{B},k}$ and $\mathbf{HZ}_k \otimes \mathbb{Q}$ are naturally isomorphic in $\mathcal{SH}_k$, and finally it follows from theorem 4.2(1) that $\mathbf{HZ}_k \otimes \mathbb{Q}$ and $\mathbf{HZ}_k^{sf} \otimes \mathbb{Q}$ are also naturally isomorphic in $\mathcal{SH}_k$. This finishes the proof. □

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