THE MORPHIC ABEL-JACOBI MAP

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ABSTRACT. The morphic Abel-Jacobi map is the analogue of the classical Abel-Jacobi map one obtains by using Lawson and morphic (co)homology in place of the usual singular (co)homology. It thus gives a map from the group of r-cycles on a complex variety that are algebraically equivalent to zero to a certain "Jacobian" built from the Lawson homology groups viewed as inductive limits of mixed Hodge structures.

In this paper, we define the morphic Abel-Jacobi map, establish its foundational properties, and then apply these results to the study of algebraic cycles. In particular, we show the classical Abel-Jacobi map (when restricted to cycles algebraically equivalent to zero) factors through the morphic version, and show that the morphic version detects cycles that cannot be detected by its classical counterpart — that is, we give examples of cycles in the kernel of the classical Abel-Jacobi map that are not in the kernel of the morphic one. We also investigate the behavior of the morphic Abel-Jacobi map on the torsion subgroup of the Chow group of cycles algebraically equivalent to zero modulo rational equivalence.

CONTENTS

1. INTRODUCTION

The classical Abel-Jacobi map provides an important tool for investigating the structure of the group of algebraic r-cycles on a smooth, projective complex variety

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X. It is a continuous homomorphism of abelian groups,

$$\Phi_r : \mathcal{Z}_r(X)_{\mathrm{hom}\sim 0} \to \mathcal{J}_r(X),$$

from the space $\mathcal{Z}_r(X)_{\mathrm{hom}\sim 0}$ of r-cycles on X that are homologically equivalent to zero (i.e., whose classes in the singular homology group $H_{2r}^{\mathrm{sing}}(X)$ vanish) to the r-th intermediate Griffiths Jacobian $\mathcal{J}_r(X)$ of X. The latter is the complex torus given as the dual of the $(r+1)$-stage of the Hodge filtration on $H^{2r+1}_{\mathrm{sing}}(X,\mathbb{C})$ modulo "periods":

$$\mathcal{J}_r(X) = \left(F^{r+1}H^{2r+1}_{\mathrm{sing}}(X,\mathbb{C})\right)^* / H^{\mathrm{sing}}_{2r+1}(X,\mathbb{Z}),$$

where $F^\bullet$ is the Hodge filtration and "periods" are those elements of $\left(F^{r+1}H^{2r+1}_{\mathrm{sing}}(X,\mathbb{C})\right)^*$ coming from classes in $H^{\mathrm{sing}}_{2r+1}(X,\mathbb{Z})$ under the map sending a closed integral chain c to the functional

$$\eta \mapsto \int_\eta c, \quad \text{for } \eta \in F^{r+1}H^{2r+1}_{\mathrm{sing}}(X,\mathbb{C}).$$

The original definition of the Abel-Jacobi map was given by Griffiths, in terms of integration: Suppose γ is an r-cycle that is homologically equivalent to zero. Then there is a $(2r+1)$-dimensional integral chain c with $\partial(c) = \gamma$ and we define

$$\Phi_r(\gamma) = \left(\eta \mapsto \int_c \eta\right) \qquad \text{mod periods.}$$

(See [23, §12] for more details.)

The aim of this paper is to define and study the "morphic Abel-Jacobi map", Φ_r^{mor}, which is the analogue of the classical Abel-Jacobi map that one obtains by replacing the singular homology and cohomology groups of a complex variety with the Lawson homology groups $L_rH_m(-)$ and morphic cohomology groups $L^tH^n(-)$. (See Section 2 for the definitions of these groups.) Just as the Lawson and morphic (co)homology groups provide refinements of their singular counter-parts, so too does the morphic Abel-Jacobi map refine the classical one. In particular, we show that the classical Abel-Jacobi map (when restricted to cycles algebraically equivalent to zero) factors as

$$\mathcal{Z}_r(X)_{\mathrm{alg}\sim 0} \xrightarrow{\Phi_r^{\mathrm{mor}}} \mathcal{J}_r^{\mathrm{mor}}(X) \to \mathcal{J}_r(X),$$

where $\mathcal{Z}_r(X)_{\mathrm{alg}\sim 0}$ denotes the space of r-cycles on X that are algebraically equivalent to zero and $\mathcal{J}_r^{\mathrm{mor}}(X)$ is the r-th "morphic Jacobian". The latter is defined analogously to $\mathcal{J}_r(X)$, using Lawson homology in place of singular cohomology. Just as is the case for the classical Abel-Jacobi map, the morphic version annihilates cycles rationally equivalent to zero. Moreover, the morphic Abel-Jacobi map is surjective, and we thus have a commutative triangle

$$\begin{array}{ccc} CH_r(X)_{\mathrm{alg}\sim 0} & \xrightarrow{\Phi_r^{\mathrm{mor}}} & \mathcal{J}_r^{\mathrm{mor}}(X) \\ & \searrow^{\Phi_r} & \downarrow \\ & & \mathcal{J}_r(X), \end{array} \tag{1.1}$$

in which the top arrow is surjective. Here, $CH_r(X)_{\mathrm{alg}\sim 0}$ is the Chow group of r-cycles on X that are algebraically equivalent to zero modulo rational equivalence.

To define the morphic Abel-Jacobi map, we rely on the general technique of Jannsen [21] for construction of Abel-Jacobi-type maps in a variety of settings.

The input to Jannsen's technique is a homology/cohomology theory for varieties equipped with suitable extra structure. His technique can be used to define the classical Abel-Jacobi map — in this case, the extra structure on the singular (co)homology groups is that of mixed Hodge structures, as provided by Deligne [8, 9]. To define the morphic Abel-Jacobi map, this extra structure stems from a result of Friedlander-Mazur [19] (later generalized by Lima-Filho [26]) that the Lawson homology groups are filtered inductive limits of mixed Hodge structures.

Using the factorization (1.1) and other properties of the morphic Abel-Jacobi map, we deduce several properties about the structure of cycles algebraically equivalent to zero on a variety X. In particular, we show that there are examples of cycles on a smooth, projective variety that vanish under the Abel-Jacobi map but not under the morphic Abel-Jacobi map. This is accomplished by constructing examples of varieties X for which the vertical map in (1.1) is not injective.

Thus, the morphic Abel-Jacobi map detects cycles that cannot be detected by its classical counterpart. In fact, we provide two types of such examples. Those of the first type arise by building on examples constructed by Friedlander [15] showing that various stages of the so-called s-filtration are non-trivial, and examples of the second type arise from examples of Schoen [34] showing that there can be an infinite amount of l-torsion in the kernel of the Abel-Jacobi map.

We also analyze the behavior of the morphic Abel-Jacobi map on torsion subgroups. The examples due to Schoen mentioned above show that the classical Abel-Jacobi map does not always induce an injection from $(CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}$ (the torsion subgroup of $CH_r(X)_{\mathrm{alg}\sim 0}$) to $\mathcal{J}_r(X)_{\mathrm{tor}}$, as was once conjectured. (Soulé and Voisin [36] also have constructed such counter-examples.) It is an intriguing question whether the morphic Abel-Jacobi map induces an isomorphism of the form

$$\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}} : (CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}} \xrightarrow{?\cong} \mathcal{J}_r^{\mathrm{mor}}(X)_{\mathrm{tor}}.$$

The map $\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}}$ is always onto, and we describe its kernel explicitly in Theorem 8.4. Conceivably, the kernel is trivial for any smooth, projective variety, so that $\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}}$ is an isomorphism for all such varieties (see Corollary 8.8). In this paper, we show that for any smooth, projective variety belonging to the class $\mathcal{C}$ defined in [17] (which includes all curves, all toric varieties, all cellular varieties, and all varieties built from these via localization, blowing up, or forming vector bundles), the kernel of $\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}}$ vanishes — see Example 8.9. We also prove, in Theorem 8.11, that $\Phi_r^{\mathrm{mor}}|_{\mathrm{tor}}$ is an isomorphism for any complex projective variety that can be defined over a number field, provided one assumes a common conjecture (namely, Conjecture 8.10) pertaining to the image of the higher Chow groups in Borel-Moore homology.

As mentioned, it was once conjectured that the classical Abel-Jacobi map is injective on torsion. The recent counter-examples of this conjecture also cast doubt on the related conjecture that the classical Abel-Jacobi map is universal among all "regular" maps from $Z_r(X)_{\mathrm{alg}\sim 0}$ to abelian varieties. (See Conjecture 7.2 for a careful formulation.) The morphic Abel-Jacobi map suggests two ways of producing counter-examples to this conjecture. First, as we prove in Corollary 5.9, the properties of the morphic Abel-Jacobi map imply that the classical Abel-Jacobi map factors as

$$\begin{array}{ccc} CH_r(X)_{\mathrm{alg}\sim 0} & \twoheadrightarrow & \mathcal{J}(N_{r+1}H^{\mathrm{sing}}_{2r+1}(X,\mathbb{Z}(r))) \\ & \searrow^{\Phi_r} & \downarrow \\ & & \mathcal{J}_r(X), \end{array} \tag{1.2}$$

where N_{r+1} refers to the subgroup of classes supported in dimension $r+1$ and the upper-right group is the "Jacobian" of the mixed Hodge structure $N_{r+1}H^{\mathrm{sing}}_{2r+1}(X,\mathbb{Z}(r))$. A counter-example to the universality of the classical Abel-Jacobi map would arise if the the vertical map in (1.2) fails to be injective, and it is conceivable that this map is not injective for certain varieties constructed by Kollár [2] — see Remark 5.10. Secondly, the fact that the vertical arrow in (1.1) can have a kernel suggests another possible source for counter-examples to the universality of the Abel-Jacobi map. Namely, such counter-examples would arise if the kernel of $\mathcal{J}^{\mathrm{mor}}_r(X) \to \mathcal{J}_r(X)$ were to admit a finite dimensional quotient, in an appropriate sense — see Theorem 7.4. Constructing an actual counter-example in either of these two ways, however, remains elusive.

The morphic Abel-Jacobi map bears an interesting relationship with the conjectural Bloch-Beilinson filtration $F^\bullet_{BB}$ of the rational Chow groups $CH_r(X)_\mathbb{Q}$ of a smooth, projective variety X. The first step in this filtration is $F^1_{BB}CH_r(X)_\mathbb{Q} = CH_r(X)_{\mathbb{Q},\mathrm{hom}\sim 0}$, and it is believed that the kernel of the classical Abel-Jacobi map Φ_r ought to provide the second step:

$$F^2_{BB}CH_r(X)_\mathbb{Q} = \ker(\Phi_r \otimes \mathbb{Q}).$$

The results mentioned above show that the inclusion $\ker(\Phi^{\mathrm{mor}}_r \otimes \mathbb{Q}) \subset \ker(\Phi_r \otimes \mathbb{Q}) \cap CH_r(X)_{\mathbb{Q},\mathrm{alg}\sim 0}$ can be strict — in fact, the quotient can have uncountable rank by Corollary 6.3. Moreover, in Theorem 10.1 we show that, assuming some common conjectures, there is a containment of the form

$$F^{\dim(X)-r}_{BB}CH_r(X)_\mathbb{Q} \subset \ker(\Phi^{\mathrm{mor}}_r \otimes \mathbb{Q}).$$

Finally, the classical Abel-Jacobi map admits a nice interpretation in terms of Deligne cohomology (more precisely, absolute Hodge cohomology), and at the end of this paper, we speculate how the morphic Abel-Jacobi map might admit a similar interpretation in terms of "absolute morphic cohomology". This cohomology theory remains conjectural, although we do sketch a possible construction of it.

We now describe the organization of this paper. Section 2 recalls various basic definitions, such as that of Lawson homology and morphic cohomology. In Section 3 we formalize what we mean by an inductive limit of mixed Hodge structures (IMHS, for short) and establish a few basic properties. Section 4 defines the IMHS for Lawson homology and establishes various functorial properties for this structure. These results are obtained by building on and modifying results of Friedlander-Mazur [19] and Lima-Filho [26]. In Section 5 we define the morphic Abel-Jacobi map and establish is foundational properties.

Starting with Section 6, we apply the morphic Abel-Jacobi map to the study of algebraic cycles. In this section we establish the examples mentioned above of cycles that vanish under the classical Abel-Jacobi map but not under the morphic one. In Section 7 we indicate how the morphic Abel-Jacobi map might lead to the construction of a counter-example to the conjectured universality of the classical

Abel-Jacobi map. Section 8 contains a description of the behavior of the morphic Abel-Jacobi map on the subgroup of torsion cycles, as discussed above.

Section 9 relates the morphic Abel-Jacobi map to a certain conjecture of Beilinson [3] that is a generalization of the classical Hodge conjecture. This conjecture of Beilinson is now known to be false, by examples constructed by Jannsen [21, 9.11]; we show how Jannsen's technique and the properties of the morphic Abel-Jacobi map lead to an even more dramatic failure of the conjecture. Section 10 discusses connections with the hypothetical Bloch-Beilinson filtration on algebraic cycles. In Section 11 we speculate about the existence of and properties enjoyed by "absolute morphic cohomology".

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2. Basics of Lawson homology

All varieties in this paper are assumed to be quasi-projective complex varieties. For such a variety U and abelian group A, we write $H^n_{\text{sing}}(U, A)$ (resp. $H_n^{\text{sing}}(U, A)$) for the singular cohomology (resp. homology) group with A coefficients of the associated space X^{an} of complex points equipped with the "analytic" (or "strong") topology. Let $H_n^{\text{BM}}(U, A)$ denote the Borel-Moore homology of U^{an} with A coefficients. When $A = \mathbb{Z}$, we omit it from the notations. Recall that $H_n^{\text{BM}}(X, A) \cong H_n^{\text{sing}}(X, A)$, if U is projective, and $H_n^{\text{BM}}(U, A) \cong H_n^{\text{sing}}(X, Y; A)$, if U is an open subvariety of a projective variety X with closed complement $Y = X - U$.

For a variety U, let $Z_r(U)$ denote the group of r-cycles on U (i.e., the free abelian group on dimension r closed, integral subvarieties) and write $CH_r(U)$ for the Chow group of r-cycles on U (i.e., the quotient of $Z_r(U)$ by the subgroup of cycles rationally equivalent to zero). An r-cycle γ on U is *algebraically equivalent to zero* if there exists a smooth, connected curve C, closed points $c_0, c_1 \in C$, and an $(r+1)$-cycle δ on $U \times C$ meeting $U \times \{c_0, c_1\}$ properly so that

$$\gamma = i_0^*(\delta) - i_1^*(\delta),$$

where i_ϵ^* is pullback along the closed immersion $U \stackrel{(id,\epsilon)}{\longrightarrow} U \times C$. Let $Z_r(U)_{\text{alg}\sim 0}$ be the subgroup of r-cycles that are algebraically equivalent to zero and write $CH_r(U)_{\text{alg}\sim 0}$ for its image in $CH_r(U)$. Equivalently, we have

$$CH_r(U)_{\text{alg}\sim 0} = \sum_{C, c_0, c_1} \text{im}\left(CH_{r+1}(U \times C) \stackrel{i_0^* - i_1^*}{\longrightarrow} CH_r(U)\right).$$

For a projective variety X, let $\mathcal{C}_{r,e}(X)$ denote the Chow variety parameterizing dimension r, degree e (with respect to some chosen embedding $X \subset \mathbb{P}^m$) effective cycles on X. Define $\mathcal{C}_r(X) = \coprod_{e \geq 0} \mathcal{C}_{r,e}(X)$ and observe that $\mathcal{C}_r(X)$ is an abelian monoid object in the category of ind-varieties, where the monoid pairing comes from the collection of morphisms

$$\mathcal{C}_{r,e}(X) \times \mathcal{C}_{r,e'}(X) \to \mathcal{C}_{r,e+e'}(X)$$

given by addition of effective cycles. The closed points of $\mathcal{C}_r(X)$ form the free abelian monoid of effective cycles on X, and the group completion of this monoid is

$Z_r(X)$. Since $\mathcal{C}_r(X)$ is an ind-variety, the set of closed points comes equipped with the analytic topology, and we also write $\mathcal{C}_r(X)$ for this topological abelian monoid. Define

$$\mathcal{Z}_r(X) = \mathcal{C}_r(X)^+ = (\mathcal{C}_r(X) \times \mathcal{C}_r(X))/\Delta(\mathcal{C}_r(X)).$$

the naive group completion of the topological abelian monoid $\mathcal{C}_r(X)$, so that $\mathcal{Z}_r(X)$ is a topological abelian group whose underlying group is $Z_r(X)$. If U is merely quasi-projective, we define the topological abelian group $\mathcal{Z}_r(U)$ by choosing a projective closure $U \subset X$ with closed complement Y, and then setting

$$\mathcal{Z}_r(U) = \mathcal{Z}_r(X)/\mathcal{Z}_r(Y).$$

It is not hard to see that the topology on $\mathcal{Z}_r(U)$ is independent of the choice of projective closure [25].

Definition 2.1. The *Lawson homology* groups of a complex variety U are

$$L_rH_n(U) = \pi_{n-2r}\mathcal{Z}_r(U).$$

According to [25, §4], the naive group completion of $\mathcal{C}_r(X)$ for a projective variety X coincides with its homotopy theoretic group completion, defined as $\Omega B(\mathcal{C}_r(X))$, where B denotes the bar construction. That is, we have natural isomorphisms

$$L_rH_n(X) \cong \pi_{n-2r+1}B(\mathcal{C}_r(X)).$$

For $U \subset X \supset Y$ as above, we also have

$$L_rH_n(U) \cong \pi_{n-2r+1}\operatorname{cone}(B(\mathcal{C}_r(Y)) \to B(\mathcal{C}_r(X))).$$

These isomorphisms will be important in section 4 for realizing the Lawson homology groups as inductive limits of mixed Hodge structures,

Two special cases of the Lawson homology groups are important to keep in mind. First, taking $r = 0$, the Dold-Thom Theorem [10] gives the isomorphism

$$L_0H_n(X) \cong H_n^{\mathrm{BM}}(X).$$

Thus, the Lawson homology groups represent a generalization of the usual Borel-Moore homology groups. Second, if one takes $n = 2r$ (so that π_0-groups are used in the definition), then it is not hard to show

$$L_rH_{2r}(X) \cong CH_r(X)/CH_r(X)_{\mathrm{alg}\sim 0},$$

the groups of r cycles modulo algebraic equivalence. When X is projective, this formula follows from the observation that two points in $\mathcal{C}_{r,e}(X)$ lie in the same connected component if and only if there exists a curve C joining them if and only if the effective cycles the points represent are part of a family of cycles over C.

For varieties U and Y with U smooth, define $Z_r(U,Y)$ to be the group of cycles consisting of formal sums of closed, integral subvarieties of $U \times Y$ each of which is equidimensional of relative dimension r over a connected component of U. (We refer the reader to [38], in which $Z_r(U,Y)$ is written $z_{equi}(Y \times U/U)(U)$, for more details on this construction.) If U is smooth and connected and Y is projective, then by [12, 1.4] we have a natural isomorphism of the form

$$Z_r(U,Y) \cong \left(\coprod_e \operatorname{Hom}(U, \mathcal{C}_{r,e}(Y))\right)^+,$$

where the superscript $+$ denotes group completion of an abelian monoid. We will have occasion to consider cycle groups $Z_r(U,Y)$ where U is a possibly singular variety and Y is projective. In this situation (and if U is connected), we take the above formula to be the definition of $Z_r(U,Y)$. (If U is not connected, set $Z_r(U,Y) = \bigoplus_i Z_r(U_i,Y)$ where $U_1,\ldots,U_m$ are the connected components of U.) Note that this definition is *not* the same definition for the group of equidimensional cycles found elsewhere, for example in [38].

If X and Y are projective, we topologize the group of cycles $Z_r(X,Y)$ by using that $\coprod_e \mathrm{Hom}(X, \mathcal{C}_{r,e}(Y))$ has the structure of an ind-variety, and thus we can equip it with the analytic topology. Define $\mathcal{Z}_r(Y,X)$ (for X connected) to be the topological abelian group obtained by naive group completion from the topological abelian monoid $\coprod_e \mathrm{Hom}(X, \mathcal{C}_{r,e}(Y))$.

Definition 2.2. The *morphic cohomology* groups of a smooth, projective variety X are given by

$$L^tH^m(X) = \pi_{2t-m}\left(\mathcal{Z}_0(X,\mathbb{P}^t)/\mathcal{Z}_0(X,\mathbb{P}^{t-1})\right),$$

where $\mathcal{Z}_0(X,\mathbb{P}^{t-1})$ is a closed subgroup of $\mathcal{Z}_0(X,\mathbb{P}^t)$ via the closed immersion $\mathbb{P}^{t-1} \subset \mathbb{P}^t$ sending $[x_0 : \cdots : x_{t-1}]$ to $[x_0 : \cdots : x_{t-1} : 0]$.

We refer the reader to [13] for the definition of the morphic cohomology groups for varieties that are not smooth and projective.

If $p : U \to V$ is a projective morphism of varieties, then pushforward of cycles induces a continuous homomorphism

$$p_* : \mathcal{Z}_r(U) \to \mathcal{Z}_r(V).$$

The induced map on homotopy groups defines the pushforward map for Lawson homology:

$$p_* : L_rH_n(U) \to L_rH_n(V).$$

Similarly, if $f : U \to V$ is flat of relative dimension e, pullback of cycles determines a continuous homomorphism

$$f^* : \mathcal{Z}_r(V) \to \mathcal{Z}_{r+e}(U)$$

and hence a pullback map

$$f^* : L_rH_n(V) \to L_{r+e}H_{n+2e}(V)$$

for Lawson homology. For morphic cohomology, there are pullback maps along arbitrary morphisms.

If Y is a closed subscheme of X and $U = X - Y$, then we have the localization long exact sequence for Lawson homology:

$$\cdots \to L_rH_m(Y) \to L_rH_m(X) \to L_rH_m(U) \to L_rH_{m-1}(Y) \to \cdots.$$

If U is smooth of pure dimension d, we have a Poincaré duality isomorphism

$$L_rH_n(U) \cong L^{d-r}H^{2d-n}(U).$$

Morphic cohomology forms a bigraded ring $L^*H^*(X)$ under a cup product operation, and Lawson homology $L_*H_*(X)$ is a $L^*H^*(X)$-module under a cap product operation:

$$- \cap - : L^tH^m(X) \otimes L_rH_n(X) \to L_{r-t}H_{n-m}(X).$$

Together, morphic and Lawson (co)homology form a twisted duality theory in the sense of Bloch-Ogus. See [14] for details.

The s-map in Lawson homology, defined in [19, 16], is a map of the form

$$s : L_rH_m(X) \to L_{r-1}H_m(X)$$

and it admits a geometric description (at least for X projective) in terms of Lefschetz pencils. The s-map can equivalently be defined via cap product with the s element, which is defined to be a generator of $L^1H^0(pt) \cong H^0_{\text{sing}}(pt) = \mathbb{Z}$. The composition of

$$L_rH_m(X) \xrightarrow{s} L_{r-1}H_m(X) \xrightarrow{s} \cdots \xrightarrow{s} L_0H_m(X) \cong H^{\text{BM}}_m(X)$$

gives a map from Lawson homology to Borel-Moore homology. For X smooth and projective, the corresponding map from morphic cohomology to singular cohomology that arises from Poincaré duality,

$$L^tH^n(X) \to H^n_{\text{sing}}(X),$$

may also be defined by viewing $\mathcal{Z}_0(X, \mathbb{P}^t)/\mathcal{Z}_0(X, \mathbb{P}^{t-1})$ as a subspace of $\text{Maps}(X, \mathcal{Z}_0(\mathbb{P}^t)/\mathcal{Z}_0(\mathbb{P}^{t-1}))$ and using that $\mathcal{Z}_0(\mathbb{P}^t)/\mathcal{Z}_0(\mathbb{P}^{t-1})$ is a $K(\mathbb{Z}, 2t)$ space. The s-maps and the maps from Lawson homology to Borel-Moore homology are natural with respect to pushforwards along projective maps and pullbacks along flat maps, and they commute with the boundary maps in localization long exact sequences.

There are also natural maps from Bloch's higher Chow groups [5] to the Lawson homology groups,

$$CH_r(X, n) \to L_rH_{2r+n}(X),$$

and the composition

$$CH_r(X, n) \to L_rH_{2r+n}(X) \to H^{\text{BM}}_{2r+n}(X)$$

is the usual map from Bloch's groups to Borel-Moore homology. For X smooth, Poincaré duality gives maps of the form

$$H^m_{\mathcal{M}}(X, \mathbb{Z}(t)) \to L^tH^m(X) \to H^m_{\text{sing}}(X),$$

where $H_{\mathcal{M}}$ denotes motivic cohomology. Such maps exist even if X is singular.

Given projective varieties X and Y and a cycle $\Gamma \in Z_e(X, Y)$ (i.e., an equidimensional correspondence) we have an induced map

$$\Gamma_* : L_rH_n(X) \to L_{r+e}H_{n+2e}(Y),$$

defined as follows: If Γ is effective, it is given by a morphism $X \to \mathcal{C}_e(Y)$ and hence determines a map

$$\mathcal{C}_r(X) \to \mathcal{C}_r(\mathcal{C}_e(Y))$$

of topological abelian monoids. Composing with the "trace map"

$$\mathcal{C}_r(\mathcal{C}_e(Y)) \to \mathcal{C}_{r+e}(Y)$$

of [18, 7.1], we obtain the map

$$\mathcal{C}_r(X) \to \mathcal{C}_{r+e}(Y)$$

of topological abelian monoids. Applying π_{n-2r} to the induced map on naive group completions gives the map Γ_*. If Γ is not effective, Γ_* is defined as the difference of the maps induced by its positive and negative parts. When X and Y are smooth, the cycle Γ determines a class in $L^tH^{2t}(X \times Y) \cong CH^t(X \times Y)/CH^t(X \times Y)_{\text{alg}\sim 0}$ (where $t = \dim(Y) - e$), and the map Γ_* defined here coincides with the map given by cap product with this class.

Since it provides much intuition about the nature of Lawson and morphic (co)homology and since we will refer to it at various points below, we conclude this section with a discussion of Suslin's Conjecture.

Conjecture 2.3 (Suslin's Conjecture, Form I). *Let $\pi : CW \to Var/\mathbb{C}$ be the morphism of sites, from the category of topological spaces having the homotopy type of CW complexes to the category of quasi-projective complex varieties, that sends a variety U to its associated analytic space U^{an}. For any smooth, quasi-projective variety U and abelian group A, there is a natural isomorphism*

$$L^t H^n(U, A) \cong \mathbb{H}^n_{Zar}(U, tr^{\leq t} \mathbb{R}\pi_* A),$$

where $\mathbb{H}_{Zar}$ denotes Zariski hypercohomology, $\mathbb{R}\pi_* A$ is the total right derived push-forward along π of the constant sheaf A on CW, and $tr^{\leq t}$ denotes good truncation in cohomological degree t.

Note the close analogy between Suslin's Conjecture and the Beilinson-Lichtenbaum conjecture, which asserts that the motivic cohomology groups with finite coefficients are given by the formula

$$H^n_{\mathcal{M}}(U, \mathbb{Z}/n(t)) = \mathbb{H}_{Zar}(U, tr^{\leq t} \mathbb{R}\epsilon_* \mu_n^{\otimes t}).$$

In fact, using a theorem of Suslin-Voevodsky [37, 9.1], which implies that morphic and motivic cohomology coincide with finite coefficients, one sees that Suslin's Conjecture is a generalization of the Beilinson-Lichtenbaum Conjecture (see [17]). Voevodsky [41] has proven the Beilinson-Lichtenbaum Conjecture when n is a power of 2, and the general case is expected to follow from recent work of Rost and Voevodsky.

By Bloch-Ogus [6], Suslin's Conjecture says that the morphic cohomology groups are the D_2-terms of the Leray spectral sequence for singular cohomology:

$$E_2^{p,q} = H^p_{Zar}(U, \mathcal{H}^q_{\text{sing}}) \Longrightarrow H^{p+q}_{\text{sing}}(U)$$

where $\mathcal{H}^q_{\text{sing}}$ denote the Zariski sheafification of $U \mapsto H^q_{\text{sing}}(U)$. Moreover, one expects that the s map ought to coincide with the usual map on D_2-terms in this spectral sequence.

It is easy to show (see [17]) that Suslin's Conjecture for morphic cohomology is equivalent to the following conjecture for Lawson homology. This version has the advantage that it can be studied and verified on a case-by-case basis.

Conjecture 2.4 (Suslin's Conjecture, Form II). *For any abelian group A and quasi-projective variety Y of dimension d, the map*

$$L_s H_n(Y, A) \to H_n^{BM}(Y, A)$$

is an isomorphism for $n \geq d + s$ and a monomorphism for $n = d + s - 1$.

From the validity of this conjecture, one deduces that the image of

$$L_s H_n(Y, A) \to H_n^{\text{BM}}(Y, A)$$

is $N_{n-s} H_n^{\text{BM}}(Y, A)$ for all Y, n, and s. Here, $N_\bullet$ denotes the niveau filtration:

$$N_k H_n^{\text{BM}}(Y, A) := \sum_{\substack{V \subset Y \\ \dim(V) \leq k}} \text{im}\left(H_n^{\text{BM}}(V, A) \to H_n^{\text{BM}}(Y, A)\right)$$

Since we will need it later, we now state and prove an (easy) special case of Suslin's Conjecture.

Proposition 2.5. *For a projective variety Y and abelian group A, the image of*

$$L_rH_n(Y,A) \to H_n^{sing}(Y,A)$$

is contained in $N_{n-r}H_n^{sing}(Y,A)$. When $n=2r+1$, we have a surjection

$$L_rH_n(Y,A) \twoheadrightarrow N_{r+1}H_{2r+1}^{sing}(Y,A).$$

For any quasi-projective variety V, if $\dim(V) \leq r+1$, then the map

$$L_rH_{2r+1}(V,A) \xrightarrow{\cong} N_{r+1}H_{2r+1}^{BM}(V,A) = H_{2r+1}^{BM}(V,A)$$

is an isomorphism.

Proof. The first assertion is given by [11, 4.3]. The surjectivity of $L_rH_{2r+1}(Y,A) \to N_{r+1}H_{2r+1}^{\text{sing}}(Y,A)$ follows by naturality for pushforwards from the final assertion.

If U is a smooth variety of dimension at most $r+1$, then the map $L_rH_{2r+1}(U,A) \to H_r^{\text{BM}}(U,A)$ is isomorphic, via Poincaré duality, to the map

$$L^1H^1(U,A) \to H^1_{\text{sing}}(U,A),$$

which is seen to be an isomorphism by using [18, 9.3] and the Five Lemma. For an arbitrary variety V with $\dim(V) \leq r+1$, let $V_s \subset V$ be the singular locus and set $U = V - V_s$. Since U is smooth, the map $L_rH_{2r+1}(U,A) \to H_{2r+1}^{\text{BM}}(U,A)$ is an isomorphism. Since $\dim(V_s) \leq r$, we have $L_rH_{2r+1}(V_s,A) = 0 = H_{2r+1}^{\text{BM}}(V_s,A)$ and $L_rH_{2r}(V_s,A) = Z_r(V_s) \otimes A = H_{2r}^{\text{BM}}(V_s,A)$. The result now follows by naturality for localization sequences and the Five Lemma. □

3. Inductive Limits of Mixed Hodge Structures

A mixed Hodge structure, MHS for short, consists of a finitely generated abelian group H together with a increasing "weight" filtration $W_\bullet$ on the finite dimensional $\mathbb{Q}$ vector space $H_\mathbb{Q} := H \otimes_\mathbb{Z} \mathbb{Q}$,

$$\cdots \subset W_n(H_\mathbb{Q}) \subset W_{n+1}(H_\mathbb{Q}) \subset \cdots,$$

and a decreasing "Hodge" filtration $F^\bullet$ on the finite dimensional $\mathbb{C}$-vector space $H_\mathbb{C} := H \otimes_\mathbb{Z} \mathbb{C}$,

$$\cdots \supset F^q(H_\mathbb{C}) \supset F^{q+1}(H_\mathbb{C}) \supset \cdots.$$

The filtrations are assumed to be complete in the sense that $\cup_q F^q(H_\mathbb{C}) = H_\mathbb{C}$, $\cap_q H^q(H_\mathbb{C}) = 0$, $\cup_n W_n(H_\mathbb{Q}) = H_\mathbb{Q}$, and $\cap_n W_n(H_\mathbb{Q}) = 0$. Since H is finitely generated, these conditions imply that both filtrations are finite. Let $\overline{F}^q(H_\mathbb{C})$ denote the complex conjugate of $F^q(H_\mathbb{C})$ with respect to the real structure $H_\mathbb{R} \subset H_\mathbb{C}$. Finally, the data are assumed to satisfy

$$Gr_n^W(H_\mathbb{C}) = F^q Gr_n^W(H_\mathbb{C}) \oplus \overline{F}^{n-q+1} Gr_n^W(H_\mathbb{C}), \quad \text{for all } n,q,$$

where $Gr_n^W(H_\mathbb{C}) = Gr_n^W(H_\mathbb{Q})_\mathbb{C} = (W_n(H_\mathbb{Q})/W_{n-1}(H_\mathbb{Q}))_\mathbb{C}$ and $F^\bullet$, $\overline{F}^\bullet$ are the induced filtrations on these subquotients. Defining

$$H^{p,q} = F^p Gr_{p+q}^W(H_\mathbb{C}) \cap \overline{F}^q Gr_{p+q}^W(H_\mathbb{C}),$$

this last condition is equivalent to the more familiar property that

$$Gr_n^W(H_\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}, \quad \text{for all } n.$$

The *Hodge type* of H refers to those pairs of integers (p,q) such that $H^{p,q} \neq 0$. A MHS H is *pure of weight* n if $W_{m-1}(H_{\mathbb{Q}}) = W_m(H_{\mathbb{Q}})$ unless $m = n$ (equivalently, if those (p,q) with $H^{p,q} \neq 0$ satisfy $p+q=n$).

A morphism of MHS's $f : H \to H'$ is a homomorphism of abelian groups f such that the induced maps $f_{\mathbb{Q}} : H_{\mathbb{Q}} \to H'_{\mathbb{Q}}$ and $f_{\mathbb{C}} : H_{\mathbb{C}} \to H'_{\mathbb{C}}$ preserve the filtrations. We will also let MHS refer to the category of all mixed Hodge structures.

The primary example of MHS's are given by the singular cohomology groups of complex varieties, as established by Deligne [8, 9]. We use the isomorphism $H_m^{\text{sing}}(X,\mathbb{Q}) \cong \text{Hom}_{\mathbb{Q}}(H^m_{\text{sing}}(X,\mathbb{Q}),\mathbb{Q})$, to define the weight filtration on $H_m^{\text{sing}}(X,\mathbb{Q})$ and similarly for the Hodge filtration on $H_m^{\text{sing}}(X,\mathbb{C})$, so that $H_m^{\text{sing}}(X)$ is a MHS. More generally, Deligne shows that the singular cohomology of a pair of varieties (X,Y), with Y closed in in X, is a MHS. In particular, cohomology with compact supports, defined by $H_c^*(U) = H^*_{\text{sing}}(X,Y)$ for a projective closure X of U with complement Y, is a MHS, and the isomorphism $H_n^{\text{BM}}(U,\mathbb{Q}) \cong \text{Hom}_{\mathbb{Q}}(H_c^n(U,\mathbb{Q}))$ makes Borel-Moore homology into a MHS. If X is smooth and projective, then $H_n^{\text{sing}}(X)$ is a pure MHS of weight $-n$.

For any integer n, let $\mathbb{Z}(n)$ denote the pure MHS whose underlying abelian group is $\mathbb{Z}$, having pure weight $-2n$ and Hodge type $(-n,-n)$. There is an evident notion of tensor product of MHS's and we define $H(n) = H \otimes \mathbb{Z}(n)$. Equivalently, the MHS $H(n)$ is obtained from H by shifting the indexing of the filtrations: $W_m(H(n)_{\mathbb{Q}}) = W_{m+2n}(H_{\mathbb{Q}})$ and $F^p(H(n)_{\mathbb{C}}) = F^{p+n}(H_{\mathbb{C}})$. We define $H_n^{\text{sing}}(X,\mathbb{Z}(m)) = H_n^{\text{sing}}(X)(-m)$ and $H^n_{\text{sing}}(X,\mathbb{Z}(m)) = H^n_{\text{sing}}(X)(m)$, and similarly for Borel-Moore homology and cohomology with compact support.

The category MHS is an abelian category. Moreover, the functors $H \mapsto W_n(H_{\mathbb{Q}})$, $H \mapsto Gr_n^W(H_{\mathbb{Q}})$, $H \mapsto F^p(H_{\mathbb{C}})$, and $H \mapsto Gr_F^p(H_{\mathbb{C}})$ (where the first two take values in category of $\mathbb{Q}$-MHS's and the latter two in the category of complex vector spaces) are exact [8, 2.3.5]. Of particular importance to us will be the Ext groups $\text{Ext}^n_{\text{MHS}}(\mathbb{Z}(0),H)$. For $n=0$, we have

$$\Gamma(H) := \text{Hom}_{\text{MHS}}(\mathbb{Z}(0),H) = H \cap W_0(H_{\mathbb{Q}}) \cap F^0(H_{\mathbb{C}}).$$

That is, $\Gamma(H)$ is the subset of H consisting of those elements that map to both $W_0(H_{\mathbb{Q}}) \subset H_{\mathbb{Q}}$ and $F^0(H_{\mathbb{C}}) \subset H_{\mathbb{C}}$. In particular, observe that $H_{\text{tor}} \subset \Gamma(H)$ for any MHS H. The group $\text{Ext}^1_{\text{MHS}}(\mathbb{Z}(0),H)$, which we will write as $\mathcal{J}(H)$, is described in Proposition 3.1 below; the groups $\text{Ext}^n_{\text{MHS}}$ vanish for $n \geq 2$ [3, 1.10].

The Lawson homology groups are, in general, not finitely generated, and thus cannot be enriched to MHS's. Rather, as established by Friedlander-Mazur [19] and Lima-Filho [26], they are filtered inductive limits of MHS's. (We will review their constructions in the next section.) Define an IMHS (inductive limit of mixed Hodge structures) to be a countable inductive limit of mixed Hodge structures with bounded filtration lengths. That is, an IMHS is system of MHS's $\{H_\alpha, \alpha \in I\}$ indexed by a filtered category I with countably many objects, such that there exist integers $M < N$ so that $W_M((H_\alpha)_{\mathbb{Q}}) = 0$, $W_N((H_\alpha)_{\mathbb{Q}}) = (H_\alpha)_{\mathbb{Q}}$, $F^N((H_\alpha)_{\mathbb{Q}}) = 0$, and $F^M((H_\alpha)_{\mathbb{Q}}) = (H_\alpha)_{\mathbb{C}}$ for all $\alpha \in I$. (This condition on the filtration lengths is fulfilled by the Lawson homology groups, as we shall see, and allows for the general results of Deligne [8, 9] concerning filtered objects to hold.) A morphism of IMHS is morphism of filtered systems of MHS's, and we write IMHS also for the category of all IMHS's.

Given an IMHS $\{H_\alpha, \alpha \in I\}$, define the abelian group $H := \varinjlim_\alpha H_\alpha$ and define filtrations $W_n(H_\mathbb{Q}) = \varinjlim_\alpha W_n((H_\alpha)_\mathbb{Q})$ and $F^q(H_\mathbb{C}) = \varinjlim_\alpha F^q((H_\alpha)_\mathbb{C})$. Since W_m and F^q are exact functors on MHS, the triple $(H, W_\bullet, F^\bullet)$ satisfies all the axioms listed above for a MHS except, of course, that H need not be finitely generated. Note that the image of the canonical map $H_\alpha \to H$, with the induced filtrations, coincides with $H_\alpha/(\ker(H_\alpha \to H_\beta))$ for some $\alpha \to \beta$ in I, and thus this image is a MHS. We see, then, that an IMHS is equivalent to a triple $(H, W_\bullet, F^\bullet)$, where H is a countable abelian group, $W_\bullet(H_\mathbb{Q})$ and $F^\bullet(H_\mathbb{C})$ are finite, complete filtrations satisfying

$$Gr_n^W(H_\mathbb{C}) = \oplus_{p+q=n} H^{p,q}$$

where

$$H^{p,q} = F^p Gr^W_{p+q}(H_\mathbb{C}) \cap \overline{F}^q Gr^W_{p+q}(H_\mathbb{C}),$$

and such that every finitely generated subgroup of H is contained in a finitely generated subgroup H' so that $(H', W_\bullet|_{H'_\mathbb{Q}}, F^\bullet|_{H'_\mathbb{C}})$ is a MHS.

We refer the reader to [20] for general properties of ind-categories associated to abelian categories. In particular, we use that

$$\operatorname{Ext}^n_{\mathrm{IMHS}}(\mathbb{Z}(0), H) = \varinjlim_\alpha \operatorname{Ext}^n_{\mathrm{MHS}}(\mathbb{Z}(0), H_\alpha)$$

where H_α ranges over all finitely generate sub-IMHS's of H (i.e., over all sub-IMHS that are actually MHS's).

In general, for a IMHS H, we write

$$\Gamma(H) = \mathrm{Hom}_{\mathrm{IMHS}}(\mathbb{Z}(0), H)$$

and

$$\mathcal{J}(H) = \operatorname{Ext}^1_{\mathrm{IMHS}}(\mathbb{Z}(0), H).$$

Since $\Gamma(H) = \varinjlim_\alpha \Gamma(H_\alpha)$, we have $\Gamma(H) = H \cap W_0(H_\mathbb{Q}) \cap F^0(H_\mathbb{C})$.

Proposition 3.1 (cf. [7, 21]). *For a IMHS H, we have*

$$\mathcal{J}(H) \cong \frac{W_0(H_\mathbb{C})}{W_0(H) + F^0 W_0(H_\mathbb{C})};$$

that is, $\mathcal{J}(H)$ is the quotient of the complex vector space $W_0(H_\mathbb{C})/F^0W_0(H_\mathbb{C})$ by the action of $W_0(H) = \ker(H \to H_\mathbb{Q}/W_0(H_\mathbb{Q}))$.

In particular, if $W_0(H_\mathbb{Q}) = H_\mathbb{Q}$, then we have an exact sequence

$$0 \to \Gamma(H) \to H \to H_\mathbb{C}/F^0(H_\mathbb{C}) \to \mathcal{J}(H) \to 0$$

and the torsion subgroup of $\mathcal{J}(H)$ is

$$\mathcal{J}(H)_{tor} \cong (H/\Gamma(H)) \otimes \mathbb{Q}/\mathbb{Z} \cong \operatorname{coker}(\Gamma(H) \otimes_\mathbb{Z} \mathbb{Q}/\mathbb{Z} \to H \otimes_\mathbb{Z} \mathbb{Q}/\mathbb{Z}).$$

If H is pure of weight -1, then

$$\mathcal{J}(H) \cong \frac{H^{-1,0} \oplus H^{-2,1} \oplus \cdots}{H} \cong H \otimes \mathbb{R}/\mathbb{Z}$$

and

$$\mathcal{J}(H)_{tor} \cong H \otimes \mathbb{Q}/\mathbb{Z}.$$

If H is pure of weight -1 and finitely generated (i.e., actually a MHS), then $\mathcal{J}(H)$ is a complex torus.

Proof. For a MHS H, Jannsen [21], building on results of Carlson [7], establishes the formula

$$\mathcal{J}(H) = \frac{W_0(H_{\mathbb{C}})}{W_0(H) + F^0 W_0(H_{\mathbb{C}})}.$$

If $W_0(H_{\mathbb{Q}}) = H_{\mathbb{Q}}$, then we clearly have an exact sequence

$$0 \to \Gamma(H) \to H \to \frac{H_{\mathbb{C}}}{F^0(H_{\mathbb{C}})} \to \mathcal{J}(H) \to 0,$$

and the formula for $\mathcal{J}(H)_{\text{tor}}$ follows from the long exact sequence for $\text{Tor}_*(-, \mathbb{Q}/\mathbb{Z})$. If H is a MHS of pure of weight -1, then $W_0(H_{\mathbb{C}}) = H_{\mathbb{C}} = \bigoplus_p H^{p,-p-1}$ and so $H_{\mathbb{C}}/F^0(H_{\mathbb{C}}) = H^{-1,0} \oplus H^{-2,1} \oplus \cdots$. Since $\overline{H^{p,-p-1}} = H^{-p-1,p}$, we have $H_{\mathbb{C}}/F^0(H_{\mathbb{C}}) \cong H_{\mathbb{R}}$ as abelian groups.

The corresponding formulas for IMHS's follow by taking filtered inductive limits, using that W_0 and F^0 are exact functors. □

For example, letting $H = H^{\text{sing}}_{2r+1}(X, \mathbb{Z}(r))$ for a projective variety X, we obtain

$$\mathcal{J}_r(X) := \mathcal{J}(H) = H_{\mathbb{C}}/(H + F^0(H_{\mathbb{C}})),$$

the r-th intermediate Griffiths Jacobian of X. (This definition coincides with that given in the introduction up to natural isomorphism.) If X is smooth, then H is pure of weight -1 so that $\mathcal{J}_r(X)$ is a complex torus isomorphic to $H^{\text{sing}}_{2r+1}(X, \mathbb{R})/H^{\text{sing}}_{2r+1}(X, \mathbb{Z})$. Letting $r = \dim(X) - 1$ and applying Poincaré duality, we have

$$\mathcal{J}_{\dim(X)-1}(X) \cong H^1_{\text{sing}}(X, \mathbb{R})/H^1_{\text{sing}}(X, \mathbb{Z}) = \text{Pic}^0(X),$$

recovering the classical Picard variety of X.

Since $\text{Ext}^n_{\text{IMHS}} = 0$ for $n \geq 2$, associated to a short exact sequence

$$0 \to H' \to H \to H'' \to 0$$

of IMHS's, we have the six-term exact sequence

$$0 \to \Gamma(H') \to \Gamma(H) \to \Gamma(H'') \to \mathcal{J}(H') \to \mathcal{J}(H) \to \mathcal{J}(H'') \to 0$$

of abelian groups. In particular, $\mathcal{J}(-)$ is a right exact functor from IMHS to abelian groups. In fact, if we topologize $\mathcal{J}(H)$ by declaring $W_0(H_{\mathbb{C}}) \twoheadrightarrow \mathcal{J}(H)$ to be a quotient map of topological spaces, where $W_0(H_{\mathbb{C}})$ is topologized by viewing it as a filtered colimit of finite dimensional subspaces, then $\mathcal{J}(-)$ takes IMHS's to topological abelian groups and it takes surjections of IMHS's to quotients in the category of topological abelian groups.

Abel-Jacobi maps, both the classical versions and the morphic ones we discuss in this paper, are obtained from the boundary map $\Gamma(H'') \to \mathcal{J}(H')$ in the above six-term exact sequence coming from certain types of short exact sequences of IMHS's [21, 9.2]. Since we will need it later, we show that the restriction of this boundary map to torsion subgroups admits an alternative description.

Lemma 3.2. *Suppose*

$$0 \to H' \to H \to H'' \to 0 \tag{3.3}$$

is a short exact sequence of IMHS's such that $W_0(H'_{\mathbb{Q}}) = H'_{\mathbb{Q}}$, $W_0(H_{\mathbb{Q}}) = H_{\mathbb{Q}}$, *and* $W_0(H''_{\mathbb{Q}}) = H''_{\mathbb{Q}}$. *Then the restriction of the boundary map coming from the long exact sequence for* $\text{Ext}^*_{IMHS}(\mathbb{Z}(0), -)$ *to torsion subgroups,*

$$\Gamma(H'')_{tor} \to \mathcal{J}(H')_{tor} \cong (H'/\Gamma(H')) \otimes \mathbb{Q}/\mathbb{Z},$$

coincides with the composition of

$$\Gamma(H'')_{tor} \rightarrowtail H''_{tor} \to H' \otimes \mathbb{Q}/\mathbb{Z} \twoheadrightarrow (H'/\Gamma(H')) \otimes \mathbb{Q}/\mathbb{Z},$$

where the second map comes from the long exact sequence for $\mathrm{Tor}_*(-, \mathbb{Q}/\mathbb{Z})$ *applied to (3.3) regarded as a short exact sequence of abelian groups.*

Proof. Observe that we have a diagram of abelian groups with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H' & \longrightarrow & H & \longrightarrow & H'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H'_{\mathbb{C}}/F^0(H'_{\mathbb{C}}) & \longrightarrow & H_{\mathbb{C}}/F^0(H_{\mathbb{C}}) & \longrightarrow & H''_{\mathbb{C}}/F^0(H''_{\mathbb{C}}) & \longrightarrow & 0, \end{array} \tag{3.4}$$

and that the kernels of the vertical maps are $\Gamma(H')$, $\Gamma(H)$, and $\Gamma(H'')$ and the cokernels of these maps are $\mathcal{J}(H')$, $\mathcal{J}(H)$, and $\mathcal{J}(H'')$. Examining the proof of Proposition 3.1 found in [7, 21], we see that the map

$$\Gamma(H'') \to \mathcal{J}(H') \cong H'_{\mathbb{C}}/(H' + F^0(H'_{\mathbb{C}}))$$

coming from the six-term exact sequence given by the Snake Lemma applied to (3.4) coincides with the boundary map coming from the six-term exact sequence for $\mathrm{Ext}^*_{\mathrm{IMHS}}(\mathbb{Z}(0), -)$. A diagram chase shows that restriction of the map coming from the Snake Lemma to torsion subgroups is also given by the boundary map in the long exact sequence for $\mathrm{Tor}_*(-\mathbb{Q}/\mathbb{Z})$. □

4. IMHS's for Lawson Homology

The cornerstone of this paper is the observation due to Friedlander-Mazur [19] that the Lawson homology groups of a projective variety X are IMHS's. The basic idea is that the Lawson groups of X are subgroups of the homology groups of $\mathcal{Z}_r(X)$, and, since $\mathcal{Z}_r(X)$ is given by group completing $\mathcal{C}_r(X)$, we have

$$H_*^{\mathrm{sing}}(\mathcal{Z}_r(X), \mathbb{Z}) \cong H_*^{\mathrm{sing}}(\mathcal{C}_r(X), \mathbb{Z}) \otimes_{H_0^{\mathrm{sing}}(\mathcal{C}_r(X), \mathbb{Z})} H_0^{\mathrm{sing}}(\mathcal{Z}_r(X), \mathbb{Z}).$$

Because $\mathcal{C}_r(X)$ is an infinite disjoint union of projective varieties, $H_*^{\mathrm{sing}}(\mathcal{C}_r(X), \mathbb{Z})$ is a countably infinite direct sum of MHS's. The tensor construction amounts to inverting all the endomorphisms of $H_*^{\mathrm{sing}}(\mathcal{C}_r(X), \mathbb{Z})$ induced by multiplication by classes of $\pi_0(\mathcal{C}_r(X))$ (which is a countable group) using the monoid product for $\mathcal{C}_r(X)$. These endomorphisms are morphisms of IMHS and hence $H_*^{\mathrm{sing}}(\mathcal{Z}_r(X))$ is an IMHS. Finally, the injection

$$\pi_*(\mathcal{Z}_r(X)) \rightarrowtail H_*^{\mathrm{sing}}(\mathcal{Z}_r(X))$$

is given rationally, in positive degrees, by the kernel of a morphism of IMHS's, and hence $\pi_q(\mathcal{Z}_r(X)) = L_r H_{2r+q}(X)$ is a sub-IMHS of $H_q^{\mathrm{sing}}(\mathcal{Z}_r(X))$ for $q \geq 1$. For $q = 0$, one declares $L_q H_{2q}(X)$ to have the trivial Hodge structure — i.e., pure of weight 0 and Hodge type $(0,0)$, so that $\Gamma(L_q H_{2q}(X)) = L_q H_{2q}(X)$.

Lima-Filho [26] has extended the ideas of Friedlander-Mazur to endow the Lawson homology groups of quasi-projective varieties with IMHS's. We follow his approach in our proof of the following theorem.

Theorem 4.1 (cf. [19, 26]). *The Lawson homology groups of complex varieties may be endowed with IMHS's such that the following properties hold.*

(1) *If X is projective, then*

$$L_rH_n(X) \rightarrowtail H^{sing}_{n-2r}(\mathcal{Z}_r(X), \mathbb{Z}(0))$$

is an injective morphism of IMHS's, where the IMHS for the target is described above.

(2) *The maps in Lawson homology given by pushforward along a projective morphism and pullback along a flat morphism are morphisms of IMHS's. Moreover, the boundary map in the long exact localization sequence for Lawson homology associated to an open embedding $U \subset X$ with closed complement $Y = X - U$ is a morphism of IMHS's.*

(3) *For a quasi-projective variety U, the natural isomorphism $L_0H_n(U) \cong H^{BM}_n(U)$ coming from the Dold-Thom theorem is an isomorphism of MHS's.*

(4) *For a quasi-projective variety U, the s map*

$$s : L_rH_n(U) \to L_{r-1}H_n(U)(-1)$$

is a morphism of IMHS's. That is, the s-map induces maps

$$s : W_m(L_rH_n(U, \mathbb{Q})) \to W_{m+2}(L_{r-1}H_n(U, \mathbb{Q}))$$

and

$$s : F^p(L_rH_n(U, \mathbb{C})) \to F^{p+1}(L_{r-1}H_n(U, \mathbb{C})).$$

(5) *For projective varieties X and W and an equidimensional correspondence $\Gamma \in Z_e(X, W)$, the induced map*

$$\Gamma_* : L_rH_n(X) \to L_{r+e}H_{n+2e}(W)$$

is a morphism of IMHS's. More generally, if $Y \subset X$ and $T \subset W$ are closed subschemes, with open complements U and V, such that the image of $\mathcal{Z}_r(Y)$ under Γ_* is contained in $\mathcal{Z}_r(T)$, then the commutative "infinite ladder" of homotopy groups associated to the commutative diagram

$$\begin{array}{ccc}
\mathcal{Z}_r(Y) & \xrightarrow{\Gamma_*} & \mathcal{Z}_{r+e}(T) \\
\subset\downarrow & & \downarrow\subset \\
\mathcal{Z}_r(X) & \xrightarrow{\Gamma_*} & \mathcal{Z}_{r+e}(W) \\
\downarrow & & \downarrow \\
\mathcal{Z}_r(U) & \xrightarrow{\Gamma_*} & \mathcal{Z}_{r+e}(V)
\end{array}$$

of topological abelian groups (whose columns are fibration sequences) consists of morphisms of IMHS's.

Remark 4.2. Parts (3) and (4) of the Theorem show that the maps from the Lawson homology groups to the Borel-Moore homology groups are morphism of IMHS's of the form

$$L_rH_n(U) \to H^{\mathrm{BM}}_n(U, \mathbb{Z}(r)).$$

Take $U = X$ to be smooth and projective (so that, in particular, $H^{\mathrm{BM}}_n(X, \mathbb{Z}(r)) = H^{\mathrm{sing}}_n(X, \mathbb{Z}(r))$). Then, for any r and n, the target of this map is a pure MHS of weight $2r - n$ and the non-trivial Hodge numbers (p, q) satisfy $r - n \leq p, q \leq r$. By part (1) of the Theorem, the non-trivial weights of the source satisfy $2r - n \leq w \leq 0$ and the non-trivial Hodge numbers (p, q) satisfy $2r - n \leq p, q \leq 0$. The properties of this map of IMHS's will be of central importance in the remainder of this paper.

The proof of the Theorem occupies the rest of this section. We begin with a construction of the IMHS for $L_rH_n(U)$ for a general quasi-projective variety U.

The basic object we deal with is a countably infinite disjoint union of complex varieties M equipped with a base point and a commutative, associative, unital pairing $M \times M \to M$ given by a morphism of ind-varieties. That is, M is an abelian monoid object in the category of ind-varieties whose connected components are actually varieties. We also require that the associated topological abelian monoid M^{an}, obtained by taking associated analytic spaces of complex points, is *good* in the sense of Quillen [19] — i.e., the canonical map $(M^{an})^+ \to \Omega B(M^{an})$ is a homotopy equivalence, where $(-)^+$ denotes naive group completion and $B(-)$ denotes the bar construction. (In what follows, we typically drop the superscript *an* and let the context determine whether M denotes an ind-variety or its associated topological space.) The property of goodness holds for all the examples we consider by the results of [25]. Let $\mathcal{M}$ denote the category consisting of such abelian monoids. A morphism $M \to M'$ in $\mathcal{M}$ is a morphism of abelian monoid objects in the category of ind-varieties. More generally, we consider simplicial objects in $\mathcal{M}$. For such a simplicial object $M_\bullet = (n \mapsto M_n)$, it follows from [19, Q2] that the map $M_\bullet^+ \to \Omega B(M)$ is a homotopy equivalence. We often assume $M_\bullet$ is connected or at least group-like, meaning that the abelian monoid $\pi_0(M_\bullet)$ is actually a group.

A simplicial object $M_\bullet$ in $\mathcal{M}$ is a filtered limit of sub-objects each of which is actually a simplicial variety. To see this, given any integer m and any $V \subset M_m$ consisting of a union of finitely many connected components of M_m, define $V_n \subset M_n$ to be the smallest union of connected components of M_n that contains the image of V_m under each of the finitely many maps $M_n \to M_m$ coming from the simplicial structure. Then one may readily verify that $n \mapsto V_n$ is a simplicial variety and that $M_\bullet$ is the union of such sub-simplicial objects.

For a simplicial ind-variety $V_\bullet$, let

$$H_*^{\text{sing}}(V_\bullet, \mathbb{Z})$$

denote the singular homology groups of $|V_\bullet^{an}|$, the geometric realization of the simplicial space $d \mapsto V_d^{an}$. By the work of Deligne [8, 9], the groups $H_n^{\text{sing}}(V_\bullet, \mathbb{Z})$ are MHS's in a natural way — that is, the functor from simplicial varieties to graded abelian groups given by taking homology of the geometric realization of the associated simplicial space factors through the category of graded MHS's. It follows that the homology groups of a simplicial object $M_\bullet$ in $\mathcal{M}$ are IMHS's in a natural way. (To see that the associated filtrations are finite, note that $H_n(M_\bullet)$ depends only on $H_p(n \mapsto H_q(M_n))$, for finitely many p, q, and $H_q(M_n)$ is a direct sum of homology groups of the form $H_q(V)$ with V a variety. Finally, the Hodge type (i, j) of $H_q(V)$ is bounded by $-q \leq i, j \leq 0$.)

We now explain how the homotopy groups $\pi_*(M_\bullet)$ of a group-like simplicial object in $\mathcal{M}$ inherit IMHS's from those of $H_*^{\text{sing}}(M_\bullet)$. First observe that if $M_\bullet$ is any group-like simplicial topological abelian monoid, the Hurewicz map

$$\pi_q(M_\bullet) \to H_q^{\text{sing}}(M_\bullet)$$

admits a canonical splitting. To see this, upon taking singular simplicial sets, it suffices to prove this result for a group-like simplicial (discrete) abelian monoid $X_\bullet$. Let $X_\bullet^+$ denote the simplicial abelian group obtained by group completing in each degree, and recall that the canonical maps $X_\bullet^+ \to \Omega B(X_\bullet)$ is a weak equivalence by [19, Q1]. Since $X_\bullet$ is group-like, the map $X_\bullet \to \Omega B(X_\bullet)$ and hence the map

$X_\bullet \to X_\bullet^+$ are also weak equivalences. We therefore may assume $X_\bullet$ is a simplicial abelian group. Then the splitting of

$$\pi_q(X_\bullet) \to H_q^{\mathrm{sing}}(X_\bullet) = \pi_q(\mathbb{Z}X_\bullet)$$

is given by the splitting of the map of simplicial sets

$$X_\bullet \rightarrowtail \mathbb{Z}(X_\bullet)$$

defined by taking sums in $X_\bullet$ in the evident manner.

For $M_\bullet$ a group-like simplicial object in $\mathcal{M}$, let

$$\theta : H_*^{\mathrm{sing}}(M_\bullet) \to H_*^{\mathrm{sing}}(M_\bullet)$$

denote the idempotent endomorphism associated to the splitting of $\pi_*(M_\bullet) \rightarrowtail H_*^{\mathrm{sing}}(M_\bullet)$ we have constructed.

Lemma 4.3. *For $M_\bullet$ a connected simplicial object in $\mathcal{M}$, the endomorphism θ is a morphism of IMHS's, so that $\pi_q(M_\bullet)$ is a summand of $H_q^{sing}(M_\bullet)$ in the abelian category IMHS.*

Proof. The assertion amounts to showing θ preserves the two filtrations and thus it suffices to establish the result after tensoring with $\mathbb{Q}$. Let $\Phi_2 : M_\bullet \to M_\bullet$ denote the squaring map (i.e., the composition of $M_\bullet \rightarrowtail M_\bullet^{\times 2} \to M_\bullet$ of the diagonal map and the monoid pairing) and write Φ_2 also for the induced morphism on rational homology groups:

$$\Phi_2 : H_*^{\mathrm{sing}}(M_\bullet, \mathbb{Q}) \to H_*^{\mathrm{sing}}(M_\bullet, \mathbb{Q}).$$

Note that Φ_2 is a morphism of $\mathbb{Q}$-IMHS's, since both the diagonal map and the product pairing are morphisms of inductive limits of simplicial varieties.

By the Milnor-Moore Theorem [28], $H_*^{\mathrm{sing}}(M_\bullet, \mathbb{Q})$ is isomorphic as a Hopf algebra to $S_\mathbb{Q}(\pi_*(M_\bullet, \mathbb{Q}))$, where $S_\mathbb{Q}(V)$ denotes the symmetric algebra of a graded vector space V. The multiplication and comultiplication making $S_\mathbb{Q}(V)$ into a Hopf algebra are induced by the addition map $V \oplus V \to V$ and diagonal map $V \rightarrowtail V \oplus V$, respectively. From this we see that Φ_2 acts as multiplication by 2^n on the summand $S^n_\mathbb{Q}(\pi_*(M_\bullet, \mathbb{Q}))$. In particular, Φ_2 is diagonalizable and $H_*^{\mathrm{sing}}(M_\bullet, \mathbb{Q})$ decomposes completely as a direct sum of eigenspaces.

We claim that the map θ is the endomorphism associated to the summand $\pi_*(M_\bullet, \mathbb{Q}) \subset H_*^{\mathrm{sing}}(M_\bullet, \mathbb{Q})$ given by viewing $\pi_*(M_\bullet, \mathbb{Q})$ as the eigenspace of Φ_2 of eigenvalue 2. This will suffice to prove the lemma, for if $H \subset H_*^{\mathrm{sing}}(M_\bullet, \mathbb{Q})$ is any finite dimensional sub-$\mathbb{Q}$-IMHS, then $\theta|_H$ coincides with

$$\frac{(\Phi_2 - 4) \circ (\Phi_2 - 8) \circ (\Phi_2 - 16) \circ \cdots \circ (\Phi_2 - 2^N) \circ (\Phi_2 - 1)}{\prod_{i=2}^N (2 - 2^i)},$$

which is a morphism of $\mathbb{Q}$-MHS's, for $N \gg 0$.

To establish our claim, it suffices to prove that θ is the zero map on

$$S^n_\mathbb{Q}(\pi_*(M_\bullet), \mathbb{Q}) \subset H_*^{\mathrm{sing}}(M_\bullet, \mathbb{Q})$$

for $n \neq 1$. This holds since the composition

$$\pi_i(M_\bullet, \mathbb{Q}) \otimes_\mathbb{Q} \pi_j(M_\bullet, \mathbb{Q}) \to H_i^{\mathrm{sing}}(M_\bullet, \mathbb{Q}) \otimes_\mathbb{Q} H_j^{\mathrm{sing}}(M_\bullet, \mathbb{Q}) \to H_{i+j}^{\mathrm{sing}}(M_\bullet \times M_\bullet, \mathbb{Q})$$
$$\to H_{i+j}^{\mathrm{sing}}(M_\bullet, \mathbb{Q}) \twoheadrightarrow \pi_{i+j}(M_\bullet, \mathbb{Q})$$

is the zero map, for all $i, j > 0$, since $M_\bullet$ is a commutative H-space. □

If X is a projective variety, then $B(\mathcal{C}_r(X))$ is a connected simplicial object in $\mathcal{M}$, where $B(-)$ denotes the bar construction. The goodness condition is verified in [25], so that $\Omega B(\mathcal{C}_r(X)))$ is homotopy equivalent to $\mathcal{Z}_r(X)$. Thus

$$L_qH_n(X) = \pi_{n-2q+1}(B(\mathcal{C}_r(X))$$

is a summand of $H^{\text{sing}}_{n-2q+1}(B(\mathcal{C}_r(X))$ in the category of IMHS's, and this endows $L_qH_n(X)$ with the structure of an IMHS. In a moment, we will show this structure coincides with that given by Friedlander-Mazur.

To extend the construction to all quasi-projective varieties, we will need the notion of a *cone* of a morphism of simplicial objects, as defined in [9, 6.3]. Specifically, if $\mathcal{A}$ is a category having finite coproducts and an object $*$ that is both terminal and initial and if $u : Y_\bullet \to X_\bullet$ is a morphism of simplicial objects in $\mathcal{A}$, then $\text{cone}_{\mathcal{A}}(u)$ is also a simplicial object in $\mathcal{A}$ with

$$\text{cone}_{\mathcal{A}}(u)_n = X_n \amalg \coprod_{i<n} Y_i \amalg *.$$

If $Y_\bullet = *$, then $\text{cone}_{\mathcal{A}}(u) \cong X$. There are a canonical maps $X_\bullet \to \text{cone}_{\mathcal{A}}(u) \to \text{cone}_{\mathcal{A}}(Y \to *)$. We will use the notion of a cone, in particular, when $\mathcal{A} = \mathcal{M}$, in which case $* = 0$ and coproduct is direct sum, and when $\mathcal{A} = pss$, the category of pointed simplicial spaces, in which case $* = pt$ and coproduct is wedge product. In these cases, $\text{cone}(Y \to *)$ is the suspension of Y, and we get long exact sequences in singular homology:

$$H^{\text{sing}}_q(Y_\bullet) \to H^{\text{sing}}_q(X_\bullet) \to H^{\text{sing}}_q(\text{cone}(u)) \to H^{\text{sing}}_{q-1}(Y_\bullet) \to \cdots .$$

In particular, the bar construction is a type of cone,

$$B(\mathcal{C}_r(X)) = \text{cone}_{\mathcal{M}}(\mathcal{C}_r(X) \to 0),$$

where $\mathcal{C}_r(X)$ and 0 are viewed as constant simplicial objects in $\mathcal{M}$.

If U is a quasi-projective variety, we choose a projective closure $U \subset X$ with closed complement $Y \subset X$, and we define

$$\begin{aligned} B(\mathcal{C}_r(X), \mathcal{C}_r(Y)) &= \text{cone}_{\mathcal{M}}(\text{cone}_{\mathcal{M}}(\mathcal{C}_r(Y) \to 0) \to \text{cone}_{\mathcal{M}}(\mathcal{C}_r(X) \to 0)) \\ &\cong \text{cone}_{\mathcal{M}}(\text{cone}_{\mathcal{M}}(\mathcal{C}_r(Y) \rightarrowtail \mathcal{C}_r(X)) \to 0) \\ &= B(\text{cone}_{\mathcal{M}}(\mathcal{C}_r(Y) \rightarrowtail \mathcal{C}_r(X))). \end{aligned}$$

(Clearly, $B(\mathcal{C}_r(X)) = B(\mathcal{C}_r(X), \mathcal{C}_r(\emptyset))$.) Then we have a homotopy equivalence

$$B(\mathcal{C}_r(X), \mathcal{C}_r(Y)) \sim B(\mathcal{C}_r(U))$$

of topological spaces so that

$$L_rH_n(U) \cong \pi_{2r+n+1}(B(\mathcal{C}_r(X), \mathcal{C}_r(Y)).$$

Since $B(\mathcal{C}_r(X), \mathcal{C}_r(Y))$ is a connected simplicial object in $\mathcal{M}$, this isomorphism makes $L_rH_n(U)$ into an IMHS, and it coincides with the IMHS defined above when $U = X$ is projective.

If $U \subset X'$ is another projective closure with closed complement $Y' \subset X'$, we claim the two induced IMHS's for $L_rH_n(U)$ coincide. To see this, note that we can reduce to the case where there is a projective morphism $p : X' \to X$ inducing the identity on U and mapping Y' to Y. Then the maps $p_* : \mathcal{C}_r(X') \to \mathcal{C}_r(X)$ and $p_* : \mathcal{C}_r(Y') \to \mathcal{C}_r(Y)$ given by pushforward of cycles induce a map

$$p_* : B(\mathcal{C}_r(X'), \mathcal{C}_r(Y')) \to B(\mathcal{C}_r(X), \mathcal{C}_r(Y))$$

of simplicial objects in $\mathcal{M}$. As with any map of connected simplicial objects in $\mathcal{M}$, the induced maps on homology groups are morphisms of IMHS's and thus the induced map on homotopy groups

$$L_rH_n(U) \cong \pi_{n-2r}B(\mathcal{C}_r(X'),\mathcal{C}_r(Y')) \xrightarrow{\cong} \pi_{n-2r}B(\mathcal{C}_r(X),\mathcal{C}_r(Y)) \cong L_rH_n(U)$$

are too.

The first part of property (2) of the Theorem is established by a slight generalization of the preceding construction. Namely, suppose $p : U \to V$ is a projective morphism. Then we can construct projective compactifications $U \subset \overline{U}$ and $V \subset \overline{V}$ with closed complements U_∞ and V_∞ such that p extends to a morphism $\overline{U} \to \overline{V}$ that sends U_∞ to V_∞. Then there is an induced map

$$p_* : B(\mathcal{C}_r(\overline{U}),\mathcal{C}_r(U_\infty)) \to B(\mathcal{C}_r(\overline{V}),\mathcal{C}_r(V_\infty))$$

of connected simplicial objects in $\mathcal{M}$ given by pushforward of cycles. The map on homotopy groups coincides with the usual pushforward map $p_* : L_rH_n(U) \to L_rH_n(V)$ in Lawson homology, which is thus a morphism of IMHS's.

Given a flat morphism $\pi : U \to V$, we may assume that U and V are connected, so that π is equidimensional, say of relative dimension e. We can construct projective closures $U \subset \overline{U}$ and $V \subset \overline{V}$, with closed complements U_∞ and V_∞, such that π extends to a morphism $\overline{\pi} : \overline{U} \to \overline{V}$ such that $\overline{\pi}^{-1}(U) = V$. By the platification par eclatement Theorem [32], we can take blow-ups and proper transforms, without affecting U, V, or π, so that $\overline{\pi}$ becomes flat, also of relative dimension e. The map $\pi^* : \mathcal{Z}_r(V) \to \mathcal{Z}_{r+e}(U)$, which defines the flat pullback map $\pi^* : L_rH_n(V) \to L_{r+e}H_{n+2e}(U)$ in Lawson homology, coincides up to homotopy equivalence with the map

$$B(\mathcal{C}_r(U_\infty),\mathcal{C}_r(\overline{U})) \xrightarrow{\pi^*} B(\mathcal{C}_r(V_\infty),\mathcal{C}_r(\overline{V})).$$

This map is a morphism of connected simplicial objects in $\mathcal{M}$ and hence defines morphisms of IMHS's upon taking homotopy groups.

To complete the proof of (2), it remains to show that the boundary map in the long exact localization sequence is a morphism of IMHS's. Replacing X with a projective closure $\overline{X}$ and Y with $\overline{X} - U$, and using naturality for pullbacks along open immersions, we may assume without loss of generality that X is projective. The long exact sequence for Lawson homology comes from the sequence

$$B(\mathcal{C}_r(Y)) \to B(\mathcal{C}_r(X)) \to B(\mathcal{C}_r(X),\mathcal{C}_r(Y)) \to BB(\mathcal{C}_r(Y))$$

which is a delooping of

$$\mathcal{Z}_r(Y) \to \mathcal{Z}_r(X) \to \mathcal{Z}_r(X)/\mathcal{Z}_r(Y) \to B(\mathcal{Z}_r(Y)).$$

In particular, the map $L_rH_n(U) \to L_rH_{n-1}(Y)$ is induced by a map of connected simplicial objects in $\mathcal{M}$ and hence is a morphism of IMHS's.

To see that property (1) holds (i.e., our definition coincides with that given by Friedlander-Mazur), define

$$\Sigma(\mathcal{C}_r(X)) = \mathrm{cone}_{pss}(\mathcal{C}_r(X) \to pt).$$

Here, the cone is taken in the category of pointed simplicial spaces, so that $\Sigma(\mathcal{C}_r(X)) = (d \mapsto \mathcal{C}_r(X) \vee \cdots \vee \mathcal{C}_r(X))$, with d copies of $\mathcal{C}_r(X)$ in degree d. Regarding $\mathcal{C}_r(X) \vee \cdots \vee \mathcal{C}_r(X)$ as a subset of $\mathcal{C}_r(X)^\times$, we obtain a natural map

$$\Sigma(\mathcal{C}_r(X)) \to B(\mathcal{C}_r(X)).$$

We also have a natural isomorphism

$$\tilde{H}_q^{\text{sing}}(\Sigma(\mathcal{C}_r(X)) \cong H_{q-1}^{\text{sing}}(\mathcal{C}_r(X)),$$

coming from the long exact sequence in homology, and it is a morphism of IMHS's by [9, 8.3.9].

The induced map $H_*^{\text{sing}}(\mathcal{C}_r(X)) \to H_{*+1}^{\text{sing}}(B(\mathcal{C}_r(X)))$, which is a morphism of IMHS's, factors as

$$H_*^{\text{sing}}(\mathcal{C}_r(X)) \to H_*^{\text{sing}}(\mathcal{Z}_r(X)) \to H_{*+1}^{\text{sing}}(B(\mathcal{C}_r(X))),$$

where $H_*^{\text{sing}}(\mathcal{C}_r(X)) \to H_*^{\text{sing}}(\mathcal{Z}_r(X))$ is the map induced by group completion. As explained at the beginning of this section, the groups $H_*^{\text{sing}}(\mathcal{Z}_r(X))$ acquire IMHS's via this group completion, and it follows that the $H_*^{\text{sing}}(\mathcal{Z}_r(X)) \to H_{*+1}^{\text{sing}}(B(\mathcal{C}_r(X)))$ is also a morphism of IMHS's. Finally, the diagram

$$\begin{array}{ccc} \pi_*(\mathcal{Z}_r(X)) & \xrightarrow{\subset} & H_*^{\text{sing}}(\mathcal{Z}_r(X)) \\ \cong\downarrow & & \downarrow \\ \pi_{*+1}(B(\mathcal{C}_r(X))) & \xrightarrow{\subset} & H_{*+1}^{\text{sing}}(B(\mathcal{C}_r(X))) \end{array}$$

commutes. The bottom arrow of this diagram gives our definition of the IMHS for $L_rH_*(X)$ and the top arrow gives the Friedlander-Mazur definition.

We now turn to the proof of part (3) of Theorem 4.1. First, we recall the Dold-Thom isomorphism [10]: Given a CW complex X and closed subcomplex Y, the maps of simplicial abelian groups

$$\mathbb{Z}\,\text{Sing}(X)/\mathbb{Z}\,\text{Sing}(Y) \to \text{Sing}(SP(X))^+/\,\text{Sing}(SP(X))^+ \to \text{Sing}(SP(X)^+/SP(Y)^+)$$

are homotopy equivalences (cf. [37, 8.3]), so that

$$H_q^{\text{sing}}(X,Y) = \pi_q(\mathbb{Z}\,\text{Sing}(X)/\mathbb{Z}\,\text{Sing}(Y)) \cong \pi_q(SP(X)^+/SP(Y)^+).$$

Here, we set $SP(X) = \coprod_n SP^n(X)$, a topological abelian monoid, and let $SP(X)^+$ denote the topological abelian group obtained by naive group completion. We prove that if X is a projective variety, Y is a closed subvariety, and $U = X - Y$, then the Dold-Thom isomorphism

$$H_q^{\text{BM}}(U) \cong H_q^{\text{sing}}(X,Y) \cong \pi_q(SP(X)^+/SP(Y)^+) = L_0H_q(U)$$

is an isomorphism of MHS's. Recall that $L_0H_q(U)$ acquires a MHS by identifying it with

$$\pi_{q+1}(B(SP(X), SP(Y))) \cong \pi_q(SP(X)^+/SP(Y)^+).$$

Define

$$\tilde{\Sigma}(X) = \text{cone}_{pss}(X_+ \to *),$$

where X_+ denotes X with a disjoint basepoint added. Then $\tilde{\Sigma}(X) = (d \mapsto (X^{\coprod d})_+)$ and there is a canonical map

$$\tilde{\Sigma}(X) \to B(SP(X)).$$

More generally, if Y is a closed subcomplex of X, define

$$\tilde{\Sigma}(X,Y) = \text{cone}_{pss}(\tilde{\Sigma}(Y) \to \tilde{\Sigma}(X)) \cong \text{cone}_{pss}(\text{cone}_{pss}(Y_+ \to X_+) \to pt).$$

As with the case $Y = \emptyset$ above, there is a canonical map $\tilde{\Sigma}(X, Y) \to B(SP(X), SP(Y))$. We may identify $H_q^{\text{sing}}(X, Y)$ with $H_q^{\text{sing}}(\text{cone}_{pss}(Y_+ \rightarrowtail X_+))$ (indeed, this isomorphism is how $H_q^{\text{sing}}(X, Y)$ acquires a MHS — see [9, §8]), so that we have an isomorphism

$$\tilde{H}_{q+1}^{\text{sing}}(\tilde{\Sigma}(X, Y)) \cong H_q^{\text{sing}}(X, Y)$$

coming from the boundary map in a long exact sequence. We claim that the composition of

$$\begin{aligned} &H_q^{\text{sing}}(X, Y) \cong H_{q+1}^{\text{sing}}(\tilde{\Sigma}(X, Y)) \to H_{q+1}^{\text{sing}}(B(SP(X), SP(Y)) \\ &\qquad \twoheadrightarrow \pi_{q+1}(B(SP(X), SP(Y)) \cong \pi_q(SP(X)^+/SP(Y)^+), \end{aligned} \tag{4.4}$$

in which the surjection is the canonical splitting constructed above, coincides with the Dold-Thom isomorphism. Indeed, it follows directly from the definitions that the map

$$H_{q+1}^{\text{sing}}(\tilde{\Sigma}(X, Y)) \to \pi_{q+1}((SP(X), SP(Y)))$$

is induced by the map of simplicial abelian groups

$$\begin{aligned} &\text{cone}(\text{cone}(\mathbb{Z}\,\text{Sing}(Y) \to 0) \to \text{cone}(\mathbb{Z}\,\text{Sing}(X) \to 0)) \\ &\qquad \to \text{cone}(\text{cone}(\text{Sing}\,SP(Y) \to 0) \to \text{cone}(\text{Sing}\,SP(X) \to 0)), \end{aligned}$$

which is a suspension (coming from the bar construction) of the Dold-Thom map

$$\text{cone}(\mathbb{Z}\,\text{Sing}(Y) \to \mathbb{Z}\,\text{Sing}(X)) \to \text{cone}(\text{Sing}\,SP(Y) \to \text{Sing}\,SP(X)).$$

If X is a projective variety and Y is a closed subvariety, then each morphism in (4.4) is a morphism of MHS. Indeed, the MHS for $H_q^{\text{sing}}(X, Y)$ is, by definition, given by

$$H_q^{\text{sing}}(X, Y) \cong H_q^{\text{sing}}(\text{cone}(Y_+ \to X_+))$$

and so the first isomorphism of (4.4) is a isomorphism of MHS's by [9, 8.3.9]. The second map is clearly induced by a morphism of simplicial varieties and is thus a morphism of MHS's. The third map is the canonical splitting map, which is a morphism of MHS's by Lemma 4.3.

Property (4) is established by Lima-Filho [26], but he uses a different definition for the IMHS's for Lawson homology. His construction turns out to be equivalent to ours, but rather than prove that, we reproduce in our context a version of his proof that the s map is a morphism of IMHS's.

Choose a projective closure X of U with closed complement Y. The s map in Lawson homology comes from the pairing

$$L_0H_2(\mathbb{P}^1) \times L_rH_n(U) \to L_rH_{n+2}(\mathbb{P}^1 \times U) \to L_rH_{n+2}(\mathbb{A}^1 \times U) \xleftarrow{\cong} L_{r-1}H_n(U),$$

in which the first map is induced by external product of cycles, the second by pullback along an open immersion, and the third by flat pullback. Namely, one fixes the element $s \in L_0H_2(\mathbb{P}^1) \cong H_2^{\text{sing}}(\mathbb{P}^1) \cong \mathbb{Z}(1)$ to be a generator of the underlying abelian group $\mathbb{Z}$. To establish (4) it thus suffices to prove this pairing is a pairing of IMHS's.

We already known that flat pullback gives a morphism of IMHS's, and so it suffices to show that the external product pairing

$$L_sH_m(W) \otimes L_rH_n(U) \to L_{s+r}H_{m+n}(V \times U)$$

is a morphism of IMHS's for an arbitrary projective variety W. (In fact, the hypothesis that W be projective is not necessary.) This pairing is induced by the two morphisms of ind-varieties

$$\mathcal{C}_s(W) \times \mathcal{C}_r(T) \to \mathcal{C}_{s+r}(W \times T),$$

where $T = Y$ or $T = X$, together with the external pairings

$$\pi_i(B(\mathcal{C}_s(W))) \otimes \pi_j(B(\mathcal{C}_r(T))) \to \pi_{i+j}(B(\mathcal{C}_s(W)) \times B(\mathcal{C}_r(T))).$$

The latter pairing injects (in the category IMHS) into

$$H_i^{\text{sing}}(B(\mathcal{C}_s(W))) \otimes H_j^{\text{sing}}(B(\mathcal{C}_r(T))) \to H_{i+j}^{\text{sing}}(B(\mathcal{C}_s(W)) \times B(\mathcal{C}_r(T))),$$

which is a morphism of IMHS's by [9, 8.2.10].

Finally, we establish property (5) by assuming, as we may, that X is connected, and by considering each term comprising the cycle Γ separately, so that we may assume Γ is given by a single morphism $X \to \mathcal{C}_{e,d}(W)$. The map on Lawson homology is then induced by the collection of morphisms of varieties

$$\mathcal{C}_{r,d'}(X) \to \mathcal{C}_{r+e,d'+d}(X),$$

as described in Section 2, giving a morphism $\mathcal{C}_r(X) \to \mathcal{C}_{r+e}(W)$ in $\mathcal{M}$. Our assumption about Y and T gives a commutative square

$$\begin{array}{ccc} \mathcal{C}_r(Y) & \longrightarrow & \mathcal{C}_r(T) \\ \downarrow & & \downarrow \\ \mathcal{C}_r(X) & \longrightarrow & \mathcal{C}_r(W) \end{array}$$

in $\mathcal{M}$. This square induces the commutative diagram of simplicial objects in $\mathcal{M}$

$$\begin{array}{ccccccc} B(\mathcal{C}_r(Y)) & \longrightarrow & B(\mathcal{C}_r(X)) & \longrightarrow & B(\mathcal{C}_r(X), \mathcal{C}_r(Y)) & \longrightarrow & BB(\mathcal{C}_r(Y)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B(\mathcal{C}_r(T)) & \longrightarrow & B(\mathcal{C}_r(W)) & \longrightarrow & B(\mathcal{C}_r(W), \mathcal{C}_r(T)) & \longrightarrow & BB(\mathcal{C}_r(Y)), \end{array}$$

and hence the induced commutative ladder obtained by taking homotopy groups consists entirely of morphisms of IMHS's.

We have now proved Theorem 4.1 in its entirety.

5. The morphic Abel-Jacobi map

Jannsen [21] has established a technique for constructing Abel-Jacobi type maps in a quite general setting. In particular, his technique applies to our situation, leading to the definition of the morphic Abel-Jacobi map.

In detail, suppose X is a quasi-projective variety and fix an integer $r \geq 0$. For each closed subvariety $Y \subset X$ of dimension r with open complement U, localization for Lawson homology gives the exact sequence

$$L_rH_{2r+1}(Y) \to L_rH_{2r+1}(X) \to L_rH_{2r+1}(U) \to L_rH_{2r}(Y) \to L_rH_{2r}(X)$$

of IMHS's. Since $L_rH_{2r+1}(Y) = 0$, $L_rH_{2r}(Y)$ is the free abelian group of integral components of Y of dimension r, and $L_rH_{2r}(X) = CH_r(X)/CH_r(X)_{\text{alg}\sim 0}$, we obtain the short exact sequence

$$0 \to L_rH_{2r+1}(X) \to L_rH_{2r+1}(U) \to Z_r^Y(X)_{\text{alg}\sim 0} \to 0, \tag{5.1}$$

where $Z_r^Y(X)$ denotes the group of r-cycles on X supported on Y and $Z_r^Y(X)_{\mathrm{alg}\sim 0}$ denotes the subgroup of such cycles that are algebraically equivalent to zero on X. This sequence is a short exact sequence of IMHS and $Z_r^Y(X)_{\mathrm{alg}\sim 0}$ has the trivial Hodge structure. Thus the boundary map in the long exact sequence for $\mathrm{Ext}^*_{\mathrm{IMHS}}(\mathbb{Z}(0), -)$ determines a map

$$Z_r^Y(X)_{\mathrm{alg}\sim 0} \to \mathrm{Ext}^1_{\mathrm{IMHS}}(\mathbb{Z}(0), L_rH_{2r+1}(X)) = \mathcal{J}(L_rH_{2r+1}(X)).$$

Explicitly, an element in $Z_r^Y(X)_{\mathrm{alg}\sim 0}$ coincides with a morphism $\mathbb{Z}(0) \to Z_r^Y(X)_{\mathrm{alg}\sim 0}$ of IMHS's and pulling back (5.1) along this morphism give an extension of IMHS's of the form

$$0 \to L_rH_{2r+1}(X) \to E \to \mathbb{Z}(0) \to 0.$$

Definition 5.2. For a quasi-projective complex variety X and integer $r \geq 0$, define the r-th *morphic Jacobian* to be

$$\mathcal{J}_r^{\mathrm{mor}}(X) = \mathcal{J}(L_rH_{2r+1}(X)) = \mathrm{Ext}^1_{\mathrm{IMHS}}(\mathbb{Z}(0), L_rH_{2r+1}(X)),$$

and define the r-th *morphic Abel-Jacobi map*

$$\Phi^{\mathrm{mor}} : \mathcal{Z}_r(X)_{\mathrm{alg}\sim 0} \to \mathcal{J}_r^{\mathrm{mor}}(X)$$

to be the map induced from the maps $Z_r^Y(X)_{\mathrm{alg}\sim 0} \to \mathrm{Ext}^1_{\mathrm{IMHS}}(\mathbb{Z}(0), L_rH_{2r+1}(X))$ above by taking the filtered colimit over all closed subschemes $Y \subset X$ of dimension r.

If X and W are projective varieties, an equidimensional correspondence $\Gamma \in Z_s(X, W)$ induces a morphism of IMHS's

$$\Gamma_* : L_rH_{2r+1}(X) \to L_{r+e}H_{2(r+e)+1}(W),$$

by Theorem 4.1 (5). Since $\mathcal{J}(-)$ is a functor on the category of IMHS's, we thus obtain the map

$$\Gamma_* : \mathcal{J}_r^{\mathrm{mor}}(X) \to \mathcal{J}_{r+e}^{\mathrm{mor}}(W).$$

Theorem 5.3. *The morphic Abel-Jacobi mapping is functorial for equidimensional correspondences between projective varieties. That is, if X and W are projective varieties and $\Gamma \in Z_e(X, W)$, then the diagram*

$$\begin{array}{ccc} Z_r(X)_{alg\sim 0} & \longrightarrow & \mathcal{J}_r^{mor}(X) \\ \Gamma_* \downarrow & & \Gamma_* \downarrow \\ Z_{r+e}(W)_{alg\sim 0} & \longrightarrow & \mathcal{J}_{r+e}^{mor}(W) \end{array}$$

commutes.

Proof. Fix a subvariety $Y \subset X$ of dimension at most r. Then the image of the subgroup $Z_r(Y) \subset Z_r(X)$ under $\Gamma_* : Z_r(X) \to Z_{r+e}(W)$ is contained in $Z_{r+e}(T)$ for some subvariety $T \subset W$ of dimension at most $r+e$. Thus we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{Z}_r(Y) & \xrightarrow{\Gamma_*} & \mathcal{Z}_{r+e}(T) \\
\subset\downarrow & & \downarrow\subset \\
\mathcal{Z}_r(X) & \xrightarrow{\Gamma_*} & \mathcal{Z}_{r+e}(W) \\
\downarrow & & \downarrow \\
\mathcal{Z}_r(U) & \xrightarrow{\Gamma_*} & \mathcal{Z}_{r+e}(V)
\end{array}$$

where $U = X - Y$, $V = W - T$, and the map on the bottom is defined by the identities $\mathcal{Z}_r(U) = \mathcal{Z}_r(X)/\mathcal{Z}_r(Y)$ and $\mathcal{Z}_r(V) = \mathcal{Z}_r(W)/\mathcal{Z}_r(T)$.

Taking homotopy groups for this diagram gives a commutative ladder of IMHS's, and upon taking limits over pairs Y, T with $\Gamma_*(Z_r(Y)) \subset Z_r(T)$, we arrive at the commutative diagram of IMHS's

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_rH_{2r+1}(X) & \longrightarrow & \varinjlim_Y L_rH_{2r+1}(X-Y) & \longrightarrow & Z_r(X)_{\mathrm{alg}\sim 0} & \longrightarrow & 0 \\
 & & \Gamma_*\downarrow & & \downarrow & & \downarrow\Gamma_* & & \\
0 & \longrightarrow & L_{r+e}H_{2r+2e+1}(W) & \longrightarrow & \varinjlim_T L_{r+e}H_{2r+2e+1}(W-T) & \longrightarrow & Z_{r+e}(W)_{\mathrm{alg}\sim 0} & \longrightarrow & 0.
\end{array}$$

The result is now evident by the functorality of the long exact sequence for $\mathrm{Ext}^*_{\mathrm{IMHS}}(\mathbb{Z}(0), -)$. □

Recall that the r-th intermediate Griffiths Jacobian of a projective variety X is

$$\mathcal{J}_r(X) := \mathcal{J}(H^{\mathrm{sing}}_{2r+1}(X, \mathbb{Z}(r)),$$

and it is the target of the (classical) Abel-Jacobi map:

$$\Phi_r : Z_r(X)_{hom\sim 0} \to \mathcal{J}_r(X).$$

If one repeats the above construction of the morphic Abel-Jacobi map Φ^{mor}_r using instead the groups $H^{\mathrm{BM}}_*(-, \mathbb{Z})$, the singular Borel-Moore homology groups of complex varieties viewed as taking values in MHS's, then one obtains the classical Abel-Jacobi map Φ_r (cf. [7], [21, 9.2]). Recall (see Remark 4.2) that the natural map $L_rH_n(U) \to H^{\mathrm{BM}}_n(U, \mathbb{Z}(r))$ is a morphisms of IMHS's, and, in particular, we have an induced map $\mathcal{J}^{\mathrm{mor}}_r(U) \to \mathcal{J}_r(U)$.

Proposition 5.4. *For any quasi-projective variety X, the diagram*

$$\begin{array}{ccc}
Z_r(X)_{alg\sim 0} & \xrightarrow{\Phi^{mor}_r} & \mathcal{J}^{mor}_r(X) \\
\downarrow & & \downarrow \\
Z_r(X)_{hom\sim 0} & \xrightarrow{\Phi_r} & \mathcal{J}_r(X)
\end{array}$$

commutes.

Proof. This follows immediately from Jannsen's construction, using that we have a commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & L_rH_n(Y) & \longrightarrow & L_rH_n(X) & \longrightarrow & L_rH_n(U) & \longrightarrow & L_rH_{n-1}(Y) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^{\mathrm{BM}}_n(Y, \mathbb{Z}(r)) & \longrightarrow & H^{\mathrm{BM}}_n(X, \mathbb{Z}(r)) & \longrightarrow & H^{\mathrm{BM}}_n(U, \mathbb{Z}(r)) & \longrightarrow & H^{\mathrm{BM}}_{n-1}(Y, \mathbb{Z}(r)) & \longrightarrow & \cdots
\end{array}$$

of IMHS's, for any closed subscheme $Y \subset X$ with open complement U. □

Example 5.5. *The morphic Abel-Jacobi map coincides with its classical counterpart for zero cycles and codimension one cycles. This is conjecturally true also for codimension two cycles.*

In detail, since $L_0H_n(X) \cong H_n^{BM}(X, \mathbb{Z}(0))$ *as MHS's for all quasi-projective varieties* U*, we have*

$$\Phi_0^{mor} = \Phi_0.$$

When X *is smooth and projective, the map* $\Phi_0^{mor} = \Phi_0$ *coincides under Poincare duality with the Albanese map* $X \to \mathrm{Alb}(X) = \mathcal{J}(H_{sing}^{2d-1}(, \mathbb{Z}(d))$ *where* $d = \dim(X)$*. See* [23, 12.11(3)] *for a description of this map in terms of integrals.*

Likewise, if U *is a variety of dimension at most* d *and we take* $r = d-1$*, then Proposition 2.5 gives the isomorphism*

$$L_{d-1}H_{2d-1}(U) \cong N_{n-d+1}H_{2d-1}^{BM}(U, \mathbb{Z}(d-1)).$$

(Recall $N_kH_n^{BM}$ *denotes the subgroup of* H_n^{BM} *consisting of classes supported in dimension* k*.) From this, we deduce* $\Phi_{d-1}^{mor} = \Phi_{d-1}$ *on a variety* X *of dimension* d*. If* X *is smooth and projective, then using Poincare duality, we have*

$$\mathcal{J}_{d-1}^{mor}(X) = \mathcal{J}_{d-1}(X) = \mathcal{J}(H^1(X, \mathbb{Z}(1)) = \mathrm{Pic}^0(X),$$

and the map

$$\Phi_{d-1} = \Phi_{d-1}^{mor} : Z^1(X)_{alg\sim 0} \to \mathrm{Pic}^0(X)$$

sends a codimension one cycle algebraically (equivalently, homologically) equivalent to zero to the corresponding point on the Picard variety of X*.*

A special case of Suslin's Conjecture 2.4 predicts that the onto map

$$L_{d-2}H_{2d-3}(X) \twoheadrightarrow N_{d-1}H_{2d-3}^{BM}(X, \mathbb{Z}(d-2))$$

is actually an isomorphism, for any quasi-projective variety X *of dimension* d*. The validity of this formula would give that* Φ_{d-2}^{mor} *and* Φ_{d-2} *coincide on* $Z_{d-2}(X)_{alg\sim 0}$*, for all such* X*.*

We see, therefore, that the cases of most interest for the morphic Abel-Jacobi map concern dimension r *cycles where* $0 < r < d-2$*.*

Example 5.6. *In particular, if we take* $X = C$ *to be a possibly singular projective curve and* $r = 0$*, then the morphic Abel-Jacobi map*

$$\Phi_0^{mor} : \mathcal{Z}_0(C)_{alg\sim 0} \to \mathcal{J}_0^{mor}(C)$$

coincides up to isomorphism with the classical Albanese map

$$\mathcal{Z}_0(C)_{deg=0} \twoheadrightarrow \mathcal{J}_0(C) = \mathrm{Alb}(C).$$

If C *is smooth, then* $\mathrm{Alb}(C) \cong \mathrm{Pic}^0(C)$*, the Jacobian variety of* C*, and this map is the canonical one identifying the Jacobian variety as degree* 0 *zero cycles on* C *modulo linear equivalence. More generally, if* $\tilde{C} \twoheadrightarrow C$ *is the normalization of a singular curve* C*, then* $\mathcal{J}_0(C)$ *is a quotient of* $\mathrm{Pic}^0(\tilde{C})$ *by a free abelian subgroup. The map* $\Phi_0^{mor} = \Phi_0$ *is uniquely determined by the existence of the commutative square of continuous surjections*

$$\begin{array}{ccc} \mathcal{Z}_0(\tilde{C})_{deg=0} & \twoheadrightarrow & \mathrm{Pic}^0(\tilde{C}) \\ \downarrow & & \downarrow \\ \mathcal{Z}_0(C)_{deg=0} & \twoheadrightarrow & \mathcal{J}_0(C). \end{array}$$

Using the facts above, we derive numerous good properties of the morphic Abel-Jacobi map. For a smooth, projective variety X, the image of $Z_r(X)_{\text{alg}\sim 0}$ under the classical Abel-Jacobi map is known (cf. [23, §12], [24]) to be a abelian variety (not merely a complex torus) and is called the *Lieberman Jacobian*, written

$$\mathcal{J}_r^a(X) := \Phi_r(Z_r(X)_{\text{alg}\sim 0}).$$

Theorem 5.7. *Let X be a (possibly singular) projective variety.*

(1) *Give $Z_r(X)_{alg\sim 0}$ the structure of a topological abelian group by identifying it with the connected component of the identity in $\mathcal{Z}_r(X)$. The morphic Abel-Jacobi map*

$$\Phi_r^{mor} : Z_r(X)_{alg\sim 0} \twoheadrightarrow \mathcal{J}_r^{mor}(X)$$

is a surjective, continuous homomorphism of topological abelian groups. Here, $\mathcal{J}_r^{mor}(X)$ is topologized by viewing it as a quotient of the complex vector space $L_rH_{2r+1}(X)_{\mathbb{C}}$, which itself is topologized as a direct limit of its finite dimensional subspaces.

(2) *The morphic Abel-Jacobi map factors through cycles modulo linear equivalence — i.e., there is a commutative diagram of the form*

$$\begin{array}{ccc} Z_r(X)_{alg\sim 0} & \twoheadrightarrow & CH_r(X)_{alg\sim 0} \\ & \searrow^{\Phi_r^{mor}} & \downarrow \\ & & \mathcal{J}_r^{mor}(X). \end{array}$$

We will also write Φ_r^{mor} for the vertical map in this diagram.

(3) *We have a commutative diagram of continuous, surjective homomorphisms of topological abelian groups*

$$\begin{array}{ccc} Z_r(X)_{alg\sim 0} & \xrightarrow{\Phi_r^{mor}} & \mathcal{J}_r^{mor}(X) \\ & \searrow^{\Phi_r} & \downarrow \\ & & \mathcal{J}_r^a(X) \end{array}$$

and a commutative diagram of abelian groups

$$\begin{array}{ccc} CH_r(X)_{alg\sim 0} & \xrightarrow{\Phi_r^{mor}} & \mathcal{J}_r^{mor}(X) \\ \downarrow & & \downarrow \\ & & \mathcal{J}^a(X) \\ & & \downarrow \\ CH_r(X)_{hom\sim 0} & \xrightarrow[\Phi_r]{} & \mathcal{J}_r(X). \end{array}$$

(4) *The morphic Abel-Jacobi map on $CH.(-)_{alg\sim 0}$ is functorial for all correspondences $\gamma \in CH_{\dim(Y)+r}(Y \times X)$, for smooth, projective varieties X and*

Y and integers $r \geq 0$ — i.e., the diagram

$$\begin{array}{ccc} CH_r(Y)_{alg\sim 0} & \longrightarrow & \mathcal{J}_r^{mor}(Y) \\ \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*} \\ CH_{r+t}(X)_{alg\sim 0} & \longrightarrow & \mathcal{J}_{r+t}^{mor}(X) \end{array}$$

commutes, for all r.

Proof. To prove (1), we use that the topology on $\mathcal{Z}_r(X)_0$ is "generated by curves". That is, letting C range over all projective curves and Γ over all equidimensional correspondences $\Gamma \in Z_r(C, X)$, we define

$$\theta : \bigoplus_{C,\Gamma} \mathcal{Z}_0(C) \to \mathcal{Z}_r(X).$$

to be the map that on the summand indexed by C, Γ sends $c \in \mathcal{Z}_0(C)$ to $\Gamma_*(c) \in \mathcal{Z}_r(X)$. Then both θ and the restriction of θ to $\bigoplus \mathcal{Z}_0(C)_0$ are quotient maps in the category of topological abelian groups. To see this, recall that $\mathcal{Z}_0(C)$ and $\mathcal{Z}_r(X)$ are topologized as quotients of the abelian monoids $\coprod_{e,e'} \mathcal{C}_{0,e}(C) \times \mathcal{C}_{0,e'}(C)$ and $\coprod_{e,e'} \mathcal{C}_{r,e}(X) \times \mathcal{C}_{r,e'}(X)$ and that θ is induced by the collection of morphisms of the form

$$\mathcal{C}_{0,e}(C) \times \mathcal{C}_{0,e'}(C) \to \mathcal{C}_{r,e+f}(X) \times \mathcal{C}_{r,e'+f'}(X)$$

induced from pairs of algebraic morphisms $C \to \mathcal{C}_{r,f}(X)$, $C \to \mathcal{C}_{r,f'}(X)$. It follows that θ is continuous, closed, and surjective. To see that θ induces a surjection from $\bigoplus \mathcal{Z}_0(C)_0$ onto $\mathcal{Z}_r(X)_0$ as well, note that an element in the target has the form $Z_1 - Z_2$ for a pair of effective cycles Z_1, Z_2 lying in the same connected component of $\mathcal{C}_{r,e}(X)$ for some e. Thus there is a curve C, closed points $c_0, c_1 \in C$, and a morphism $C \to \mathcal{C}_{r,e}(X)$ such that $c_i \in C$ maps to Z_i, for $i = 1, 2$, so that $\theta(c_1 - c_2) = Z_1 - Z_2$.

Likewise, the IMHS $L_r H_{2r+1}(X)$ is generated by curves in the sense that

$$\bigoplus_{C,\Gamma} H_1^{\text{sing}}(C, \mathbb{Z}(0)) \cong \bigoplus_{C,\Gamma} L_0 H_1(C) \to L_r H_{2r+1}(X) \tag{5.8}$$

is surjective. To see this, recall that we have

$$L_r H_{2r+1}(X) = \pi_1(\mathcal{Z}_r(X)) \cong H_1^{\text{sing}}(\mathcal{Z}_r(X)) \cong \varinjlim_e H_1^{\text{sing}}(\mathcal{C}_{r,e}(X)).$$

By the Lefschetz Theorem of Andreotti-Frankel [1], the map $H_1^{\text{sing}}(C) \to H_1^{\text{sing}}(\mathcal{C}_{r,e}(X))$ is surjective for some (possibly singular) curve $C \subset \mathcal{C}_{r,e}(X)$, and the inclusion $C \rightarrowtail \mathcal{C}_{r,e}(X)$ determines a correspondence. This shows that the image of $H_1^{\text{sing}}(\mathcal{C}_{r,e}(X)) \to L_r H_{2r+1}(X)$ is contained in the image of (5.8), which establishes the claim.

Since $\mathcal{J}(-)$ is a left-exact functor, the induced map

$$\bigoplus_{C,\Gamma} \mathcal{J}_0(C) \twoheadrightarrow \mathcal{J}_r^{\text{mor}}(X)$$

is a quotient map of topological abelian groups. Since the morphic Abel-Jacobi map is natural for equidimensional correspondences, we have a commutative square

$$\begin{array}{ccc} \bigoplus_{C,\Gamma} \mathcal{Z}_0(C)_0 & \twoheadrightarrow & \bigoplus_{C,\Gamma} \mathcal{J}_0(C) \\ \downarrow & & \downarrow \\ \mathcal{Z}_r(X)_0 & \longrightarrow & \mathcal{J}_r^{\mathrm{mor}}(X). \end{array}$$

We know the vertical arrows are quotients maps of topological abelian groups, and the top arrow is surjective by Example 5.6. It follows that the bottom arrow is also a continuous surjection.

To prove (2), we need to show $\Phi_r^{\mathrm{mor}}(\gamma) = 0$ if $\gamma \in Z_r(X)$ is rationally equivalent to 0. For such a γ, writing $\gamma = \gamma^+ - \gamma^-$ for effective cycles γ^+, γ^-, we have that γ^+ can be joined to γ^- by a sequence of "elementary" linear equivalences — i.e., whose given by morphisms from $\mathbb{P}^1$ to $\mathcal{C}_{r,e}(X)$. In other words, γ will lie in the image of $\Gamma_* : Z_0(\mathbb{P}^1)_0 \to Z_r(X)_0$ for some correspondence $\Gamma \in Z_r(\mathbb{P}^1, X)$. The result now follows from Theorem 5.3 since $\mathcal{J}_0^{\mathrm{mor}}(\mathbb{P}^1) = \mathcal{J}_0(\mathbb{P}^1) = 0$.

Result (3) follows from Proposition 5.4 and part (2).

Every cycle class in $CH_{\dim(Y)+r}(Y \times X)$ is linearly equivalent to an equidimensional correspondence [39, 3.2], and thus part (4) follows from Theorem 5.3 and part (2). □

For a projective variety X, we have from Proposition 2.5 that the image of the morphism of IMHS's

$$L_r H_{2r+1}(X) \to H_{2r+1}^{\mathrm{sing}}(X, \mathbb{Z}(r))$$

is $N_{r+1}H_{2r+1}^{\mathrm{sing}}(X, \mathbb{Z}(r))$. Recall that for any abelian group A, the group $N_k H_n^{\mathrm{sing}}(X, A)$ consists of those classes supported on subschemes of dimension k. Note that we have $N_k H_n^{\mathrm{sing}}(X, \mathbb{Q}) = N_k H_n^{\mathrm{sing}}(X, \mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $H_n^{\mathrm{sing}}(X, \mathbb{Z})$ is finitely generated, we have that $N_k H_n^{\mathrm{sing}}(X, \mathbb{Z})$ is the kernel of $H_n^{\mathrm{sing}}(X, \mathbb{Z}) \to H_n^{\mathrm{BM}}(X - Y, \mathbb{Z})$ for a suitable closed subvariety Y. It follows that $N_k H_n^{\mathrm{sing}}(X, \mathbb{Z})$ is a sub-MHS of $H_n^{\mathrm{sing}}(X, \mathbb{Z})$. Define $N_k H_n^{\mathrm{sing}}(X, \mathbb{Z}(r)) = N_r H_n^{\mathrm{sing}}(X, \mathbb{Z}(0)) \otimes \mathbb{Z}(r)$, a sub-MHS of $H_n^{\mathrm{sing}}(X, \mathbb{Z}(r))$.

We obtain, in particular, a factorization

$$\mathcal{J}_r^{\mathrm{mor}}(X) \twoheadrightarrow \mathcal{J}(N_{r+1}H_{2r+1}^{\mathrm{sing}}(X, \mathbb{Z}(r))) \twoheadrightarrow \mathcal{J}_r^a(X)$$

of the canonical map. In light of Theorem 5.7, we thus obtain the following new result about the classical Abel-Jacobi map.

Corollary 5.9. *For a projective variety X, the restriction of the classical Abel-Jacobi map to cycles algebraically equivalent to zero factors as*

$$CH_r(X)_{alg\sim 0} \twoheadrightarrow \mathcal{J}(N_{r+1}H_{2r+1}^{sing}(X, \mathbb{Z}(r))) \twoheadrightarrow \mathcal{J}_r^a(X).$$

If X is, in addition, smooth, we have an exact sequence

$$\begin{aligned} H_{2r+1}(X, \mathbb{Z}(r))_{tor} \to (H_{2r+1}(X, \mathbb{Z}(r))/N_{r+1}H_{2r+1}(X, \mathbb{Z}(r)))_{tor} \to \\ \mathcal{J}(N_{r+1}H_{2r+1}^{sing}(X, \mathbb{Z}(r))) \to \mathcal{J}_r^a(X) \to 0. \end{aligned}$$

Proof. The exact sequence comes from the long exact sequence for $\mathrm{Ext}^*_{\mathrm{MHS}}(\mathbb{Z}(0), -)$, using the fact that

$$\Gamma(H) = \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(0), H) = H_{\mathrm{tor}}$$

for a MHS H of pure weight -1. □

Remark 5.10. If X is smooth and projective, the surjection

$$\mathcal{J}(N_{r+1}H^{\text{sing}}_{2r+1}(X,\mathbb{Z}(r))) \twoheadrightarrow \mathcal{J}^a_r(X)$$

is, in fact, a morphism of abelian varieties. This follows from Lemma 7.3 in Section 7 below. As discussed in that section, the Corollary thus indicates a possible counter-example to the conjectured universality of the classical Abel-Jacobi map on $CH_r(X)_{\text{alg}\sim 0}$. Namely, if the map

$$H_{2r+1}(X,\mathbb{Z}(r))_{\text{tor}} \to (H_{2r+1}(X,\mathbb{Z}(r))/N_{r+1}H_{2r+1}(X,\mathbb{Z}(r)))_{\text{tor}}$$

fails to be surjective for some smooth, projective variety X, then such universality fails. We know of no such examples, but see Section 7 for further discussion.

6. Using the morphic Abel-Jacobi map to detect cycles

Recall that a cycle is "Abel-Jacobi equivalent" to zero if it is homologically equivalent to zero and lies in the kernel of the classical Abel-Jacobi map (equivalently, if it lies in the kernel of the Deligne cycle class map). Theorem 5.7 suggests that we can find cycles that are algebraically equivalent to zero, Abel-Jacobi equivalent to zero, and yet not "morphic Abel-Jacobi" equivalent to zero, meaning not in the kernel of Φ^{mor}_r. Indeed, such cycles will exist precisely when the map

$$\mathcal{J}^{\text{mor}}_r(X) \twoheadrightarrow \mathcal{J}^a_r(X)$$

has a non-trivial kernel. In this section we give examples of varieties X and integers r for which this holds by building upon examples constructed by Friedlander. Additional examples, arising from the work of C. Schoen [34], where the above map has a non-trivial kernel will be discussed in Section 8.

Theorem 6.1 (Friedlander)**.** [15, 4.5] *Fix an integer $r \geq 1$. Suppose W is a smooth complete intersection in projective space, $\dim(W) = 2r+2$, and the image of $CH_{r+1}(W)_{\mathbb{Q}} \to H^{sing}_{2r+2}(W,\mathbb{Q})$ has rank at least 2. Let Y be a sufficiently general member of a Lefschetz pencil of codimension one complete intersections in W of sufficiently large degree. Then there exists an r-cycle γ on Y such that*

$$\gamma \in \ker\left(L_rH_{2r}(Y)\xrightarrow{s^r}L_0H_{2r}(Y) \cong H^{sing}_{2r}(Y)\right)$$

but

$$\gamma \notin \ker\left(L_rH_{2r}(Y)\xrightarrow{s^{r-1}}L_1H_{2r}(Y)\right) \otimes \mathbb{Q}.$$

That is, there is a $\gamma \in L_rH_{2r}(Y)$ that is homologically equivalent to 0 but that remains non-zero, even modulo torsion, in the penultimate stage of the sequence of maps

$$L_rH_{2r}(Y)\xrightarrow{s}L_{r-1}H_{2r}(Y)\xrightarrow{s}\cdots\xrightarrow{s}L_1H_{2r}(Y)\xrightarrow{s}L_0H_{2r}(Y) \cong H^{sing}_{2r}(Y).$$

We construct the examples we seek by building on those of Friedlander's Theorem:

Theorem 6.2. *Pick an integer $r \geq 2$ and a variety Y, smooth of dimension $2r+1$, as in Friedlander's Theorem 6.1. Let $X = Y \times C$ where C is a smooth, projective curve of genus at least one. Then the kernel of*

$$\mathcal{J}^{mor}_r(X) \twoheadrightarrow \mathcal{J}^a_r(X)$$

contains the quotient of a non-zero complex vector space by a countable subgroup — in particular, it contains uncountable many non-torsion elements.

Proof. Recall that

$$\mathrm{Griff}_r(Y) = \ker(L_r H_{2r}(Y) \to L_0 H_{2r}(Y)),$$

is the Griffiths group of r-cycles homologically equivalent to 0 modulo algebraic equivalence, and note $\mathrm{Griff}_r(Y)_{\mathbb{Q}} \neq 0$. Exterior product gives a map

$$\times : \mathrm{Griff}_r(Y) \otimes L_0 H_1(C) \to L_r H_{2r+1}(X)$$

and the composition of this map with $L_r H_{2r+1}(X) \to H^{\mathrm{sing}}_{2r+1}(X)$ is zero, since it coincides with the composition of

$$\mathrm{Griff}_r(Y) \otimes L_0 H_1(C) \to H^{\mathrm{sing}}_{2r}(Y) \otimes H^{\mathrm{sing}}_1(C) \to H^{\mathrm{sing}}_{2r+1}(X).$$

On the other hand, the basic formulas satisfied by cup and cap product imply that the composition of

$$L_r H_{2r}(Y) \otimes L_0 H_1(C) \otimes L^1 H^1(C) \xrightarrow{\times \otimes \pi_2^*} L_r H_{2r+1}(X) \otimes L^1 H^1(X)$$
$$\xrightarrow{\cap} L_{r-1} H_{2r}(X) \xrightarrow{(\pi_1)_*} L_{r-1} H_{2r}(Y)$$

coincides with the composition of

$$L_r H_{2r}(Y) \otimes L_0 H_1(C) \otimes L^1 H^1(C) \xrightarrow{\cong} L_r H_{2r}(Y) \otimes L^1 H^1(C) \otimes L^1 H^1(C)$$
$$\xrightarrow{id \otimes \cup} L_r H_{2r}(Y) \otimes L^2 H^2(C) \xrightarrow{id \otimes \pi_*} L_r H_{2r}(Y) \otimes L^1 H^0(pt) \xrightarrow{\cap} L_{r-1} H_{2r}(Y).$$

The map $L^1 H^1(C) \otimes L^1 H^1(C) \xrightarrow{\cup} L^2 H^2(C)$ is onto and the map $\pi_* : L^2 H^2(C) \to L^1 H^0(pt) \cong \mathbb{Z}$ is an isomorphism, since C is a smooth projective curve of genus at least one. Moreover, the map

$$L_r H_{2r}(Y) \cong L_r H_{2r}(Y) \otimes L^1 H^0(pt) \xrightarrow{\cap} L_{r-1} H_{2r}(Y)$$

is the s-map. In Friedlander's example, there are elements of $\mathrm{Griff}_r(Y) \subset L_r H_{2r}(Y)$ that are not in the kernel of the s-map (even tensor $\mathbb{Q}$).

It follows that the image of

$$\mathrm{Griff}_r(Y, \mathbb{Q}) \otimes L_0 H_1(C, \mathbb{Q}) \to L_r H_{2r+1}(X, \mathbb{Q})$$

is a non-zero $\mathbb{Q}$-IMHS of pure weight -1 that maps to zero in $H^{\mathrm{sing}}_{2r+1}(X, \mathbb{Q}(r))$. Letting

$$K = \ker\left(L_r H_{2r+1}(X) \to H^{\mathrm{sing}}_{2r+1}(X, \mathbb{Z}(r))\right),$$

we have that K is a IMHS having weights $-1 \leq w \leq 0$ and $W_{-1}(K_{\mathbb{Q}}) \neq 0$. Applying the long exact sequence for $\mathrm{Ext}^*_{\mathrm{IMHS}}(\mathbb{Z}(0), -)$ to the short exact sequence of IMHS's

$$0 \to K \to L_r H_{2r+1}(X) \to N_{r+1} H^{\mathrm{sing}}_{2r+1}(X, \mathbb{Z}(r)) \to 0$$

gives the exact sequence

$$N_{r+1} H^{\mathrm{sing}}_{2r+1}(X)_{\mathrm{tor}} \to \mathcal{J}(K) \to \mathcal{J}^{mor}_r(X) \to \mathcal{J}(N_{r+1} H^{\mathrm{sing}}_{2r+1}(X, \mathbb{Z}(r))) \to 0.$$

Here, $\Gamma(N_{r+1} H^{\mathrm{sing}}_{2r+1}(X, \mathbb{Z}(r))) = N_{r+1} H^{\mathrm{sing}}_{2r+1}(X)_{\mathrm{tor}}$ since $N_{r+1} H^{\mathrm{sing}}_{2r+1}(X, \mathbb{Z}(r))$ is pure of weight -1.

Since the map $\mathcal{J}^{\mathrm{mor}}_r(X) \twoheadrightarrow \mathcal{J}^a_r(X)$ factors through $\mathcal{J}(N_{r+1} H^{\mathrm{sing}}_{2r+1}(X, \mathbb{Z}(r)))$, we have that $\mathcal{J}(K)/\left(N_{r+1} H^{\mathrm{sing}}_{2r+1}(X)_{\mathrm{tor}}\right)$ injects into the kernel of $\mathcal{J}^{\mathrm{mor}}_r(X) \twoheadrightarrow \mathcal{J}^a_r(X)$. Finally, Proposition 3.1 shows that $\mathcal{J}(K)$ is the quotient of a non-trivial complex vector space (namely, $K_{\mathbb{C}}/F^0(K_{\mathbb{C}})$, which is non-zero since $W_{-1}(K_{\mathbb{Q}}) \neq 0$) by a countable group (namely, the image of K), and thus $\mathcal{J}(K)/\big(N_{r+1}H^{\text{sing}}_{2r+1}(X)_{\text{tor}}\big)$ is also the quotient of a non-zero complex vector space by a countable subgroup. □

Corollary 6.3. *For Y and r as in the Theorem, there exist uncountably many non-torsion elements in $CH_r(Y)$ that are algebraically equivalent to 0, lie in the kernel of the Abel-Jacobi map, but do not lie in the kernel of the morphic Abel-Jacobi map. That is,*

$$\ker(\Phi_r) \cap CH_r(X)_{alg\sim 0}/\ker(\Phi_r^{mor})$$

has uncountable rank.

7. On Universality

Definition 7.1. For a a smooth, projective complex variety X and an abelian variety A (resp., a complex torus A), a function

$$f : CH_r(X)_{\text{alg}\sim 0} \to A(\mathbb{C})$$

is *regular* (resp., *analytic*) if given any smooth, connected, projective variety T of dimension d, base point $t_0 \in T$, and correspondence $\Gamma \in CH_{r+d}(T \times X)$, the map

$$T(\mathbb{C}) \xrightarrow{t \mapsto t - t_0} CH_0(T)_{\text{alg}\sim 0} \xrightarrow{\Gamma_*} CH_r(X)_{\text{alg}\sim 0} \xrightarrow{f} A(\mathbb{C})$$

is induced by a morphism of varieties (resp., is holomorphic).

Conjecture 7.2 (Universality of Abel-Jacobi map)**.** *(cf.* [23]*) For a smooth, projective complex variety X, the classical Abel-Jacobi map*

$$\Phi_r : CH_r(X)_{alg\sim 0} \twoheadrightarrow \mathcal{J}^a_r(X)$$

is universal among regular functions. That is, given an abelian variety A and a regular function

$$f : CH_r(X)_{alg\sim 0} \to A,$$

the diagram

$$\begin{array}{ccc} CH_r(X)_{alg\sim 0} & \xrightarrow{\Phi_r} & \mathcal{J}^a_r(X) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \overline{f}} & \\ & A & \end{array}$$

can be completed to a commutative diagram by some (necessarily unique) morphism of abelian varieties $\overline{f}$.

In this section, we indicate two possible ways in which the morphic Abel-Jacobi map might lead to a counter example of the above conjecture. The first way has already been mentioned — namely, as shown in Corollary 5.9, the morphic Abel-Jacobi map implies the existence of a factorization

$$CH_r(X)_{\text{alg}\sim 0} \twoheadrightarrow \mathcal{J}(N_{r+1}H^{\text{sing}}_{2r+1}(X, \mathbb{Z}(r))) \twoheadrightarrow \mathcal{J}^a_r(X).$$

Now, the map $\mathcal{J}(N_{r+1}H^{\text{sing}}_{2r+1}(X, \mathbb{Z}(r))) \to \mathcal{J}^a_r(X)$ will have a kernel if and only if there is a non-torsion class $\alpha \in H_{2r+1}(X, \mathbb{Z})$ such that some non-zero multiple of α belongs to $N_{r+1}H_{2r+1}(X, \mathbb{Z})$, but α itself does not belong to $N_{r+1}H_{2r+1}(X, \mathbb{Z})$. In other words, the Universality Conjecture will fail for X provided such an α exists.

We do not know of any examples of such elements, but Kollár [2] (see also [36, §2]) has constructed a smooth, projective three-fold Y and a non-torsion class

$b \in H_2^{\mathrm{sing}}(Y, \mathbb{Z})$ such that b is not algebraic (i.e., $b \notin N_1 H_2^{\mathrm{sing}}(Y), \mathbb{Z}))$ but some non-zero multiple of b is algebraic. In other words, Kollár provides a non-torsion counter-example to the "integral Hodge conjecture". (The original counter-examples to this conjecture were torsion.) Now, letting C be a smooth, projective curve of positive genus, setting $X = Y \times C$, and choosing $c \in H_1(C, \mathbb{Z})$ to be a generator, we can form the external product

$$a = b \times c \in H_3^{\mathrm{sing}}(X, \mathbb{Z}).$$

Then a is non-zero, a non-zero multiple of a belongs to $N_2 H_3^{\mathrm{sing}}(X)$, and yet there is no readily apparent reason why a should belong to $N_2 H_3^{\mathrm{sing}}(X)$.

Concerning the second kind of potential counter-example of Universality, we recall the commutative diagram

$$\begin{array}{ccc} Z_r(X)_{\mathrm{alg}\sim 0} & \xrightarrow{\Phi_r^{\mathrm{mor}}} & \mathcal{J}_r^{\mathrm{mor}}(X) \\ & \searrow^{\Phi_r} & \downarrow \\ & & \mathcal{J}_r^a(X) \end{array}$$

and the examples of Theorem 6.2 which show $\mathcal{J}_r^{\mathrm{mor}}(X) \twoheadrightarrow \mathcal{J}_a^r(X)$ can have a kernel. This does not immediately lead to a counter-example, because the group $L_r H_{2r+1}(X)$ may well fail to be finitely generated or of pure weight -1 and hence $\mathcal{J}_r^{\mathrm{mor}}(X) = \mathcal{J}(L_r H_{2r+1}(X))$ is typically not be an abelian variety. The Universality conjecture does, however, imply a very strong condition on the IMHS given as the kernel of $L_r H_{2r+1}(X) \to H_{2r+1}^{\mathrm{sing}}(X, \mathbb{Z}(r))$, as we now show.

Lemma 7.3. *Let X be a smooth, projective variety and suppose*

$$L_r H_{2r+1}(X) \twoheadrightarrow H$$

is a quotient of IMHS's such that H is finitely generated (i.e., is a MHS) and has pure weight -1. Then $\mathcal{J}(H)$ admits the structure of an abelian variety so that the composition of

$$CH_r(X)_{alg\sim 0} \xrightarrow{\Phi_r^{mor}} \mathcal{J}_r^{mor}(X) \twoheadrightarrow \mathcal{J}(H)$$

is regular.

Proof. According to [23, 12.23], it suffices to show that the onto map

$$\pi : CH_r(X)_{\mathrm{alg}\sim 0} \twoheadrightarrow \mathcal{J}(H)$$

is analytic, where $\mathcal{J}(H)$ acquires the structure of a complex torus via the surjection $H_{\mathbb{C}}/F^0(H_{\mathbb{C}}) \twoheadrightarrow H_{\mathbb{C}}/(H + F^0(H_{\mathbb{C}})) = \mathcal{J}(H)$. Let T and Γ be as in Definition 7.1. Since the morphic Abel-Jacobi map is natural for correspondences the composition of

$$T \to CH_0(T)_{\mathrm{alg}\sim 0} \xrightarrow{\Gamma_*} CH_r(X)_{\mathrm{alg}\sim 0} \to \mathcal{J}(H),$$

whose holomorphicity we seek to establish, coincides with the composition of

$$T \to CH_0(T)_{\mathrm{alg}\sim 0} \xrightarrow{\Phi_0^{\mathrm{mor}}} \mathcal{J}_0^{\mathrm{mor}}(T) \xrightarrow{\Gamma_*} \mathcal{J}_r^{\mathrm{mor}}(X) \twoheadrightarrow \mathcal{J}(H).$$

Note that $\mathcal{J}_0^{\mathrm{mor}}(T) = \mathcal{J}_0(T)$, $\Phi_0^{\mathrm{mor}} = \Phi_0$, and the map $T \to \mathcal{J}_0(T)$ coincides with the classical Albanese map by [23, 12.11(3)], and thus is holomorphic. The composition of

$$\mathcal{J}_0^{\mathrm{mor}}(T) \xrightarrow{\Gamma_*} \mathcal{J}_r^{\mathrm{mor}}(X) \twoheadrightarrow \mathcal{J}_r(H)$$

is holomorphic since it is induced from the composition of

$$L_0H_1(T)\xrightarrow{\Gamma_*}L_rH_{2r+1}(X) \twoheadrightarrow H$$

which is a morphism of MHS's. □

Theorem 7.4. *If, for a smooth, projective variety X, the IMHS*

$$K_rH_{2r+1}(X) := \ker(L_rH_{2r+1}(X) \to H_{2r+1}^{sing}(X, \mathbb{Z}(r))$$

admits a quotient IMHS that is finitely generated (i.e., is a MHS), non-torsion, and of pure weight -1, then the Universality of the Abel-Jacobi map fails for X.

Proof. The existence of such a quotient $K_rH_{2r+1}(X) \twoheadrightarrow H''$ implies that the surjective morphisms of IMHS's

$$L_rH_{2r+1}(X) \twoheadrightarrow N_{r+1}H_{2r+1}^{\text{sing}}(X, \mathbb{Z}(r))$$

factors as

$$L_rH_{2r+1}(X) \twoheadrightarrow H \twoheadrightarrow N_{r+1}H_{2r+1}^{\text{sing}}(X, \mathbb{Z}(r))$$

for some MHS H such that $H \twoheadrightarrow N_{r+1}H_{2r+1}^{\text{sing}}(X, \mathbb{Z}(r))$ has kernel H''. Then

$$N_{r+1}H_{2r+1}^{\text{sing}}(X, \mathbb{Z}(r))_{\text{tor}} \to \mathcal{J}(H'') \to \mathcal{J}(H) \to \mathcal{J}(N_{r+1}H_{2r+1}^{\text{sing}}(X, \mathbb{Z}(r))) \to 0$$

is exact and $\mathcal{J}(H'') \to \mathcal{J}(H)$ is not the zero map since $\mathcal{J}(H'')$ is uncountable. The result now follows from the Lemma. □

8. On torsion cycles

In this section, we describe the restriction of the morphic Abel-Jacobi map to torsion subgroups,

$$\Phi_r^{\text{mor}}|_{\text{tor}} : (CH_r(X)_{\text{alg}\sim 0})_{\text{tor}} \to \mathcal{J}_r^{\text{mor}}(X)_{\text{tor}}.$$

In particular, we show this map is surjective and give an explicit description of its kernel. In some simple cases, we prove unconditionally that this map is, in fact, an isomorphism. We show it is an isomorphism for projective varieties definable over a number field, assuming a common conjecture. We also show in this section how the examples due to Schoen [34] of varieties X for which the group $CH_r(X)_{\text{alg}\sim 0}$ contains an infinite amount of l-torsion lead to additional examples where the map $\mathcal{J}_r(X)^{\text{mor}} \to \mathcal{J}_r(X)$ has a non-trivial kernel.

Recall that there is a natural map

$$CH_r(X, 1) \to L_rH_{2r+1}(X). \tag{8.1}$$

The description of the kernel of $\Phi_r^{\text{mor}}|_{\text{tor}}$ depends on the image of this map, which is not well understood. Theorem 8.4 below characterizes the morphic Abel-Jacobi map on torsion subgroups in terms of the image of (8.1), using the following lemma.

Lemma 8.2. *For a quasi-projective variety U, the image of*

$$CH_r(U, 1) \to L_rH_{2r+1}(U)$$

is contained in $\Gamma(L_rH_{2r+1}(U))$.

Proof. Every class in $CH_r(U,1)$ is supported on a closed subscheme of dimension at most $r+1$. By naturality, it thus suffices to assume $\dim(U) \leq r+1$. In this case, we have that the map

$$L_rH_{2r+1}(U) \to H^{\mathrm{BM}}_{2r+1}(U, \mathbb{Z}(r))$$

is an isomorphism of IMHS's by Proposition 2.5 and Remark 4.2, and thus it suffices to prove that the natural map

$$CH_r(U,1) \to H^{\mathrm{BM}}_{2r+1}(U, \mathbb{Z}(r))$$

lands in $\Gamma(H^{\mathrm{BM}}_{2r+1}(U, \mathbb{Z}(r)))$. (In fact, this holds in all degrees.)

It suffices to prove this upon tensoring with $\mathbb{Q}$. Let $U \rightarrowtail V$ be a closed embedding with V smooth of dimension d. Then the map in question is isomorphic to

$$H^n_{\mathcal{M},U}(V, \mathbb{Q}(t)) \to H^n_{\mathrm{sing},U}(V, \mathbb{Q}(t)),$$

where $n = 2d - 2r - 1$ and $t = d - r$ and the subscripts U denote cohomology with supports. The image of this map is contained in

$$\Gamma(H^n_{\mathrm{sing},U}(V, \mathbb{Q}(t))) \cong \Gamma(H^{\mathrm{BM}}_{2r+1}(U, \mathbb{Q}(r)))$$

by [21, §8]. □

Remark 8.3. We presume that assertion of Lemma 8.2 holds in all degrees — i.e., the image of

$$CH_r(U,n) \to L_rH_{2r+n}(U)$$

is contained in $\Gamma(L_rH_{2r+n}(U))$, for all $r, n \geq 0$. We know of a proof of this statement when U is projective, but omit it since it is not needed for the rest of this paper. (But see the discussion in Section 11.)

For a quasi-projective variety V, define

$$L_rH_{2r+1}(V)_{\mathcal{M}} := \mathrm{im}\left(CH_r(V,1) \to L_rH_{2r+1}(V)\right).$$

(The subscript $\mathcal{M}$ stands for "motivic".) By Lemma 8.2, $L_rH_{2r+1}(V)_{\mathcal{M}}$ is a sub-IMHS of $L_rH_{2r+1}(V)$ having trivial Hodge structure, and hence the short exact sequence of IMHS's

$$0 \to L_rH_{2r+1}(V)_{\mathcal{M}} \to L_rH_{2r+1}(V) \to L_rH_{2r+1}(V)/L_rH_{2r+1}(V)_{\mathcal{M}} \to 0$$

induces a short exact sequence of abelian groups

$$0 \to L_rH_{2r+1}(V)_{\mathcal{M}} \to \Gamma(L_rH_{2r+1}(X)) \to \Gamma\left(L_rH_{2r+1}(V)/L_rH_{2r+1}(V)_{\mathcal{M}}\right) \to 0$$

and an isomorphism

$$\mathcal{J}^{\mathrm{mor}}_r(V) = \mathcal{J}(L_rH_{2r+1}(V)) \cong \mathcal{J}\left(L_rH_{2r+1}(V)/L_rH_{2r+1}(V)_{\mathcal{M}}\right).$$

Theorem 8.4. *Let X be a projective variety and r any integer. There is a natural exact sequence*

$$\Gamma\left(L_rH_{2r+1}(X)/L_rH_{2r+1}(X)_{\mathcal{M}}\right) \otimes \mathbb{Q}/\mathbb{Z} \to \left(CH_r(X)_{alg\sim 0}\right)_{tor} \xrightarrow{\Phi^{mor}_r} \mathcal{J}^{mor}_r(X)_{tor} \to 0,$$

where

$$\Gamma\left(L_rH_{2r+1}(V)/L_rH_{2r+1}(V)_{\mathcal{M}}\right) \cong \mathrm{coker}\left(CH_r(X,1) \to \Gamma\left(L_rH_{2r+1}(X)\right)\right).$$

Proof. To simplify the notation, we set

$$H(V) := L_rH_{2r+1}(V)/L_rH_{2r+1}(V)_{\mathcal{M}},$$

for a quasi-projective variety V. We use a theorem of Suslin-Voevodsky [37, 9.1], which implies that the natural map

$$CH_r(V,n;\mathbb{Q}/\mathbb{Z}) \to L_rH_{2r+n}(V,\mathbb{Q}/\mathbb{Z})$$

is an isomorphism for all n and V. The Suslin-Voevodsky result, together with the long exact sequences in Chow groups and Lawson homology obtained from $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, yields a short exact sequence

$$0 \to H(V) \to H(V)_{\mathbb{Q}} \to (CH_r(V)_{\mathrm{alg}\sim 0})_{\mathrm{tor}} \to 0 \tag{8.5}$$

and hence an isomorphism

$$H(V) \otimes \mathbb{Q}/\mathbb{Z} \cong (CH_r(V)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}.$$

Let Y be a closed subvariety of X of dimension r. Using $CH_r(Y,1) = 0 = L_rH_{2r+1}(Y)$, $Z_r^Y(X) = CH_r(Y) \cong L_rH_{2r}(Y)$, and the localization sequences for Chow groups and Lawson homology, one obtains the exact sequence of IMHS's

$$0 \to H(X) \to H(X-Y) \to CH_r^Y(X)_{\mathrm{alg}\sim 0} \to 0. \tag{8.6}$$

(Here $CH_r^Y(X)_{\mathrm{alg}\sim 0}$ denotes the subgroup of $CH_r(X)$ of cycles supported on Y and algebraically equivalent to 0 on X.) The boundary map in the associated six-term exact sequence for $\mathrm{Ext}^*_{\mathrm{IMHS}}(\mathbb{Z}(0),-)$ has the form

$$CH_r^Y(X)_{\mathrm{alg}\sim 0} \to \mathcal{J}(H(X)) \cong \mathcal{J}_r^{\mathrm{mor}}(X). \tag{8.7}$$

Using that

$$0 \to L_rH_{2r+1}(X) \to L_rH_{2r+1}(X-Y) \to Z_r^Y(X)_{\mathrm{alg}\sim 0} \to 0$$

surjects onto (8.6), it follows directly from the definition of Φ_r^{mor} that (8.7) is the restriction of Φ_r^{mor} to $CH_r^Y(X)_{\mathrm{alg}\sim 0} \subset CH_r(X)_{\mathrm{alg}\sim 0}$. Using now Lemma 3.2, we see that the restriction of Φ_r^{mor} to $(CH_r^Y(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}$ coincides with the composition of

$$(CH_r^Y(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}} \to H(X) \otimes \mathbb{Q}/\mathbb{Z} \twoheadrightarrow \Big(H(X)/\Gamma(H(X))\Big) \otimes \mathbb{Q}/\mathbb{Z} \cong \mathcal{J}_r^{\mathrm{mor}}(X)_{\mathrm{tor}},$$

where the first map is the boundary map in the long exact sequence for $\mathrm{Tor}_*(-,\mathbb{Q}/\mathbb{Z})$ applied to (8.6).

We now take inductive limits over all dimension r subvarieties Y of X. We have $\varinjlim_Y CH_r(X-Y) = 0$, and hence, using (8.5), we see that $\varinjlim_Y H(X-Y)$ is uniquely divisible. As a consequence, the boundary map $\varinjlim_Y (CH_r^Y(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}} \to H(X) \otimes \mathbb{Q}/\mathbb{Z}$ is an isomorphism.

We have shown that the restriction of Φ_r^{mor} to $(CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}$ factors as

$$(CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}} \xrightarrow{\cong} H(X) \otimes \mathbb{Q}/\mathbb{Z} \twoheadrightarrow \Big(H(X)/\Gamma(H(X))\Big) \otimes \mathbb{Q}/\mathbb{Z} \cong \mathcal{J}_r^{\mathrm{mor}}(X)_{\mathrm{tor}},$$

and the Theorem follows. □

Corollary 8.8. *If, for a given projective variety X and integer r, the cokernel of the map*

$$CH_r(X,1) \to \Gamma(L_rH_{2r+1}(X))$$

is divisible modulo torsion — for example, if

$$CH_r(X,1;\mathbb{Q}) \to \Gamma(L_rH_{2r+1}(X,\mathbb{Q}))$$

is onto — then the morphic Abel-Jacobi map induces an isomorphism on torsion subgroups:

$$\Phi_r^{mor}|_{tor} : (CH_r(X)_{alg\sim 0})_{tor} \xrightarrow{\cong} \mathcal{J}_r(X)_{tor}.$$

Example 8.9. *Recall that $L_rH_{2r+1}(X)$ has weights $-1 \leq w \leq 0$, and thus the hypothesis of Corollary 8.8 is met if $Gr_0^W(L_rH_{2r+1}(X,\mathbb{Q})) = 0$. This holds, for example, if X is smooth and projective and the map*

$$L_rH_{2r+1}(X,\mathbb{Q}) \to H_{2r+1}^{sing}(X,\mathbb{Q}(r))$$

is injective. Using the results of [17, §6]*, we see that we have an isomorphism*

$$\Phi_r^{mor}|_{tor} : (CH_r(X)_{alg\sim 0})_{tor} \xrightarrow{\cong} \mathcal{J}_r^{mor}(X)_{tor}$$

if X is smooth, projective, and belongs to the class $\mathcal{C}$ defined in op. cit. Moreover, in this case, we have $\mathcal{J}_r^{mor}(X) = \mathcal{J}_r(X)$ and $\Phi_r^{mor} = \Phi_r$, so that the classical Abel-Jacobi map induces an isomorphism

$$\Phi_r|_{tor} : (CH_r(X)_{alg\sim 0})_{tor} \xrightarrow{\cong} \mathcal{J}_r^{mor}(X)_{tor}$$

for such X.

The class $\mathcal{C}$ includes all curves, all linear varieties (for example, all toric varieties), and all cellular varieties, and it is closed under localization, blow-ups, and the formation of bundles, in the sense made precise in [17, 6.9]*.*

To obtain more interesting examples of varieties satisfying the hypothesis of Corollary 8.8, we need to assume a well-known conjecture:

Conjecture 8.10. *(See, e.g.,* [21, 5.20]*.) If U is a smooth, quasi-projective complex variety that can be defined over $\mathbb{Q}$, then the canonical map*

$$CH_r(U,n;\mathbb{Q}) \to \Gamma(H_{2r+n}^{BM}(U,\mathbb{Q}(r)))$$

is a surjection for all n, r.

Theorem 8.11. *Let X be a projective complex variety that can be defined over a number field. Assume Conjecture 8.10 holds when $r = 0$ and $n = 2$ for all smooth U definable over $\mathbb{Q}$. Then the morphic Abel-Jacobi map induces an isomorphism on torsion subgroups:*

$$\Phi_r^{mor}|_{tor} : (CH_r(X)_{alg\sim 0})_{tor} \xrightarrow{\cong} \mathcal{J}_r(X)_{tor}.$$

Proof. Suppose X is definable over a number field k. By [21, 5.21], if Conjecture 8.10 holds (for a fixed r and n), then the same statement holds for any smooth U definable over k (for the same r and n). Let S be a (possibly singular) complex projective surface that can be defined over k and choose a closed embedding $S \rightarrowtail \mathbb{P}^N$, defined over k, with open complement U. Then one easily sees [21, 8.9] that the surjectivity of $CH_0(U,2;\mathbb{Q}) \to \Gamma(H_2^{\mathrm{BM}}(U,\mathbb{Q}(0)))$ (which we are assuming, since U is definable over k) implies the surjectivity of

$$CH_0(S,1;\mathbb{Q}) \to \Gamma(H_1^{\mathrm{sing}}(S,\mathbb{Q}(0))).$$

We claim now that $\Gamma(L_rH_{2r+1}(X,\mathbb{Q}))$ is generated by classes coming via correspondences $\gamma \in CH_{r+2}(S \times X)$ from classes in $\Gamma(L_0H_1(S,\mathbb{Q})) \cong \Gamma(H_1^{\mathrm{sing}}(S,\mathbb{Q}(0)))$, where S ranges over all projective surface definable over k. This will prove the Theorem, since the diagram

$$\begin{array}{ccc} CH_0(S,1;\mathbb{Q}) & \xrightarrow{\gamma_*} & CH_r(X,1;\mathbb{Q}) \\ \downarrow & & \downarrow \\ L_0H_1(S,\mathbb{Q}) & \xrightarrow{\gamma_*} & L_rH_{2r+1}(X,\mathbb{Q}) \end{array}$$

commutes.

To establish the claim, recall from Theorem 4.1 that $L_rH_{2r+1}(X,\mathbb{Q})$ is isomorphic to the IMHS

$$H_1^{\text{sing}}(\mathcal{Z}_r(X),\mathbb{Q}) \cong H_1^{\text{sing}}(\mathcal{C}_r(X),\mathbb{Q}) \otimes_{H_0^{\text{sing}}(\mathcal{C}_r(X),\mathbb{Q})} H_0^{\text{sing}}(\mathcal{Z}_r(X),\mathbb{Q}),$$

where $\mathcal{C}_r(X) = \coprod_e \mathcal{C}_{r,e}(X)$ is the Chow monoid of r-cycles on X. From this, we see that $\Gamma(L_rH_{2r+1}(X,\mathbb{Q}))$ is generated by classes coming via the "universal correspondence" from classes in $\Gamma(H_1(\mathcal{C}_{r,e}(X),\mathbb{Q}(0)))$. Since X is definable over k, so is $\mathcal{C}_{r,e}(X)$, for all e. The Andreotti-Frankel Theorem [1] shows that, for each e, there is a surface $S_e \subset \mathcal{C}_{r,e}$, so that the map

$$H_1(S_e,\mathbb{Q}) \xrightarrow{\cong} H_1(\mathcal{C}_{r,e}(X),\mathbb{Q})$$

is an isomorphism. This shows that $\Gamma(L_rH_{2r+1}(X,\mathbb{Q}))$ is generated by classes coming via correspondence from classes in $\Gamma(H_1^{\text{sing}}(S_e,\mathbb{Q}(0)))$.

In fact, we claim that the Andreotti-Frankel Theorem allows us to take S_e to be definable over k, and hence the result follows. Indeed, let Y be any projective variety definable over k, say of dimension n. The Andreotti-Frankel Theorem amounts to the assertion that a smooth, affine complex variety of dimension d has the homotopy type of a d-dimensional CW complex Thus, if we take Y_{n-1} to a subvariety of Y that is definable over k, that contains the singular locus of Y and all components of Y having dimension less than two, and for which $U = Y - Y_{n-1}$ is affine, then we have $H_2^{\text{BM}}(U,\mathbb{Q}) = H_1^{\text{BM}}(U,\mathbb{Q}) = 0$ by Poincaré duality, so long as $n > 2$. Thus $H_1^{\text{sing}}(Y_{n-1},\mathbb{Q}) \cong H_1^{\text{sing}}(Y,\mathbb{Q})$, for $n > 2$, and the claim is established by induction on $n \geq 2$. □

At one time, it was conjectured that the classical Abel-Jacobi map itself ought to induce an isomorphism on torsion subgroups:

$$\Phi_r|_{\text{alg}\sim 0,\text{tor}} : (CH_r(X)_{\text{alg}\sim 0})_{\text{tor}} \xrightarrow{?\cong} \mathcal{J}_r^a(X)_{\text{tor}}, \tag{8.12}$$

for X smooth and projective. More generally, it was conjectured that the map

$$\Phi_r|_{\text{tor}} : (CH_r(X)_{\text{hom}\sim 0})_{\text{tor}} \to \mathcal{J}_r(X)_{\text{tor}} \tag{8.13}$$

ought to be injective.

The map (8.12) is always surjective. In codimension one (i.e., $r = \dim(X) - 1$), the injectivity of (8.12) and (8.13) are easily seen to hold; in codimension two ($r = \dim(X) - 2$), the injectivity of these maps is a consequence of the Merkurjev-Suslin Theorem [27] (see [30, 10.3]); and for zero-cycles ($r = 0$), the injectivity of these maps is a theorem of Roitman [33, 4]. The injectivity of (8.12), however, is now known to fail in general. Schoen [34] has constructed examples of a smooth, projective complex variety X of dimension d, definable over a field of transcendence degree one over $\mathbb{Q}$, for which $CH_r(X)_{\text{alg}\sim 0}$ has infinite l-torsion for all $0 < r < d-2$ and some prime l. Since $\mathcal{J}_r(X)_{\text{tor}} \cong (\mathbb{Q}/\mathbb{Z})^\nu$ for some integer ν, the Abel-Jacobi map cannot be injective on $(CH_r(X)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}$ for such varieties. Soulé and Voisin [36] have also shown that (8.12) can fail to be injective for four-folds with $r = 1$. In earlier work, Totaro [40] constructed examples of a smooth, projective variety X, definable over a number field, for which (8.13) fails to be injective. In light of Theorem 8.11, it is interesting to note that there are no known examples of the failure of the injectivity of (8.12) for varieties definable over number fields.

We now show how Schoen's examples gives rise to additional examples of varieties X for which $\mathcal{J}_r^{\mathrm{mor}}(X) \twoheadrightarrow \mathcal{J}_r^a(X)$ has a kernel. We first recall the details of Schoen's construction:

Theorem 8.14 (Schoen [34]). *Suppose k is an algebraically closed subfield of $\mathbb{C}$, W is a smooth, projective k-variety, E is an elliptic curve over $\mathbb{C}$ whose j-invariant does not belong to k. Then for any integer $r \geq 0$, the map*

$$CH_r(W) \otimes CH_0(E)_{tor} \to CH_r(W \times_k E)$$

given by external product of cycles is injective.

Remark 8.15. Schoen's theorem is actually more general that this — one may replace $k \subset \mathbb{C}$ with any extension of algebraically closed fields, even those of positive characteristic.

Schoen also proves [35] that for any algebraically closed field k of characteristic zero, there exist a smooth, projective three-fold W (in fact, W can be taken to be an abelian variety) such that $CH_1(W) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ has infinite corank for some prime l. Taking $k = \overline{\mathbb{Q}}$ and picking E to be an elliptic curve over $\mathbb{C}$ whose j-invariant does not belong to k, then since $CH_0(E)_{\mathrm{tor}} \cong (\mathbb{Q}/\mathbb{Z})^2 \cong \bigoplus_l (\mathbb{Q}_l/\mathbb{Z}_l)^2$, we see that $CH_r(W) \otimes CH_0(E)_{\mathrm{tor}}$ has infinite l-torsion for some prime l.

Theorem 8.16. *Suppose k, W, E are as in Theorem 8.14 and that $CH_r(W) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ has infinite corank for some prime l. Then the kernel of*

$$\mathcal{J}_r^{mor}(W \times_k E) \to \mathcal{J}_r(W \times_k E)$$

contains the quotient of a non-zero complex vector space by a countable subgroup. In particular, there exists a complex abelian four-fold V, definable over an algebraically closed field of transcendence degree one over $\mathbb{Q}$, such that the kernel of

$$\mathcal{J}_1^{mor}(V) \to \mathcal{J}_1(V)$$

is uncountable.

Proof. Observe that elements of $CH_0(E)_{\mathrm{tor}}$ are algebraically equivalent to 0 and hence we have an isomorphism

$$CH_0(E)_{\mathrm{tor}} \cong L_0H_1(E) \otimes \mathbb{Q}/\mathbb{Z}.$$

Moreover, the kernel of $CH_r(W) \to CH_r(W_{\mathbb{C}}) \to L_rH_{2r}(W_{\mathbb{C}})$ is divisible (since $CH_r(W)/\mathrm{alg} \sim 0 \cong L_rH_{2r}(W_{\mathbb{C}})$) and thus

$$CH_r(W) \otimes CH_0(E)_{\mathrm{tor}} \cong L_rH_{2r}(W_{\mathbb{C}}) \otimes L_0H_1(E) \otimes \mathbb{Q}/\mathbb{Z}.$$

The image of the injection in Schoen's Theorem is contained in $(CH_r(W_{\mathbb{C}} \times_{\mathbb{C}} E)_{\mathrm{alg}\sim 0})_{\mathrm{tor}}$ and there is a natural surjection

$$L_rH_{2r+1}(W_{\mathbb{C}} \times_{\mathbb{C}} E) \otimes \mathbb{Q}/\mathbb{Z} \twoheadrightarrow (CH_r(W_{\mathbb{C}} \times_{\mathbb{C}} E)_{\mathrm{alg}\sim 0})_{\mathrm{tor}},$$

coming from (8.5) in the proof of Theorem 8.4. Moreover, the injection in Schoen's Theorem coincides with the composition of

$$\begin{aligned} CH_r(W) \otimes CH_0(E)_{\rm tor} &\cong L_rH_{2r}(W_{\mathbb{C}}) \otimes L_0H_1(E) \otimes \mathbb{Q}/\mathbb{Z} \\ &\to L_rH_{2r+1}(W_{\mathbb{C}} \times_{\mathbb{C}} E) \otimes \mathbb{Q}/\mathbb{Z} \twoheadrightarrow (CH_r(W_{\mathbb{C}} \times_{\mathbb{C}} E)_{{\rm alg}\sim 0})_{\rm tor}, \end{aligned}$$

where the first map is induced by external product in Lawson homology, and thus the map

$$L_rH_{2r}(W_{\mathbb{C}}) \otimes L_0H_1(E) \otimes \mathbb{Q}/\mathbb{Z} \to L_rH_{2r+1}(W_{\mathbb{C}} \times_{\mathbb{C}} E) \otimes \mathbb{Q}/\mathbb{Z}$$

is injective.

We claim that the map

$$\mathrm{Griff}_r(W_{\mathbb{C}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} L_0H_1(E, \mathbb{Q}) \to L_rH_{2r+1}(W_{\mathbb{C}} \times_{\mathbb{C}} E, \mathbb{Q})$$

is not the zero map. As in the proof of Theorem 6.2, this will suffice to complete the proof. Note that $CH_r(W) \otimes \mathbb{Q}_l/\mathbb{Z}_l \cong L_rH_{2r}(W_{\mathbb{C}}) \otimes \mathbb{Q}_l/\mathbb{Z}_l$. The hypotheses imply that the map

$$\mathrm{Griff}_r(W_{\mathbb{C}}) \otimes L_0H_1(E) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to L_rH_{2r}(W_{\mathbb{C}}) \otimes L_0H_1(E) \otimes \mathbb{Q}_l/\mathbb{Z}_l$$

is not the zero map, and hence that

$$\mathrm{Griff}_r(W_{\mathbb{C}}) \otimes L_0H_1(E) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to L_rH_{2r+1}(W_{\mathbb{C}} \times_{\mathbb{C}} E) \otimes \mathbb{Q}_l/\mathbb{Z}_l$$

is non-zero. □

Note that Schoen's examples give *torsion* cycles that lie in the kernel of the classical Abel-Jacobi map and are algebraically equivalent to zero. The previous result shows that these torsion cycles entail the existence of *non-torsion* classes in

$$\big(\ker(\Phi_r) \cap CH_r(X)_{{\rm alg}\sim 0}\big) / \ker(\Phi_r^{\rm mor}).$$

9. Relationship to a Conjecture of Beilinson

Beilinson [3, Conjecture 6] once conjectured that the canonical map

$$CH_r(U, n-2r)_{\mathbb{Q}} \to \Gamma(H_n^{\rm BM}(U, \mathbb{Q}(r)) \tag{9.1}$$

is surjective for every smooth, quasi-projective variety U. (Beilinson's formulation involved rational K-theory and singular cohomology, but the one stated here is equivalent.) When $U = X$ is projective, then since $\Gamma(H_n^{\rm BM}(X, \mathbb{Z}(r)) = 0$ unless $n = 2r$, this assertion is equivalent to the classical Hodge conjecture.

The general conjecture, however, is now known to be false for arbitrary varieties in the case $n = 2r+1$, thanks to an argument of Jannsen [21, 9.11], which shows: Given a smooth, projective variety X and integer r for which the rational Abel-Jacobi map $\Phi_r \otimes \mathbb{Q} : CH_r(X, \mathbb{Q})_{{\rm hom}\sim 0} \to \mathcal{J}_r(X)_{\mathbb{Q}}$ fails to be injective, the map

$$CH_r(U, 1) \to \Gamma(H_{2r+1}^{\rm BM}(U, \mathbb{Z}(r)))$$

fails to be surjective for some open subvariety U of X. Failures of injectivity of $\Phi_r \otimes \mathbb{Q}$ are common — for example, this map is not injective when $r = 0$ and X is a smooth, projective surface with $p_g(X) > 0$, as shown by Mumford [29]. (The more restrictive conjecture that the map (9.1) is surjective for varieties definable over a number field remains open.) Jannsen basic argument applies in our context as well to establish the following related result.

Theorem 9.2. *The maps*

$$CH_r(U,1) \to \Gamma(L_rH_{2r+1}(U)) \quad \text{and} \quad \Gamma(L_rH_{2r+1}(V)) \to \Gamma(H^{BM}_{2r+1}(V,\mathbb{Z}(r)))$$

fail to be surjective, even upon tensoring with $\mathbb{Q}$, for some smooth, quasi-projective varieties U and V

Proof. We follow the approach of Jannsen in [21, §9].

Given a smooth, projective variety X and closed subscheme W of dimension r, set $U = X - W$. We have a commutative diagram with exact rows

$$\begin{array}{ccccc}
CH_r(U,1) & \longrightarrow & Z^W_r(X)_{\text{alg}\sim 0} & \longrightarrow & CH_r(X)_{\text{alg}\sim 0} \\
\downarrow & & \| & & \downarrow \Phi^{\text{mor}}_r \\
\Gamma(L_rH_{2r+1}(U)) & \longrightarrow & Z^W_r(X)_{\text{alg}\sim 0} & \longrightarrow & \mathcal{J}^{\text{mor}}_r(X) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(H^{\text{BM}}_{2r+1}(U,\mathbb{Z}(r))) & \longrightarrow & Z^W_r(X)_{\text{hom}\sim 0} & \longrightarrow & \mathcal{J}_r(X)
\end{array}$$

where $Z^W_r(X)_{\text{alg}\sim 0}$ is the subgroup of $Z_r(X)_{\text{alg}\sim 0}$ consisting of classes supported on W. The middle and bottom rows of this diagram are parts of the long exact sequences for $\text{Ext}^*_{\text{IMHS}}(\mathbb{Z}(0),-)$ coming from the short exact sequences of IMHS's

$$0 \to L_rH_{2r+1}(X) \to L_rH_{2r+1}(U) \to Z^W_r(X)_{\text{alg}\sim 0} \to 0 \tag{9.3}$$

and

$$0 \to H^{\text{BM}}_{2r+1}(X,\mathbb{Z}(r)) \to H^{\text{BM}}_{2r+1}(U,\mathbb{Z}(r)) \to Z^W_r(X)_{\text{hom}\sim 0} \to 0. \tag{9.4}$$

The commutativity of the upper-left square follows from the commutativity of

$$\begin{array}{ccccc}
CH_r(U,1) & \longrightarrow & CH_r(W) & \longrightarrow & CH_r(X) \\
\downarrow & & \downarrow & & \downarrow \\
L_rH_{2r+1}(U) & \longrightarrow & L_rH_{2r}(W) & \longrightarrow & L_rH_{2r}(X).
\end{array}$$

(Recall that $CH_r(W) \cong L_rH_{2r}(W) \cong Z^W_r(X)$.) The upper-right square commutes by definition of Φ^{mor}_r. The bottom two squares commute since there is a map of short exact sequences from (9.3) to (9.4).

Take X to be an example of a smooth, projective complex variety such that the kernel of

$$\mathcal{J}^{\text{mor}}_r(X) \to \mathcal{J}_r(X)$$

contains torsion-free elements, such as is provided by Theorem 6.2 or Theorem 8.16. By the surjectivity of Φ^{mor}_r, there is a non-torsion class α in $CH_r(X)_{\text{alg}\sim 0}$ such that $\Phi^{\text{mor}}_r(\alpha) \neq 0$ but $\Phi_r(\alpha) = 0$. The class α is supported on some closed subvariety W of dimension r, so that α lifts to an element $\tilde{\alpha} \in Z^W_r(X)_{\text{alg}\sim 0}$. An easy diagram chase shows that

$$\Gamma(L_rH_{2r+1}(V)) \to \Gamma(H^{\text{BM}}_{2r+1}(V,\mathbb{Z}(r)))$$

cannot be surjective (even modulo torsion, since α is non-torsion).

The other assertion follows in a similar manner, starting with a non-torsion element α in the kernel of Φ^{mor}_r that is supported on W. Such elements can be constructed using the general method given in [21, §10]. In fact, as mentioned above, Mumford proved that the Abel-Jacobi map for $r = 0$ fails to be injective modulo torsion when X is a smooth, projective surface with $p_g(X) > 0$. This leads to examples where

$$CH_0(U,1) \to \Gamma(H_1^{\mathrm{BM}}(U,\mathbb{Z}(0)))$$

fails to be onto. Since $L_0H_1(X) \cong H_1^{\mathrm{BM}}(X,\mathbb{Z}(0))$, and $\Phi_0^{\mathrm{mor}} = \Phi_0$, our assertion is an immediate consequence. But, using Jannsen's construction, one gets examples where

$$CH_r(U,1) \to \Gamma(L_rH_{2r+1}(U))$$

fails to be surjective for $r > 0$ too. □

In contrast to Theorem 9.2, it is reasonable to expect $CH_r(U,1) \to \Gamma(L_rH_{2r+1}(U))$ to be onto, at least upon tensoring with $\mathbb{Q}$, if either U is definable over a number field or U is smooth and projective. Compare with Corollary 8.8 and Conjecture 8.10.

10. Relationship with Bloch-Beilinson Filtration

Bloch and Beilinson have independently proposed a filtration on the rational Chow groups of a smooth, projective complex variety X, which we will write as

$$CH^j(X)_{\mathbb{Q}} = F^0_{BB}CH^j(X) \supset F^1_{BB}CH^j(X) \supset F^2_{BB}CH^j(X) \supset \cdots .$$

(Here, the subscript $\mathbb{Q}$ denotes tensoring with $\mathbb{Q}$, but we omit it in the notation $F^t_{BB}CH^j(X)$.) This filtration is conjectured to enjoy the following properties:

1. $F^1_{BB}CH^j(X) = CH^j(X)_{\mathbb{Q},\mathrm{hom}\sim 0}$
2. $F^2_{BB}CH^j(X) = \ker(\Phi^i \otimes \mathbb{Q} : CH^j(X)_{\mathbb{Q},\mathrm{hom}\sim 0} \to \mathcal{J}^j(X)_{\mathbb{Q}})$
3. $F^{j+1}_{BB}CH^j(X) = 0$
4. $F^tCH^j(X) \cdot F^sCH^i(X) \subset F^{t+s}CH^{j+i}(X)$
5. The filtration $F^\bullet CH^j(X)$ is natural for pushforward and pullback
6. The associated quotient group $Gr^t_{BB}CH^j(X) = F^t_{BB}CH^j(X)/F^{t+1}_{BB}CH^j(X)$ depends only on the Grothendieck motive $h^{2j-t}(X)$. (For this conjectural property to make sense, one must assume Grothendieck's standard conjecture C, that the Kunneth components of the class of the diagonal are algebraic.)

We refer the reader to [22] for more details.

In this section we discuss the connection between the Bloch-Beilinson filtration on the morphic Abel-Jacobi map. A motivating idea is to imagine that the kernel of the morphic Abel-Jacobi map forms the second stage in a "morphic" filtration of $CH^j(X)$,

$$CH^j(X,\mathbb{Q}) = F^0_{\mathrm{mor}}CH^j(X) \supset F^1_{\mathrm{mor}}CH^j(X) \supset F^2_{\mathrm{mor}}CH^j(X) \supset \cdots ,$$

where

$$F^1_{\mathrm{mor}}CH^j(X) = CH^j(X)_{\mathbb{Q},\mathrm{alg}\sim 0}$$
$$F^2_{\mathrm{mor}}CH^j(X) = \ker(\Phi^j_{\mathrm{mor}} \otimes \mathbb{Q} : CH^j(X)_{\mathbb{Q},\mathrm{hom}\sim 0} \to \mathcal{J}^j_{\mathrm{mor}}(X)_{\mathbb{Q}}),$$

and such that analogues of some of the conjectural properties of the Bloch-Beilinson filtration also hold. (Here, to conform with the notation of the Bloch-Beilinson filtration, we use cohomological indexing: define $\Phi^j_{\mathrm{mor}} : CH^j(X)_{\mathrm{alg}\sim 0} \to \mathcal{J}^j_{\mathrm{mor}}(X) := \mathcal{J}(L^jH^{2j-1}(X))$ to be the map obtained from Φ^{mor}_r by use of the identifications $CH^j(X)_{\mathrm{alg}\sim 0}(X) = CH_r(X)_{\mathrm{alg}\sim 0}$ and $L^jH^{2j-1}(X) \cong L_rH_{2r+1}(X)$, where $r + j = \dim(X)$.)

It is unclear what to expect, if anything, as the analogue of property (6), but perhaps one should conjecture that the associated quotient group

$$Gr^r_{\text{mor}}CH^j(X) := F^r_{\text{mor}}CH^j(X)/F^{r+1}_{\text{mor}}CH^j(X)$$

depends only on the motive $h^{2j-t}_{alg}(X)$ in the category of pure motives modulo algebraic equivalence. (Note that it is conjectured by Murre [31] that the motive of X modulo algebraic equivalence admits such a decomposition, but not necessarily canonically.)

Moreover, it seems reasonable to expect that $F^t_{\text{mor}} \subset F^t_{BB}$ for all t, since it holds for $t \leq 2$, so that we would have a diagram of inclusions

$$\begin{array}{ccccccc} F^0_{BB}CH^j(X) & \hookleftarrow & F^1_{BB}CH^j(X) & \hookleftarrow & F^2_{BB}CH^j(X) & \hookleftarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ F^0_{\text{mor}}CH^j(X) & \hookleftarrow & F^1_{\text{mor}}CH^j(X) & \hookleftarrow & F^2_{\text{mor}}CH^j(X) & \hookleftarrow & \cdots . \end{array}$$

Note that Corollary 6.3 shows that the containment

$$F^2_{\text{mor}}CH^j(X) \subset F^1_{\text{mor}}CH^j(X) \cap F^2_{BB}CH^j(X)$$

can be proper, so that $F^\bullet_{\text{mor}}$ cannot merely be obtained by intersecting $F^\bullet_{BB}$ with $CH^j(X)_{\mathbb{Q},\text{alg}\sim 0}$.

Regardless of the existence of F^t_{mor} for $t \geq 3$, we can relate F^2_{mor} with the Bloch-Beilinson filtration $F^\bullet_{BB}$ (provided the later exists), using a result of Jannsen.

Theorem 10.1. *Assume the Bloch-Beilinson filtration exists and let X be a smooth, projective complex variety. Assume also that Grothendieck's standard conjecture B holds for X. Then*

$$F^j_{BB}CH^j(X) \subset F^2_{mor}CH^j(X) = \ker(\Phi^j_{mor} \otimes \mathbb{Q}),$$

for all $j \geq 2$.

Proof. According [22, 6.4], we have

$$F^j_{BB}CH^j(X) \subset \left\langle F^j_{BB} \right\rangle_0 CH^j(X)$$

where

$$\left\langle F^j_{BB} \right\rangle_0 CH^j(X) := \sum_{T,\gamma} \operatorname{im}(\gamma_* : F^j_{BB}CH_0(T) \to F^j_{BB}CH^j(X))$$

with T ranging over all smooth, projective complex varieties and γ ranging over all correspondences in $CH^j(T \times X)$. Since the morphic Abel-Jacobi map on zero cycles coincides with the classical one, we have

$$F^j_{BB}CH_0(T) \subset F^2_{BB}CH_0(T) = F^2_{\text{mor}}CH_0(T) \cap CH_0(T)_{\mathbb{Q},\text{alg}\sim 0},$$

and the result follows from part (4) of Theorem 5.7. □

For example, recall from Theorem 8.16 that Schoen's results provide examples of a smooth, projective four-dimensional (abelian) variety X such that

$$\mathcal{J}^{\text{mor}}_1(X)_{\mathbb{Q}} \twoheadrightarrow \mathcal{J}^a_1(X)_{\mathbb{Q}}$$

has a non-trivial kernel — specifically, this kernel is $\mathcal{J}(K^3H^5(X))_{\mathbb{Q}}$ where

$$K^3H^5(X) := \ker(L^3H^5(X) \to H^5(X, \mathbb{Z}(3))).$$

In other words, we have

$$F^1_{\mathrm{mor}}CH^3(X) \cap F^2_{BB}CH^3(X)/F^2_{\mathrm{mor}}CH^3(X) \cong \mathcal{J}(K^3H^5(X))_{\mathbb{Q}} \neq 0.$$

In particular, we have inclusions

$$F^3_{BB}CH^3(X) \subset F^2_{\mathrm{mor}}CH^3(X) \subset F^2_{BB}CH^3(X) \cap F^1_{\mathrm{mor}}CH^3(X) \subset F^2_{BB}CH^3(X),$$

so that $Gr^2_{BB}CH^3(X)$ contains $\mathcal{J}(K^3H^5(X))_{\mathbb{Q}}$ as a sub-quotient. Recall that $Gr^2_{BB}CH^3(X)$ is conjectured to depend only on the Grothendieck motive $h^4(X)$. It would be interesting to formulate, at least conjecturally, a motivic description of $\mathcal{J}(K^3H^5(X))$, or possibly even $K^3H^5(X)$ itself, that "explains" why it only depends on $h^4(X)$. We note that Jannsen proves (assuming that the Bloch-Beilinson filtration exists) that the quotient

$$F^2_{BB}CH^3(X)/F^2_{BB}CH^3(X)\cap F^1_{\mathrm{mor}}CH^3(X) \cong (F^2_{BB}CH^3(X)+F^1_{\mathrm{mor}}CH^3(X))/F^1_{\mathrm{mor}}CH^3(X)$$

depends only on the motive $h^4(X)/N^1h^4(X)$, where $N^\bullet$ refers to the filtration by coniveau. Moreover,the quotient $Gr^2_{BB}CH^3(X)$ should depend only on $h^4(X)/N^2h^4(X)$ — in fact, one conjectures that

$$Gr^2_{BB}CH^3(X) = \mathrm{Ext}^2_{\mathcal{MM}}(1, h^4(X)) \cong \mathrm{Ext}^2_{\mathcal{MM}}(1, h^4(X)/N^2h^4(X)).$$

A possibly outlandish conjecture, then, is that we have

$$\mathcal{J}(K^3H^5(X))_{\mathbb{Q}} \overset{?}{\cong} \mathrm{Ext}^2_{\mathcal{MM}}(1, N^1h^4(X)/N^2h^4(X)).$$

We now describe a method of filtering $F^2_{BB}CH^j(X)/F^2_{\mathrm{mor}}CH^j(X)$ that arises from the factorization

$$L_rH_{2r+1}(X) \xrightarrow{s} L_{r-1}H_{2r+1}(X)(-1) \xrightarrow{s} \cdots \xrightarrow{s} L_0H_{2r+1}(X)(-r) \cong H^{\mathrm{sing}}_{2r+1}(X, \mathbb{Z}(r))$$

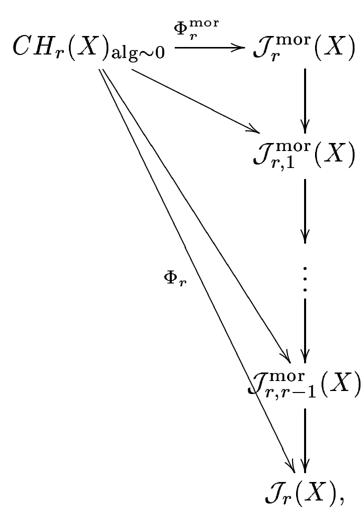
of the canonical map from Lawson homology to singular homology. Applying $\mathcal{J}(-)$ to this sequence gives a sequence maps of topological abelian groups

$$\mathcal{J}^{\mathrm{mor}}_r(X) = \mathcal{J}^{\mathrm{mor}}_{r,0}(X) \to \mathcal{J}^{\mathrm{mor}}_{r,1}(X) \to \cdots \to \mathcal{J}^{\mathrm{mor}}_{r,r}(X) = \mathcal{J}_r(X),$$

where we define

$$\mathcal{J}^{\mathrm{mor}}_{r,i}(X) := \mathcal{J}\left(L_{r-i}H_{2r+1}(X)(-i)\right).$$

We thus have a diagram of continuous maps of topological abelian groups



in which the oblique maps

$$\Phi^{\rm mor}_{r,i} : CH_r(X)_{\rm alg\sim 0} \to \mathcal{J}^{\rm mor}_{r,i}(X), \quad 0 \le i \le r,$$

are defined by composition. (Note that $\Phi^{\rm mor}_{r,0} = \Phi^{\rm mor}_r$ and $\Phi^{\rm mor}_{r,r} = \Phi_r$.) Taking kernels of these maps therefore determines a filtration of $\ker(\Phi_r)/\ker(\Phi^{\rm mor}_r)$. Suslin's Conjecture 2.4 predicts, however, that some of the stages in this filtration are trivial — namely, we expect that

$$\mathcal{J}^{\rm mor}_{r,d-r-1}(X) \cong \mathcal{J}^{\rm mor}_{r,d-r}(X) \cong \cdots \cong \mathcal{J}^{\rm mor}_{r,r}(X) = \mathcal{J}_r(X),$$

for a projective variety X of dimension d. Moreover, Suslin's conjecture also predicts that the image of the injection $L_sH_{d+s-1}(X) \rightarrowtail H^{\rm sing}_{d+s-1}(X)$ is $N_{d-1}H^{\rm sing}_{d+s-1}(X)$, and this leads to the expectation that

$$\mathcal{J}^{\rm mor}_{r,d-r-2}(X) \cong \mathcal{J}(N_{d-1}H^{\rm sing}_{2r+1}(X, \mathbb{Z}(r))).$$

Finally, a consequence of the Beilinson-Lichtenbaum Conjecture is that $H^{\rm sing}_n(X)/N_{d-1}H^{\rm sing}_n(X)$ is torsion free, for all n. In particular, we expect a short exact sequence of MHS's of pure weight -1

$$0 \to N_{d-1}H^{\rm sing}_{2r+1}(X, \mathbb{Z}(r)) \to H^{\rm sing}_{2r+1}(X, \mathbb{Z}(r)) \to Q \to 0$$

so that $\Gamma(Q) = Q_{\rm tor} = 0$, and hence $\mathcal{J}^{\rm mor}_{r,d-r-2}$ should inject into $\mathcal{J}_r(X)$.

Taking these expectations into account, we arrive at the filtration defined as follows:

Definition 10.2. For a smooth, projective variety X of pure dimension d and an integer $r \le d$, define a descending filtration $\tilde{F}^\bullet$ of

$$F^2_{BB}CH_r(X)/F^2_{\rm mor}CH_r(X)$$

by

$$\begin{aligned}
\tilde{F}^0 &= F^2_{BB}CH_r(X) = \ker(\Phi_r : CH_r(X)_{\rm alg\sim 0} \to \mathcal{J}_r(X)) \\
\tilde{F}^1 &= \ker(\Phi^{\rm mor}_{r,d-r-3} : CH_r(X)_{\rm alg\sim 0} \to \mathcal{J}^{\rm mor}_{r,d-r-3}(X)) \\
\tilde{F}^2 &= \ker(\Phi^{\rm mor}_{r,d-r-4} : CH_r(X)_{\rm alg\sim 0} \to \mathcal{J}^{\rm mor}_{r,d-r-4}(X)) \\
&\;\vdots \\
\tilde{F}^{d-r} &= F^2_{\rm mor}CH_r(X) = \ker(\Phi^{\rm mor}_r : CH_r(X)_{\rm alg\sim 0} \to \mathcal{J}^{\rm mor}_r(X)).
\end{aligned}$$

Theorem 6.2 gives examples where

$$F^2_{\rm mor}CH_r(X)/F^{d-r+1}_{\rm mor}CH_r(X) \neq 0$$

where $r \ge 2$ and $d = 2r + 2$.

We can enrich this filtration a bit, provided we assume Suslin's Conjecture 2.4, a special case of which predicts that, for a variety Y of dimension at most r, we have $L_{r-i}H_{2r+1}(Y) = 0$ and $L_{r-i}H_{2r}(Y) \cong H^{\rm BM}_{2r}(Y) = Z_r(Y)$. Using the long exact sequences for Lawson homology, it would thus follow from Suslin's Conjecture that the procedure used to define $\Phi^{\rm mor}_r$ and Φ_r also produces an Abel-Jacobi-type map

$$\ker\left(Z_r(X) \to L_rH_{2r}(X) \xrightarrow{s^i} L_{r-i}H_{2r}(X)\right) \to \mathcal{J}^{\rm mor}_{r,i}(X),$$

for each $0 \leq i \leq r$. Moreover, it can be shown that these maps would enjoy many of the same properties as do Φ_r^{mor} and Φ_r, and, in particular, they would kill cycles rationally equivalent to zero and thus induce maps

$$\tilde{\Phi}_{r,i}^{\mathrm{mor}} : CH_r(X)_{s^i\sim 0} \to \mathcal{J}_{r,i}^{\mathrm{mor}}(X)$$

where we define

$$CH_r(X)_{s^i\sim 0} = \ker\left(CH_r(X) \to L_rH_{2r}(X) \xrightarrow{s^i} L_{r-i}H_{2r}(X) \right).$$

Note that $\tilde{\Phi}_{r,0}^{\mathrm{mor}} = \Phi_r^{\mathrm{mor}}$. Still assuming Suslin's conjecture, we would also have $CH_r(X)_{s^i\sim 0} = CH_r(X)_{\mathrm{hom}\sim 0}$ and $\tilde{\Phi}_{r,i}^{\mathrm{mor}} = \Phi_r$ for $i \geq d-r-1$, and there would be a commutative diagram

$$\begin{array}{ccc}
CH_r(X)_{\mathrm{alg}\sim 0} & \twoheadrightarrow & \mathcal{J}_r^{\mathrm{mor}}(X) \\
\big\downarrow & & \big\downarrow \\
CH_r(X)_{s\sim 0} & \longrightarrow & \mathcal{J}_{r,1}^{\mathrm{mor}}(X) \\
\big\downarrow & & \big\downarrow \\
CH_r(X)_{s^2\sim 0} & \longrightarrow & \mathcal{J}_{r,2}^{\mathrm{mor}}(X) \\
\big\downarrow & & \big\downarrow \\
\vdots & & \vdots \\
\big\downarrow & & \big\downarrow \\
CH_r(X)_{s^{d-r-2}\sim 0} & \longrightarrow & \mathcal{J}_{r,d-r-2}^{\mathrm{mor}}(X) \\
\big\downarrow & & \big\downarrow \\
CH_r(X)_{\mathrm{hom}\sim 0} & \longrightarrow & \mathcal{J}_r(X).
\end{array}$$

The left-hand column in this diagram gives the so-called s-filtration on the Griffiths group, $\mathrm{Griff}_r(X) = CH_r(X)_{\mathrm{hom}\sim 0}/CH_r(X)_{\mathrm{alg}\sim 0}$ (cf. [19, 15]). By taking kernels of the various compositions in the above diagram, we obtain, conjecturally, a refinement of the s-filtration.

11. Towards an absolute morphic cohomology

In this section we provide a description of what might be termed "absolute morphic cohomology". The properties and even the very definition of this theory should be taken as conjectural, although we foresee no glaring obstacle in carrying out the program suggested in this section. The rest of this paper has dealt with Lawson homology — the reason we are switching to the dual theory of morphic cohomology is merely to facilitate comparison with absolute Hodge cohomology (i.e., Deligne cohomology). For smooth varieties, they are equivalent.

To explain what "absolute morphic cohomology" might mean, recall, first of all, that (ordinary) morphic cohomology is defined, for a projective variety X, by

$$L^qH^n(X) = \pi_{2q-n}\left(\mathcal{Z}_0(X, \mathbb{P}^q)/\mathcal{Z}_0(X, \mathbb{P}^{q-1})\right).$$

(See Section 2 for details.) Moreover, the group $\mathcal{Z}_0(X, \mathbb{P}^q)$ is given as the group completion of a monoid object in the category of ind-varieties. Thus, although we have not carried out the details yet, the construction of IMHS's for Lawson homology presented in this paper ought to carry over to endow the morphic cohomology groups with IMHS's, such that Lawson homology and morphic cohomology form a Bloch-Ogus duality theory in the abelian category of IMHS's. In particular, the Poincare duality isomorphism $L^qH^n(X) \cong L_{d-q}H_{2d-n}(X)$ should be an isomorphism of IMHS's. Viewing morphic and Lawson (co)homology as taking values in IMHS's is analogous to the fact that singular (co)homology of complex varieties forms a Bloch-Ogus duality theory in the abelian category of MHS's.

Beilinson [3] had the insight that Deligne cohomology can be defined by first endowing the complex that gives the singular cohomology of a variety U with the structure of a so-called "mixed Hodge complex" (equivalently, an object in the derived category of MHS's) and then applying the hyper-Ext functor $\mathrm{Ext}_{\mathrm{MHS}}(\mathbb{Z}(0), -)$. Let $\mathcal{D}(U)$ denote the complex so formed — it satisfies the key property that

$$h^n(\mathcal{D}(U)) \cong H^n(U, \mathbb{Z}(0))$$

as MHS's, where h^n denotes taking homology in the category MHS. We also define $\mathcal{D}(U,t) = \mathcal{D}(U) \otimes \mathbb{Z}(t)$ to get

$$h^n(\mathcal{D}(U)(t)) \cong H^n_{\mathrm{sing}}(U, \mathbb{Z}(t)).$$

Beilinson defines the "absolute Hodge cohomology" of X by the formula

$$H^n_{\mathcal{D}}(U, \mathbb{Z}(t)) = \mathrm{Ext}^n_{\mathrm{MHS}}(\mathbb{Z}(0), \mathcal{D}(U,t))$$

(These groups give not exactly Deligne cohomology, but rather a truncated version of it.) Since $\mathrm{Ext}^n_{\mathrm{MHS}}$ vanishes for $n \geq 2$, there is a canonical exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H^{n-1}_{\mathrm{sing}}(U, \mathbb{Z}(t))) \to H^n_{\mathcal{D}}(U, \mathbb{Z}(t)) \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(0), H^n_{\mathrm{sing}}(U, \mathbb{Z}(t))) \to 0.$$

Taking $U = X$ smooth and projective and setting $n = 2t$ gives the sequence

$$0 \to \mathcal{J}^t(X) \to H^{2t}_{\mathcal{D}}(X, \mathbb{Z}(t)) \to \Gamma(H^{2t}(X, \mathbb{Z}(t))) \to 0.$$

Note that $\Gamma(H^{2t}(X, \mathbb{Z}(t)) \subset H^{2t}(X)$ is expected, according to the Hodge conjecture, to be the group of cohomology classes associated codimension t cycles on X.

There is a natural transformation of cohomology theories

$$H^n_{\mathcal{M}}(X, \mathbb{Z}(t)) \to H^n_{\mathcal{D}}(X, \mathbb{Z}(t)$$

whose composition with

$$H^n_{\mathcal{D}}(X, \mathbb{Z}(t) \twoheadrightarrow \Gamma(H^n_{\mathrm{sing}}(X, \mathbb{Z}(t)) \subset H^n_{\mathrm{sing}}(X, \mathbb{Z}(t))$$

is the canonical map from motivic to singular cohomology. In particular, we have the Deligne cycle class map

$$CH^t(X) \cong H^{2t}_{\mathcal{M}}(X, \mathbb{Z}(t)) \to H^{2t}_{\mathcal{D}}(X, \mathbb{Z}(t)),$$

and thus an induced map

$$\Phi^t : CH^t(X)_{\mathrm{hom}\sim 0} \to \mathcal{J}^t(X),$$

which coincides with the classical (cohomological) Abel-Jacobi map.

We now describe the properties we expect for the conjectural "absolute morphic cohomology":

Absolute morphic cohomology $H^n_{\mathcal{L}}(-, \mathbb{Z}(t))$ (the $\mathcal{L}$ stands for "Lawson") should form part of a Bloch-Ogus duality theory, with absolute Lawson homology, $H_m^{\mathcal{L}}(-, \mathbb{Z}(r))$, being the dual theory. The absolute morphic cohomology groups should fit into short exact sequences of the form

$$0 \to \mathrm{Ext}^1_{\mathrm{IMHS}}(\mathbb{Z}(0), L^t H^{n-1}) \to H^n_{\mathcal{L}}(X, \mathbb{Z}(t)) \to \Gamma(L^t H^n(X)) \to 0.$$

Moreover, there should be natural maps

$$H^n_{\mathcal{M}}(X, \mathbb{Z}(t)) \to H^n_{\mathcal{L}}(X, \mathbb{Z}(t)) \to H^n_{\mathcal{D}}(X, \mathbb{Z}(t)),$$

and similar maps on homology groups, which form morphisms of Bloch-Ogus duality theories. The compositions of these maps should give the usual maps from motivic to Deligne (co)homology. Moreover, the composition of

$$H^n_{\mathcal{M}}(X, \mathbb{Z}(t)) \to H^n_{\mathcal{L}}(X, \mathbb{Z}(t)) \to \Gamma(L^t H^n(X)) \subset L^t H^n(X)$$

should be the usual map from motivic cohomology to morphic cohomology. In particular, that this suggests the map

$$H^n_{\mathcal{M}}(X, \mathbb{Z}(t)) \to L^t H^n(X)$$

lands in $\Gamma(L^t H^n(X))$ — see Lemma 8.2 and Remark 8.3.

Taking $n = 2t$ gives the short exact sequence

$$0 \to \mathcal{J}^t_{\mathrm{mor}}(X) \to H^{2t}_{\mathcal{L}}(X, \mathbb{Z}(t)) \to L^t H^{2t}(X) \to 0,$$

since $L^t H^{2t}(X)$ is pure of weight 0 and Hodge type $(0,0)$, where we define $\mathcal{J}^t_{\mathrm{mor}}(X) := \mathrm{Ext}^1_{\mathrm{IMHS}}(\mathbb{Z}(0), L^t H^{2t-1}(X))$. From this we obtain a map $CH^t(X)_{\mathrm{alg}\sim 0} \to \mathcal{J}^t_{\mathrm{mor}}(X)$, which, using the identifications $CH^t(X) = CH_{d-t}(X)$ and $\mathcal{J}^t_{\mathrm{mor}}(X) \cong \mathcal{J}^{\mathrm{mor}}_{d-t}(X)$, ought to coincide with the morphic Abel-Jacobi map. Since the morphic Abel-Jacobi map is surjective, we are able to make the identification

$$H^{2t}_{\mathcal{L}}(X, \mathbb{Z}(t)) = CH^t(X)/\ker(\Phi^t_{\mathrm{mor}}).$$

In particular, this suggests that the functor $X \mapsto CH^t(X)/\ker(\Phi^t_{\mathrm{mor}})$ is part of a Bloch-Ogus duality theory, an assertion we have not attempted to verify directly.

Many of the groups $H^n_{\mathcal{L}}(X, \mathbb{Z}(t))$ for $n \neq 2t$ would be, of course, rather mysterious. But, if we accept Suslin's Conjecture for morphic cohomology (Conjecture 2.3), we deduce the conjectural isomorphism

$$H^n_{\mathcal{L}}(X, \mathbb{Z}(t)) \overset{?}{\cong} H^n_{\mathcal{D}}(X, \mathbb{Z}(t)), \quad \text{for } n \leq t.$$

To make the discussion of this section seem like more than a pipe-dream, we now sketch a definition of absolute morphic cohomology for a projective variety X. Actually, since it fits better with the notation and results of the rest of this paper, we instead sketch a definition of the dual theory of absolute Lawson homology, $H^{\mathcal{L}}_n(X, \mathbb{Z}(t))$. The construction of absolute morphic cohomology should not be much more complicated.

Recall that the Lawson homology of X is given by a summand, in the category IMHS, of the homology of $B(\mathcal{C}_r(X))$. This summand is defined via an idempotent endomorphism we called θ. To define absolute Lawson homology, first observe that, since $B(\mathcal{C}_r(X))$ is a filtered colimit of simplicial varieties, the construction of Beilinson gives that the chain complex $\mathbb{Z}\,\mathrm{Sing}_*(B(\mathcal{C}_r(X)))$, which computes the homology of $B(\mathcal{C}_r(X))$, can be endowed with the structure of a filtered limit of mixed Hodge complexes. (Actually, Beilinson works in the derived category, and so some care should be taken at this point.) The idempotent endomorphism θ we constructed actually is defined on the level of chain complexes, and thus θ likely defines an idempotent endomorphism of $\mathbb{Z}\,\mathrm{Sing}_*(B(\mathcal{C}_r(X)))$ in the category of inductive limits of mixed Hodge complexes. Let $\mathcal{L}(X,t)$ denote the summand given by the cokernel of this idempotent endomorphism — it is the Lawson homology analogue of Beilinson's complex $\mathcal{D}(X,t)$. The key property the complex $\mathcal{L}(X,t)$ ought to possess is

$$h_n(\mathcal{L}(X,t)) \cong L_tH_n(X), \quad \text{for all } n,$$

as IMHS's.

Finally, granting the construction of $\mathcal{L}(X,t)$, we would define

$$H_n^{\mathcal{L}}(X,\mathbb{Z}(t)) := \mathrm{Ext}^n_{\mathrm{IMHS}}(\mathbb{Z}(0),\mathcal{L}(X,t)).$$

The desired short exact sequences

$$0 \to \mathrm{Ext}^1_{\mathrm{IMHS}}(\mathbb{Z}(0),L_tH_n(X)) \to H_n^{\mathcal{L}}(X,\mathbb{Z}(t)) \to \Gamma(L_tH_n(X)) \to 0,$$

(the homological versions of those mentioned above) would arise immediately from the isomorphism $h_n(\mathcal{L}(X,t)) \cong L_tH_n(X)$ and the fact that $\mathrm{Ext}^m_{\mathrm{IMHS}}$ vanishes for $m \geq 2$.

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