

$K(1)$ -Local Homotopy Theory, Iwasawa Theory and Algebraic K -Theory

Stephen A. Mitchell*

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Introduction

The *Iwasawa algebra* Λ is a power series ring $Z_\ell[[T]]$, ℓ a fixed prime. It arises in number theory as the pro-group ring of a certain Galois group, and in homotopy theory as a ring of operations in ℓ -adic complex K -theory. Furthermore, these two incarnations of Λ are connected in an interesting way by algebraic K -theory. The main goal of this paper is to explore this connection, concentrating on the ideas and omitting most proofs.

Let F be a number field. Fix a prime ℓ and let F_∞ denote the ℓ -adic cyclotomic tower – that is, the extension field formed by adjoining the ℓ^n -th roots of unity for all $n \geq 1$. The central strategy of Iwasawa theory is to study number-theoretic invariants associated to F by analyzing how these invariants change as one moves up the cyclotomic tower. Number theorists would in fact consider more general towers, but we will be concerned exclusively with the cyclotomic case. This case can be viewed as analogous to the following geometric picture: Let X be a curve over a finite field \mathbb{F} , and form a tower of curves over X by extending scalars to the algebraic closure of \mathbb{F} , or perhaps just to the ℓ -adic cyclotomic tower of \mathbb{F} . This analogy was first considered by Iwasawa, and has been the source of many fruitful conjectures.

As an example of a number-theoretic invariant, consider the norm inverse limit A_∞ of the ℓ -torsion part of the class groups in the tower. Then A_∞ is a profinite module over the pro-group ring $\Lambda'_F = Z_\ell[[G(F_\infty/F)]]$. Furthermore, $\Lambda'_F = \Lambda_F[\Delta_F]$, where $\Lambda_F \cong \Lambda$, and Δ_F is a cyclic group of order dividing $\ell - 1$ (if ℓ is odd) or dividing 2 (if $\ell = 2$). The beautiful fact about the Iwasawa algebra is that finitely-generated modules over it satisfy a classification theorem analogous to the classification theorem for modules over a principal ideal domain. The difference is that isomorphisms must be replaced by pseudo-isomorphisms; these are the homomorphisms with finite kernel and cokernel. Then the game is to see how modules such as A_∞ fit into this classification scheme. For a survey of Iwasawa theory, see [13].

The Iwasawa algebra also arises in homotopy theory. Let \mathcal{K} denote the periodic complex K -theory spectrum, and $\widehat{\mathcal{K}}$ denote its ℓ -adic completion in the sense of Bousfield. Then the ring of degree zero operations $[\widehat{\mathcal{K}}, \widehat{\mathcal{K}}]$ is isomorphic to $\Lambda' = \Lambda[\Delta]$, where Λ is again a power series ring over Z_ℓ , and Δ is cyclic of order $\ell - 1$ (if ℓ is odd) or of order 2 (if $\ell = 2$). The isomorphism comes about by regarding Λ' as the pro-group ring of the group of ℓ -adic Adams operations. Hence the classification theory for Λ -modules can be applied to $\widehat{\mathcal{K}} \cdot X = \widehat{\mathcal{K}}^0 X \oplus \widehat{\mathcal{K}}^1 X$, at least when X is a spectrum with $\widehat{\mathcal{K}} \cdot X$ finitely-generated over Λ .

One can go further by passing to $L_{K(1)}\mathcal{S}$, the Bousfield $K(1)$ -localization of the stable homotopy category \mathcal{S} . This is a localized world in which all spectra with vanishing $\widehat{\mathcal{K}}$ have been erased. The category $L_{K(1)}\mathcal{S}$ is highly algebraic; for example, for ℓ odd its objects are determined up to a manageable ambiguity by $\widehat{\mathcal{K}} \cdot X$ as Λ' -module. This suggests studying $L_{K(1)}\mathcal{S}$ from the perspective of Iwasawa theory. Call the objects X with $\widehat{\mathcal{K}} \cdot X$ finitely-generated as Λ -module $\widehat{\mathcal{K}}$ -finite. The $\widehat{\mathcal{K}}$ -finite

objects are the ones to which Iwasawa theory directly applies; they can be characterized as the objects whose homotopy groups are finitely-generated Z_ℓ -modules, and as the objects that are weakly dualizable in the sense of axiomatic stable homotopy theory. Within this smaller category there is a notion of pseudo-equivalence in $L_{K(1)}\mathcal{S}$ analogous to pseudo-isomorphism for Λ -modules, an analogous classification theorem for objects, and an Iwasawa-theoretic classification of the thick subcategories [15].

Algebraic K-theory provides a link from the number theory to the homotopy theory. Let $R = \mathcal{O}_F[\frac{1}{\ell}]$, and let KR denote the algebraic K-theory spectrum of R . By deep work of Thomason [44], the famous Lichtenbaum–Quillen conjectures can be viewed as asserting that the ℓ -adic completion KR^\wedge is essentially $K(1)$ -local, meaning that for some $d \geq 0$ the natural map $KR^\wedge \rightarrow L_{K(1)}KR$ induces an isomorphism on π_n for $n \geq d$; here $d = 1$ is the expected value for number rings. Since the Lichtenbaum–Quillen conjectures are now known to be true for $\ell = 2$, by Voevodsky’s work on the Milnor conjecture, Rognes–Weibel [41] and Østvær [39], it seems very likely that they are true in general. In any case, it is natural to ask how $L_{K(1)}KR$ fits into the classification scheme alluded to above.

The first step is to compute $\widehat{\mathcal{K}} KR$. Let M_∞ denote the Galois group of the maximal abelian ℓ -extension of F_∞ that is unramified away from ℓ . We call M_∞ the *basic Iwasawa module*.

1 Theorem 1 [9,33] Let ℓ be any prime. Then there are isomorphisms of Λ' -modules

$$\widehat{\mathcal{K}}^n KR \cong \begin{cases} \Lambda' \otimes_{\Lambda'_F} Z_\ell & \text{if } n = 0 \\ \Lambda' \otimes_{\Lambda'_F} M_\infty & \text{if } n = -1 . \end{cases}$$

For ℓ odd this theorem depends on Thomason [44]. For $\ell = 2$ it depends on the work of Rognes–Weibel and Østvær cited above. Theorem 1 leads to a complete description of the homotopy-type $L_{K(1)}KR$, and hence also KR^\wedge in cases where the Lichtenbaum–Quillen conjecture is known.

It is known that M_∞ is a finitely-generated Λ_F -module. Many famous conjectures in number theory can be formulated in terms of its Iwasawa invariants, including the Leopoldt conjecture, the Gross conjecture, and Iwasawa’s μ -invariant conjecture. Consequently, all of these conjectures can be translated into topological terms, as conjectures about the structure of $\widehat{\mathcal{K}} KR$ or the homotopy-type of $L_{K(1)}KR$.

As motivation for making such a translation, we recall a theorem of Soulé. One of many equivalent forms of Soulé’s theorem says that A_∞ contains no negative Tate twists of Z_ℓ . This is a purely number-theoretic assertion. The only known proof, however, depends in an essential way on higher K-theory and hence on homotopy theory. It reduces to the fact that the homotopy groups $\pi_{2n}KR$ are finite for $n > 0$. There are at least two different ways of proving this last assertion – one can work with either the plus construction for $BGL(R)$, or Quillen’s Q -construction – but they both ultimately reduce to finiteness theorems for general linear group homology

due to Borel, Borel–Serre, and Raghunathan. These are in essence analytic results. One can therefore view Soule’s theorem as a prototype of the strategy:

$$\begin{aligned} &(\text{analytic input}) \Rightarrow (\text{estimates on the homotopy-type } KR) \\ &\Rightarrow (\text{number-theoretic output}) \end{aligned}$$

Notice, however, that only the bare homotopy groups of KR have been exploited to prove Soule’s theorem. The homotopy-type of KR contains far more information, since it knows everything about the basic Iwasawa module M_∞ . For example, the homotopy groups alone cannot decide the fate of the algebraic Gross conjecture; as explained in Sect. 3.6.1, this is a borderline case one step beyond Soule’s theorem, for which one additional bit of structure would be needed. For Iwasawa’s μ -invariant conjecture, on the other hand, knowledge of π_*KR alone is of little use. But certain crude estimates on the homotopy-type of KR would suffice, and the conjecture is equivalent to the assertion that $\widehat{\mathcal{K}}^{-1}KR$ is ℓ -torsionfree.

There is a curious phenomenon that arises here. By its definition, the spectrum KR has no homotopy groups in negative degrees. But $K(1)$ -localizations are never connective, and much of the number theory is tied up in the *negative* homotopy groups of $L_{K(1)}KR$. For example, the Leopoldt conjecture is equivalent to the finiteness of $\pi_{-2}L_{K(1)}KR$. Even the Gross conjecture mentioned above involves the part of $\pi_0L_{K(1)}KR$ that doesn’t come from KR , and indirectly involves $\pi_{-1}L_{K(1)}S^0 \cong Z_\ell$. It is tempting to think of KR as a sort of homotopical L -function, with $L_{K(1)}KR$ as its analytic continuation and with functional equation given by some kind of Artin–Verdier–Brown–Comenetz duality. (Although in terms of the generalized Lichtenbaum conjecture on values of ℓ -adic L -functions at integer points – see Sect. 3.6 – the values at negative integers are related to *positive* homotopy groups of $L_{K(1)}KR$, while the values at positive integers are related to the negative homotopy groups!) Speculation aside, Theorem 1.1 shows that all of these conjectures are contained in the structure of $\widehat{\mathcal{K}}^{-1}KR$.

Another goal of this paper is to explain some examples of the actual or conjectural homotopy-type KR in cases where it can be determined more or less explicitly. For example, if one assumes not only the Lichtenbaum–Quillen conjecture, but also the Kummer–Vandiver conjecture, then one can give a fairly explicit description of the ℓ -adic homotopy-type of $K\mathbb{Z}$. Most interesting of all at present is the case $\ell = 2$, since in that case the Lichtenbaum–Quillen conjecture itself is now known. Then KR^\wedge can be described completely in Iwasawa-theoretic terms; and if the Iwasawa theory is known one obtains an explicit description of the homotopy-type KR^\wedge .

Organization of the paper: Sect. 3.2 introduces the theory of modules over the Iwasawa algebra. Sect. 3.3 is an overview of $K(1)$ -local homotopy theory, including the $\widehat{\mathcal{K}}$ -based Adams spectral sequence and the structure of the category $L_{K(1)}\mathcal{S}$. Here we take the viewpoint of axiomatic stable homotopy theory, following [17, 18]. But we also refine this point of view by exploiting the Iwasawa algebra [15]. In Sect. 3.4 we study the Iwasawa theory of the ℓ -adic cyclotomic tower of a number

field, using a novel étale homotopy-theoretic approach due to Bill Dwyer. Sect. 3.5 is a brief discussion of algebraic K-theory spectra and the Lichtenbaum–Quillen conjectures for more general schemes. Although our focus in this paper is on $\mathcal{O}_F[\frac{1}{\ell}]$, it will be clear from the discussion that many of the ideas apply in a much more general setting. In Sect. 3.6 we explain some standard conjectures in number theory, and show how they can be reinterpreted homotopically in terms of the spectrum KR . We also discuss the analytic side of the picture; that is, the connection with L-functions. In particular, we state a generalized Lichtenbaum conjecture on special values of L-functions, and prove one version of it. Sect. 3.7 is a study of the example $K\mathbb{Z}$. Finally, Sect. 3.8 is devoted to KR at the prime 2. Here we start from scratch with $K\mathbb{Z}[\frac{1}{2}]$ and Bökstedt’s $JK\mathbb{Z}$ construction, and then study KR in general.

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3.2 The Iwasawa Algebra

In this section we introduce the Iwasawa algebra Λ and some basic properties of its modules, as well as the related algebra Λ' . General references for this material include [38] and [46]. We also define Tate twists, and discuss some properties of modules over Λ' in the case $\ell = 2$ that are not so well known.

3.2.1 Definition of Λ and Λ'

Let Γ' denote the automorphism group of \mathbb{Z}/ℓ^∞ . Thus Γ' is canonically isomorphic to the ℓ -adic units Z_ℓ^\times , with the isomorphism $c : \Gamma' \rightarrow Z_\ell^\times$ given by $\gamma(x) = c(\gamma)x$ for $\gamma \in \Gamma'$, $x \in \mathbb{Z}/\ell^\infty$. In fact, if A is any abelian group isomorphic to \mathbb{Z}/ℓ^∞ , we again have a canonical isomorphism $\Gamma' \cong \text{Aut } A$, defined in the same way. In particular, Γ' is canonically identified with the automorphism group of the group of ℓ -power roots of unity in any algebraically closed field of characteristic different from ℓ . Note also that there is a unique product decomposition $\Gamma' = \Gamma \times \Delta$, where Γ corresponds under c to the units congruent to 1 mod ℓ (resp. 1 mod 4) if ℓ is odd (resp. $\ell = 2$), and Δ corresponds to the $\ell - 1$ -st roots of unity (resp. ± 1) if ℓ is odd (resp. $\ell = 2$). The restriction of c to Δ is denoted ω and called the *Teichmüller character*.

We write Λ for $Z_\ell[[\Gamma]]$ and Λ' for $Z_\ell[[\Gamma']]$. Let γ_0 be a topological generator of Γ . To be specific, we take $c(\gamma_0) = 1 + \ell$ if ℓ is odd, and $c(\gamma_0) = 5$ if $\ell = 2$. Then it was observed by Serre that there is an isomorphism of profinite rings

$$Z_\ell[[\Gamma]] \cong Z_\ell[[T]],$$

such that $\gamma_0 \mapsto T + 1$. Note that $\Lambda' = \Lambda[\Delta]$.

The ring Λ is a regular noetherian local domain of Krull dimension two. In particular, it has global dimension two, and every module over it admits a projective resolution of length at most two. It is also complete with respect to its maximal ideal $\mathcal{M} = (\ell, T)$, with residue field \mathbb{Z}/ℓ , and therefore is a profinite topological ring.

The height one prime ideals are all principal, and are of two types. First there is the prime ℓ , which plays a special role. Second, there are the prime ideals generated by the irreducible distinguished polynomials $f(T)$. Here a polynomial $f \in \mathbb{Z}_\ell[T]$ is *distinguished* if it is monic and $f(T) = T^n \bmod \ell, n = \deg f$. Note that each height one prime is now equipped with a canonical generator; we will occasionally not bother to distinguish between the ideal and its generator.

Modules over the Iwasawa Algebra

3.2.2

An *elementary cyclic module* is a Λ -module that is either free of rank one or of the form Λ/q^i , where q is either ℓ or an irreducible distinguished polynomial. A finitely-generated Λ -module E is *elementary* if it is a direct sum of elementary cyclic modules. The primes q and exponents i that appear are uniquely determined by E , up to ordering.

A *pseudo-isomorphism* is a homomorphism of Λ -modules with finite kernel and cokernel. The main classification theorem then reads:

Theorem 2 Let M be a finitely-generated Λ -module. Then there is an elementary module E and a pseudo-isomorphism $\phi : M \rightarrow E$. Up to isomorphism, E is uniquely determined by M .

2

Remark 3 If M is a Λ -torsion module, then one can also find a pseudo-isomorphism $\psi : E \rightarrow M$. In general, however, this isn't true; if M is Λ -torsionfree but not free, there is no pseudo-isomorphism from a free module to M .

3

Recall that the *support* of a module M , denoted $\text{Supp } M$, is the set of primes q such that $M_q \neq 0$. Note that $\text{Supp } M$ is *closed under specialization*, i.e., if $q \in \text{Supp } M$ and $q \subset q'$, then $q' \in \text{Supp } M$. For example, the nonzero finite modules are the modules with support $\{\mathcal{M}\}$, while the Λ -torsion modules are the modules with $(0) \notin \text{Supp } M$. Now let N be a finitely-generated torsion module and let q_1, \dots, q_m be the height one primes in $\text{Supp } M$. These are, of course, precisely the q 's that appear in the associated elementary module E . Thus we can write

$$E = E_1 \oplus \dots \oplus E_m \qquad E_i = \bigoplus_{j=1}^{r_i} \Lambda/q_i^{s_{ij}}.$$

The data $(q_1; s_{11}, \dots, s_{1r_1}), \dots, (q_m; s_{m1}, \dots, s_{mr_m})$ constitute the *torsion invariants* of N . The *multiplicity* of q_i in N is $n_i = \sum_j s_{ij}$, the *frequency* is r_i , and the *lengths* are the exponents s_{ij} . The *divisor* of N is the formal sum $\mathcal{D}(N) = \sum n_i q_i$. Sometimes we

write $\langle q, N \rangle$ for the multiplicity of q in N . In the case of the prime ℓ , the multiplicity is also denoted $\mu(N)$. If M is an arbitrary finitely-generated Λ -module, the above terms apply to its Λ -torsion submodule tM .

Thus one can study finitely-generated Λ -modules M up to several increasingly coarse equivalence relations:

- Up to isomorphism. This can be difficult, but see for example [20].
- Up to pseudo-isomorphism. At this level M is determined by its Λ -rank and torsion invariants.
- Up to divisors. If we don't know the torsion invariants of M , it may still be possible to determine the divisor $\sum n_i q_i$ of tM . This information can be conveniently packaged in a *characteristic series* for tM ; that is, an element $g \in \Lambda$ such that $g = u \prod q_i^{n_i}$ for some unit $u \in \Lambda$.
- Up to support. Here we ask only for the Λ -rank of M and the support of tM .

We mention a few more interesting properties of Λ -modules; again, see [38] for details and further information.

A finitely-generated Λ -module M has a unique maximal finite submodule, denoted M^0 .

4 Proposition 4 M has projective dimension at most one if and only if $M^0 = 0$.

Now let M^* denote the Λ -dual $\text{Hom}_\Lambda(M, \Lambda)$.

5 Proposition 5 If M is a finitely-generated Λ -module, then M^* is a finitely-generated free module. Furthermore, the natural map $M \rightarrow M^{**}$ has kernel tM and finite cokernel.

We call the cokernel of the map $M \rightarrow M^{**}$ the *freeness defect* of M ; it is zero if and only if M/tM is free. In fact:

6 Proposition 6 Suppose N is finitely-generated and Λ -torsionfree, and let $N \rightarrow F$ be any pseudo-isomorphism to a free module (necessarily injective). Then F/N is Pontrjagin dual to $\text{Ext}_\Lambda^1(N, \Lambda)$, and is isomorphic to the freeness defect of N .

In particular, then, F/N is independent of F and the choice of pseudo-isomorphism.

Call a finitely-generated Λ -module L *semi-discrete* if Γ acts discretely on L . Thus L is finitely-generated as Z_ℓ -module with Γ_n , the unique subgroup of index ℓ^n of Γ , acting trivially on L for some n . The semi-discrete modules fit into the pseudo-isomorphism theory as follows: Let $\omega_n = (1 + T)^{\ell^n} - 1$. Then $\omega_n = v_0 v_1 \dots v_n$ for certain irreducible distinguished polynomials v_i ; in fact, the v_i 's are just the cyclotomic polynomials. We call these the *semi-discrete primes* because they are pulled back from $Z_\ell[\mathbb{Z}/\ell^n]$ under the map induced by the natural epimorphism

from Γ to the discrete group \mathbb{Z}/ℓ^n . Clearly L is semi-discrete if and only if it has support in the semi-discrete primes and each v_i occurs with length at most one. Finally, note that for any finitely-generated Λ -module M , the ascending chain

$$M^\Gamma \subset M^{\Gamma_1} \subset M^{\Gamma_2} \subset \dots$$

terminates. Hence M has a maximal semi-discrete submodule M^δ , with $M^\delta = M^{\Gamma_n}$ for $n \gg 0$.

Tate Twisting

3.2.3

Let M be a Λ -module. The n -th Tate twist $M(n)$ has the same underlying Z_ℓ -module, but with the Γ -action twisted by the rule

$$\gamma \cdot x = c(\gamma)^n \gamma x.$$

If Z_ℓ has trivial Γ -action, then clearly

$$M(n) = M \otimes_{Z_\ell} (Z_\ell(n))$$

as Λ -modules. Thinking of Λ as the power series ring $Z_\ell[[T]]$, we can interpret this twisting in another way. Given any automorphism ϕ of Λ – where we mean automorphism as topological ring – any module M can be twisted to yield a new module M_ϕ in which $\lambda \cdot x = \phi(\lambda)x$. In particular, any linear substitution $T \mapsto cT + d$ with $c, d \in Z_\ell$, c a unit and $d = 0 \pmod{\ell}$ defines such an automorphism. Among these we single out the case $c = c_0$, $d = c_0 - 1$. This automorphism will be denoted τ and called the *Tate automorphism*, the evident point being that $M(1) = M_\tau$.

Note that τ permutes the height one primes. In particular, we write $\tau_n = \tau^n(T)$. Written as an irreducible distinguished polynomial, $\tau_n = (T - (c_0^n - 1))$. These are the *Tate primes*, which will play an important role in the sequel.

Modules over Λ'

3.2.4

Suppose first that ℓ is odd. Then the entire theory of finitely-generated Λ -modules extends in a straightforward way to Λ' -modules. The point is that $\Lambda' = \Lambda[\Delta]$, and Δ is finite of order prime to ℓ . Hence Λ' splits as a direct product of topological rings into $\ell - 1$ copies of Λ , and similarly for its category of modules. Explicitly, there are idempotents $e_i \in Z_\ell[\Delta] \subset \Lambda'$, $0 \leq i \leq \ell - 2$, such that for any module M we have $M = \bigoplus e_i M$ as Λ' -modules, with Δ acting on $e_i M$ as ω^i . Hence we can apply the structure theory to the summands $e_i M$ independently.

We also have $\text{Spec } \Lambda' = \coprod_{i=0}^{\ell-2} \text{Spec } \Lambda$. If $q \in \text{Spec } \Lambda$ and $n \in \mathbb{Z}$, we write (q, n) for the prime q in the n -th summand, n of course being interpreted modulo $\ell - 1$. Tate-twisting is defined as before, interpreting Λ' as $Z_\ell[[\Gamma']]$. In terms of the product decomposition above, this means that τ' permutes the factors by $(q, n) \mapsto (\tau(q), n+1)$. The *extended Tate primes* are defined by $\tau'_n = (\tau_n, n)$. Sometimes we will want to twist the Δ -action while leaving the Γ -action alone; we call this Δ -twisting.

Semi-discrete primes are defined just as for Λ . Note that the semi-discrete primes are invariant under Δ -twisting.

If $\ell = 2$, the situation is considerably more complicated, and indeed does not seem to be documented in the literature. The trouble is that now Δ has order 2. Thus we still have that Λ' is a noetherian, profinite, local ring, but it does not split as a product of Λ 's, and modules over it can have infinite projective dimension. Letting σ denote the generator of Δ , we will sometimes use the notation e_n as above for the elements $1 + \sigma$ (n even) or $1 - \sigma$ (n odd), even though these elements are not idempotent. The maximal ideal is $(2, T, 1 - \sigma)$, and there are two minimal primes $(1 \pm \sigma)$. The prime ideal spectrum can be written as a pushout

$$\begin{array}{ccc} \text{Spec } \Lambda/2 & \rightarrow & \text{Spec } \Lambda \\ \downarrow & & \downarrow \\ \text{Spec } \Lambda & \rightarrow & \text{Spec } \Lambda' \end{array} .$$

The extended Tate twist τ' is still simple enough; it merely acts by τ combined with an exchange of the two $\text{Spec } \Lambda$ factors.

The pseudo-isomorphism theory is more complicated. For modules with vanishing μ -invariant, we have the following:

7 Theorem 7 Let M be a finitely-generated Λ' -module with $\mu(M) = 0$. Then M is pseudo-isomorphic to a module of the form

$$E = E^+ \oplus E^- \oplus E^f,$$

where E^+ (resp. E^-) is an elementary Λ -module supported on the distinguished polynomials, with σ acting trivially (resp. acting as -1), and E^f is Λ -free.

This is far from a complete classification, however, since we need to analyze E^f . Let Λ^l denote Λ with σ acting as -1 , and let Λ unadorned denote Λ with the trivial σ action. Next, let L_n denote $\Lambda \oplus \Lambda$ with σ acting by the matrix

$$\begin{pmatrix} 1 & T^n \\ 0 & -1 \end{pmatrix}$$

and let L_n^* denote its Λ -dual. Note that L_0 is free over Λ' of rank one, and hence $L_0 \cong L_0^*$, but with that exception no two of these modules are isomorphic.

We pause to remark that these modules L_n, L_n^* arise in nature, in topology as well as in number theory. In topology the module L_1^* occurs as the 2-adic topological K-theory of the cofibre of the unit map $S^0 \rightarrow \widehat{\mathcal{K}}$; that is, the first stage of an Adams resolution for the sphere (see Sect. 3.3.4 below). The modules L_1, L_2 occur in the Iwasawa theory of 2-adic local fields; for example, the torsion-free quotient of the basic Iwasawa module M_∞ associated to \mathbb{Q}_2 is isomorphic to L_2 (see [36]).

Theorem 8 Let N be a finitely-generated Λ -free Λ' -module. Then N is isomorphic as Λ' -module to a direct sum of modules of the form $\Lambda, \Lambda^f, L_n (n \geq 0)$, and L_n^* ($n \geq 1$), with the number of summands of each type uniquely determined by N . Furthermore

- (a) Such modules N of fixed Λ -rank are classified by their Tate homology groups $H_*(\sigma; N)$, regarded as modules over the principal ideal domain $\Lambda/2$;
- (b) Any pseudo-isomorphism between two such modules is an isomorphism.

This gives a complete classification up to pseudo-isomorphism in the case $\mu = 0$. The proof of Theorem 8 given in [36] is *ad hoc*; it would be nice to have a more conceptual proof.

K(1)-local Homotopy Theory

K -theoretic localization has been studied extensively by Bousfield, who indeed invented the subject. Many of the results in this section either come directly from Bousfield's work or are inspired by it. The most important references for us are [3, 4] and [5]; we will not attempt to give citations for every result below.

We will also take up the viewpoint of axiomatic stable homotopy theory, following Hovey, Palmieri and Strickland [18]. Further motivation comes from the elegant Hovey–Strickland memoir [17], in which $K(n)$ -localization for arbitrary n is studied. In the latter memoir the case $n = 1$ would be regarded as the trivial case; even in the trivial case, however, many interesting and nontrivial things can be said. In particular we will use Iwasawa theory to study the finer structure of $L_{K(1)}\mathcal{S}$. Although the connection with Iwasawa theory has been known since the beginning [4] [40], it seems not to have been exploited until now.

We begin in Sect. 3.3.1 with a discussion of ℓ -adic completion. All of our algebraic functors will take values in the category of Ext - ℓ -complete abelian groups, so we have included a brief introduction to this category. The short Sect. 3.3.2 introduces $\widehat{\mathcal{K}}$ and connects its ring of operations with the Iwasawa algebra. Some important properties of $K(1)$ -localization are summarized in Sect. 3.3.3.

Section 3.3.4 discusses the $\widehat{\mathcal{K}}$ -based Adams spectral sequence. In fact the spectral sequence we use is the so-called modified Adams spectral sequence, in a version for which the homological algebra takes place in the category of compact Λ' -modules.

Section 3.3.5 studies the structure of the category $L_{K(1)}\mathcal{S}$, beginning with some remarks concerning the small, dualizable, and weakly dualizable objects of $L_{K(1)}\mathcal{S}$. This last subcategory is the category in which we will usually find ourselves; it coincides with the thick subcategory generated by $\widehat{\mathcal{K}}$ itself, and with the subcategory of objects X such that $\widehat{\mathcal{K}}X$ is a finitely-generated Λ' -module. Most of the results here are special cases of results from [17], and ultimately depend on the work of Bousfield cited above. We also briefly discuss the Picard group of

$L_{K(1)}\mathcal{S}$ [16]. In Sect. 3.3.6, we summarize some new results from [15], with ℓ odd, including a spectrum analogue of the classification of Iwasawa modules, and an Iwasawa-theoretic classification of thick subcategories of the subcategory of weakly dualizable objects.

We remark that according to an unpublished paper of Franke [12], for ℓ odd $L_{K(1)}\mathcal{S}$ is equivalent to a certain category of chain complexes. We will not make any use of this, however. Nor will we make any use of Bousfield’s united K -theory [5], but the reader should be aware of this homologically efficient approach to 2-primary K -theory.

3.3.1 ℓ -Adic Completion of Spectra

Fix a spectrum E . A spectrum Y is said to be E -acyclic if $E \wedge Y \cong *$, and E -local if $[W, Y] = 0$ for all E -acyclic W . There is a Bousfield localization functor $L_E : \mathcal{S} \rightarrow \mathcal{S}$ and a natural transformation $Id \rightarrow L_E$ such that $L_E X$ is E -local and the fibre of $X \rightarrow L_E X$ is E -acyclic [3].

The ℓ -adic completion of a spectrum is its Bousfield localization with respect to the mod ℓ Moore spectrum $M\mathbb{Z}/\ell$. We usually write X^\wedge for $L_{M\mathbb{Z}/\ell} X$. Note that the $M\mathbb{Z}/\ell$ -acyclic spectra are the spectra with uniquely ℓ -divisible homotopy groups. The ℓ -adic completion can be constructed explicitly as the homotopy inverse limit $\text{holim}_n X \wedge M\mathbb{Z}/\ell^n$, or equivalently as the function spectrum $\mathcal{F}(\mathcal{N}, X)$, where $\mathcal{N} = \Sigma^{-1}M\mathbb{Z}/\ell^\infty$. Resolving \mathcal{N} by free Moore spectra, it is easy to see that there is a universal coefficient sequence

$$0 \rightarrow \text{Ext}(\mathbb{Z}/\ell^\infty, [W, X]) \rightarrow [W, X^\wedge] \rightarrow \text{Hom}(\mathbb{Z}/\ell^\infty, [\Sigma^{-1}W, X]) \rightarrow 0.$$

To see what this has to do with ℓ -adic completion in the algebraic sense, we need a short digression.

An abelian group A is *Ext- ℓ complete* if

$$\text{Hom}(\mathbb{Z}[1/\ell], A) = 0 = \text{Ext}(\mathbb{Z}[1/\ell], A).$$

Thus the full subcategory \mathcal{E} of all Ext- ℓ complete abelian groups is an abelian subcategory closed under extensions. In fact it is the smallest abelian subcategory containing the ℓ -complete abelian groups. The category \mathcal{E} enjoys various pleasant properties, among which we mention the following:

- An Ext- ℓ complete abelian group has no divisible subgroups.
- The objects of \mathcal{E} have a natural \mathbb{Z}_ℓ -module structure.
- \mathcal{E} is closed under arbitrary inverse limits.
- If A, B are in \mathcal{E} , then so are $\text{Hom}(A, B)$ and $\text{Ext}(A, B)$.
- For any abelian group A , $\text{Hom}(\mathbb{Z}/\ell^\infty, A)$ and $\text{Ext}(\mathbb{Z}/\ell^\infty, A)$ are Ext- ℓ complete.
- The functor $eA = \text{Ext}(\mathbb{Z}/\ell^\infty, A)$ is idempotent with image \mathcal{E} . The kernel of the evident natural transformation $A \rightarrow eA$ is $\text{Div } A$, the maximal divisible subgroup of A .
- If A is finitely-generated, eA is just the usual ℓ -adic completion A^\wedge .

Here it is important to note that although an Ext- ℓ complete abelian group has no divisible subgroups, it may very well have divisible *elements*. A very interesting example of this phenomenon arises in algebraic K-theory, in connection with the so-called “wild kernel”; see [1].

We remark that although \mathcal{E} is not closed under infinite direct sums, it nevertheless has an intrinsic coproduct for arbitrary collections of objects: $\coprod A_\alpha = e(\oplus A_\alpha)$. Hence \mathcal{E} has arbitrary colimits, and these are constructed applying e to the ordinary colimit.

Finally, we note that any profinite abelian ℓ -group is Ext- ℓ complete, and furthermore if A, B are profinite then $Hom_{cont}(A, B)$ is Ext- ℓ complete. To see this, let $A = \lim A_\alpha, B = \lim B_\beta$ with A_α, B_β finite, and recall that the continuous homomorphisms are given by

$$\lim_{\beta} \text{colim}_{\alpha} Hom(A_{\alpha}, B_{\beta}) .$$

Any group with finite ℓ -exponent is Ext- ℓ complete, whence the claim. Thus the category of profinite abelian ℓ -groups and continuous homomorphisms embeds as a non-full subcategory of \mathcal{E} , compatible with the intrinsic Hom and Ext . This ends our digression.

Returning to the topology, the universal coefficient sequence shows that if X, Y are ℓ -complete, $[X, Y]$ is Ext- ℓ complete. Now let \mathcal{S}_{tor} denote the full subcategory of ℓ -torsion spectra; that is, the spectra whose homotopy groups are ℓ -torsion groups. Then the functors

$$F(\mathcal{N}, -) : \mathcal{S}_{tor} \rightarrow \mathcal{S}^{\wedge} \qquad (-) \wedge \mathcal{N} : \mathcal{S}^{\wedge} \rightarrow \mathcal{S}_{tor}$$

are easily seen to be mutually inverse equivalences of categories. In fact, these functors are equivalences of abstract stable homotopy theories in the sense of [18]. To make sense of this, it is crucial to distinguish between various constructions performed in the ambient category \mathcal{S} and the *intrinsic* analogues obtained by reflecting back into $\mathcal{S}_{tor}, \mathcal{S}^{\wedge}$. For example, \mathcal{S}^{\wedge} is already closed under products and function spectra, so the ambient and intrinsic versions coincide, whereas intrinsic coproducts and smash products must be defined by completing the ambient versions. In \mathcal{S}_{tor} it is the reverse: For smash product and coproducts the ambient and intrinsic versions coincide, while intrinsic products and function spectra must be defined by $(\prod X_{\alpha}) \wedge \mathcal{N}$ and $\mathcal{F}(X, Y) \wedge \mathcal{N}$, respectively. Note also that \mathcal{N} is the unit in \mathcal{S}_{tor} .

One also has to be careful about “small” objects. In the terminology of [18], an object W of \mathcal{S}^{\wedge} is *small* if the natural map

$$\oplus [W, X_{\alpha}] \rightarrow [W, \coprod X_{\alpha}]$$

is an isomorphism. The symbol \coprod on the right refers to the intrinsic coproduct – that is, the completed wedge. With this definition, the completed sphere $(S^0)^{\wedge}$ is not small. The problem is that an infinite direct sum of Z_{ℓ} ’s is not Ext- ℓ complete. As we have just seen, however, the category \mathcal{E} has its own intrinsic coproduct, and

it is easy to see that the functor $[(S^0)^\wedge, -]$ does commute with intrinsic coproducts in this sense. Thus there are two variants of “smallness” in the ℓ -complete world. To avoid confusion, however, we will keep the definition given above, and call objects that commute with intrinsic coproducts *quasi-small*. It follows immediately that an object is quasi-small if and only if it is \mathcal{F} -small in the sense of [18]. We also find that the small objects in \mathcal{S}^\wedge are the finite ℓ -torsion spectra.

3.3.2 ℓ -Adic Topological K -Theory

Let $\widehat{\mathcal{K}}$ denote the ℓ -completion of the periodic complex K -theory spectrum. We will use the notation $\widehat{\mathcal{K}}X = \widehat{\mathcal{K}}^0X \oplus \widehat{\mathcal{K}}^1X$. Then the ring of operations $\widehat{\mathcal{K}}^*\widehat{\mathcal{K}}$ is completely determined by $\widehat{\mathcal{K}}^*\widehat{\mathcal{K}}$, and has an elegant description in terms of the Iwasawa algebra ([27]; see also [30]).

9 Proposition 9 $\widehat{\mathcal{K}}^0\widehat{\mathcal{K}}$ is isomorphic to Λ' . The isomorphism is uniquely determined by the correspondence $\psi^k \leftrightarrow \gamma$ with $c(\gamma) = k$ ($k \in \mathbb{Z}$, ℓ prime to k). Furthermore, $\widehat{\mathcal{K}}^1\widehat{\mathcal{K}} = 0$.

Using the idempotents $e_i \in Z_\ell[\Delta]$, we obtain a splitting

$$\widehat{\mathcal{K}} \cong \bigvee_{i=0}^{\ell-2} e_i \widehat{\mathcal{K}} .$$

The zero-th summand is itself a $(2\ell - 2)$ -periodic ring spectrum, called the *Adams summand* and customarily denoted $E(1)^\wedge$; furthermore, $e_i \widehat{\mathcal{K}} \cong \Sigma^{2i} E(1)^\wedge$. Smashing with the Moore spectrum yields a similar decomposition, whose Adams summand $e_0 \mathcal{K} \wedge M\mathbb{Z}/\ell$ is usually denoted $K(1)$ – the first Morava K -theory.

Note also that $\widehat{\mathcal{K}}^0 S^{2n} = Z_\ell(n)$. More generally, $\widehat{\mathcal{K}}^0 S^{2n} \wedge X = \widehat{\mathcal{K}}^0 X(n)$. Thus Tate twisting corresponds precisely to double suspension.

One striking consequence of Proposition 9 is that the theory of Iwasawa modules can now be applied to K -theory. This will be one of the major themes of our paper.

3.3.3 $K(1)$ -Localization

We will be working almost exclusively in the $K(1)$ -local world. We cannot give a thorough introduction to this world here, but we will at least mention a few salient facts.

1. $L_{K(1)}X = (L_{\mathcal{K}}X)^\wedge$. The functor $L_{\mathcal{K}}$ is *smashing* in the sense that $L_{\mathcal{K}}X = X \wedge L_{\mathcal{K}}S^0$, but completion is not smashing and neither is $L_{K(1)}$.
2. A spectrum X is $K(1)$ -acyclic if and only if $\widehat{\mathcal{K}}^*X = 0$.
3. $K(1)$ -local spectra X manifest a kind of crypto-periodicity (an evocative term due to Bob Thomason). Although X itself is rarely periodic, each reduction $X \wedge M\mathbb{Z}/\ell^n$ is periodic, with the period increasing as n gets larger. A similar periodicity is familiar in number theory, for example in the Kummer congruences.

4. The crypto-periodicity has two striking consequences: (i) If X is both $K(1)$ -local and connective, then X is trivial; and (ii) The functor $L_{K(1)}$ is invariant under connective covers, in the sense that if $Y \rightarrow X$ has fibre bounded above, then $L_{K(1)}Y \rightarrow L_{K(1)}X$ is an equivalence. Hence the $K(1)$ -localization of a spectrum depends only on its “germ at infinity” (another evocative term, due to Bill Dwyer this time).
5. The $K(1)$ -local sphere fits into a fibre sequence of the form

$$L_{K(1)}S^0 \rightarrow \widehat{\mathcal{K}} \xrightarrow{\psi^{q-1}} \widehat{\mathcal{K}}$$

for ℓ odd, or

$$L_{K(1)}S^0 \rightarrow KO^\wedge \xrightarrow{\psi^{q-1}} KO^\wedge$$

for $\ell = 2$. Here q is chosen as follows: If ℓ is odd, q can be taken to be any topological generator of \mathbb{Z}_ℓ^\times . In terms of the Iwasawa algebra, we could replace $\psi^q - 1$ by $e_0 T$. If $\ell = 2$, any $q = \pm 3 \pmod 8$ will do; the point is that $q, -1$ should generate \mathbb{Z}_2^\times . In the literature $q = 3$ is the most popular choice, but $q = 5$ fits better with our conventions on Γ . One can identify $\widehat{KO}^0 \widehat{KO}$ canonically with Λ , and then $\psi^q - 1$ corresponds to T . In any event, the homotopy groups of $L_{K(1)}S^0$ can be read off directly from these fibre sequences.

6. There is an equivalence of stable homotopy categories

$$L_{\mathcal{K}} \mathcal{S}_{tor} \cong L_{K(1)} \mathcal{S}$$

given by the same functors discussed earlier in the context of plain ℓ -adic completion. The expression on the left is unambiguous; $L_{\mathcal{K}}(\mathcal{S}_{tor})$ and $(L_{\mathcal{K}} \mathcal{S})_{tor}$ are the same.

The $\widehat{\mathcal{K}}$ -Based Adams Spectral Sequence

The functors $\widehat{\mathcal{K}}^n$ take values in the category of compact Λ' -modules and continuous homomorphisms. This puts us in the general setting of “compact modules over complete group rings”, a beautiful exposition of which can be found in [38], Chapter V, Sect. 3.2. In particular, there is a contravariant equivalence of categories

$$(\text{discrete torsion } \Lambda'\text{-modules}) \cong (\text{compact } \Lambda'\text{-modules})$$

given by (continuous) Pontrjagin duality. This fits perfectly with the equivalence

$$L_{\mathcal{K}} \mathcal{S}_{tor} \cong L_{K(1)} \mathcal{S}$$

mentioned earlier, since there are universal coefficient isomorphisms

$$\widehat{\mathcal{K}}^n X \cong (\mathcal{K}_{n-1} X \wedge \mathcal{N})^\# ,$$

where $\#$ denotes Pontrjagin duality.

Furthermore, homological algebra in either of the equivalent categories above is straightforward and pleasant; again, see [38] for details. The main point to bear in mind is that *Hom* and *Ext* will always refer to the continuous versions; that is, to *Hom* and *Ext* in the category of compact Λ' -modules (or occasionally in the category of discrete torsion Λ' -modules). We note also that if M is finitely-generated over Λ and N is arbitrary, the continuous $Ext_{\Lambda'}^p(M, N)$ is the same as the ordinary *Ext* ([38], 5.2.22).

The Adams spectral sequence we will use is the so-called “modified” Adams spectral sequence, as discussed for example in [6] and [17], except that we prefer to work with compact Λ' -modules rather than with discrete torsion modules or with comodules. The “modified” spectral sequence works so beautifully here that we have no need for its unmodified antecedent, and consequently we will drop “modified” from the terminology.

Call $W \in L_{K(1)}\mathcal{S}$ *projective* if (i) $\widehat{\mathcal{K}} \cdot W$ is projective as compact Λ' -module; and (ii) for any $X \in L_{K(1)}\mathcal{S}$, the natural map

$$[X, W] \rightarrow Hom_{\Lambda'}(\widehat{\mathcal{K}} \cdot W, \widehat{\mathcal{K}} \cdot X)$$

is an isomorphism. There is the obvious analogous notion of injective object in $L_{\mathcal{K}}\mathcal{S}_{tors}$, and clearly W is projective if and only if $W \wedge \mathcal{N}$ is injective. It is not hard to show:

10

Proposition 10

- (a) If $M = M^0 \oplus M^1$ is any $\mathbb{Z}/2$ -graded projective compact Λ' -module, there is a projective spectrum W with $\widehat{\mathcal{K}} \cdot W \cong M$.
- (b) $L_{K(1)}\mathcal{S}$ has enough projectives, in the sense that for every X there exists a projective W and a map $X \rightarrow W$ inducing a surjection on $\widehat{\mathcal{K}}$.

Hence for any X we can iterate the construction of (b) to obtain an *Adams resolution*

$$\begin{array}{ccccccc} X & \leftarrow & X_1 & \leftarrow & X_2 & \leftarrow & \dots \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\ W_0 & & W_1 & & W_2 & & \dots \end{array}$$

where the triangles are cofibre sequences and the horizontal arrows shift dimensions and induce zero on $\widehat{\mathcal{K}}^*$. If Y is another object, applying the functor $[Y, -]$ yields an exact couple and a spectral sequence with

$$E_2^{s,t} = Ext_{\Lambda'}^s(\widehat{\mathcal{K}} \cdot X, \widehat{\mathcal{K}} \cdot \Sigma^t Y)$$

and abutment $[\Sigma^{t-s} Y, X]$. This is the (modified) Adams spectral sequence. It could be displayed as a right half-plane cohomology spectral sequence, but the custom in homotopy theory is to put $t - s$ on the horizontal axis and s on the vertical axis. This yields a display occupying the upper half-plane, with the differential d^r going up r and to the left 1, and with the i -th column of E_∞ corresponding to the associated graded module of $[\Sigma^i Y, X]$. Here the filtration on $[Y, X]$ is the obvious one obtained from the tower: Filtration n consists of the maps that lift to X_n .

It is easy to check that from E_2 on, the spectral sequence does not depend on the choice of Adams resolution. The filtration is also independent of this choice, and in fact has an alternate, elegant description: Let $\mathcal{A}^n[Y, X]$ denote the subgroup of maps that factor as a composite of n maps each of which induces the zero homomorphism on $\widehat{\mathcal{K}}$. This is the *Adams filtration*, and it coincides with the filtration obtained from any Adams resolution.

It remains to discuss convergence. In fact the Adams spectral sequence converges *uniformly* to the associated graded object of the Adams filtration. By “uniformly” we mean that there is a fixed d such that $\mathcal{A}^d[Y, X] = 0$, and the spectral sequence collapses at E_d . The simplest way for this to happen is to have every d -fold composite of maps in the tower $X_{n+d} \rightarrow \dots \rightarrow X_n$ be null; then we say the tower is *uniformly d -convergent*. This would be useful even if d depended on X, Y , but in the present situation we will have d depending only on whether $\ell = 2$ or ℓ odd.

If ℓ is odd, every compact Λ' -module has projective dimension ≤ 2 , and hence every $X \in L_{K(1)}\mathcal{S}$ admits an Adams resolution of length ≤ 2 . So we can trivially take $d = 3$, and furthermore the spectral sequence is confined to the lines $s = 0, 1, 2$, with d_2 the only possible differential. In general, this d_2 is definitely nonzero.

If $\ell = 2$, then Λ' has infinite global dimension, and most objects will not admit a finite Adams resolution. Nevertheless, it is easy to see that the Adams tower is uniformly convergent with $d \leq 6$. For if $\widehat{\mathcal{K}}X$ is Λ -projective, then one can use $W_0 = X \wedge C_\eta$ as the first term of an Adams resolution, where C_η is the mapping cone of $\eta \in \pi_1 S^0$. Then the cofibre X_1 is again Λ -projective, so one can iterate the process to obtain an Adams resolution in which the maps $X_n \rightarrow X_{n-1}$ are all multiplications by η . Since $\eta^4 = 0$, we can take $d = 4$ in this case. If $\widehat{\mathcal{K}}X$ is only \mathbb{Z}_2 -projective, then any choice of X_1 is Λ -projective, and we can take $d = 5$. Finally, if X is arbitrary then any choice of X_1 is \mathbb{Z}_2 -projective, and one can take $d = 6$. (Compare [17], Proposition 6.5.)

As an illustration of the Adams spectral sequence, we compute the Z_ℓ -ranks of $\pi_* X$ in terms of $\widehat{\mathcal{K}}X$. The following proposition is valid at any prime ℓ .

Proposition 11 Suppose $\widehat{\mathcal{K}}X$ is finitely-generated over Λ' . Then

11

$$\begin{aligned} \text{rank}_{Z_\ell} \pi_{2m} X &= \text{rank}_\Lambda e_m \widehat{\mathcal{K}}^0 X + \langle \tau'_m, \widehat{\mathcal{K}}^0 X \rangle + \langle \tau'_m, \widehat{\mathcal{K}}^1 X \rangle \\ \text{rank}_{Z_\ell} \pi_{2m+1} X &= \text{rank}_\Lambda e_m \widehat{\mathcal{K}}^1 X + \langle \tau'_m, \widehat{\mathcal{K}}^1 X \rangle + \langle \tau'_{m+1}, \widehat{\mathcal{K}}^0 X \rangle. \end{aligned}$$

The proof is straightforward, part of the point being that all the groups above the 1-line in the Adams E_2 -term are finite, and even when $\ell = 2$ only finitely many of these survive in each topological degree.

When the terms involving, say, $\widehat{\mathcal{K}}^0$ vanish, we can turn these formulae around to get

$$\begin{aligned} \langle \tau'_m, \widehat{\mathcal{K}}^1 X \rangle &= \text{rank}_{Z_\ell} \pi_{2m} X \\ \text{rank}_\Lambda e_m \widehat{\mathcal{K}}^1 X &= \text{rank}_{Z_\ell} \pi_{2m+1} X - \text{rank}_{Z_\ell} \pi_{2m} X. \end{aligned}$$

3.3.5 **The Structure of $L_{K(1)}\mathcal{S}$: General Results**

To begin, we have:

12 Theorem 12 $L_{K(1)}\mathcal{S}$ has no proper nontrivial localizing subcategories.

This means that a natural transformation of cohomology theories on $L_{K(1)}\mathcal{S}$ is an isomorphism provided that it is an isomorphism on a single nontrivial object. Next, recall that a full subcategory \mathcal{C} of a stable homotopy category is *thick* if it is closed under retracts and under cofibrations (meaning that if $X \rightarrow Y \rightarrow Z$ is a cofibre sequence and any two of X, Y, Z lie in \mathcal{C} , so does the third). We write $Th(X)$ for the thick subcategory generated by an object X ; objects of this category are said to be *X-finite*. Recall also that $W \in L_{K(1)}\mathcal{S}$ is *small* if for any collection of objects X_α , the natural map

$$\oplus [W, X_\alpha] \rightarrow \left[W, \coprod X_\alpha \right]$$

is an isomorphism. Here the coproduct on the right is the intrinsic coproduct; that is, the $K(1)$ -localization of the wedge.

13 Theorem 13 The following are equivalent:

- (a) X is small.
- (b) $\widehat{\mathcal{K}}X$ is finite.
- (c) X is $M\mathbb{Z}/\ell$ -finite.
- (d) $X = L_{K(1)}F$ for some finite ℓ -torsion spectrum F .

Next recall (again from [18]) that an object X is *dualizable* if the natural map

$$\mathcal{F}(X, S) \wedge Y \rightarrow \mathcal{F}(X, Y)$$

is an equivalence for all Y . Here the smash product on the left is the intrinsic smash product; that is, the $K(1)$ -localization of the ordinary smash product. This property has the alternate name “strongly dualizable”, but following [18] we will say simply that X is dualizable. If we were working in the ordinary stable homotopy category \mathcal{S} , the small and dualizable objects would be the same, and would coincide with the finite spectra. In $L_{K(1)}\mathcal{S}$, however, the dualizable objects properly contain the small objects:

14 Theorem 14 The following are equivalent:

- (a) X is dualizable.
- (b) $\widehat{\mathcal{K}}X$ is a finitely-generated Z_ℓ -module.
- (c) X is quasi-small.

Recall that quasi-small means that the map occurring in the definition of small object becomes an isomorphism after *Ext*- ℓ -completion of the direct sum in its

source; this is equivalent to being \mathcal{F} -small in the sense of [18]. Of course the localization of any finite spectrum is dualizable, but these are only a small subclass of all dualizable objects. We will return to this point in the next section.

Let $DX = \mathcal{F}(X, S)$. An object X is *weakly dualizable* if the natural map $X \rightarrow D^2X$ is an equivalence. Now there is another kind of duality in stable homotopy theory: Brown–Comenetz duality, an analogue of Pontrjagin duality. In $L_{K(1)}\mathcal{S}$ we define it as follows [17]: Fix an object $A \in L_{K(1)}\mathcal{S}$ and consider the functor $X \mapsto (\pi_0A \wedge X \wedge \mathcal{N})^\#$, where $(-)^\#$ denotes the ℓ -primary Pontrjagin dual $Hom(-, \mathbb{Z}/\ell^\infty)$. This functor is cohomological, and therefore by Brown representability there is a unique spectrum dA representing it; we call dA the *Brown–Comenetz dual* of A . It is easy to see that $dA = \mathcal{F}(A, dS)$, and so understanding Brown–Comenetz duality amounts to understanding dS . It turns out that in $L_{K(1)}\mathcal{S}$ life is very simple, at least for ℓ odd: dS is just $L_{K(1)}S^2$. Hence Brown–Comenetz duality is just a Tate-twisted form of functional duality, with the appearance of the twist strongly reminiscent of number theory. If $\ell = 2$ the situation is, as usual, somewhat more complicated. Let V denote the cofibre of $\varepsilon : \Sigma M\mathbb{Z}/2 \rightarrow S^0$, where ε is one of the two maps of order 4. Then the Brown–Comenetz dual of $L_{K(1)}S^0$ is $\Sigma^2 L_{K(1)}V$.

As another example, note that the universal coefficient isomorphism given at the beginning of Sect. 3.3.4 says precisely that $d\widehat{\mathcal{K}} \cong \Sigma\widehat{\mathcal{K}}$.

Some parts of the following theorem are special cases of the results of [17]; the rest can be found in [15].

Theorem 15 The following are equivalent:

- (a) X is weakly dualizable.
- (b) X is $\widehat{\mathcal{K}}$ -finite.
- (c) $\widehat{\mathcal{K}}X$ is finitely-generated over Λ .
- (d) $\pi_n X$ is finitely-generated over Z_ℓ for all n .
- (e) The natural map $X \rightarrow d^2X$ is an equivalence.

A $K(1)$ -local spectrum is *invertible* if $X \wedge Y \cong S$ for some Y . Here the smash product is of course the intrinsic smash product, and S is the $K(1)$ -local sphere. The set (and it is a set) of weak equivalence classes of such X forms a group under smash product, called the Picard group (see [16]). It is not hard to show that X is invertible if and only if $\widehat{\mathcal{K}}X$ is free of rank one as Z_ℓ -module. For simplicity we restrict our attention to the subgroup of index two of $Pic L_{K(1)}\mathcal{S}$ consisting of the X with $\widehat{\mathcal{K}}^1 X = 0$, denoted $Pic^0 L_{K(1)}\mathcal{S}$. There is a natural homomorphism

$$\phi : Pic^0 L_{K(1)}\mathcal{S} \rightarrow Pic \Lambda'$$

where $Pic \Lambda' \cong Z_\ell^\times$ is the analogous Picard group for the category of Λ' -modules. It is not hard to show that ϕ is onto at all primes ℓ , and is an isomorphism if ℓ is odd. The subtle point is that ϕ has a nontrivial kernel when $\ell = 2$. In fact, $Ker \phi$ has order two and is generated by the object $L_{K(1)}V$ defined above.

As an example for ℓ odd we mention the Δ -twists of the sphere. Let χ be a power of the Teichmuller character; in other words, χ is pulled back along the projection

$\Gamma' \rightarrow \Delta$. Let S_χ^0 denote the corresponding invertible spectrum. Thus Γ acts trivially on $\widehat{\mathcal{K}}^0 S_\chi^0$, but Δ acts via χ . We call these spectra Δ -twists of the sphere. It turns out that they arise in nature in an interesting way. Since $\Delta \cong \text{Aut } \mathbb{Z}/\ell$, Δ acts on the suspension spectrum of the classifying space $B\mathbb{Z}/\ell$. Hence the idempotents e_i defined above can be used to split $B\mathbb{Z}/\ell$, in exactly the same way that we used them to split $\widehat{\mathcal{K}}$. Then for $0 \leq i \leq \ell - 2$, $e_i L_{K(1)} B\mathbb{Z}/\ell \cong S_{\omega^i}^0$. For $i = 0$ one concludes that $L_{K(1)} B\Sigma_\ell \cong L_{K(1)} S^0$. For $i \neq 0 \pmod{\ell - 1}$, however, $S_{\omega^i}^0$ cannot be the localization of a finite spectrum.

Note that maps out of invertible spectra can be viewed as ℓ -adic interpolations of ordinary homotopy groups.

3.3.6 Iwasawa Theory for $\widehat{\mathcal{K}}$ -Finite Spectra

If $X \in \text{Th}(\widehat{\mathcal{K}})$ then we can apply the classification theory for Λ -modules. In this section we assume ℓ is odd. Most of the results below are from [15], where details and further results can be found.

Let \mathcal{C} be a thick subcategory of $L_{K(1)}\mathcal{S}$. A map $X \rightarrow Y$ is a \mathcal{C} -equivalence if its fibre lies in \mathcal{C} . In the case when \mathcal{C} is the category of small objects, we call a \mathcal{C} -equivalence a *pseudo-equivalence*. An object X of $L_{K(1)}\mathcal{S}$ is *elementary* if (i) $X \cong X_0 \vee X_1$, where $\widehat{\mathcal{K}}^1 X_0 = 0 = \widehat{\mathcal{K}}^0 X_1$, and (ii) $\widehat{\mathcal{K}} X$ is elementary as Λ -module. An elementary object X is determined up to equivalence by $\widehat{\mathcal{K}} X$ as Λ' -module. Hence if E is a $\mathbb{Z}/2$ -graded elementary Λ' -module, it makes sense to write \mathcal{M}_E for the corresponding elementary object.

16 Theorem 16 Any $\widehat{\mathcal{K}}$ -finite X is pseudo-equivalent to an elementary spectrum \mathcal{M}_E .

There need not be any pseudo-equivalence $X \rightarrow \mathcal{M}_E$ or $\mathcal{M}_E \rightarrow X$. In general, one can only find a third object Y and pseudo-equivalences

$$X \rightarrow Y \leftarrow \mathcal{M}_E .$$

The next result provides a simple but interesting illustration.

17 Theorem 17 X is the $K(1)$ -localization of a finite spectrum if and only if X is pseudo-equivalent to a finite wedge of spheres.

Notice that the collection of localizations of finite spectra does not form a thick subcategory. It is easy to see why this fails: The point is that $L_{K(1)} S^0$ has nontrivial negative homotopy groups, and no nontrivial element of such a group can be in the image of the localization functor. In particular, $\pi_{-1} L_{K(1)} S^0 = \mathbb{Z}_\ell$, and the cofibre of a generator is not the localization of any finite spectrum. Oddly enough, this cofibre comes up in connection with the Gross conjecture from algebraic number theory (see below).

It is possible to classify thick subcategories of the $\widehat{\mathcal{K}}$ -finite spectra. Define the *support* of X , denoted $\text{Supp } X$, to be the union of the supports of the Λ' -modules

$\widehat{\mathcal{K}}^n X$, $n \in \mathbb{Z}$. Note that $Supp X$ is invariant under Tate twisting and hence will typically contain infinitely many height one primes. On the other hand, it is clear that $Supp X$ is generated by $Supp \widehat{\mathcal{K}}^0 X \amalg Supp \widehat{\mathcal{K}}^1 X$ under Tate twisting.

A subset A of $Spec A'$ will be called *fit* if it is closed under specialization and under Tate twisting. For any subset A , we let \mathcal{C}_A denote the full subcategory of $L_{K(1)}\mathcal{S}$ consisting of objects whose support lies in the fit subset generated by A . If A is given as a subset of $Spec A$ only, we interpret this to mean taking closure under Δ -twisting as well. In other words, we take the fit subset generated by all (q, i) , $q \in A$. It is easy to see that \mathcal{C}_A is a thick subcategory. Note for example:

- If $A = \{\mathcal{M}\}$, \mathcal{C}_A consists of the small objects;
- If A is the collection of all irreducible distinguished polynomials, \mathcal{C}_A consists of the dualizable objects.

Theorem 18 Let \mathcal{C} be a thick subcategory of $Th(\widehat{\mathcal{K}})$. Then there is a unique fit set of primes A such that $\mathcal{C} = \mathcal{C}_A$.

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Some further examples:

- Let \mathcal{C} be the collection of objects X with finite homotopy groups. Then A is generated by the complement of the set of extended Tate primes (in the set of height one primes).
- Let \mathcal{C} be the collection of objects X with almost all homotopy groups finite. Then A is generated by the set of all height one primes.
- Let $\mathcal{C} = Th(L_{K(1)}S^0)$. Then A is generated by the set of extended Tate primes.
- Let \mathcal{C} be the thick subcategory generated by the invertible spectra. Then A is generated by the set of linear distinguished polynomials.

The semi-discrete primes are also of interest. Note, for example, that if G is a finite group then $\widehat{\mathcal{K}} BG$ is semi-discrete; this follows from the famous theorem of Atiyah computing $\widehat{\mathcal{K}} BG$ in terms of the representation ring of G . In the next two propositions, the spectra occurring are implicitly localized with respect to $K(1)$.

Proposition 19 Let A be the set of semi-discrete primes.

19

- a) \mathcal{C}_A is generated by the suspension spectra BC , C ranging over finite cyclic ℓ -groups.
- b) Fix a prime $p \neq \ell$. Then \mathcal{C}_A is generated by the spectra $K\mathbb{F}_q$, q ranging over the powers of p .

Proposition 20 For any S -arithmetic group G , the classifying space BG is in \mathcal{C}_A , where A is the set of semi-discrete primes.

20

This last proposition is an easy consequence of well-known theorems of Borel-Serre; the point is that BG has a finite filtration whose layers are finite wedges of suspensions of classifying spaces of finite groups.

3.4 Iwasawa Theory

We assume throughout this section that ℓ is odd.

We begin by introducing our notation for various objects and Iwasawa modules associated to the ℓ -adic cyclotomic tower (Sect. 3.4.1). In Sect. 3.4.2 we use Dwyer's étale homotopy theory approach to prove many of the classical theorems of Iwasawa theory.

3.4.1 Notation

We regret having to subject the reader to a barrage of notation at this point, but we might as well get it over with, and at least have all the notation in one place for easy reference. Fix the number field F and odd prime ℓ . Recall that r_1 and r_2 denote respectively the number of real and complex places of F .

Warning: Some of our notation conflicts with standard usage in number theory. The main example is that for us, A_∞ and A'_∞ refer to norm inverse of ℓ -class groups, not the direct limits. Thus our A_∞, A'_∞ would usually be denoted X_∞, X'_∞ in the number theory literature.

The Cyclotomic Tower

By the *cyclotomic tower* we mean the ℓ -adic cyclotomic tower defined as follows: F_0 is the extension obtained by adjoining the ℓ -th roots of unity to F . We let $d = d_F$ denote the degree of F_0 over F ; note that d divides $\ell - 1$. The Galois group $G(F_0/F)$ will be denoted Δ_F ; it is a cyclic group of order d .

F_∞ is the extension obtained by adjoining all the ℓ -power roots of unity to F . The Galois group $G(F_\infty/F_0)$ will be denoted Γ_F ; it is isomorphic as profinite group to Z_ℓ . More precisely, the natural map $\Gamma_F \rightarrow \Gamma_{\mathbb{Q}}$ is an isomorphism onto a closed subgroup of index ℓ^m for some $m = m_F$. Note that when ℓ -th roots of unity are adjoined, we may have accidentally adjoined ℓ^j -th roots for some finite j as well. The Galois group $G(F_\infty/F)$ will be denoted Γ'_F ; it splits uniquely as $\Gamma' \times \Delta_F$.

It follows that there is a unique sequence of subextensions F_n/F_0 whose union is F_∞ , and with F_n of degree ℓ over F_{n-1} for all $n \geq 1$. Our *cyclotomic tower* over F is this tower $F \subset F_0 \subset F_1 \subset \dots$

For $0 \leq n \leq \infty$, we write \mathcal{O}_n for \mathcal{O}_{F_n} and R_n for $\mathcal{O}_{F_n}[1/\ell]$. Further variations of this obvious notational scheme will be used without comment.

Let Λ_F and Λ'_F denote respectively the pro-group rings of Γ_F and Γ'_F . Thus $\Lambda_F \subset \Lambda$, and at the same time Λ_F is abstractly isomorphic to Λ as profinite ring; in particular, Λ_F is a power series ring over Z_ℓ . We need a separate notation, however, for its power series generator. Set

$$T_F = (1 + T)^{\ell^m} - 1,$$

where $m = m_F$ is as above. Then Λ_F is identified with $Z_\ell[[T_F]] \subset Z_\ell[[T]] = \Lambda$.

Some Important Modules Over the Iwasawa Algebra of F

Starting from basic algebraic objects attached to number rings – class groups, unit groups and so on – we can construct associated Λ'_F -modules in two ways: (i) By taking ℓ -adic completions at each level of the cyclotomic tower and passing to the inverse limit over the appropriate norm maps, thereby obtaining a profinite Λ'_F -module; or (ii) passing to the direct limit over the appropriate inclusion-induced maps in the cyclotomic tower – in some cases, after first tensoring with \mathbb{Z}/ℓ^∞ – thereby obtaining, typically, a discrete torsion Λ'_F -module.

The basic examples:

Primes over ℓ : Let S denote the set of primes dividing ℓ in \mathcal{O}_F , and let s denote the cardinality of S . Each $\beta \in S$ is ramified in the cyclotomic tower, and furthermore there is some finite j such that all $\beta \in S_j$ are totally ramified in $\mathcal{O}_\infty/\mathcal{O}_j$. Thus S_∞ , the set of all primes over ℓ in \mathcal{O}_∞ , is finite, and the permutation representation of Γ'_F given by $Z_\ell S_\infty$ is the same thing as the norm inverse limit of the representations $Z_\ell S_n$. As with any permutation representation, there is a canonical epimorphism to the trivial module. Let B_∞ denote the kernel. Thus there is a short exact sequence of semidiscrete Λ'_F -modules

$$0 \rightarrow B_\infty \rightarrow Z_\ell S_\infty \rightarrow Z_\ell \rightarrow 0 .$$

The letter B is chosen to suggest the Brauer group, since it follows from class field theory that B is naturally isomorphic to $Hom(\mathbb{Z}/\ell^\infty, Br R)$, the Tate module of the Brauer group of R .

Class groups: Let A (resp. A') denote the ℓ -torsion subgroup of the class group of \mathcal{O}_F (resp. of R). Passing to the norm inverse limit with the A 's yields profinite Λ'_F -modules

$$A_\infty = \lim_n A_n \qquad A'_\infty = \lim_n A'_n .$$

Passing to the direct limit yields discrete torsion Λ'_F -modules

$$\mathcal{A}_\infty = \text{colim}_n A_n \qquad \mathcal{A}'_\infty = \text{colim}_n A'_n .$$

Note that there is a short exact sequence of the form $0 \rightarrow J_\infty \rightarrow A_\infty \rightarrow A'_\infty \rightarrow 0$, where J_∞ is a certain quotient of $Z_\ell S_\infty$ and in particular is semidiscrete.

Unit groups: Let E (resp. E') denote the ℓ -adic completion of the unit group \mathcal{O}_F^\times (resp. R^\times). Do not confuse this construction with taking units of the associated local rings.

Passing to the norm inverse limit yields profinite Λ'_F -modules

$$E_\infty = \lim_n (\mathcal{O}_n^\times)^\wedge \qquad E'_\infty = \lim_n (R_n^\times)^\wedge .$$

Let \mathcal{E} (resp. \mathcal{E}') denote $\mathcal{O}_F^\times \otimes \mathbb{Z}/\ell^\infty$ (resp. $R^\times \otimes \mathbb{Z}/\ell^\infty$). Passing to the direct limit in the cyclotomic tower yields discrete torsion Λ'_F -modules

$$\mathcal{E}_\infty = \text{colim}_n \mathcal{O}_n^\times \otimes \mathbb{Z}/\ell^\infty \qquad \mathcal{E}'_\infty = \text{colim}_n R_n^\times \otimes \mathbb{Z}/\ell^\infty .$$

Local unit groups: For each prime $\beta \in S$, let $F_{\mathcal{P}}$ denote the local field obtained by \mathcal{P} -adic completion. Let

$$U = \prod_{\beta \in S} (F_{\beta}^{\times})^{\wedge},$$

where as usual $(-)^{\wedge}$ denotes ℓ -adic completion. Let U_{∞} denote the norm inverse limit of the U_n 's. In view of the above remarks on primes over ℓ , the number of factors in the product defining U_n stabilizes to s_{∞} . This leads easily to the following description of the norm inverse limit U_{∞} : For each fixed prime β over ℓ in \mathcal{O}_F , let $U_{\infty, \beta}$ denote the norm inverse limit of the completed unit groups for the ℓ -adic cyclotomic tower over the completion F_{β} . Let $\Lambda'_{F_{\beta}}$ denote the pro-group ring analogous to Λ'_F . Then as Λ'_F -modules we have

$$U_{\infty} \cong \bigoplus_{\beta \in S} \Lambda'_F \otimes_{\Lambda'_{F_{\beta}}} U_{\infty, \beta}.$$

ℓ -extensions unramified away from ℓ : Let M_n denote the Galois group of the maximal abelian ℓ -extension of F_n that is unramified away from ℓ . Passing to the inverse limit over n yields M_{∞} , the maximal abelian ℓ -extension of F_{∞} that is unramified away from ℓ . Note that M_{∞} is a profinite Γ'_F -module by conjugation, and hence a profinite Λ'_F -module.

Algebraic and étale K -groups: One can play the same game using the algebraic K -groups K_*R_n in the cyclotomic tower. If the Lichtenbaum–Quillen conjecture are true, however, these do not yield much new. Taking norm inverse limits of ℓ -completed groups to illustrate, the reason is that these groups become more and more periodic (conjecturally) as $n \rightarrow \infty$, while in low degrees K_*R is essentially determined by the class group, unit group, and Brauer group – all of which we have already taken into account above.

In étale K -theory, however, there is one small but very useful exception. Equivalently, we can take $K(1)$ -localized K -theory $\pi_*L_{K(1)}KR$, and in any case we are in effect just looking at étale cohomology $H^*(R; Z_{\ell}(n))$ for $* = 1, 2$. In particular there is a short exact sequence

$$0 \rightarrow H^2(R; Z_{\ell}(1)) \rightarrow K_0^{\text{ét}}R \rightarrow Z_{\ell} \rightarrow 0,$$

where the first term in turn fits into a short exact sequence

$$0 \rightarrow A' \rightarrow H^2(R; Z_{\ell}(1)) \rightarrow B \rightarrow 0.$$

Writing $L_n = H^2(R_n; Z_{\ell}(1))$, we may again pass to an inverse limit in the cyclotomic tower, yielding a short exact sequence of Λ'_F -modules

$$0 \rightarrow A'_{\infty} \rightarrow L_{\infty} \rightarrow B_{\infty} \rightarrow 0.$$

As we will see below, L_{∞} is in many ways better behaved than A'_{∞} .

Iwasawa Theory for the Cyclotomic Extension

The beautiful fact is that virtually all of the profinite modules considered in the previous section are finitely-generated Λ_F -modules. (In the case of the algebraic *K*-groups, this would follow from the Lichtenbaum–Quillen conjecture; can it be shown directly?) Then one can ask: What are the Λ_F -ranks? What are the torsion invariants?

Finding the torsion invariants explicitly is an extremely difficult problem, but one can ask for qualitative information of a more general nature. In this section we will sketch an approach based on Poincaré–Artin–Verdier duality and étale homotopy theory. Some of this material comes from unpublished joint work of Bill Dwyer and the author, but the key ideas below are due to Dwyer, and the author is grateful for his permission to include this work here. While the method comes with a certain cost in terms of prerequisites, it yields many of the classical results (compare e.g. [19, 38, 46]) in an efficient, conceptual fashion.

We point out that number theorists usually study much more general Z_ℓ -extensions, not just the cyclotomic one. It would be interesting to apply this approach to the more general setting. Also, for us the ℓ -adic cyclotomic extension never means just the Z_ℓ -extension it contains; we invariably adjoin all the ℓ -power roots of unity and work with modules over Λ'_F . On the other hand, since it is usually easy to descend from F_0 to F , we will sometimes assume for simplicity that F contains the ℓ -th roots of unity.

Duality and the Seven-term Exact Sequence

We begin by considering the natural map

$$i : \coprod_{\beta|\ell} \text{Spec } F_\beta \rightarrow \text{Spec } R ,$$

in the étale topology. Let us abbreviate the target of i as Y , and the source as ∂Y . One can define homology groups $H_*(\partial Y; Z_\ell(n))$ by considering the s components of ∂Y separately and simply taking Galois homology as in [38]. Then local class field theory says that ∂Y behaves like a nonorientable 2-manifold with s components, in that there are natural local duality isomorphisms

$$H_k(\partial Y; Z_\ell(n)) \cong H^{2-k}(\partial Y; Z_\ell(1-n)) .$$

One could define homology groups for Y in a similar *ad hoc* way. Let Ω_F denote the maximal ℓ -extension of F_∞ that is unramified away from ℓ . Then the étale cohomology for Y is the same as the profinite group cohomology of the Galois group $G(\Omega_F/F)$, so we could define homology by the same device. But we also want to define relative homology for the pair $Y, \partial Y$, and here the *ad hoc* approach begins to break down. Instead, we will use pro-space homology as in [7]. It is then

possible to regard $H_*(Y, \partial Y; Z_\ell(n))$ as the homology of the “cofibre” of i . Then $(Y, \partial Y)$ behaves like a nonorientable 3-manifold with boundary, in that Artin–Verdier duality (or rather a special case of it usually called Poincaré duality) yields duality isomorphisms

$$H_k(Y; Z_\ell(n)) \cong H^{3-k}(Y, \partial Y; Z_\ell(1 - n))$$

and similarly with the roles of H_* , H^* reversed. A general discussion and proof of Artin–Verdier duality can be found in [28]. The proof makes use of local duality and a relatively small dose of global class field theory, including the Hilbert classfield.

Bearing in mind that $H_2(\partial Y; Z_\ell) = 0$ by local duality, we get a seven-term exact sequence (with trivial Z_ℓ coefficients understood)

$$0 \rightarrow H_2 Y \rightarrow H_2(Y, \partial Y) \rightarrow H_1(\partial Y) \rightarrow H_1 Y \rightarrow H_1(Y, \partial Y) \rightarrow Z_\ell S \rightarrow Z_\ell \rightarrow 0 .$$

Taking into account both local and Artin–Verdier duality, and using notation from the previous section, we have at each stage of the cyclotomic tower a seven-term exact sequence

$$0 \rightarrow D_n \rightarrow E'_n \rightarrow U'_n \rightarrow M_n \rightarrow L_n \rightarrow Z_\ell S_n \rightarrow Z_\ell \rightarrow 0 .$$

Here we have abbreviated $H_2(Y_n)$ as D_n ; this group is the *Leopoldt defect* as will be explained below. Recall that E'_n , U'_n are ℓ -completed global and local unit groups, M_n is the Galois group of the maximal abelian ℓ -extension unramified away from ℓ , L_n is the interesting part of the zero-th étale K-theory, and S_n is the set of primes dividing ℓ .

Taking norm inverse limits we obtain a seven-term exact sequence of profinite A'_F -modules

$$0 \rightarrow D_\infty \rightarrow E'_\infty \rightarrow U'_\infty \rightarrow M_\infty \rightarrow L_\infty \rightarrow Z_\ell S_\infty \rightarrow Z_\ell \rightarrow 0 .$$

Note that we deduce at once a standard result of class field theory, namely a short exact sequence

$$0 \rightarrow U'_\infty / E'_\infty \rightarrow M_\infty \rightarrow A'_\infty \rightarrow 0 ,$$

We will see later that $D_\infty = 0$, justifying the notation. If the Leopoldt conjecture holds (see Sect. 3.6), then all the D_n 's vanish as well. Thus an optimistic title for this section would be “Duality and the six-term exact sequence”.

The Local Case

We digress to consider the local case, as both warm-up and input for the global case.

Let K be a finite extension of \mathbb{Q}_ℓ , ℓ odd. Then in a notation that should be self-explanatory, we have K_∞ , A'_K , $M_{K,\infty}$, $U'_{K,\infty}$, etc. For example, $M_{K,\infty}$ is the Galois group of the maximal abelian ℓ -extension of K_∞ . Local class field theory gives isomorphisms

$$M_{K,m} \cong U'_{K,m} ,$$

$m \leq \infty$. Our goal is to determine explicitly the structure of $M_{K,\infty}$ as Λ'_K -module. For simplicity we will assume $\mu_\ell \subset K$, and to avoid notational clutter we will drop the subscript K from Λ_K , etc. First, there is a Serre spectral sequence

$$H_p(\Gamma; H_q K_\infty) \Rightarrow H_{p+q} K .$$

From this we obtain at once a short exact sequence

$$0 \rightarrow (M_\infty)_\Gamma \rightarrow H_1 K \rightarrow Z_\ell \rightarrow 0 .$$

Since $H_1 K \cong H^1(K; Z_\ell(1)) \cong U'$ by local duality, this shows that $(M_\infty)_\Gamma$ has Z_ℓ -rank d , where $d = [K : \mathbb{Q}_\ell]$.

Second, there is a universal coefficient spectral sequence

$$\text{Ext}_\Lambda^p (H_q K_\infty, \Lambda(1)) \Rightarrow H_{2-p-q} K_\infty .$$

Notice here that the Tate twist $\Lambda(1)$ is isomorphic to Λ as Λ -modules, but the twist is necessary in order to come out with the right Λ -module structure on the E_∞ -term. In fact the natural abutment of the spectral sequence is $H^{p+q}(K; \Lambda(1))$, but this group is isomorphic to the homology group above by local duality plus a form of Shapiro's lemma. This spectral sequence yields at once a short exact sequence

$$0 \rightarrow \text{Ext}_\Lambda^1 (Z_\ell, \Lambda(1)) \rightarrow M_\infty \rightarrow \text{Hom}_\Lambda (M_\infty, \Lambda(1)) \rightarrow 0 .$$

Since Λ -duals are always free, combining these results yields:

Proposition 21 There are isomorphisms of Λ -modules

21

$$U'_\infty \cong M_\infty \cong \Lambda^d \oplus Z_\ell(1) .$$

Now recall that for a number field F we defined a norm inverse limit of local units U'_∞ , using all the primes over ℓ . Again assuming for simplicity that F contains the ℓ -th roots of unity, we have:

Corollary 22 If F is a number field,

22

$$U'_\infty \cong \Lambda_F^{2r_2} \oplus \left(\bigoplus_{\beta \in S} \Lambda_F \otimes_{\Lambda_{F_\beta}} Z_\ell(1) \right) .$$

The Global Case

We consider four miniature spectral sequences associated to the cyclotomic tower. They are miniature in the sense that they are first quadrant spectral sequences occupying a small rectangle near the origin. The first three collapse automatically. Since we are concerned mainly with the Λ_F -structure here, we will ignore the Δ_F module structure for the time being. To avoid notational clutter, in this section we will drop the subscript F from Γ_F, Λ_F , etc.

First spectral sequence: There is a Serre spectral sequence

$$H_p(\Gamma; H_q Y_\infty) \Rightarrow H_{p+q} Y_0 .$$

From it we obtain immediately:

23 Proposition 23 There are short exact sequences

a)

$$0 \rightarrow (D_\infty)_\Gamma \rightarrow D_0 \rightarrow (M_\infty)^\Gamma \rightarrow 0$$

and

b)

$$0 \rightarrow (M_\infty)_\Gamma \rightarrow M_0 \rightarrow Z_\ell \rightarrow 0 .$$

In particular, as a corollary of (b) we get:

24 Corollary 24 M_∞ is a finitely-generated Λ -module.

Proof By a version of Nakayama's lemma, it suffices to show $(M_\infty)_\Gamma$ is finitely-generated as Z_ℓ -module. Hence by (b), it suffices to show $M_0 = H_1(F_0; Z_\ell)$ is finitely-generated as Z_ℓ -module. This in turn reduces to showing $H^1(F_0; \mu_\ell)$ is finite. But there is a Kummer exact sequence

$$0 \rightarrow R_0^\times / \ell \rightarrow H^1(F_0; \mu_\ell) \rightarrow A_0[\ell] \rightarrow 0 ,$$

where $A_0[\ell]$ denotes the elements annihilated by ℓ , so this follows from the finite-generation of the unit group and finiteness of the class group.

Second spectral sequence: There is a relative Serre spectral sequence

$$H^p(\Gamma; H_q(Y_\infty, \partial Y_\infty)) \Rightarrow H_{p+q}(Y_0, \partial Y_0) .$$

From it we obtain immediately:

25 Proposition 25

a) $(L_\infty)_\Gamma = L_0$.

b) There is a short exact sequence

$$0 \rightarrow (E'_\infty)_\Gamma \rightarrow E'_0 \rightarrow (L_\infty)^\Gamma \rightarrow 0 .$$

In particular, as a corollary of (a) we get:

26 Corollary 26 L_∞, A'_∞ , and hence A_∞ are finitely-generated Λ -modules.

This follows because L_0 is a finitely-generated Z_ℓ -module – thanks to the fact that both the class group and S_∞ are finite.

The analogue of Proposition 25a for A'_∞ is false; this is one reason that L_∞ is easier to work with. In fact Proposition 25a yields an interesting corollary. For a fixed $\beta \in S_0$, let s_β denote the number of primes over β in S_∞ .

Corollary 27 Let ℓ^v denote the minimum value of s_β , $\beta \in S$. Then the cokernel of the natural map $\phi : (A'_\infty)_\Gamma \rightarrow A'_0$ is cyclic of order ℓ^v . 27

In particular, ϕ is onto if and only if at least one $\beta \in S_0$ is nonsplit (i.e., inert or ramified) in F_1/F_0 .

Proof There are isomorphisms

$$\text{Coker } \phi \cong \text{Ker } ((B_\infty)_\Gamma \rightarrow B_0) \cong \text{Coker } ((Z_\ell S_\infty)^\Gamma \rightarrow Z_\ell) \cong \mathbb{Z}/\ell^v,$$

where the first follows from Proposition 25a and the snake lemma, the second is elementary, and the third is obvious.

Remark 28 The kernel of ϕ is isomorphic to the image of the boundary map $B_\infty^\Gamma \rightarrow (A'_\infty)_\Gamma$, or equivalently the cokernel of $L_\infty^\Gamma \rightarrow B_\infty^\Gamma$. This seems harder to analyze, although it follows for example that if $s_0 = 1$ then ϕ is injective. If $s_\infty = 1$ then $B_\infty = 0$ and ϕ is an isomorphism. 28

Next we illustrate the significance of part (b). Note that $(E'_\infty)_\Gamma$ is the subgroup of *universal norms* in E'_0 ; that is, the units which are in the image of the norm map from every level of the tower. (Clearly $(E'_\infty)_\Gamma$ is contained in the universal norms, and a \lim^1 argument shows every universal norm comes from E'_∞ .) Hence if $s = 1$ and $(A'_\infty)^\Gamma = 0$, every element of E'_0 is a universal norm. We also get a theorem of Kuz'min on the Gross–Sinnott kernel (see [24], Theorem 3.3): Let \mathcal{N}_F denote the subgroup of E'_0 consisting of elements that are local universal norms at every prime over ℓ .

Corollary 29 There is a canonical short exact sequence 29

$$0 \rightarrow (E'_\infty)_\Gamma \rightarrow \mathcal{N}_F \rightarrow (A'_\infty)^\Gamma \rightarrow 0.$$

Proof This is immediate from the commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & (E'_\infty)_\Gamma & \rightarrow & E'_0 & \rightarrow & L_\infty^\Gamma & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & (U'_\infty)_\Gamma & \rightarrow & U'_0 & \rightarrow & (Z_\ell S_\infty)^\Gamma & \rightarrow & 0. \end{array}$$

The second spectral sequence also shows that $(E'_\infty)^\Gamma = 0$, but this was obvious *a priori*.

Third spectral sequence: There is a universal coefficient spectral sequence

$$\text{Ext}_{\Lambda}^p(H_q Y_{\infty}, \Lambda(1)) \Rightarrow H_{3-p-q}(Y_{\infty}, \partial Y_{\infty}) .$$

Notice here that the Tate twist $\Lambda(1)$ is isomorphic to Λ as Λ -modules, but the twist is necessary in order to come out with the right Λ -module structure on the E_{∞} -term. In fact the natural abutment of the spectral sequence is $H^{p+q}(Y_0; \Lambda(1))$, but this group is isomorphic to the homology group above by Artin–Verdier duality plus a form of Shapiro’s lemma. Recalling that Λ has global dimension two, we obtain at once:

30 Proposition 30

a) There is a short exact sequence

$$0 \rightarrow Z_{\ell}(1) \rightarrow E'_{\infty} \rightarrow \text{Hom}_{\Lambda}(M_{\infty}, \Lambda)(1) \rightarrow 0$$

b) There is an isomorphism $L_{\infty} \cong \text{Ext}_{\Lambda}^1(M_{\infty}, \Lambda)(1)$

c) $\text{Ext}_{\Lambda}^2(M_{\infty}, \Lambda) = 0$. Hence M_{∞} has no nonzero finite submodules, and has projective dimension at most one.

Since any module of the form $\text{Ext}_{\Lambda}^1(N, \Lambda)$ is a Λ -torsion module, part (b) shows at once:

31 Corollary 31 L_{∞} is a Λ -torsion module. Hence A_{∞} and A'_{∞} are also Λ -torsion modules.

Next we have:

32 Corollary 32 $E'_{\infty} \cong \Lambda^{r_2} \oplus Z_{\ell}(1)$. Moreover, M_{∞} has Λ -rank r_2 .

Proof Any module of the form $\text{Hom}_{\Lambda}(N, \Lambda)$ with N finitely-generated is a free module. By Proposition 30a we conclude that $E'_{\infty} \cong Z_{\ell}(1) \oplus \Lambda^t$ for some t . Furthermore t is the Z_{ℓ} -rank of $(E'_{\infty})_{\Gamma}$. Now since L_{∞} is a finitely-generated Λ -torsion module, $(L_{\infty})^{\Gamma}$ and $(L_{\infty})_{\Gamma}$ have the same Z_{ℓ} -rank; hence $(L_{\infty})^{\Gamma}$ has rank $s_0 - 1$. Then Proposition 25b shows that $(E'_{\infty})_{\Gamma}$ has rank r_2 . This completes the proof of the first statement. The second is then immediate from Proposition 30a.

33 Corollary 33 The freeness defect of M_{∞}/tM_{∞} is Pontrjagin dual to the maximal finite submodule of A'_{∞} .

Proof By Proposition 2.4, the freeness defect is Pontrjagin dual to $\text{Ext}_{\Lambda}^1(M_{\infty}/tM_{\infty}, \Lambda)$, and this latter group is easily seen to be isomorphic to the maximal finite submodule of $\text{Ext}_{\Lambda}^1(M_{\infty}, \Lambda)$. Now use Proposition 30b.

We also get an important corollary on the Leopoldt defect groups D_n .

Corollary 34

34

- a) The norm inverse limit $D_\infty = H_2(Y_\infty; Z_\ell)$ vanishes.
- b) $D_0 = M^T$.
- c) The direct limit \mathcal{D}_∞ is just D_n for large enough n , and is in fact isomorphic to the maximal semidiscrete submodule M_∞^δ .

Proof

- a) Note that we have now computed the Λ -rank of every term in the seven-term exact sequence except D_∞ . Counting ranks then forces $\text{rank}_\Lambda D_\infty = 0$; in other words, D_∞ is a Λ -torsion module. But the seven-term exact sequence also shows that it embeds in $E'_\infty/Z_\ell(1)$, and so is Λ -torsionfree. Hence it is zero.
- b) This is now immediate from Proposition 23a.
- c) This follows from the validity of (b) at each level of the cyclotomic tower.

Fourth spectral sequence:

Reversing the roles of Y and $(Y, \partial Y)$ in the third spectral sequence yields our last spectral sequence:

$$\text{Ext}_\Lambda^p (H_q(Y_\infty, \partial Y_\infty), \Lambda(1)) \Rightarrow H_{3-p-q} Y_\infty .$$

Some of the information from this spectral sequence is already known: We find that $\text{Hom}_\Lambda(L_\infty, \Lambda) = 0$, confirming that L_∞ is a Λ -torsion module; we find that $\text{Ext}_\Lambda^2(E'_\infty, \Lambda) = 0$, confirming that E'_∞ has no finite submodules. The remaining parts of the spectral sequence yield an exact sequence

$$0 \rightarrow \text{Ext}_\Lambda^1(L_\infty, \Lambda(1)) \rightarrow M_\infty \rightarrow \text{Hom}_\Lambda(E'_\infty, \Lambda(1)) \xrightarrow{d_2} \text{Ext}_\Lambda^2(L_\infty, \Lambda(1)) \rightarrow 0 .$$

and an isomorphism $Z_\ell \cong \text{Ext}_\Lambda^1(E'_\infty, \Lambda(1))$.

The isomorphism yields no new information, but we get something interesting from the exact sequence. First we should justify the surjectivity of the indicated d_2 . This can be seen by direct inspection, or as follows: We have seen in Corollary 33 that the freeness defect of M_∞/tM_∞ is Pontrjagin dual to the maximal finite submodule of L_∞ , which in turn is dual to the indicated Ext^2 . Hence d_2 is surjective. Thus we have:

Corollary 35

35

- a) The torsion submodule tM_∞ of M_∞ is isomorphic to $\text{Ext}_\Lambda^1(L_\infty, \Lambda(1))$. Hence there is a short exact sequence

$$0 \rightarrow \text{Ext}_\Lambda^1(B_\infty, \Lambda(1)) \rightarrow tM_\infty \rightarrow \text{Ext}_\Lambda^1(A'_\infty, \Lambda(1)) \rightarrow 0 .$$

- b) The torsion-free quotient of M_∞ embeds with finite index in the free module $\text{Hom}_\Lambda(E'_\infty, \Lambda(1))$.

One can also get the following theorem of Iwasawa, in effect by the ordinary universal coefficient theorem:

36 Theorem 36 There is a short exact sequence of Λ' -modules

$$0 \rightarrow (\mathcal{A}'_\infty)^\#(1) \rightarrow M_\infty \rightarrow (\mathcal{E}'_\infty)^\#(1) \rightarrow 0.$$

3.5 Algebraic K-Theory Spectra

We continue to fix a prime ℓ .

Let X be a sufficiently nice scheme. By this we mean that X is a separated noetherian regular scheme of finite Krull dimension and with all residue fields of characteristic different from ℓ , and that we reserve the right to impose additional hypotheses as needed. In the present context, it is common to assume also that X has finite étale cohomological dimension for ℓ -torsion sheaves, written $cd_{\text{ét}}^\ell X < \infty$. We will, however, avoid this last hypothesis as much as possible.

Now let $\mathbb{H}_{\text{ét}}^\cdot(X, \underline{K})$ denote the Thomason–Jardine hypercohomology spectrum associated to the algebraic K -theory presheaf on the étale site of X . Up to connective covers, the ℓ -adic completion of this spectrum is equivalent to the Dwyer–Friedlander étale K -theory spectrum of X . See [22, 44] or [34] for details. The key fact about $\mathbb{H}_{\text{ét}}^\cdot(X, \underline{K})$ is that it admits a conditionally convergent right half-plane cohomology spectral sequence

$$E_2^{p,-q} = H_{\text{ét}}^p\left(X; Z_\ell\left(\frac{q}{2}\right)\right) \Rightarrow \pi_{q-p}\mathbb{H}_{\text{ét}}^\cdot(X, \underline{K})^\wedge.$$

Here $Z_\ell(\frac{q}{2})$ is to be interpreted as zero if q is odd, and as always our étale cohomology groups are continuous étale cohomology groups in the sense of Jannsen [21]. The condition $cd_{\text{ét}}^\ell X < \infty$ is often invoked to ensure actual convergence of the spectral sequence, but it is not a necessary condition.

There is a natural augmentation map $\eta : KX \rightarrow \mathbb{H}_{\text{ét}}^\cdot(X, \underline{K})$. The Dwyer–Friedlander spectrum-level version of the Lichtenbaum–Quillen conjecture can then be stated as follows:

37 Conjecture 37 Let X be a sufficiently nice scheme. Then for some $d \geq 0$, the completed augmentation

$$\eta^\wedge : (KX)^\wedge \rightarrow \mathbb{H}_{\text{ét}}^\cdot(X, \underline{K})^\wedge$$

induces a weak equivalence on $d - 1$ -connected covers.

In a monumental paper [44], Thomason proved the $K(1)$ -local Lichtenbaum–Quillen conjecture.

Theorem 38 (Thomason) Let X be a sufficiently nice scheme. Then $L_{K(1)}\eta$ is a weak equivalence .

38

For Thomason, “sufficiently nice” includes several further technical hypotheses, including $cd_{\acute{e}t}^\ell X < \infty$. He also assumes $\sqrt{-1} \in X$ in the case $\ell = 2$. But for $\ell = 2$ the Lichtenbaum–Quillen conjecture itself has now been proved in many cases [41] [39], and in those cases the assumption $\sqrt{-1} \in X$ can be dropped.

Now $(\mathbb{H}_{\acute{e}t}(X, \underline{K}))^\wedge$ is *essentially K(1)-local*, meaning that the map to its $K(1)$ -localization induces a weak equivalence on some connected cover. Hence the Lichtenbaum–Quillen conjecture can be re-interpreted as follows:

Conjecture 39 Let X be a sufficiently nice scheme. Then $(KX)^\wedge$ is essentially $K(1)$ -local.

39

Thomason’s theorem (or the actual Lichtenbaum–Quillen conjecture, when known) also has the corollary:

Corollary 40 $\widehat{\mathcal{K}}^* KX \cong \widehat{\mathcal{K}}^* \mathbb{H}_{\acute{e}t}(X, \underline{K})$.

40

Since $\widehat{\mathcal{K}}^* \mathbb{H}_{\acute{e}t}(X, \underline{K})$ is computable, this leads to explicit computations of $\widehat{\mathcal{K}}^* KX$ ([9, 10, 33]).

It is natural to ask how $L_{K(1)}KX$ fits into the $K(1)$ -local world described in Sect. 3.3. We will assume that X satisfies the $K(1)$ -local Lichtenbaum–Quillen conjecture. Thus X could be any scheme satisfying the hypotheses of Thomason’s theorem, or any scheme for which the actual Lichtenbaum–Quillen conjecture is known. In many interesting cases, $L_{K(1)}KX$ belongs to the thick subcategory of $\widehat{\mathcal{K}}$ -finite spectra:

Proposition 41 Suppose that (a) $H_{\acute{e}t}^i(X; \mathbb{Z}/\ell(m))$ is finite for every i, m ; and (b) the descent filtration on $\pi_n(\mathbb{H}_{\acute{e}t}(X, \underline{K}))^\wedge$ terminates for each n . Then $L_{K(1)}KX$ is $\widehat{\mathcal{K}}$ -finite.

41

Proof By (a), $H_{\acute{e}t}^i(X; \mathbb{Z}_\ell(m))$ is a finitely-generated \mathbb{Z}_ℓ -module for all i, m . Hence by (b) and the descent spectral sequence, the same is true of $\pi_n(\mathbb{H}_{\acute{e}t}(X, \underline{K}))^\wedge$. Using Theorem 15 and the fact that $(\mathbb{H}_{\acute{e}t}(X, \underline{K}))^\wedge$ is essentially $K(1)$ -local, it follows that $L_{K(1)}\mathbb{H}_{\acute{e}t}(X, \underline{K})$ is $\widehat{\mathcal{K}}$ -finite. Since $L_{K(1)}KX \cong L_{K(1)}\mathbb{H}_{\acute{e}t}(X, \underline{K})$ by assumption, this completes the proof.

There are many examples of schemes X satisfying conditions (a) and (b). The most important for our purposes is $X = \text{Spec } R$, where $R = \mathcal{O}_F[\frac{1}{\ell}]$ as in Sect. 3.4.1. Condition (b) holds even when $\ell = 2$ and F has a real embedding; in this case descent filtration 5 vanishes. See for example [35], Proposition 2.10.

Given such an X , we can then analyze $L_{K(1)}KX$ up to pseudo-equivalence, or up to some weaker mod \mathcal{C} equivalence in the sense of Sect. 3.3.6. The case $X = \text{Spec } R$ is the topic of the next section.

K-Theoretic Interpretation of some Conjectures in Iwasawa Theory

3.6

A fundamental problem of Iwasawa theory is to determine the pseudo-isomorphism-type of M_∞ . In Corollary 32 we determined the rank; the much more difficult question that remains is to determine the torsion invariants. Several classical conjectures from number theory – the Leopoldt conjecture and Iwasawa’s μ -invariant conjecture, for example – have interpretations in terms of the basic Iwasawa module M_∞ . Using a theorem of the author and Bill Dwyer, we show how to reinterpret these conjectures in terms of the homotopy-type of KR (Sects. 3.6.1, 3.6.2). In the case of totally real fields, these conjectures have an analytic interpretation also, in terms of ℓ -adic L-functions. In Sect. 3.6.3 we indicate how to make the connection between the algebraic and analytic points of view, and discuss the generalized Lichtenbaum conjecture.

3.6.1

Conjectures Concerning the Semi-discrete Primes

As a first step we consider the multiplicity of the extended Tate primes $\tau'_n = (\tau_n, n)$ in M .

42 Conjecture 42: (*Basic Conjecture I*)

$$\langle \tau'_n, M_\infty \rangle = \begin{cases} 0 & \text{if } n \neq 1 \\ s - 1 & \text{if } n = 1 \end{cases}$$

Note that since M_∞ has no nonzero finite submodules, for $n \neq 1$ the conjecture has the more transparent reformulation

$$\text{Hom}_{\Lambda'_F}(Z_\ell(n), M_\infty) = 0 .$$

In the case $n = 1$ it is known (see below) that

$$\text{rank}_{Z_\ell} \text{Hom}_{\Lambda'_F}(Z_\ell(n), M_\infty) = s - 1 .$$

Hence the content of the conjecture is that $e_1 M_\infty$ does not involve any elementary factors of the form Λ_F / τ_1^k for $k > 1$.

I don’t know where this conjecture was first formulated, although it seems to be standard; see [24]. For the case $n \neq 1$ it is equivalent to a conjecture in Galois (or étale) cohomology of Schneider [42]. For $n = 1$ it is an algebraic version of a conjecture of Gross [14]. In all cases there are analytic analogues of the conjecture for totally real fields, expressed in terms of special values of ℓ -adic L-functions. The link between the algebraic and analytic versions is provided by Iwasawa’s Main Conjecture as proved by Wiles [47], as will be discussed briefly below.

If we consider the Basic Conjecture for all the F_m 's at once, it can be formulated in another way. If S is a set of height one primes of Λ , and M, N are finitely-generated Λ -modules, write

$$M \sim_S N$$

if M and N have the same elementary summands at all primes in S . Equivalently, the torsion submodules are isomorphic after localizing at any prime in S . Let S_δ denote the set of semi-discrete primes in Λ , and let $\langle S_\delta \rangle$ denote the fit subset it generates.

Conjecture 43: (*Basic Conjecture II*)

43

$$M_\infty \sim_{\langle S_\delta \rangle} B_\infty(1) .$$

Equivalently, $(M_\infty(-n))^\delta = 0$ for $n \neq 1$, and $M_\infty(-1) \sim_{S_\delta} B_\infty$.

It is easy to see that Conjecture II is equivalent to Conjecture I for all the F_n 's. It can also be interpreted as giving the torsion invariants of M_∞ at the Tate twists of the semi-discrete primes; it says that v_i only occurs 1-twisted with length one, and with frequency determined in a simple way by the splitting behaviour of the primes over ℓ in the cyclotomic tower.

The conjecture has an equivalent formulation in terms of class groups:

Conjecture 44: (*Basic Conjecture III*)

44

For all $n \in \mathbb{Z}$, $\langle r'_n, A'_\infty \rangle = 0$. Equivalently (if we consider all F_m at once), for all $n \in \mathbb{Z}$ we have

$$(A'_\infty(-n))^\delta = 0 .$$

The equivalence of I–II with III is immediate from the formula for tM_∞ given in Corollary 35.

We will see that these conjectures incorporate versions of the Leopoldt and Gross conjectures, and that the case $n > 1$ has been proved by Soulé. First, however, we bring topological K -theory back into the picture.

The following theorem was proved by Bill Dwyer and the author in [9]; see also [33].

Theorem 45

45

$$\begin{aligned} \widehat{\mathcal{K}}^0 KR &\cong \Lambda' \otimes_{\Lambda'_F} Z_\ell \\ \widehat{\mathcal{K}}^{-1} KR &\cong \Lambda' \otimes_{\Lambda'_F} M_\infty \end{aligned}$$

This theorem yields a description of the homotopy-type $L_{K(1)}KR$, and hence conjecturally of KR itself. First of all – since we are working at an odd prime ℓ –

one can always find a certain residue field \mathbb{F} of R such that $KR \rightarrow K\mathbb{F}$ is a retraction at ℓ . More explicitly, one chooses \mathbb{F} so that if \mathbb{F}_∞ is the ℓ -adic cyclotomic extension of \mathbb{F} , then $G(\mathbb{F}_\infty/\mathbb{F}) = \Gamma'_F$. The existence of such residue fields follows from the Tchebotarev density theorem. Let $K^{red}R$ denote the fibre of this map – the *reduced K -theory spectrum* of R . Then $K^{red}R$ is the fibre of a map between wedges of copies of $\Sigma^{-1}\widehat{\mathcal{K}}$, where the map is given by a length one resolution of M_∞ as Λ'_F -module.

Theorem 45 also shows that the Basic Conjecture may be translated into a conjecture about the action of the Adams operations on $\widehat{\mathcal{K}}^{-1}KR$. It can also be formulated in terms of the homotopy groups of $L_{K(1)}KR$. For the case $n \neq 1$ we have:

46 **Proposition 46** For $n \neq 1$, the following are equivalent:

- a) The Basic Conjecture for n
- b) $\widehat{\mathcal{K}}^{2n-1}KR$ has no nonzero fixed points for the Adams operations.
- c) $\pi_{2n-2}L_{K(1)}KR$ is finite.

The equivalence of (a) and (b) is immediate from Theorem 45; note that (b) is just another way of saying that $\langle \tau'_n, \widehat{\mathcal{K}}^{-1}KR \rangle = 0$. The equivalence of (b) and (c) is immediate from the Adams spectral sequence; cf. Proposition 11. The case $n = 1$ will be discussed further below.

In terms of the category of $\widehat{\mathcal{K}}$ -finite spectra discussed in Sect. 3.3.5, we can reformulate the Basic Conjecture in its second version as follows: Fix a residue field \mathbb{F} of R as above. Now let β_1, \dots, β_s denote the primes over ℓ in R_0 , and let d_i denote the number of primes over β_i in R_∞ . Finally, let \mathbb{F}_i denote the extension of \mathbb{F} of degree d_i .

47 **Proposition 47** Assume for simplicity that $\mu_\ell \subset F$. Let A denote the complement of the set of Tate-twisted semi-discrete primes (in the set of height one primes of Λ'), and let $\mathcal{C} = \mathcal{C}_A$. Then Basic Conjecture II is equivalent to the existence of a mod \mathcal{C} equivalence

$$L_{K(1)}KR \sim_{\mathcal{C}} L_{K(1)} \left(K\mathbb{F} \vee \Sigma \overline{K\mathbb{F}}_1 \vee \Sigma K\mathbb{F}_2 \vee \dots \vee \Sigma K\mathbb{F}_s \vee \left(\bigvee^{\tau_2} \Sigma^{-1} \widehat{\mathcal{K}} \right) \right),$$

where $\overline{K\mathbb{F}}_1$ denotes the cofibre of the natural map $K\mathbb{F} \rightarrow K\mathbb{F}_1$. (The ordering of the β_i 's is immaterial.)

We next consider various special cases of the Basic Conjecture.

Soulé's Theorem: This is the case $n \geq 2$ of the Basic Conjecture, which was proved in [43]. A proof can be given as follows: It suffices to show that $\pi_{2n}L_{K(1)}KR$ is finite for $n \geq 1$. By theorems of Borel and Quillen the groups $\pi_{2n}KR$ are finite for $n \geq 1$, so if the Lichtenbaum–Quillen conjecture holds for R we are done. But in fact all

we need is that $\pi_{2n}KR$ maps onto $\pi_{2n}L_{K(1)}KR$, and then the surjectivity theorem of Dwyer and Friedlander [9] completes the proof.

The Leopoldt Conjecture. The Leopoldt conjecture asserts that the map $\phi : E' \rightarrow U'$ from global to local ℓ -adically completed units is injective. The kernel D of ϕ is the Leopoldt defect defined earlier. There are several interesting equivalent versions (see [38]):

Theorem 48 The following are equivalent: 48

- 1) The Leopoldt conjecture for F
- 2) $M_\infty^{F'} = 0$
- 3) F has exactly $r_2 + 1$ independent Z_ℓ -extensions

Note this is the case $n = 0$ of Basic Conjecture I.

In topological terms, we then have:

Proposition 49 The following are equivalent: 49

- 1) The Leopoldt conjecture for F
- 2) There are no nonzero fixed points for the action of the Adams operations on $\widehat{\mathcal{K}}^{-1}KR$.
- 3) $\pi_{-2}L_{K(1)}KR$ is finite.

Here we encounter a recurring and tantalizing paradox: On the one hand, the algebraic K -groups of R vanish by definition in negative degrees; on the face of it, then, they are useless for analyzing condition (3) above in the spirit of Soulé’s theorem. On the other hand, we know that $L_{K(1)}KR$ is determined by any of its connective covers, and hence if Lichtenbaum–Quillen conjecture holds it is determined, in principle, by KR . The problem lies in making this determination explicit. Smashing with a Moore spectrum $M\mathbb{Z}/\ell^v$ makes the homotopy groups periodic, thereby relating negative homotopy to positive homotopy, but it is difficult to get much mileage of out this.

The Gross Conjecture. The case $n = 1$ of Basic Conjecture I is an algebraic version of the Gross conjecture [14]. For simplicity we will assume $\mu_\ell \subset R$, so that $\Lambda'_F = \Lambda_F$. Since the original Gross conjecture concerns totally real fields, this might seem like a strange assumption, but in the version to be discussed here the assumption can be eliminated by a simple descent. The algebraic Gross conjecture says that τ_1 occurs with multiplicity $s - 1$ in M_∞ . The multiplicity is at least $s - 1$ by Corollary 35a. More precisely, we have:

Proposition 50 τ_1 occurs in M_∞ with frequency $s - 1$. In other words, 50

$$\text{rank}_{Z_\ell} \text{Hom}_{\Lambda_F}(Z_\ell(1), M_\infty) = s - 1 .$$

Proof Let ν denote the frequency. Using Theorem 45, Proposition 11 and Corollary 32, we have

$$\text{rank}_{Z_\ell} \pi_1 L_{K(1)} KR = r_2 + \nu .$$

But $\pi_1 L_{K(1)} KR \cong \pi_1 KR^\wedge \cong (R^\times)^\wedge$, and $(R^\times)^\wedge$ has rank $r_2 + s - 1$ as desired. (This is just Soulé’s étale cohomology argument translated into topological terms.)

Now the homotopy groups by themselves can only detect the frequencies of the τ_n ’s, not the multiplicities. Nevertheless, there is a curious homotopical interpretation of the Gross conjecture that we now explain.

Let ξ denote a generator of $\pi_{-1} L_{K(1)} S^0 \cong Z_\ell$, and recall that ξ has Adams filtration one.

51 Theorem 51 The Gross conjecture holds for R if and only if multiplication by ξ :

$$\pi_1 L_{K(1)} KR \rightarrow \pi_0 L_{K(1)} KR$$

has rank $s - 1$.

Proof The algebraic Gross conjecture holds for R if and only if

$$\text{rank}_{Z_\ell} \text{Hom}_{\Lambda_F} (M_\infty, \Lambda_F / \tau_1^2) = 2r_2 + s - 1 .$$

Now let X denote the cofibre of $\xi : L_{K(1)} S^{-1} \rightarrow L_{K(1)} S^0$. Then X can be described as the unique object of $L_{K(1)} \mathcal{S}$ with $\widehat{\mathcal{K}}^1 X = 0$ and $\widehat{\mathcal{K}}^0 X = \Lambda / T^2$ (with trivial Δ action). The usual Adams spectral sequence argument shows that the Z_ℓ -rank of $[\Sigma X, L_{K(1)} KR]$ is the same as the rank of the Hom term appearing in the lemma above. Now consider the exact sequence

$$\pi_2 L_{K(1)} KR \xrightarrow{\xi} \pi_1 L_{K(1)} KR \rightarrow [\Sigma X, L_{K(1)} KR] \rightarrow \pi_1 L_{K(1)} KR \xrightarrow{\xi} \pi_0 L_{K(1)} KR .$$

Since $\pi_2 L_{K(1)} KR \cong (K_2 R)^\wedge$ is finite, we see that $[\Sigma X, L_{K(1)} KR]$ has the desired rank $2r_2 + s - 1$ if and only if $\xi : \pi_1 L_{K(1)} KR \rightarrow \pi_0 L_{K(1)} KR$ has rank $s - 1$.

52 Remark 52 Recall here that $\pi_0 L_{K(1)} KR \cong K_0^{\text{ét}} R$, and that there is a short exact sequence

$$0 \rightarrow L \rightarrow \pi_0 L_{K(1)} KR \rightarrow Z_\ell \rightarrow 0$$

with $L \cong H_{\text{ét}}^2(R; Z_\ell(1))$ and $\text{rank}_{Z_\ell} L = s - 1$. Since ξ has Adams filtration one, the image of multiplication by ξ lies in L .

Iwasawa's μ -Invariant Conjecture

3.6.2

Recall that if M is a finitely-generated Λ -module, $\mu(M)$ denotes the multiplicity of the prime ℓ in the associated elementary module E ; in other words, it measures the number of Λ/ℓ^a 's occurring in E , weighted by the exponents a .

Conjecture 53: (*Iwasawa's μ -invariant conjecture*) For any number field F , $\mu(M_\infty) = 0$. Equivalently, $\mu(A_\infty) = 0$.

53

The A_∞ version of the conjecture was motivated by the analogy with curves over a finite field; see [38] or [13]. Note that since $M_\infty^0 = 0$, for M_∞ the conjecture is equivalent to the statement that M_∞ is ℓ -torsion-free.

Proposition 54 The following are equivalent:

54

- 1) Iwasawa's μ -invariant conjecture
- 2) $\widehat{\mathcal{K}}^{-1}KR$ is ℓ -torsion-free
- 3) (Here we assume $\mu_\ell \subset R$ for simplicity.) If \mathcal{C} denotes the thick subcategory of dualizable $K(1)$ -local spectra, $L_{K(1)}KR$ is equivalent mod \mathcal{C} to a wedge of r_2 copies of $\Sigma^{-1}\widehat{\mathcal{K}}$.

The equivalence of (1) and (2) is immediate from Theorem 45, together with the fact that M_∞ has no finite submodules. The equivalence of (2) and (3) is an easy consequence of Theorem 45, the fact that M_∞ has Λ -rank r_2 , and the characterization of dualizable objects in Sect. 3.3.5.

Totally Real Fields

3.6.3

When the number field F is totally real, the Basic Conjectures can be formulated in terms of special values of ℓ -adic L -functions. Since the connection is not always easy to extract from the literature, we give a brief discussion here. See [38] or [13] for further information.

Algebraic Aspects

On the algebraic side, the Iwasawa theory of totally real fields simplifies somewhat for the following reason. Let F be such a field, let F_0 denote as usual the extension obtained by adjoining the ℓ -th roots of unity, and let F_0^+ denote the fixed field of complex conjugation – that is, the unique element σ of order 2 in Δ_F – acting on F_0 . Then σ acts on the various Iwasawa modules M_∞, A'_∞ , etc. associated to F as above, and these split into ± 1 -eigenspaces: $M_\infty = M_\infty^+ \oplus M_\infty^-$, and so on. In terms of the idempotents e_i of $Z_\ell\Delta_F$, we are merely sorting the summands e_iM according to the parity of i . Now it is not hard to show that every unit of \mathcal{O}_0 is the product of a unit of \mathcal{O}_0^+ and root of unity. The next result then follows easily from Corollary 32.

55 Proposition 55 $E_{\infty}^{-} = (E'_{\infty})^{-} = Z_{\ell}(1)$, and therefore M_{∞}^{+} is a Λ_F -torsion module.

Here the second statement follows from Corollary 35; note the Tate twist there that reverses the ± 1 -eigenspaces of σ .

As an illustration we compare the Iwasawa modules tM_{∞}^{+} and A_{∞}^{-} . Define J_{∞} by the short exact sequence

$$0 \rightarrow J_{\infty} \rightarrow A_{\infty} \rightarrow A'_{\infty} \rightarrow 0,$$

and note that there is an exact sequence of Λ'_F -modules

$$0 \rightarrow E_{\infty} \rightarrow E'_{\infty} \rightarrow Z_{\ell}S_{\infty} \rightarrow J_{\infty} \rightarrow 0.$$

Taking (-1) -eigenspaces, the proposition yields:

56 Corollary 56 $(Z_{\ell}S_{\infty})^{-} \cong J_{\infty}^{-}$.

Now recall from Sect. 3.2.3 the divisor \mathcal{D} of a module, and note that \mathcal{D} is additive on short exact sequences. Hence we have the divisor equation

$$\mathcal{D}(A_{\infty}^{-}) = \mathcal{D}((A'_{\infty})^{-}) + \mathcal{D}(B_{\infty}^{-}).$$

Finally, for a Λ'_F -module N let \check{N} denote N with the twisted Γ'_F -action $\gamma \cdot x = c(\gamma)\gamma^{-1}x$; note this twist takes $Z_{\ell}(n)$ to $Z_{\ell}(1 - n)$. Then Corollary 35 yields the important fact:

57 Corollary 57 $\mathcal{D}(A_{\infty}^{-}) = \mathcal{D}(tM_{\infty}^{+})$.

A conjecture of Greenberg (see [13]) asserts that A_{∞}^{+} is finite. Assuming this conjecture, the contributions of the units and the class group to M_{∞} can be neatly separated into the $+$ and $-$ summands. In fact the torsion-free and torsion parts would then also separate into the $+$ and $-$ summands, except for the part coming from B_{∞} . Even without the Greenberg conjecture, this splitting into $+$ and $-$ summands is very useful; see for example the reflection principle as discussed in [46], Sect. 10, or [38], XI, Sect. 4.

L-functions

On the analytic side, the totally real fields are distinguished by their interesting ℓ -adic L -functions. Now an L -function typically involves a choice of Dirichlet character, or more generally a representation of the Galois group, so we emphasize from the outset that we are only going to consider a very special case: characters of Δ_F . These are the characters ω^i , where ω is the Teichmüller character and $0 \leq i \leq d$ ($d = [F_0 : F]$).

The following theorem was first proved for abelian fields F by Leopoldt and Kubota. It was proved in general, for arbitrary abelian L -functions, by Deligne and Ribet.

Theorem 58 Let F be a totally real field, and assume ℓ is odd. For each character $\chi = \omega^i$ of Δ_F with i even, there is a unique continuous function

$$L_\ell(s, \chi) : Z_\ell - \{1\} \rightarrow \mathbb{Q}_\ell$$

such that for all $n \geq 1$

$$L_\ell(1 - n, \chi) = L(1 - n, \chi \omega^{-n}) \prod_{\beta | \ell} (1 - \chi \omega^{-n}(\beta) N(\beta)^{n-1}).$$

Moreover, there are unique power series $g_i(T) \in \Lambda_F$ such that

$$L_\ell(1 - s, \chi) = \begin{cases} g_i(c_0^s - 1) & \text{if } i \neq 0 \\ g_i(c_0^s - 1)/(c_0^s - 1) & \text{if } i = 0. \end{cases}$$

Several remarks are in order. The L -function appearing on the right of the first equality is a classical complex L -function. The indicated values, however, are known to lie in $\mathbb{Q}(\mu_{\ell-1})$ and hence may be regarded as lying in \mathbb{Q}_ℓ . One could try to define L_ℓ for odd χ or arbitrary F by the same interpolation property, but then the classical L -function values on the righthand side would be zero; indeed the functional equation for such L -functions (see [37], pp. 126–7 for a short and clear overview of this equation) shows that for $n \geq 1$ $L(1 - n, \omega^j)$ is nonzero if and only if F is totally real and $j = n \pmod{2}$. Thus L_ℓ would be identically zero.

Turning to the Euler factors, recall that $N(\beta)$ is the cardinality of the associated residue field and that for any character χ , $\chi(\beta)$ is defined as follows: Let F^χ denote the fixed field of the kernel of χ , so that χ is pulled back from a faithful character of $G(F^\chi/F)$. If β is unramified in F^χ/F , we set $\chi(\beta) = \chi(\sigma_\beta)$, where σ_β is the associated Frobenius element. In practical terms, this means that $\chi(\beta) = 1$ if and only if β splits completely in F^χ . If β is ramified, we set $\chi(\beta) = 0$.

Iwasawa’s Main Conjecture – motivated by the analogy with curves over a finite field – predicted that the power series g_i were twisted versions of characteristic series for the appropriate eigenspaces of A_∞ . The conjecture was proved by Wiles [47], and can also be formulated in terms of M_∞ ; the two versions are related by Corollary 57.

Theorem 59 With the notation of the preceding theorem, the power series $g_i(T)$ is a characteristic series for the Λ_F -module $e_i M_\infty$.

Analytic Versions of the Basic Conjecture

Combining these two results, we can now translate Basic Conjecture I for F_0 into a statement about zeros of $L_\ell(s, \chi)$. The translation is not perfect; for example, as it stands we can only hope to get information about the even eigenspaces $e_i M_\infty$. If we assume Greenberg’s conjecture, however, the torsion in the odd eigenspaces all comes from $B_\infty(1)$. Let us consider case by case:

The case $n > 1$: Note that τ_n is in the support of $e_i M$ – or in other words, $Z_\ell(n)$ occurs in $e_i M$ – if and only if the characteristic series for $e_i M$ vanishes at $c_0^n - 1$. By Wiles theorem and the second part of the Deligne–Ribet theorem, this in turn is equivalent to the vanishing of $L_\ell(1 - n, \omega^i)$. Assuming Greenberg’s conjecture, we get the clean statement that Basic Conjecture I for $n > 1$ is equivalent to the nonvanishing of the $L_\ell(1 - n, \omega^i)$. Now observe that the Euler factors appearing in the definition of $L_\ell(s, \chi)$ are units in Z_ℓ when $n > 1$. Hence $L_\ell(1 - n, \omega^i) = 0$ if and only if $L(s, \chi\omega^{-n}) = 0$. But as noted above, for $n \geq 1$, $L(1 - n, \omega^j)$ is nonzero if and only if F is totally real and $j = n \pmod{2}$. Here we have $j = i - n$ with i even. This yields Basic Conjecture I for $n > 1$; that is, Soulé’s theorem – at least for the even eigenspaces $e_i M$.

The case $n = 1$: In this case a typical Euler factor has the form $1 - \chi\omega^{-1}(\mathcal{P})$, and hence will vanish precisely when β splits completely in $F^{\chi\omega^{-1}}$. Let m_χ denote the total number of such primes β . Note that the classical L -function factor does not vanish. Then it is natural to conjecture that $L_\ell(s, \chi)$ has a zero of order m_χ at $s = 0$. I will call this the analytic Gross conjecture, even though it is only a part of a special case of the general conjecture made by Gross in [14]; the general version not only considers much more general characters but also predicts the exact value of the leading coefficient. In view of Wiles’ theorem, we see at once that the analytic Gross conjecture is equivalent to the algebraic Gross conjecture given earlier, but restricted to the even eigenspaces $e_i M$. Once again, if we assume Greenberg’s conjecture, the algebraic Gross conjecture for the odd eigenspaces is automatic.

The case $n = 0$: Here we are just outside the range where $L_\ell(1 - n, \chi)$ is specified by Theorem 58. On the other hand, Wiles’ theorem tells us that if χ is nontrivial then $L_\ell(1, \chi)$ is defined, and is nonzero if and only if the algebraic Leopoldt conjecture holds for $e_i M$ ($\chi = \omega^i$). If $\chi = \omega^0$ is the trivial character, so that $L_\ell(s, \chi)$ is the ℓ -adic zeta function $\zeta_\ell(s)$, then a theorem of Colmez says that the algebraic Leopoldt conjecture for $e_0 M$ is equivalent to the assertion that $\zeta_\ell(s)$ has a simple pole at $s = 1$. Indeed this last equivalence would follow immediately from Wiles’ theorem, but Wiles’ proof uses Colmez’ theorem, so such an argument would be circular. In any case, we can now formulate an analytic Leopoldt conjecture – nonvanishing of $L_\ell(1, \chi)$ when χ is nontrivial, and the simple pole when χ is trivial – and the analytic form is equivalent to the algebraic Leopoldt conjecture for the even eigenspaces $e_i M_\infty$.

The case $n < 0$: Again we are outside the range where $L_\ell(1 - n, \chi)$ is specified. But in this case the classical values $L(1 - n, \chi)$ are obviously nonzero, by the Euler product formula. So (as far as I know) it is a reasonable conjecture that $L_\ell(1 - n, \chi)$ is also nonzero. Again, this corresponds to Basic Conjecture I for $n < 0$ and even eigenspaces.

The Generalized Lichtenbaum Conjecture

Finally, this is a good place to mention the generalized Lichtenbaum conjecture – or theorem, depending on which version of it is considered. Here is a version that

is known: For $x, y \in \mathbb{Q}_\ell$, write $x \sim_\ell y$ if $v_\ell x = v_\ell y$. Recall that $d = [F_0 : F]$, and let $e_i, 0 \leq i < d$, denote the idempotents in $Z_\ell[\Delta_F]$ corresponding to the characters ω^i .

Theorem 60 Let F be a totally real field, and assume ℓ is odd. Then for $n > 1$ and $i = n \bmod 2$,

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$$L(1 - n, \omega^i) \sim_\ell \frac{|e_{-i}\pi_{2n-2}L_{K(1)}KR_0|}{|e_{-i}\pi_{2n-1}L_{K(1)}KR_0|}.$$

Remark 61 If the Lichtenbaum–Quillen conjecture is true for R_0 , then the homotopy groups in the fraction can be replaced by the corresponding ℓ -completed K -groups of R_0 . These homotopy groups coincide with the étale cohomology groups $H_{\text{ét}}^2(R_0; Z_\ell(n))$ (the numerator) and $H_{\text{ét}}^1(R_0; Z_\ell(n))$ (the denominator). In this form, a more general version of Theorem 60 is given in [24].

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We sketch the proof, leaving the details to the reader. We use the abbreviation $\pi_m = \pi_m L_{K(1)}KR$. Of course we should first show that the groups appearing in the fraction are finite, so that the theorem makes sense. For the numerator this is clear from Soulé’s theorem, even before applying the idempotent e_{-i} . For the denominator we have:

Lemma 62

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$$e_{-i}\pi_{2n-1} \cong \begin{cases} Z_\ell/(c_0^n - 1) & \text{if } i + n = 0 \bmod d \\ 0 & \text{otherwise.} \end{cases}$$

The proof is similar to some of the arguments below, but easier. Now using Wiles’ theorem we have

$$L(1 - n, \omega^i) \sim_\ell L_\ell(1 - n, \omega^{i+n}) \sim_\ell \begin{cases} g_{i+n}(c_0^n - 1) & \text{if } i + n \neq 0 \bmod d \\ g_{i+n}(c_0^n - 1)/(c_0^n - 1) & \text{if } i + n = 0 \bmod d. \end{cases}$$

Furthermore, for any power series g prime to τ_n we have $g(c_0^n - 1) \sim_\ell |Z_\ell(n)g|$. The next lemma is an interesting exercise in Λ -modules:

Lemma 63 Suppose M, N are finitely-generated Λ -torsion modules with disjoint support, and $M^0 = 0 = N^0$. Then $\text{Ext}_\Lambda^1(M, N)$ is finite, and $|\text{Ext}_\Lambda^1(M, N)|$ depends only on the divisor of M . In fact, if f is a characteristic series for M ,

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$$|\text{Ext}_\Lambda^1(M, N)| = |N/fN|.$$

We conclude that

$$g_{i+n}(c_0^n - 1) \sim_\ell \left| \text{Ext}_{\Lambda_F}^1(e_{i+n}M_\infty, Z_\ell(n)) \right| = \left| e_{-i}\text{Ext}_{\Lambda_F}^1(M_\infty, Z_\ell(n)) \right| \\ = \left| e_{-i}\text{Ext}_{\Lambda'}^1(\widehat{\mathcal{K}} \cdot KR_0, \widehat{\mathcal{K}} \cdot S^{2n-2}) \right| = \left| e_{-i}\pi_{2n-2} \right|.$$

Taking into account the first lemma, Theorem 60 follows.

Note that taking $i = 0$ yields the more familiar formula for the Dedekind zeta function:

$$\zeta_F(1 - n) \sim_\ell \frac{|\pi_{2n-2}L_{K(1)}KR|}{|\pi_{2n-1}L_{K(1)}KR|}.$$

for $n > 1$ even. If we formulate the theorem in terms of the ℓ -adic L -function, we can conjecturally get values at positive integers also:

64 **Corollary 64** Let F be a totally real field, and assume ℓ is odd. Suppose also that Basic Conjecture I holds for F_0 . Then for all $n \neq 0, 1$ and $i = n \bmod 2$,

$$L_\ell(1 - n, \omega^{i+n}) \sim_\ell \frac{|e_{-i}\pi_{2n-2}L_{K(1)}KR_0|}{|e_{-i}\pi_{2n-1}L_{K(1)}KR_0|}.$$

The proof is the same as before. Note that we cannot take $n = 0$ because $e_0\pi_{-1}L_{K(1)}KR_0 \cong \pi_{-1}L_{K(1)}KR$ always contains a copy of Z_ℓ in Adams filtration one coming from $\pi_{-1}L_{K(1)}S^0$ under the unit map $S^0 \rightarrow KR$; this Z_ℓ is also detected by mapping to a suitable residue field. We cannot take $n = 1$ because the primes over ℓ contribute to the ranks of π_0 and π_1 .

3.7 The K-Theory of \mathbb{Z}

Since \mathbb{Q} is totally real, the material of the previous section applies to it. Furthermore, for any prime ℓ there is a unique prime over ℓ in \mathbb{Q}_∞ ; in other words, $s_\infty = 1$. Hence $B_\infty = 0$, and $A_\infty = A'_\infty = L_\infty$, simplifying the analysis further. Note also that $\Gamma'_\mathbb{Q} = \Gamma'$, and hence $\Lambda'_\mathbb{Q} = \Lambda'$.

Many of the conjectures mentioned above are known for abelian fields, and in particular \mathbb{Q} , where the proofs are often easier. For example, the Leopoldt and Gross conjectures, and Iwasawa's μ -invariant conjecture, are known for abelian fields. Iwasawa's Main Conjecture, which was proved by Mazur–Wiles in the abelian case and by Wiles in general, has a much easier proof for the case \mathbb{Q} ; see [46]. Furthermore the L -function values appearing in the generalized Lichtenbaum conjecture can be computed *a priori* in a more elementary form (see [46], Sect. 3.4):

65 **Proposition 65** $L(1 - n, \chi) = \frac{-B_{n,\chi}}{n}$, where $B_{n,\chi}$ is the n -th generalized Bernoulli number.

We recall that $B_{n,\omega^0} = B_n$; here B_n is the usual Bernoulli number, in the notation for which $B_n = 0$ if $n > 1$ is odd.

In the case $F = \mathbb{Q}$ there is an older and stronger version of the Greenberg conjecture, namely the Kummer–Vandiver conjecture. This conjecture asserts that for any prime ℓ , ℓ does not divide the order of the class group of \mathbb{Q}_0^+ . Recall that for ℓ odd, \mathbb{Q}_0 means \mathbb{Q} with the ℓ -th roots of unity adjoined, while the “+” means take the maximal real subfield. It is known to be true for all primes ℓ up to eight million or so. In our notation, the conjecture says that $A_0^+ = 0$.

Proposition 66 Suppose the Kummer–Vandiver conjecture holds for the prime ℓ . Then $A_n^+ = 0$ for all n , and hence $A_\infty^+ = 0$. 66

The proof is immediate, since $B_\infty = 0$ and $A_n = A'_n$ for all n ; hence $(A_\infty)_{\Gamma_n} = A_n$ for all n by Proposition 25a.

Theorem 67 Suppose the Kummer–Vandiver conjecture holds for ℓ . Then if M_∞ is the basic Iwasawa module associated to $R = \mathbb{Z}[1/\ell]$, we have 67

$$e_i M_\infty \cong \begin{cases} \Lambda & \text{if } i \text{ odd} \\ \Lambda/g_i & \text{if } i \text{ even} \end{cases}$$

where g_i is the power series associated to the ℓ -adic L -function $L_\ell(s, \omega^i)$ as in Theorem 58. Furthermore, $e_0 M_\infty = 0$.

Proof If i is odd, then the Kummer–Vandiver conjecture and the formula for tM given in Corollary 35 imply that M_∞^- is Λ -torsionfree of rank $r_2 \mathbb{Q}_0 = \frac{\ell-1}{2}$. Moreover, by Corollary 35 the freeness defect vanishes and hence M_∞^- is actually free of rank r_2 , and indeed M_∞^- is the twisted Λ -dual of E'_∞ . A standard argument with Dirichlet’s unit theorem then shows that all the odd characters occur; that is, $e_i M$ is free of rank one for each odd i .

If i is even, then $e_i M_\infty$ is Λ -torsion. The Main Conjecture then shows that $e_i M_\infty$ has characteristic series g_i , but in this case a more elementary argument proves a much stronger statement, namely that $e_i M_\infty \cong \Lambda/g_i$ ([46], Theorem 10.16; this step does not depend on the Kummer–Vandiver conjecture). When $i = 0$, g_0 is the numerator of the ℓ -adic zeta function, which has a simple pole at $s = 1$. Hence $g_0(0)$ is a unit in \mathbb{Z}_ℓ , g_0 is a unit in Λ , and $\Lambda/g_0 = 0$. Again, see [46] for further details.

Now write $B_m/m = c_m/d_m$, where c_m and d_m are relatively prime.

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Theorem 68 Assume the Lichtenbaum–Quillen conjecture and the Kummer–Vandiver conjecture at all primes. Then for $n \geq 2$ the K -groups of \mathbb{Z} are given as follows, where $n = 2m - 2$ or $n = 2m - 1$ as appropriate:

$n \bmod 8$	π_n
0	0
1	$\mathbb{Z} \oplus \mathbb{Z}/2$
2	$\mathbb{Z}/c_m \oplus \mathbb{Z}/2$
3	$\mathbb{Z}/4d_m$
4	0
5	\mathbb{Z}
6	\mathbb{Z}/c_m
7	$\mathbb{Z}/2d_m$

Proof The 2-primary part of this result will be discussed in the next section. So we assume ℓ is odd, and show that the theorem holds at ℓ .

Given the structure of M_∞ , and assuming the Lichtenbaum–Quillen conjecture, there are two ways to proceed. First, one can easily compute the étale cohomology groups occurring in the descent spectral sequence by using the universal coefficient spectral sequence

$$\text{Ext}_{\Lambda'}^p(H_q^{\text{ét}} R, Z_\ell(m)) \Rightarrow H_{\text{ét}}^{p+q}(R; Z_\ell(m)) ;$$

the descent spectral sequence collapses and one can read off the result from the Kummer–Vandiver conjecture. Alternatively, in the spirit of this paper, one can use the $\widehat{\mathcal{K}}$ -based Adams spectral sequence. This is what we will do; in any case the calculations involved are almost identical because in this simple situation the Adams and descent spectral sequences are practically the same thing.

Now we know that as Λ' -modules

$$\widehat{\mathcal{K}}^n K\mathbb{Z} \cong \begin{cases} M_\infty & \text{if } n = -1 \\ Z_\ell & \text{if } n = 0 . \end{cases}$$

Assuming the Lichtenbaum–Quillen conjecture, $K\mathbb{Z}^\wedge$ is essentially $K(1)$ -local, so we can use the $\widehat{\mathcal{K}}$ -based Adams spectral sequence to compute its homotopy. Displayed in its customary upper half-plane format, the Adams spectral sequence will have only two nonzero rows, namely filtrations zero and one. Hence the spectral sequence collapses, and there are no extensions because the bottom row is always Z_ℓ -torsionfree. First of all we have

$$E_2^{2m,0} = \text{Hom}_{\Lambda'}(Z_\ell, Z_\ell(m)) \cong \begin{cases} Z_\ell & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E_2^{2m+1,0} = \text{Hom}_{\Lambda'}(M_\infty(-1), Z_\ell(m)) \cong \begin{cases} Z_\ell & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd.} \end{cases}$$

The (-1) -twist occurs because we set up our Adams spectral sequence using $\widehat{\mathcal{K}}^\cdot$ where $\cdot = 0, 1$; since $\widehat{\mathcal{K}}^{-1}K\mathbb{Z}[\frac{1}{\ell}] = M_\infty$ we then have $\widehat{\mathcal{K}}^1K\mathbb{Z}[\frac{1}{\ell}] = M_\infty(-1)$. This accounts for all the non-torsion in the theorem. Next we have

$$E_2^{2m-1,1} = \text{Ext}_{\Lambda'}(Z_\ell, Z_\ell(m)) \cong \begin{cases} Z_\ell/(c_0^m - 1) & \text{if } m = 0 \text{ mod } \ell - 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall our convention that $c_0 = 1 + \ell$, although the choice doesn't really matter. The Clausen–von Staudt theorem then implies that the denominator of the indicated Bernoulli number and $((1 + \ell)^m - 1)$ have the same powers of ℓ . This accounts for everything in degrees $n = 3 \text{ mod } 4$, as well as the absence of ℓ -torsion in degrees $n = 1 \text{ mod } 4$. Now let m be even. Then

$$E_2^{2m,1} = \text{Ext}_{\Lambda'}^1(M_\infty(-1), Z_\ell(m)) = 0$$

because $e_i(M_\infty(-1))$ is Λ -free for i even. This proves the theorem in degrees $n = 0 \text{ mod } 4$. Finally,

$$E_2^{2m-2,1} = \text{Ext}_{\Lambda'}^1(\Lambda/g_m, Z_\ell(m)).$$

If $m = 0 \text{ mod } \ell - 1$, then $\Lambda/g_m = 0$. Since then ℓ does not divide c_m , by the Clausen–von Staudt theorem, the theorem is proved for this case. If $m \neq 0 \text{ mod } \ell - 1$, the Ext group above is isomorphic to $Z_\ell/g_m(c_0^m - 1)$, and hence is cyclic of order ℓ^v , where $v = v_\ell(g_m(c_0^m - 1))$. But

$$g_m(c_0^m - 1) \sim_\ell L_\ell(1 - m, \omega^m) \sim_\ell L(1 - m, \omega^0) = -\frac{B_m}{m}.$$

This completes the proof.

Theorem 68 has the following converse (cf. [25]):

Theorem 69 Suppose $(K_{4n}\mathbb{Z})_\ell = 0$ for $2n \leq \ell - 3$. Then the Kummer–Vandiver conjecture holds at ℓ .

Proof sketch: Suppose the Kummer–Vandiver conjecture fails at ℓ . Then for some even i , $0 \leq i \leq \ell - 3$, $e_i(M_\infty(-1))$ is not free (either because of a torsion submodule, or because the torsion-free part is not free). It follows that

$$E_2^{2m,1} = \text{Ext}_{\Lambda'}^1(e_i(M_\infty(-1)), Z_\ell(m)) \neq 0 \text{ for all } m = i \text{ mod } \ell - 1.$$

Taking $i = 2n$, we conclude that $(K_{4n}\mathbb{Z})_\ell \neq 0$. (Here we do not need the Lichtenbaum–Quillen conjecture, because of the Dwyer–Friedlander surjectivity theorem [9].)

We conclude by discussing the conjectural homotopy-type of $K\mathbb{Z}[\frac{1}{\ell}]$. Consider first $L_{K(1)}K\mathbb{Z}[\frac{1}{\ell}]$. For a Λ -module M , write $M[i]$ for the Λ' -module obtained by letting Δ act on M as ω^i . Then we have seen above that

$$\widehat{\mathcal{K}}^0 K\mathbb{Z}[\frac{1}{\ell}] \cong Z_\ell[0]$$

and

$$\widehat{\mathcal{K}}^{-1} K\mathbb{Z}[\frac{1}{\ell}] \cong \Lambda[1] \oplus \Lambda[3] \oplus \dots \oplus \Lambda[\ell - 2] \oplus (\Lambda/g_2)[2] \oplus \dots \oplus (\Lambda/g_{\ell-3})[\ell - 3].$$

Since these Λ' -modules have projective dimension one, $L_{K(1)}K\mathbb{Z}[\frac{1}{\ell}]$ splits into wedge summands corresponding to the indicated module summands. The $\widehat{\mathcal{K}}^0$ term contributes a copy of $L_{K(1)}S^0$. Each $\Lambda[i]$ contributes a desuspended Adams summand of $\widehat{\mathcal{K}}$; taken together, these free summands contribute a copy of ΣKO^\wedge . Finally, let X_i denote the fibre of $g_i : \Sigma^{-1}e_i\widehat{\mathcal{K}} \rightarrow \Sigma^{-1}e_i\widehat{\mathcal{K}}$. Then we have

$$L_{K(1)}K\mathbb{Z}[\frac{1}{\ell}] \cong L_{K(1)}S^0 \vee \Sigma KO^\wedge \vee X_2 \vee \dots \vee X_{\ell-3}.$$

Of course this description is only “explicit” to the extent that one knows the g_i ’s explicitly, at least up to units. But even our assumption of the Kummer–Vandiver conjecture does not yield this information. There is an auxiliary conjecture – see [46], Corollary 10.17 and its proof – that implies g_i is a unit times a certain linear distinguished polynomial. In any event, if the Lichtenbaum–Quillen conjecture holds at ℓ we conclude:

$$K\mathbb{Z}[\frac{1}{\ell}]^\wedge \cong j^\wedge \vee \Sigma bo^\wedge \vee Y_2 \vee \dots \vee Y_{\ell-3},$$

where j^\wedge is the completed connective J-spectrum, bo is the (-1) -connected cover of KO , and the Y ’s are the (-1) -connected covers of the X ’s.

The spectrum j^\wedge has the following algebraic model: Let p be any prime that generates the ℓ -adic units; such primes exist by Dirichlet’s theorem on arithmetic progressions. Then $j^\wedge \cong K\mathbb{F}_p^\wedge$, and the retraction map $K\mathbb{Z}[\frac{1}{\ell}]^\wedge \rightarrow j^\wedge$ in the conjectural equivalence above would correspond to the mod p reduction map in algebraic K -theory.

We remark also that when ℓ is a regular prime, all the X ’s and Y ’s are contractible. Even when ℓ is irregular, the proportion of nontrivial Y ’s tends to be low. For example, when $\ell = 37$ – the smallest irregular prime – there will be just one nontrivial summand Y_{32} , while for $\ell = 691$ – the first prime to appear in a Bernoulli numerator, if the Bernoulli numbers are ordered as usual – there are two: Y_{12}, Y_{200} . These assertions follow from the tables in [46], p. 350.

Homotopy-type of KR at the Prime 2

In this section we work at the prime 2 exclusively. If the number field F has at least one real embedding – and to avoid trivial exceptions, we will usually assume that it does – then R has infinite étale cohomological dimension for 2-torsion sheaves. This makes life harder. It is also known, however, that the higher cohomology all comes from the Galois cohomology of \mathbb{R} , and after isolating the contribution of the reals, one finds that life is not so hard after all.

On the topological side, the element of order 2 in Γ' causes trouble in a similar way. In particular, the ring of $\widehat{\mathcal{K}}$ -operations Λ' has infinite global dimension. Perhaps the best way around this problem would be to work with Bousfield’s united K-theory [5], which combines complex, real and self-conjugate K-theory so as to obtain, loosely speaking, a ring of operations with global dimension two. We hope to pursue this approach in a future paper, but we will not use united K-theory here.

The Construction JKR

We begin by turning back the clock two or three decades. After Quillen’s landmark work on the K-theory of finite fields, it was natural to speculate on the K-theory of \mathbb{Z} . The ranks of the groups were known, as well as various torsion subgroups at 2: (i) a cyclic subgroup in degrees $n = 7 \bmod 8$ coming from the image of the classical J -homomorphism in $\pi_* S^0$; (ii) a similar cyclic subgroup in degrees $n = 3 \bmod 8$, of order 16 and containing the image of J with index 2; and (iii) subgroups of order 2 in degrees $n = 1, 2 \bmod 8$, again coming from $\pi_* S^0$ and detected by the natural map $K\mathbb{Z} \rightarrow bo$.

The simplest guess compatible with this data is the following (I first heard this, or something like it, from Mark Mahowald): Define $JK\mathbb{Z}[\frac{1}{2}]$ by the following homotopy fibre square:

$$\begin{array}{ccc}
 JK\mathbb{Z}[\frac{1}{2}] & \rightarrow & bo^\wedge \\
 \downarrow & & \downarrow \\
 K\mathbb{Z}_3^\wedge & \xrightarrow{\theta} & bu^\wedge
 \end{array}$$

where θ is Quillen’s Brauer lift, as extended to spectra by May. Then the algebraic data was consistent with the conjecture that $K\mathbb{Z}[\frac{1}{2}]^\wedge \cong JK\mathbb{Z}[\frac{1}{2}]$. But on the face of it, there is not even an obvious map from $K\mathbb{Z}[\frac{1}{2}]^\wedge$ to $JK\mathbb{Z}[\frac{1}{2}]$. There are natural maps from $K\mathbb{Z}[\frac{1}{2}]$ to $K\mathbb{F}_3$ and bo , but no apparent reason why these maps should be homotopic when pushed into bu .

Nevertheless, Bökstedt showed in [2] that there is a natural map from $K\mathbb{Z}[\frac{1}{2}]^\wedge$ to $JK\mathbb{Z}[\frac{1}{2}]$. The clearest way to construct such a map is to appeal to the work of Suslin.

Choose an embedding of the 3-adic integers \mathbb{Z}_3 into \mathbb{C} , and form the commutative diagram of rings

$$\begin{array}{ccc} \mathbb{Z}[\frac{1}{2}] & \rightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{Z}_3 & \rightarrow & \mathbb{C} \end{array} .$$

Applying the completed K -theory functor yields a strictly commutative diagram of spectra

$$\begin{array}{ccc} K\mathbb{Z}[\frac{1}{2}]^\wedge & \rightarrow & K\mathbb{R}^\wedge \\ \downarrow & & \downarrow \\ K\mathbb{Z}_3^\wedge & \rightarrow & K\mathbb{C}^\wedge \end{array} .$$

But by the well-known work of Suslin, $K\mathbb{R}^\wedge$ is bo^\wedge , $K\mathbb{C}^\wedge$ is bu^\wedge , and there is a commutative diagram

$$\begin{array}{ccc} K\mathbb{Z}_3^\wedge & \rightarrow & K\mathbb{C}^\wedge \\ \cong \downarrow & \nearrow \theta & \\ K\mathbb{F}_3^\wedge & & \end{array} ,$$

in which the vertical map is an equivalence. Here the Brauer lift θ should be chosen to be compatible with the embedding of \mathbb{Z}_3 ; alternatively, one could even use the above diagram to define the Brauer lift. In any case, this yields the desired map $\phi : K\mathbb{Z}[\frac{1}{2}]^\wedge \rightarrow JK\mathbb{Z}[\frac{1}{2}]$.

Of course this map is not canonical, as it stands. *A priori*, it depends on the choice of embedding of the 3-adic integers and on the choice of Brauer lift θ . Furthermore, the choice of the prime 3 was arbitrary to begin with; in fact, one could replace 3 by any prime $p = \pm 3 \pmod{8}$. The alternatives 3 and $-3 \pmod{8}$ definitely yield nonequivalent spectra $K\mathbb{F}_p^\wedge$, but the homotopy-type of $JK\mathbb{Z}[\frac{1}{2}]$ is the same.

3.8.2 The 2-Adic Lichtenbaum–Quillen Conjecture for $\mathbb{Z}[1/2]$

Building on work of Voevodsky, Rognes and Weibel [41] proved the algebraic form of the 2-adic Lichtenbaum–Quillen conjecture for number rings; that is, they computed π_*KR^\wedge in terms of the étale cohomology groups $H_{\text{ét}}^*(R; \mathbb{Z}_2(n))$. In the case $R = \mathbb{Z}[\frac{1}{2}]$, the computation shows that $\pi_*K\mathbb{Z}[\frac{1}{2}]^\wedge \cong \pi_*JK\mathbb{Z}[\frac{1}{2}]$, but not that the isomorphism is induced by the map ϕ defined above. On the other hand, Bökstedt also showed that ϕ is surjective on homotopy groups. Since the groups in question are finitely-generated \mathbb{Z}_2 -modules, it follows that the naive guess is in fact true:

70 Theorem 70 There is a weak equivalence $K\mathbb{Z}[\frac{1}{2}]^\wedge \xrightarrow{\cong} JK\mathbb{Z}[\frac{1}{2}]$.

This result has many interesting consequences. For example, it leads immediately to a computation of the mod 2 homology of $GL\mathbb{Z}[\frac{1}{2}]$, since one can easily compute $H_*\Omega_0^\infty JK\mathbb{Z}[\frac{1}{2}]$; cf. [32]. Note this also gives the 2-primary part of Theorem 68. We also have:

Corollary 71 $K\mathbb{Z}[\frac{1}{2}]^\wedge$ is essentially $K(1)$ -local; in fact, $K\mathbb{Z}[\frac{1}{2}]^\wedge \rightarrow L_{K(1)}K\mathbb{Z}[\frac{1}{2}]$ induces an equivalence on (-1) -connected covers. 71

This is immediate since $JK\mathbb{Z}[\frac{1}{2}]$ has the stated properties.

Before stating the next corollary, we need to discuss the connective J -spectrum at 2. Perhaps the best definition is to simply take the (-1) -connected cover of $L_{\mathcal{K}}S^0$; or, for present purposes, the (-1) -connected cover of $L_{K(1)}S^0$. As noted earlier, for any $q = \pm 3 \pmod 8$ there is a noncanonical equivalence $L_{K(1)}S^0 \cong JO(q)^\wedge$, where $JO(q)$ is the fibre of $\psi^q - 1 : KO \rightarrow KO$. The (-1) -connected cover $jo(q)^\wedge$ has homotopy groups corresponding to the image of the classical J -homomorphism and the Adams μ -family in degrees $n = 1, 2 \pmod 8$, with one small discrepancy: There are extra $\mathbb{Z}/2$ summands in π_0 and π_1 . These elements can be eliminated via suitable fibrations, although there is no real need to do so. Nevertheless, to be consistent with received notation we will let \tilde{j} denote the (-1) -connected cover of $L_{K(1)}S^0$, while j^\wedge will denote the 2-adic completion of the traditional connective j -spectrum, in which the two spurious $\mathbb{Z}/2$'s have been eliminated. These are both ring spectra, and the natural map $j^\wedge \rightarrow \tilde{j}$ is a map of ring spectra.

We remark that these spectra have interesting descriptions in terms of algebraic K -theory: We have $\tilde{j} \cong KO\mathbb{F}_q^\wedge$, where $KO\mathbb{F}_q$ is the K -theory spectrum associated to the category of nondegenerate quadratic spaces over \mathbb{F}_q ([11], p. 84ff and p. 176ff), while $j^\wedge \cong KND\mathbb{F}_q^\wedge$, where ND refers to the property “determinant times spinor norm equals 1” ([11], p. 68).

Corollary 72 Let X be any module spectrum over $K\mathbb{Z}[\frac{1}{2}]^\wedge$. For example, X could be the completed K -theory spectrum of any $\mathbb{Z}[\frac{1}{2}]$ -algebra (not necessarily commutative), or scheme over $Spec\mathbb{Z}[\frac{1}{2}]$, or category of coherent sheaves over such a scheme. 72

Then X has a natural module structure over the 2-adic connective J -spectrum j^\wedge ; in fact, it has a natural module structure over \tilde{j} .

This result follows from the previous corollary. By taking connective covers in the diagram

$$\begin{array}{ccc} S^0 & \rightarrow & L_{K(1)}K\mathbb{Z}[\frac{1}{2}] \\ \downarrow & \nearrow & \\ L_{K(1)}S^0 & & \end{array}$$

we get unique maps of ring spectra $j^\wedge \rightarrow K\mathbb{Z}[\frac{1}{2}]^\wedge$ or $\tilde{j} \rightarrow K\mathbb{Z}[\frac{1}{2}]^\wedge$ factoring the unit map.

73 **Corollary 73** After localization at 2, the induced homomorphism $\pi_* S^0 \rightarrow K_* \mathbb{Z}$ factors through $\pi_* j^\wedge$.

This corollary was proved for all ℓ in [29], by a completely different method independent of the Lichtenbaum–Quillen conjecture. It is also known that the homotopy of j^\wedge injects into $K_* \mathbb{Z}$, with only the following exception:

74 **Corollary 74** Let $Im J_n \subset \pi_n S^0$ denote the image of the classical J -homomorphism. Then for $n > 1$ and $n = 0, 1 \pmod 8$, $Im J_n$ maps to zero in $K_n \mathbb{Z}$.

This is immediate since $\pi_{8k} JK\mathbb{Z} = 0$ for $k > 0$, and $Im J_{8k+1} = \eta Im J_{8k}$. As far as I know, the only other proof of this fact is the original proof of Waldhausen [45].

The 2-Adic Lichtenbaum–Quillen Conjecture for General \mathbb{R}

3.8.3

In the general case a more sophisticated, systematic construction is required. One approach is the étale K -theory spectrum of Dwyer and Friedlander [7]. Their spectrum $K_{\text{ét}} R$ can be thought of as a kind of twisted bu^\wedge -valued function spectrum on $Spec R$, where the latter is thought of as a space in its étale topology. In fact this “space” is in essence the classifying space of the Galois group $G(\Omega/F)$, where Ω is the maximal algebraic extension of F unramified away from 2; the only subtlety is that one must take into account the profinite topology on this Galois group.

Consider first the case $R = \mathbb{Z}[\frac{1}{2}]$. Then the 2-adic cohomological type of $Spec \mathbb{Z}[\frac{1}{2}]$ turns out to be very simple [8]. Define a homomorphism ξ from the free product $(\mathbb{Z}/2) * \mathbb{Z}$ to $G(\Omega/F)$ by sending the involution to complex conjugation and sending the free generator to any lift of $\gamma_q \in \Gamma_{\mathbb{Q}}$, where $q = \pm 3 \pmod 8$ as discussed above. This yields a map of classifying spaces

$$B\xi : \mathbb{R}P^\infty \vee S^1 \rightarrow \left(Spec \mathbb{Z}[\frac{1}{2}] \right)_{\text{ét}}$$

inducing an isomorphism on cohomology with locally constant 2-torsion coefficients. Hence we can replace $(Spec \mathbb{Z}[\frac{1}{2}])_{\text{ét}}$ with $\mathbb{R}P^\infty \vee S^1$ as the domain of our twisted function spectrum. The upshot is that up to connective covers there is a homotopy fibre square (suppressing 2-adic completions)

$$\begin{array}{ccc} K_{\text{ét}} \mathbb{Z}[\frac{1}{2}] & \rightarrow & bu^{h\mathbb{Z}/2} \\ \downarrow & & \downarrow \\ bu^{h\mathbb{Z}} & \rightarrow & bu \end{array}$$

corresponding to the pushout diagram of spaces

$$\begin{array}{ccc} * & \rightarrow & \mathbb{R}P^\infty \\ \downarrow & & \downarrow \\ S^1 & \rightarrow & \mathbb{R}P^\infty \vee S^1 . \end{array}$$

Here $(-)^{hG}$ denotes the homotopy-fixed point construction $F^G(EG, -)$. Since $-$ up to connective covers $-$ we have $bu^{h\mathbb{Z}/2} = bo$ and $bu^{h\mathbb{Z}} = K\mathbb{F}_q$, we recover the spectrum $JK\mathbb{Z}[\frac{1}{2}]$.

For general R the cohomological type of $(Spec R)_{\acute{e}t}$ is not so simple; we have to just take $K_{\acute{e}t}R$ as it comes.

Another approach, due in different versions to Thomason and Jardine, starts from algebraic K -theory as a presheaf of spectra on the Grothendieck site $(Spec R)_{\acute{e}t}$. This leads to the étale hypercohomology spectrum $\mathbb{H}_{\acute{e}t}(Spec R; \underline{K})$ discussed in Sect. 3.5.

The strong form of the Lichtenbaum–Quillen conjecture was proved by Rognes–Weibel for purely imaginary fields, and by Østvær [39] for fields with a real embedding:

Theorem 75 The natural map $KR^\wedge \rightarrow \mathbb{H}_{\acute{e}t}(Spec R; \underline{K})^\wedge$ induces a weak equivalence on 0-connected covers. **75**

As before, we get the corollary:

Corollary 76 KR^\wedge is essentially $K(1)$ -local. In fact, the natural map $KR^\wedge \rightarrow L_{K(1)}KR$ induces an equivalence on 0-connected covers. **76**

The map on π_0 in the corollary is injective, but gives an isomorphism if and only if there is a unique prime dividing 2 in \mathcal{O}_F .

To analyze the homotopy-type of KR^\wedge , then, it suffices to analyze the homotopy-type of $L_{K(1)}KR = L_{K(1)}\mathbb{H}_{\acute{e}t}(Spec R; \underline{K})^\wedge$. Now the topological K -theory of KR is given by the same formula as before:

$$\widehat{\mathcal{K}}^i KR \cong \begin{cases} \Lambda' \otimes_{\Lambda'_F} M_\infty & \text{if } i = -1 \\ \Lambda' \otimes_{\Lambda'_F} \mathbb{Z}_2 & \text{if } i = -0. \end{cases}$$

The difference is that when F has at least one real embedding, both $\widehat{\mathcal{K}}^0$ and $\widehat{\mathcal{K}}^{-1}$ have infinite projective dimension as Λ' -modules, which complicates the analysis. To get around this we isolate the real embeddings, since these are the source of the homological difficulties. Define $K^{rel}R$ by the fibre sequence

$$K^{rel}R \rightarrow KR \rightarrow \prod_{i=1}^{r_1} bo.$$

We can then compute the topological K -theory of the relative term. Although the basic Iwasawa module M_∞ will have infinite projective dimension, it fits into a canonical short exact sequence of Λ'_F -modules

$$0 \rightarrow M_\infty \rightarrow N_\infty \rightarrow \Lambda_F^{r_1} \rightarrow 0,$$

where N_∞ has projective dimension one as Λ'_F -module. We then have:

Theorem 77

$$\widehat{\mathcal{K}}^i K^{rel} R \cong \begin{cases} \Lambda' \otimes_{\Lambda'_F} N_\infty & \text{if } i = -1 \\ 0 & \text{if } i = -0 \end{cases}$$

Then the homotopy-type of $K^{rel} R$ is completely determined by the Λ'_F -module N_∞ , in exactly the same way that the homotopy-type of $K^{red} R$ is completely determined by M_∞ in the odd-primary case. Finally, one can explicitly compute the connecting map $\prod^{r_1} bo \rightarrow \Sigma K^{rel} R$, yielding a complete description of the homotopy-type of KR .

Theorem 77 cannot be proved from the fibre sequence defining $K^{rel} R$ alone; this only gives $\widehat{\mathcal{K}}^* K^{rel} R$ up to an extension, and it is essential to determine this extension explicitly. The proof makes use of an auxiliary Grothendieck site associated to $Spec R$, defined by Zink [48]. In effect, one partially compactifies $Spec R$ by adjoining the real places as points at infinity. Up to connective covers $K^{rel} R$ is the hypercohomology of a relative K -theory presheaf on the Zink site, and this description leads to the computation above.

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