# ON THE CANONICAL ELEMENT CONJECTURE II

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In [H] M. Hochster has assigned a canonical element $\eta_A \in H^n_m(\mathrm{syz}^n(K))$ with each $n$-deminsional local ring $(A, m, K)$ (i.e., commutative noetherian local ring $A$ with a unique maximal ideal $m$ and $K = A/m$) and has conjectured $\eta_A \neq 0$ for every such $A$. In elementary terms this conjecture means the following [H]:

For every free resolution $F$

$$\to A^{s_i} \to A^{s_{i-1}} \to \cdots \to A^{s_0} \to K \to 0$$

of $K$ and for every system of parameters $x_1, \cdots, x_n$ of $A$, if $\phi$ is any map of complexes $K_\bullet(x; A) \to F$ which lifts the quotient surjection $A/(x_1, \ldots, x_n) \to K$, then $\phi_n : K_n(X; A) \to A^{s_n}$ is non-zero.

Hochster has shown that the above conjecture (henceforth C.E.C.) is equivalent to the direct summand conjecture ([H]). Moreover his work together with the work done by this author [D1] establishes the fact that the C.E.C. is equivalent to the improved new intersection conjecture. For several other implications (and equivalences) of the C.E.C. the reader is referred to [H] and [D1]. The equicharacteristic case of the conjecture was proved by Hochster in [H] and some special cases were established in [D1] and [D2].

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The central idea of this paper is to a great degree a follow-up of the idea evolved in [D1]. Hence, for convenience of the reader, in section 1, we put forth some of the main results of [D1] and a theorem of Schenzel which will be used in the sequel.

In section 2, we prove our main two theorems. Without any loss of generality we can assume $A$ is a complete normal domain.

**Theorem 1** (2.5). *Let $J_i = \operatorname{ann}_A H_m^{n-i}(A)$, and let $I = J_1 J_2 \cdots J_r$. Choose any $x \in mI$. Then $A/xA$ satisfies the C.E.C.*

Theorem 1 shows that even in the mixed characteristic for every positive integer $n$, there are plenty of local rings of dimension $n$ satisfying the C.E.C. Moreover it also made me raise the following question whose answer I still do not know:

**Question.** Given a local ring $(A, m, k)$, can we find a finite injective map $f : A \to B$, $B$ is a local ring such that $B \simeq S/yS$ where $S$ is local, $y \neq 0$ and $y \in \prod_{i<\dim S} \operatorname{ann} H_m^i(S)$? When $A$ is Cohen-Macaulay on the punctured spectrum, the question really boils down to whether $S$ is also the same?

**Theorem 2** (2.6). *Let $x_n \in I$, $I$ as in Theorem 1. Write $B = A[[X]]/(X^2)$. Then for any system of parameters $x_1, \ldots, x_{n-1}$, $x_n$ in $A$, $x_1^t$, $x_2^t, \ldots, x_{n-1}^t$, $x_n^t + x$ satisfies the C.E.C. in $B$ (here $x$ denotes the image of $X$ in $B$) for every positive integer $t$.*

Recall that in order to prove the C.E.C. it is enough to choose one system of parameters $x_1, \ldots, x_n$ and show that $x_1^t, \ldots, x_n^t$ satisfies the conjecture for every positive integer $t$.

Up until this time, I was getting several results using induction on $\dim A$. Now I am getting direct proofs (without using any induction). The most notable is the following result:

**Theorem 3** (3.2). *Write $S_i$ to denote the ith syzygy of $K$ in a minimal free resolution of $K$. Then $\theta_i : \operatorname{Ext}^i(K, S_i) \to H^i_m(S_i)$ is non-zero, for $0 \le i < \dim A$.*

Recall that the conjecture asserts $\theta_n$ is non-zero, when $n = \dim A$.

We will use the above notations throughout the rest of the paper.

## Section 1

We put forward several results from [D1] and a theorem of Shenzel which will be used in section 2 and section 3. For their proofs we refer to [D1], [D2] and [Sc].

**1.1 Proposition.** *Let $F_\bullet :\to A^{s_i} \to A^{s_{i-1}} \to \cdots \to A^{s_0} \to 0$ be a free complex of finitely generated modules with $H_0(F_\bullet) = M$. Let $N \subset M$ be a submodule of $M$. Then we can construct a free complex $L_\bullet :\to A^{d_i} \to A^{d_{i-1}} \to \cdots \to A^{d_0} \to 0$ with $H_0(L_\bullet) = N$ and a map $\phi_\bullet : L_\bullet \to F_\bullet$ such that the mapping cone of $\phi_\bullet$ is a free resolution of $M/N$. Moreover, in this case $H_i(L_\bullet) \simeq H_i(F_\bullet)$ for $i > 0$. When $F_\bullet$ is minimal, so is $L_\bullet$.*

For a proof see [D1].

**1.2**. Let $x_1, \ldots, x_n$ be a system of parameters (henceforth s.o.p.) of $A$: Write $K_\bullet(\underline{x}; A)$ to denote the corresponding Koszul complex, $\{d_i\}$ to denote the boundary maps, $H_i$ to denote $H_i(\underline{x}; A)$ and $G_1 = \operatorname{coker} d_2$. Denote the truncated Koszul complex of $K_\bullet(\underline{x}; A)$

$$\cdots \to A^{\binom{n}{i}} \xrightarrow{d_i} A^{\binom{n}{i-1}} \to \cdots \to A^{\binom{n}{2}} \xrightarrow{d_2} A^n \to 0$$

by $K'_\bullet(\underline{x}; A)$. Note $H_1 \subset G_1$ and $H_0(K'_\bullet) = G_1$. Then by the above proposition we have a free complex $L_\bullet$ with $H_0(L_\bullet) = H_1$ and a map $\phi_\bullet : L_\bullet \to K'_\bullet$ such that the mapping cone of $\phi_\bullet$ is a free resolution of $\underline{x}A$ (here $\underline{x}A$ stands for the ideal $(x_n, \ldots, x_n)$ in $A$).

**Theorem.** *$\{x_n, \ldots, x_n\}$ satisfies the C.E.C.* $\Leftrightarrow \operatorname{Im}\phi_n \neq A$.

For a proof see [D1].

**1.3**. We consider the Koszul complex $K(x_1, \ldots, x_{n-1}; A)$ which we abbreviate as $K_\bullet(\underline{x}_{n-1}; A)$. Write $H = (0 : x_n)A/(x_1, \ldots, x_{n-1})$. By proposition (1.1) we construct a free complex $P_\bullet$ and map $\psi_\bullet : P_\bullet \to K_\bullet(\underline{x}_{n-1}; A)$ such that (i) $H_0(P_\bullet) = H$, (ii) $\tilde{\psi}_i : H_i(P_\bullet) \to H_i(K_\bullet(\underline{x}_{n-i}; A))$ is an isomorphism for $i > 0$; for $i = 0$, it is the natural injection.

**Theorem.** *$\{x_1, \ldots, x_n\}$ satisfies C.E.C.* $\Leftrightarrow \operatorname{Im}\psi_{n-1} \neq A$.

For a proof see [D2].

**1.4.** Let $x_1, \ldots, x_n$ be a s.o.p. of $A$.

We consider the following quotients $(x_1^t, \ldots, x_{i-1}^t) : x_i^t/(x_1^t, \ldots, x_{i-1}^t)$ for $i = 1, 2, \ldots, n$ and $t \geq 1$. We define $r_x(A)$ to be the intersection of the annihilators of all these modules for $i = 1, \ldots, n$ and $t \geq 1$. Let $J_i$ denote the annihilator of $H_m^{n-i}(A)$, and let $I = J_1 J_2 \cdots J_r$, where $n - r =$ depth $A$. The theorem below is due to P. Schenzel.

**Theorem.** *With the notations as above. We have the following conclusions*

*(a)* $I \subset r_x(A)$, *and*

*(b)* $r_x(A)^t \subseteq J_1 \cap J_2 \cap \cdots \cap J_r$ *for* $t = \binom{n}{[n/2]}$.

For a proof see [Sc].

## Section 2

We assume, without any loss of generality that $A$ is a complet local normal demain of dimension $n(> 2)$, depth $A = n - r$, $r > 0$. The notations are the same as in section 1.

Since $A$ is normal, $A_P$ is Cohen-Macaulay for every prime ideal $P$ of height $\leq 2$ and this in turn implies $ht\, J_i \geq 3$. Hence $ht\, I \geq 3$.

**(2.1) Lemma.** *Let $x_1, \ldots, x_n$ be a s.o.p. such that $x_n \in I$. Then we have the following exact sequences*

$$(1) \qquad 0 \to H_i(x_1, \ldots, x_{n-1}) \to H_i(x_1, \ldots, x_n) \to H_{i-1}(x_1, \ldots, x_{n-1}) \to 0$$

*for all $i > 1$ and*

$$(2) \qquad 0 \to H_1(x_1, \ldots, x_{n-1}) \to H_1(x_1, \ldots, x_n) \to (0 : x_n)A/\underline{x}_{n-1} \to 0$$

*for $i = 1$. (We drop $A$ from the notation $H_i(x_1, \ldots, x_t; A)$ and $\underline{x}_{n-1}$ denotes the ideal $(x_1, \ldots, x_{n-1})A$).*

*Proof.* By Theorem (1.4) $x_n$ kills all $H_i(x_1, \ldots, x_t)$ for $i \geq 1$ and $t \leq n$. Hence we get the above result from the following exact sequence:

$$0 \to \frac{H_i(x_1, \ldots, x_{n-1})}{x_n H_i(x_1, \ldots, x_{n-1})} \to H_i(x_1, \ldots, x_n) \to (0 : x_n)H_{i-1}(x_1, \ldots, x_{n-1}) \to 0$$

for $i \geq 1$.

**Corollary.** *If $x_n \in I$, then $\ell(H_i(x_1, \ldots, x_{n-1})) < \infty$ for $i \geq 1$.*

**(2.2) Proposition.** *Let $F_\bullet$, $V_\bullet$ be two complexes of finitely generated free modules and let $\phi_\bullet : F_\bullet \to V_\bullet[d - n + 2]$ (shift of $V_\bullet$ by $d - n + 2$, $d > n - 2$) be a map. Assume $H_0(F_\bullet) \neq 0$, $\ell(H_i(F_\bullet)) < \infty$ for $i \geq 0$ and $V_{d+i} = 0$ for $i \geq 0$. then* $\operatorname{Im} \phi_{n-2} \subset m \cdot V_d$.

*Proof.* We have the following commutative diagram:

$$\begin{array}{ccccccccccc}
F_{n-1} & \xrightarrow{d_{n-1}} & F_{n-2} & \xrightarrow{d_{n-2}} & F_{n-3} & \to \cdots \to & F_1 & \xrightarrow{d_1} & F_0 & \to 0 \\
\downarrow & & \downarrow{\scriptstyle \phi_{n-2}} & & \downarrow{\scriptstyle \phi_{n-3}} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & \\
0 & \to & V_d & \to & V_{d-1} & \to \cdots \to & V_{d-n+3} & \to & V_{d-n+2} & \to
\end{array}$$

Let $e_1, \ldots, e_t$ be a basis of $F_{n-2}$. If $\operatorname{Im} \phi_{n-2} \not\subset mV_d$, we can assume without any loss of generality $\phi_{n-2}(e_1)$ is a part of a basis of $V_d$. Then we can make a linear change of coordinates in $F_{n-2}$ so that $\phi_{n-2}(e_1)$-th component of $\phi_{n-2}(e_i)$ in $G_d$ is 0. This implies that $d_{n-1}(F_{n-1}) \subset$ free submodule generated by $e_2, \ldots, e_t$. Hence in $\operatorname{coker} d^*_{n-2}$, $e^*_1$ is killed by some power of the maximal ideal.

Let us consider the complex

$$0 \to F^*_0 \to F^*_1 \to \ldots \to F^*_{n-3} \to F^*_{n-2} \to 0$$

$$[F^*_i = \operatorname{Hom}_A(F_i, A)]$$

We denote it by $L_\bullet$, writing $L_i = F^*_{n-2-i}$. Then $\ell(H_i(L_\bullet)) < \infty$ for $i > 0$ and $M = H_0(L_\bullet)$ has a minimal generator killed by a power of $m$.

Now we consider the complex $\overline{L}_\bullet = L_\bullet \otimes_A A/pA$ where $p =$ characteristic of $A/m$. Write $\overline{M} = M/pM$. It is clear then that $\ell(H_i(\overline{L}_\bullet)) < \infty$ for $i > 0$, and $H_0(\overline{L}_\bullet) = \overline{M}$ has a minimal generator killed by some power of the maximal idel of $\overline{A}(= A/pA)$. But this contradicts the fact that the C.E.C. holds in $\overline{A}$. Thus $\operatorname{Im} \phi_{n-2} \subset mV_d$.

**Corollary.** (Notation from (1.2)) *Any minimal free resolution $T_\bullet$ of $G_1$ will contain a copy of $K_n(\underline{x}; A)(= A)$ as a component of $T_{n-1}$.*

*Proof.* We consider the complex $K'_\bullet(\underline{x}; A)(1.2)$; and write it as $K'_\bullet$. Recall $H_i(K'_\bullet) = H_i(K'_\bullet) = H_{i+1}$ for $i > 0$ and $H_0(K'_\bullet) = G_1$. We construct a free complex $L_\bullet$ (as in (1.1)) and a map $\phi_\bullet : L_\bullet \to K'_\bullet[1]$ such that (i) $H_0(L_\bullet) = H_2$, (ii) $\tilde{\phi}_i$ induces an isomorphism between $H_i(L_\bullet)$ and $H_i(K'_\bullet[1])$ for $i > 0$ and $\tilde{\phi}_0$ is the natural injection $H_2 \hookrightarrow \operatorname{coker} d_2$. Then, by the above proposition, $\operatorname{Im} \phi_{n-2} \subset mA$. Hence when we extract a minimal resolution $T_\bullet$ of $G_1$ from the mapping cone of $\phi_\bullet$, the copy of $K_n(\underline{x}; A)(= A)$ survives in $T_\bullet$ as a component of $T_{n-1}$.

Let us denote the copy by $K_n$. We have the following theorem (with the above notations).

**Theorem.** *Let $F_\bullet$ denote a minimal free resolution of $H_1$ and let $\psi_\bullet : F_\bullet \to T_\bullet$ lift eh natural injection $H_1 \hookrightarrow G_1$. Then $x_1, \dots, x_n$ satisfirs the* C. E. C. *if and only if the projection of* $\operatorname{Im} \psi_{n-1}$ *onto $K_n$ is contained in $mK_n$.*

*Proof.* The proof follows easily from Theorem (1.2) and the above corollary. We note that Theorem (1.2) implies the fact that the C. E. C. holds for $x_1, \dots, x_n$ if and only if $K_n = K_n(\underline{x}; A)(= A)$ survives in a free minimal resolution of $A/\underline{x}$.

**2.3** We now state a result of Shamash and derive a corollary to fit our purpose.

**Theorem** (Shamash). *Let $(F_\bullet, d_\bullet)$ be a minimal free resultion of a finitely generated module $M$. Let $x \in m(\operatorname{ann}_A M)$ be a non-zero element. Then there exists a family of endomorphisms $\{c_n\}_{n\geq}$ of $(F_\bullet, d_\bullet)$ (regarded as a differential graded module) such that*

*(i) degree of $c_n = 2n-1$, $c_0 = d_\bullet$.*

*(ii) $c_0c_1 + c_1c_0 = x$, $\sum_{i=0}^{n} c_i c_{n-i} = 0$ for $n \geq 2$*

*(iii)* $\operatorname{Im} c_n \subset m^n F_\bullet$

*and (iv) $M$ has a minimal free resolution over $\overline{A}(= A/xA)$ given by $(G_\bullet, d_\bullet)$ where $G_i = \overline{A}^{t_i} \oplus \overline{A}^{t_{i-2}} \oplus \cdots \oplus A^{t_{i-2k}} \oplus \dots$, and if $v = (v_k)_{k\geq 0} \in G_i$ then $d_i(v) = \sum_{k\geq 0} \left( \sum_{0\leq j\leq k} \overline{c_j(v_k)} \right)$ ("$-$" denotes the image modulo $x$).*

**Corollary.** (Notations as above) *Write $\overline{F}_\bullet = F_\bullet \otimes \overline{A}$. Then $H_0(\overline{F}_\bullet) = M$, $H_1(\overline{F}_\bullet) = M$ and $H_i(\overline{F}_\bullet) = 0$ for $i > 1$; Moreover $\exists$ a map $\alpha_\bullet : (G_\bullet, d_\bullet) \to (\overline{F}_\bullet, -\overline{d}_\bullet)[1]$ such that $\alpha_0$ induces the identity map between $H_0(G_\bullet)$ and $H_1(\overline{F}_\bullet)$ and* $\operatorname{Im} \alpha_\bullet \subset m\overline{F}_\bullet$.

We define $\alpha_i((v_k)) = \sum \overline{c_k(v_k)}$. The assertion now follows from (i), (ii) and (iii) above.

**2.4 Proposition.** *Let $x_1, \ldots, x_{n-1}, x_n$ be a s.o.p. for $A$. Suppose $\ell(H_i(x_1, \ldots, x_{n-1})) < \infty$ for $\forall i \geq 1$. Write $H = (0 : x_n)A/\underline{x}_{n-1}$. Then $A = K_{n-1}(x_1, \ldots, x_{n-1}; A)$ survives in a minimal free resolution $L_\bullet$ of $A/(x_1, \ldots, x_{n-1})$. If $x_n$ is such that $x_n \in m \operatorname{ann}_A H$, then $\overline{x}_1, \ldots, \overline{x}_{n-1}$ satisfies the* C.E.C. *over $\overline{A}(= A/x_nA)$. (Here $\overline{x}_i$ denotes the image of $x_i$ in $\overline{A}$).*

*Proof.* By Proposition (1.1), we construct a minimal free complex of finitely generated modules and a map $\phi_\bullet : P_\bullet \to K_\bullet(\underline{x}_{n-1}; A)[1]$ such that $\tilde{\phi}_i$ induces an isomorphism between $H_i(P_\bullet)$ and $H_{i+1}(\underline{x}_{n-1}; A)$ for $i \geq 0$. Actually we have the following commutative diagram:

$$\begin{array}{ccccccccccccc}
 & \to & P_{n-2} & \to & P_{n-3} & \to \cdots \to & P_1 & \to & P_0 & \to & 0 \\
 & & \downarrow{\scriptstyle \phi_{n-2}} & & \downarrow{\scriptstyle \phi_{n-3}} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \\
0 & \to & A & \to & A^{n-1} & \to \cdots \to & A^{\binom{n-1}{2}} & \to & A^{n-1} & \to & A & \to & 0
\end{array}$$

Since $\ell(H_i(x_1, \ldots, x_{n-1})) < \infty$ for $i \geq 1$, we have $\ell(H_i(P_\bullet)) < \infty$ for $i \geq 0$; hence by proposition (2.2) we have $\operatorname{Im} \phi_{n-2} \subset m$. This implies that when we extract a minimal free resolution $L_\bullet$ of $A/\underline{x}_{n-1}$ from the mapping cone of $\phi_\bullet$, $K_{n-1}(\underline{x}_{n-1}; A)(= A)$ survives as a component of $L_{n-1}$.

Let $F_\bullet$ be a minimal free resolution of $H$ over $A$ and let $\psi_\bullet : F_\bullet \to L_\bullet$ be a lift of the natural injection $H \hookrightarrow A/\underline{x}_{n-1}$.

Write $\overline{A} = A/x_nA$, $\overline{L}_\bullet = L_\bullet \otimes_A \overline{A}$, $\overline{F}_\bullet = F_\bullet \otimes_A \overline{A}$ and $\overline{\psi}_\bullet = \psi_\bullet \otimes \operatorname{Id} \overline{A}$. Note that $\overline{\psi}_1$ induces an isomosphism between $H_1(\overline{F}_\bullet)$ and $H_1(\overline{L}_\bullet)(= H)$ and $H_i(\overline{F}_\bullet) = H_i(\overline{L}_\bullet) = 0$ for $i \geq 2$.

Let $G_\bullet$ denote a minimal resolution of $H$ over $\overline{A}$.

We then have the following commutative diagram:

$$\begin{array}{ccccccccccccccccc}
\to G_{n-1} & \to & G_{n-2} & \to & G_{n-3} & \to \cdots \to & G_1 & \to & G_0 & \to & H & \to & & & 0 & & \\
\downarrow{\scriptstyle \alpha_{n-1}} & & \downarrow{\scriptstyle \alpha_{n-2}} & & \downarrow{\scriptstyle \alpha_{n-3}} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & & & & & & & \\
\to \overline{F}_n & \to & \overline{F}_{n-1} & \to & \overline{F}_{n-2} & \to \cdots \to & \overline{F}_2 & \xrightarrow{\overline{d}_2} & \overline{F}_1 & \to & \overline{F}_0 & \to & & & H & & \to 0 \\
\downarrow{\scriptstyle \overline{\psi}_n} & & \downarrow{\scriptstyle \overline{\psi}_{n-1}} & & \downarrow{\scriptstyle \overline{\psi}_{n-2}} & & \downarrow{\scriptstyle \overline{\psi}_2} & & \downarrow{\scriptstyle \overline{\psi}_1} & & \downarrow{\scriptstyle \overline{\psi}_0} & & & & & & \\
\to \overline{L}_n & \to & \overline{L}_{n-1} & \to & \overline{L}_{n-2} & \to \cdots \to & \overline{L}_2 & \to & \overline{L}_1 & \to & \overline{L}_0 & \to & & & \overline{A}/(\overline{x}_1, \cdots, \overline{x}_{n-1}) & & \to 0
\end{array}$$

where $\{\alpha_i\}$ lifts the injection $H \hookrightarrow \operatorname{coker} \overline{d}_2$. Since $x_n \in m \operatorname{ann}_A H$, by Corollary (2.3), $\operatorname{Im} \alpha_i \subset m\overline{F}_{i+1} \forall i \geq 0$. Write $\theta_i = \overline{\psi}_{i+1} \circ \alpha_i$ for $i \geq 1$. Since $\tilde{\theta}_i$ induces an isomorphism between $H_i(G_\bullet)$ and $H_{i+1}(\overline{L}_\bullet)$ for $i \geq 0$ and $\operatorname{Im} \theta_{n-1} \subset m\overline{L}_{n-1}$, $\overline{K_{n-1}(\underline{x}_{n-1}; A)}(= \overline{A})$ survives in a minimal free resolution of $\overline{A}/(\overline{x}_1, \ldots, \overline{x}_{n-1})$. Thus we are done by Theorem (2.2).

**2.5 Theorem 1.** *Let $A$ be a complete local normal domain of dimension $n$ and depth $n - r$, $r > 0$. Let $J_i = \operatorname{ann}_A H_m^{n-i}(A)$ and let $I = J_1 J_2 \cdots J_r$. Take any non-zero element $x \in mI$. Then $A/xA$ satisfies the* C. E. C..

*Proof.* Take any s.o.p. $\overline{x}_1, \ldots, \overline{x}_{n-1}$ of $A/xA$ and let $x_i$ denote a lift of $\overline{x}_i$ for $1 \leq i \leq n-1$. Then $x_1, \ldots, x_{n-1}, x$ is a s.o.p. for $A$. Since $x \in I$, $\ell(H_i(x_1, \ldots, x_{n-1}; A)) < \infty$ for $\forall i \geq 1$. Write $H = (0; x)A/\underline{x}_{n-1}$. Since $x \in mI$ and $IH = 0$ we have $x \in m \operatorname{ann}_A H$. Thus all the conditions of proposition (2.4) are satisfied and we are done.

**2.6 Theorem 2.** *Let $A$ be as in Theorem 1. Let $x_1, \ldots, x_n$ be a s.o.p. of $A$ such that $x_n \in I$. Then $x_1^t, \ldots, x_{n-1}^t$, $x_n^t + x$ satisfies the* C. E. C. *over* $A[[X]]/X^2 A[[X]]$ *(here $x$ denotes the image of $X$ in* $A[[X]]/X^2 A[[X]]$*) for $t \geq 1$.*

*Proof.* It will be clear from the proof that we can take $t = 1$. By Lemma (2.1) we know that $\ell(H_i(x_1, \ldots, x_{n-1})) < \infty$. We write $R = A[[X]]$. Then $x_1, \ldots, x_{n-1}$,

$x_n+X$, $X$ form a s.o.p. for $R$. We have

$$\begin{aligned} H_i(x_1,\dots,x_{n-1},x_n+X;R) &= \frac{H_i(x_1,\dots,x_{n-1})[[X]]}{(x_n+X)H_i(x_1,\dots,x_{n-1})[X]]} \\ &= H_i(x_1,\dots,x_{n-1})\otimes_A \frac{A[[X]]}{(x_n+X)} = H_i(x_1,\dots,x_{n-1}); \end{aligned}$$

moreover action of $X$ on $H_i(x_1,\dots,x_{n-1},x_n+X;R)$ is that of $-x_n$. Hence $XH_i(x_1,\dots,x_{n-1},x_n+X)=0$ for $\forall i\geq 1$ and $\ell H_i(x_1,\dots,x_{n-1},x_n+X;R))<\infty$. Write $H=(0:X)R/(x_1,\dots,x_{n-1},x_n+X)$. Then $H=(0:x_n)A/(x_1,\dots,x_{n-1})=(0:X^2)R/(x_1,\dots,x_{n-1},x_n+X)$. Note that $X^2\in m_R\operatorname{ann}_R H$ where $m_R$ denotes the maximal ideal of $R$.

Thus all the requirements of Proposition (2.4) are satisfied and we are done.

*Remark.* In order to prove the C. E. C. for $A$, it is enough to prove that $x_1,\dots,x_{n-1},(x_n+x)^2$ satisfies the C. E. C. in $R/X^2R$.

**2.7** In this section I am going to describe two results which are not directly linked to the main body of this paper, but it came up while working out the idea involved.

Let $x_1,\dots,x_{n-1},x_n$ be a s.o.p. of $A$ such that $x_n\in I$.

**Result 1.** We have, for $t>1$

(a) $H_i(x_1,\dots,x_{n-1},x_n^t)\simeq H_i(x_1,\dots,x_{n-1})\oplus H_{i-1}(x_1,\dots,x_{n-1})$ for $i>1$

and (b) $H_1(x_1,\dots,x_n^t)\simeq H_1(x_1,\dots,x_{n-1})\oplus(0:x_n)A/\underline{x}_{n-1}$.

*Proof.* Let $f_\bullet:K_\bullet(x_1,\dots,x_{n-1},x_n^t;A)\to K_\bullet(x_1,\dots,x_{n-1},x_n^{t+1})$ lift the map $\tilde{x}_n:A/(x_1,\dots,x_n^t)\to A/(x_1,\dots,x_n^{t+1})$ given by

$$\tilde{x}_n(\overline{1})=\overline{x}_n.$$

We know

$$f_i(e_{j_1}\wedge\cdots\wedge e_{j_i})=x_ne_{j_1}\wedge\cdots\wedge e_{j_i},\quad j_i\neq n$$

and

$$f_i(e_{j_1} \wedge \cdots \wedge e_n) = e_{j_1} \wedge \cdots \wedge e_n$$

This induces the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \frac{H_i(x_1,\dots,x_{n-1})}{x_n^t H_i(x_1,\dots,x_{n-1})} & \to & H_i(x_1,\dots,x_n^t) & \to & (0:x_n^t)H_{i-1}(x_1,\dots,x_{n-1}) & \to 0 \\ & & \downarrow{\scriptstyle x_n} & & \downarrow{\scriptstyle \tilde{f}_i} & & \downarrow{\scriptstyle \text{natural injection}} & \\ 0 & \to & \frac{H_i(x_1,\dots,x_{n-1})}{x_n^{t+1} H_i(x_1,\dots,x_{n-1})} & \to & H_i(x_1,\dots,x_n^{t+1}) & \to & (0:x_n^{t+1})H_{i-1}(x_1,\dots,x_{n-1}) & \to 0 \end{array}$$

Since $x_n \in I$, the above commutative diagram gives rise to the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & H_i(x_1,\dots,x_{n-1}) & \to & H_i(x_1,\dots,x_n^t) & \to & H_{i-1}(x_1,\dots,x_{n-1}) & \to 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \tilde{f}_i} & & \downarrow{\scriptstyle \text{id}} & \\ 0 & \to & H_i(x_1,\dots,x_{n-1}) & \to & H_i(x_1,\dots,x_n^{t+1}) & \to & H_{i-1}(x_1,\dots,x_{n-1}) & \to 0 \end{array}$$

This shows that id splits and hence when $t > 1$ we have

$$H_i(x_1,\dots,x_n^t) \simeq H_i(x_1,\dots,x_{n-1}) \oplus H_{i-1}(x_1,\dots,x_{n-1})$$

for $i \leq 1$.

For $i = 1$, note that $H = (0:x_n)A/(x_1,\dots,x_{n-1}) = (0:x_n^2)A/(x_1,\dots,x_n)$ $(0:x_n^3)A/(x_1,\dots,x_{n-1}) = \cdots$.

Then arguing as above we get $H_1(x_1,\dots,x_n^t) \simeq H_1(x_1,\dots,x_{n-1}) \oplus H$.

**Result 2.** Let $x_1,\dots,x_{n-1},x_n$ be a s.o.p. for $A$ and let $x_n \in I$. We write $J = (x_1,\dots,x_{n-1})$. Then $H^i_m(A) \xrightarrow{\sim} H^i_J(A)$ for $i < n-1$ and $H^{n-1}_m(A)$ is isomorphic to a submodule of $H^{n-1}_J(A)$.

*Proof.* Let $g_\bullet : K_\bullet(x_1^t,\dots,x_n^t;A) \to K_\bullet(x_1^{t+1},\dots,x_n^{t+1};A)$ be a natural lift of the map $x_1\widetilde{x_2\cdots}x_n : A/(x_1^t,\dots,x_n^t) \to A/(x_1^{t+1},\dots,x_n^{t+1})$ given by $x_1\widetilde{x_2\cdots}x_n(\overline{1}) = \overline{x}_1\overline{x}_2 \bullet\cdots\bullet \overline{x}_n$. (Recall $g_p(e_{i_1}\wedge\cdots\wedge e_{i_p}) = \pm x_{j_1}x_{j_2}\cdots x_{j_{n-p}}e_{i_1}\wedge\cdots\wedge e_{i_p}$). Then $g_\bullet$ induces the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & H_i(x_1^t,\dots,x_{n-1}^t) & \to & H_i(x_1^t,\dots,x_n^t) & \to & H_{i-1}(x_1^t,\dots,x_{n-1}^t) & \to 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow & & \downarrow & \\ 0 & \to & H_i(x_1^{t+1},\dots,x_{n-1}^{t+1}) & \to & H_i(x_1^{t+1},\dots,x_n^{t+1}) & \to & H_{i-1}(x_1^{t+1},\dots,x_{n-1}^{t+1}) & \to 0. \end{array}$$

Taking the direct limit, we get

$$H_m^{n-i}(A) \simeq H_J^{n-i}(A) \quad \text{for} \quad i > 1$$

and

$$H_m^{n-1}(A) \hookrightarrow H_J^{n-1}(A) \quad \text{for} \quad i = 1.$$

## Section 2

**3.1 Discussion.** Let $\to A^{t_i} \to A^{t_{i-1}} \to \cdots \to A^{t_1} \to A^{t_0} \to A/m = K$ be minimal free resolution of $K$ over $A$. Let $S_i$ denote the $i$th syzygy of $K$ in the above resolution. We have the following exact sequences

$$0 \to S_n \to A^{t_{n-1}} \to S_{n-1} \to 0, \quad 0 \to S_{n-1} \to A^{t_{n-2}} \to S_{n-2} \to 0, \ldots$$

$$0 \to S_i \to A^{t_{i-1}} \to S_{i-1} \to 0, \cdots, 0 \to S_1 \to A \to K \to 0.$$

The above exact sequences give rise to the following commutative diagram:

$$\begin{array}{ccccccccc} K & \to & \mathrm{Ext}^1(K,S_1) & \to & \mathrm{Ext}^2(K,S_2) & \to\cdots\to & \mathrm{Ext}^{n-1}(K,S_{n-1}) & \to & \mathrm{Ext}^n(K,S_n) \\ \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta-2} & & \downarrow{\scriptstyle \theta_{n-1}} & & \downarrow{\scriptstyle \theta_n} \\ H_m^0(K) & \to & H_m^1(S_1) & \to & H_m^2(S_2) & \to\cdots\to & H_m^{n-1}(S_{n-1}) & \to & H_m^n(S_n) \end{array}$$

(all the horizontal maps are connecting homomorphisms obtained from the short exact sequences). The C. E. C. asserts that $\theta_n$ (class $\mathrm{Id}_{S_n}$ in $\mathrm{Ext}^n(k, S_n)) = \eta_A$ is non-zero. One can check that $\eta_A$ is nothing but the image of $1 \in K$ from the upper left-hand corner in this long commutative diagram. We now prove the following theorem:

**3.2 Theorem.** *With the notations as above, $\theta_i(\mathrm{Id}_{S_i})$ is non-zero in $H_m^i(S_i)$ for $0 \le i \le n-1$. ($\mathrm{Id}_{S_i}$ stands for the identity map on $S_i$.)*

*Proof.* If possible let $\theta_i(\mathrm{Id}_{S_i})$ be $0$ in $H_m^i(S_i)$. Recall that $H_m^i(S_i) = \varinjlim_t \mathrm{Ext}^i(A/m^t, S_i)$. Hence for $t >> 0$, if $\alpha_t$ denotes the directed map $\mathrm{Ext}^i(K, S_i) \to \mathrm{Ext}^i(A/m^t, S_i)$, then $\alpha_t(\mathrm{Id}_{S_i}) = 0$; we fix one such $t$. Let $\to A^{r_i} \to A^{r_{i-1}} \to \cdots \to A^{r_0} \to A \to A/m^t$ be a minimal free resolution of $A/m^t$. We then have the following commutative diagram

$$(1)\qquad \begin{array}{ccccccccccc} \to A^{r_i} & \xrightarrow{\delta_i} & A^{r_{i-1}} & \xrightarrow{\delta_{i-1}} & \cdots & \to A^{r_0} & \to A & \to & A/m^t & \to 0 \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i-1}} & & & \downarrow{\scriptstyle \phi_0} & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & \\ \to A^{t_i} & \xrightarrow{\beta_i} & A^{t_{i-1}} & \xrightarrow{\beta_{i-1}} & \cdots & \to A^{t_0} & \to A & \to & A/m = K & \to 0 \end{array}$$

where $\{\phi_i\}$ lifts the natural surjection $A/m^t \to A/m = k$. Write $T_i = \mathrm{syz}^i(A/m^t)$ and $S_i = \mathrm{syz}^i(A/m)$. From (1) we get the following diagram

$$\begin{array}{cccccccccc} 0 & \to T_i & \xrightarrow{\delta} & A^{r_{i-1}} & \to \cdots & \to A^{r_0} & \to A & \to A/m^t & \to 0 \\ & \downarrow{\scriptstyle \tilde{\phi}_i} & & \downarrow{\scriptstyle \phi_{i-1}} & & \downarrow{\scriptstyle \phi_0} & \downarrow{\scriptstyle \mathrm{id}} & \downarrow & \\ 0 & \to S_i & \to & A^{t_{i-1}} & \to \cdots & \to A^{t_0} & \to A & \to A/m & \to 0 \end{array}$$

here $\tilde{\phi}_i = \phi_i \big| T_i$. Now $\alpha_t(\mathrm{Id}\, S_i) = 0 \Rightarrow \exists h : A^{r_{i-1}} \to S_i$ such that $h \circ \delta = \tilde{\phi}_i$. Hence we can assume that $\tilde{\phi}_i = 0$ and this imples that we can take $\phi_s$ to be the 0-maps for $s \geq i$ in (1). Write $G_{i-1} = \mathrm{coker}\, \beta^*_{i-1}$, $\tilde{G}_{i-1} = \mathrm{coker}\, \delta^*_{i-1}$, $H_{i-1} = \mathrm{Ext}^{i-1}_A(K, A)$ and $\tilde{H}_{i-1} = \mathrm{Ext}^{i-1}_A(A/m^t, A)$. Applying $\hom(-, A)$ to (1) we get the following commutative diagram of short exact sequences

$$(2)\qquad \begin{array}{cccccc} 0 & \to \mathrm{Ext}^{i-1}(K, A) & \to G_{i-1} & \to & \mathrm{Im}\, \beta^*_i & \to 0 \\ & \downarrow & \downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle 0} & \\ 0 \to & \tilde{H}_{i-1} & \to \tilde{G}_{i-1} & \xrightarrow{\eta} & \mathrm{Im}\, \delta^*_i & \to 0 \end{array}$$

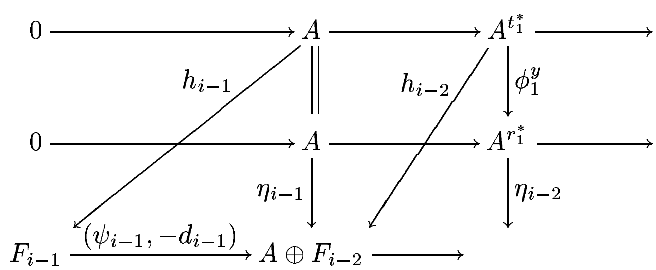
The above diagram implies that $\tilde{\phi}$ factors through $\tilde{H}_{i-1}$. Moreover we have another commutative diagram:

$$(3)\qquad \begin{array}{ccccccc} 0 & \to A & \to A^{t^*_1} & \to \cdots & \to A^{t^*_{i-1}} & \to G_{i-1} & \to 0 \\ & \| & \mathrm{id} \downarrow{\scriptstyle \phi^*_1} & & \downarrow{\scriptstyle \phi^*_{i-1}} & \downarrow{\scriptstyle \tilde{\phi}} & \\ 0 & \to A & \to A^{r^*}_1 & \to \cdots & \to A^{r^*_{i-1}} & \to \tilde{G}_{i-1} & \to 0 \end{array}$$

We denote the free part of the bottom complex by $G \left( G_j = A^{r^*_{i-1-j}}, r^*_0 = 1 \right)$. Now by proposition (1.1) we can construct a minimal free complex of finitely generated modules $F_\bullet = \{A^{s_j}, d_j\}$ and a map $\psi_\bullet : F_\bullet \to G_\bullet$ such that $H_0(F_\bullet) = \tilde{H}_{i-1}$, $\widetilde{\psi_0}$ induces the natural injection of $\tilde{H}_{i-1}$ into $\tilde{G}_{i-1}$ and $\psi_j$ induces an isomorphism between $H_j(F_\bullet)$ and $H_j(G_\bullet)$ for $j \geq 1$, so that the mapping cone of $\psi_\bullet$ gives a free resolution of $\operatorname{Im} \delta_i^*$.

**Claim.** Because of (2) and (3), $\operatorname{Im} \psi_{i-1} = A$.

Write $Mc\psi_\bullet$ to denote the mapping cone of $\psi^\bullet$, and let $\eta_\bullet$ be the canonical injection of $G_\bullet$ into $Mc\psi_\bullet$. Since $\eta \circ \tilde{\psi}_0 = 0$ ((2)) and $Mc\psi_\bullet$ is a free resolution of $\operatorname{Im} \delta_i^*$, $\eta_\bullet \cdot \phi_\bullet^*$ is homotopically equivalent to 0. Let $\{h_j\}$ denote the corresponding homotopy maps. Note that the map on the extreme left column of (3) is identity on $A$, and we have the following commutative diagram:




Since $\eta_{i-1} = (\operatorname{Id} A, 0)$ and the above diagram is commutative we are forced to get $\operatorname{Im} \psi_{i-1} = A$.

Now let us consider the following commutative diagram

$$
(4) \qquad \begin{array}{ccccccccccccc}
F_i & \xrightarrow{d_i} & F_{i-1} & \xrightarrow{d_{i-1}} & F_{i-2} & \to \cdots \to & F_1 & \to & F_0 & \to & \tilde{H}_{i-1} & \to & 0 \\
\downarrow & & \downarrow{\scriptstyle \psi_{i-1}} & & \downarrow{\scriptstyle \psi_{i-2}} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & A^{r_1^*} & \to \cdots \to & A^{r_{i-2}^*} & \to & A^{r_{i-1}^*} & \to & \tilde{G}_{i-1} & \to & 0
\end{array}
$$

Since $\operatorname{Im} \psi_{i-1} = A$, we can choose a basis $e_1, \ldots, e_d$ of $F_{i-1}$ such that $\psi_{i-1}(e_1) = 1$ and $\psi_{i-1}(e_j) = 0$ for $i < j \leq d$. This implies that (i) $d_i(F_i)$ is contained in the free submodule generated by $e_2, \ldots, e_d$, and (ii) image of $e_1^*$ in coker $d_{i-1}^*$ is killed by a power of $m$. Since $\ell(H_i(F_\bullet)) < \infty$, by applying $\operatorname{Hom}(-, A)$ to the top row of (4) we get

$$0 \to F_0^* \to F_1^* \to \cdots \to F_{i-2}^* \xrightarrow{d_{i-1}^*} F_{i-1}^* \to M \to 0 \tag{5}$$

such that $M(= \operatorname{coker} d_{i-1}^*)$ has a minimal generator killed by a power of $m$ and all the homologies of this complex are of finite length. Since $i-1 \leq n-2$, by tensoring (5) with $\bar{A} = A/pA$ $(0 < p =$ characteristic of $A/m)$, we arrive at a contradiction (due to the fact that the C. E. C. holds in $\bar{A}$, characteristic of $\bar{A} = p > 0)$.

*Remark.* As a corollary to proposition (1.1) one can derive the following theorem in [S]: $\dim Tor_i^A(K,K) \geq \binom{s}{i}$, $s = \dim_K m/m^2$. Actually the method of proposition (1.1) and (1.2) show, in a very direct manner, that the Koszul complex on a minimal set of generators of $m$ imbeds into the minimal resolution of $A/m$.

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