Algebraic D-groups and Differential Galois theory

Anand Pillay*
Dept. Mathematics
University of Illinois at Urbana-Champaign

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Abstract

We discuss various relationships between the algebraic D-groups of Buium [3], and differential Galois theory. In the first part we give another exposition of our general differential Galois theory, which is somehat more explicit than [8], and where "generalized logarithmic derivatives" on algebraic groups play a central role. In the second part we prove some results with a "constrained Galois cohomological flavour". For example, if G and H are connected algebraic D-groups over an algebraically closed differential field F, and G and H are isomorphic over some differential field extension of F, then they are isomorphic over some Picard-Vessiot extension of F. Suitable generalizations to isomorphisms of algebraic D-varieties are also given.

1 Introduction

We work throughout with (differential) fields of characteristic zero. In [8] the notion of a generalised differential Galois extension (or generalizes strongly normal extension) of a differential field was introduced, generalizing Kolchin's theory of strongly normal extensions, which in turn generalized the Picard-Vessiot theory. The idea was to systematically replace algebraic groups over

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the constants by "finite-dimensional differential algebraic groups", to obtain new classes of extensions of differential fields with a good Galois theory. This idea (almost obvious from the model-theoretic point of view) was implicit in Poizat [11] who gave a model-theoretic treatment of the strongly normal theory. However the "correct" definition of a generalized differential Galois extension needed some additional fine-tuning. Nevertheless, our exposition of this general theory in [8] was overly model-theoretic, and possibly remained somewhat obscure to differential algebraists. We try to remedy this in the current paper by concentrating on the *differential equations* which have a good Galois theory, very much in the spirit of section 7, Chapter IV of Kolchin's book [4]. The key notion is that of a "generalized logarithmic derivative" on an algebraic group G over a differential field K (a certain kind differential rational map from G to its Lie algebra). We will see that such a generalized logarithmic derivative is essentially equivalent to an algebraic D-group structure on G (in the sense of Buium [3]). Our resulting exposition of the generalized differential Galois theory will be equivalent to that in [8] when the base field K is *algebraically closed*. The general situation (K not necessarily algebraically closed) can be treated using analogues of the "V-primitives" from IV.10 of [4], and we leave the details to others.

Let me now say a little more about the generalized logarithmic derivatives, and how they tie up with the Picard-Vessiot/strongly normal theory. Let us fix a differential field K, and assume for now that the field C_K of constants of K is algebraically closed. A linear differential equation over K, in vector form, is $\partial y = Ay$ where y is a $n \times 1$ column vector of unknowns and A is an $n \times n$ matrix over K. Looking for a fundamental matrix of solutions, one is led to the equation on GL_n: $\partial Y = AY$, where Y is a $n \times n$ matrix of unknowns ranging over GL_n, which we can write as $\partial(Y)Y^{-1} = A$. Now the map $Y \to \partial(Y)Y^{-1}$ is the classical logarithmic derivative, a first order differential crossed homomorphism from GL_n into its Lie algebra, which is surjective when viewed in a differentially closed overfield of K. A Picard-Vessiot extension of K for the original equation is then a differential field extension $L = K(g)$, where $g \in GL_n$ is a solution of $\partial(Y)Y^{-1} = A$, and $C_L = C_K$. Such an extension exists, and is unique up to K-isomorphism. The group of (differential) automorphisms of L over K has the structure of an algebraic subgroup of $GL_n(C_K)$, and there is a Galois correspondence.

In place of GL_n one can consider an arbitrary algebraic group G defined over K (not necessarily linear and not necessarily defined over the constants

of K). By a *generalized logarithmic derivative* on G we will mean a first order differential rational crossed homomorphism μ from G to $L(G)$, defined over K, such that μ is "geometrically surjective", namely surjective when viewed in a differentially closed overfield, and such that $Ker(\mu)$, a *finite-dimensional differential algebraic subgroup* of G, is Zariski-dense in G. The analogue of a linear differential equation over K will then be an equation
(*) $\mu(x) = a$
where x ranges over G and $a \in L(G)(K)$.
Under an additional technical condition on the data, analogous to the requirement that the field of constants of K be algebraically closed, we can define the notion of a differential Galois extension L of K for the equation (*), prove its existence and uniqueness, identify the Galois group, and obtain a Galois correspondence. In the case where G is defined over the constant field C_K *and* μ is the standard logarithmic derivative of Kolchin, we recover Kolchin's strongly normal extensions (see Theorem 6, section 7, Chapter IV of [4]).

For G an algebraic group over the differential field K, an algebraic D-group structure on G is precisely an extension of the derivation ∂ on K to a derivation on the structure sheaf of G, respecting the group operation. Algebraic D-groups belong entirely to algebraic geometry, and Buium [3] points out that there is an equivalence of categories between the category of algebraic D-groups and the category of ∂_0-groups, finite-dimensional *differential algebraic* groups. The latter category belongs to Kolchin's *differential algebraic geometry*. On the other hand, there is essentially a one-one correspondence between algebraic D-group structures on G and generalized logarithmic derivatives on G. So our general differential Galois theory is in a sense subsumed by the very concept of an algebraic D-group.

Details of the above will be given in sections 2 and 3, including a "Tannakian" approach and an examination of different manifestations of the differential Galois group.

In section 4 we will give another relation between algebraic D-groups and the Picard-Vessiot theory: if two algebraic D-groups over an algebraically closed differential field are isomorphic (as D-groups) over some differential field extension of K, then such an isomorphism can be found defined over a Picard-Vessiot extension of K. This uses Kolchin's differential Lie algebra, and strengthens the somewhat artifical results from [9]. We will also give some related results on isomorphisms between algebraic D-varieties, using differential jets (higher dimensional versions of differential tangent spaces).

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2 Algebraic D-groups

We will briefly describe the algebraic D-groups of Buium in a matter suitable for our purposes. We also introduce the "generalized logarithmic derivative" on an algebraic group induced by a given D-group structure. We guess that the latter is implicit (or maybe explicit) in [3]. We also refer to [6] for a discussion of related themes.

Let us fix an ordinary differential field (K,∂). For convenience we also give ourselves a differential field extension $\mathcal{U},\partial)$ of (K,∂) which is "universal" with respect to (K,∂). Namely, $\mathcal{U}$ has cardinality $\kappa > |K|$, and for any differential subfield $F < \mathcal{U}$ of cardinality $< \kappa$ and differential extension L of K of cardinality $\leq \kappa$ there is an embedding (as differential fields) of L into $\mathcal{U}$ over K. C_K denotes the field of constants of K, and $\mathcal{C}$ the field of constabts of $\mathcal{U}$.

We begin by recall the tangent bundle of an algebraic variety or group over K (in which the derivation on K plays no role)

Let X be an algebraic variety over K (maybe reducible). The tangent bundle $T(X)$ of X is another algebraic variety over K, with a canonical surjective morphism π (over K) to X, and is defined locally by equations: $\sum_i \partial P/\partial x_i(x_1,..,x_n)v_i = 0$ for P polynomials over K generating the ideal of X over K. If $X = G$ is an algebraic group over K, then $T(G)$ has the structure of an algebraic group over K such that the canonical projection to G is an (algebraic) group homomorphism. The group operation on $T(G)$ is obtained by differentiating the group operation of G. That is, if $f(-,-)$ is the group operation on G and (g,u) and (h,v) are in $T(G)$ then the product

$(g,u)\cdot(h,v)$ equals $(g\cdot h, df_{g,h}(u,v))$. Note that if λ^g, ρ^g denote left and right multiplication by g in G, then we have:
(*) In $T(G)$, $(g,u)\cdot(h,v) = (g\cdot h, d(\lambda^g)_h(v) + d(\rho^h)_g(u))$.

We will denote $T(G)_e$, the tangent space of G at the identity, by $L(G)$ (for the Lie algebra of G). Then $L(G)$ with its usual vector space group structure, is a normal subgroup of $T(G)$, and we have the exact sequence $0 \to L(G) \to T(G) \to G \to \{e\}$ of algebraic groups (over K). $i : L(G) \to T(G)$ is the natural inclusion, and $\pi : T(G) \to G$ is the canonical surjection above.

Note that G acts on $L(G)$ (denoted $(g,a) \to a^g$) by the adjoint map (differentiating conjugation by $g \in G$ at the identity). And this "coincides" with the action of $T(G)$ on the normal subgroup $L(G)$ by conjugation: for $a \in L(G)$ and $g \in G$, $a^g = xax^{-1}$ for any $x \in T(G)$ such that $\pi(x) = g$.

For $a \in L(G)$, we let l_a and r_a denote the left and right invariant vector fields on G determined by a. Namely for $g \in G$, $l_a(g) = d(\lambda^g)_e(a)$ and $r_a(g) = d(\rho^g)_e(a)$.

A K-rational splitting of $T(G)$ as a semidirect product of G and $L(G)$, is given by either of the equivalent pieces of data: (a) a K-rational homomorphic section $s : G \to T(G)$ (that is $\phi \circ s = id$), or (b) a K-rational crossed homomorphism h from $T(G)$ onto $L(G)$ such that $h \circ i = id$. (By definition $\alpha : T(G) \to L(G)$ is a crossed homomorphism if $\alpha(xy) = \alpha(x) + x\alpha(y)x^{-1}$.)

Note that the set of these K-rational splittings has the structure of a commutative group. For example, using the data in (a), if s_1, s_2 are K-rational homomorphic sections of the tangent bundle, and $s_i(g) = (g, u_i)$ for $i = 1, 2$ then $(s_1 + s_2)(g) = (g, u_1, +u_2)$. The identity element is just the 0-section $s_0(g) = (g, 0)$, which IS a K-rational homomorphic section. Let $P_K(G)$ denote this commutative group of K-rational splittings of $T(G)$. We denote the crossed homomorphism from $T(G)$ onto $L(G)$ corresponding to the identity of $P_K(G)$ by $h_0 : T(G) \to L(G)$, and the crossed homomorphism corresponding to the homomorphic section s by h_s.

Remark 2.1 *(i)* $h_0(g,u) = d(\rho^{g^{-1}})_g(u)$.
(ii) More generally, if s is a K-rational homomorphic section of $T(G) \to G$, then $h_s(g,u) = d(\rho^{g^{-1}})_g(u - s(g))$.

Proof. This follows directly from the formula (*) for multiplication in $T(G)$.

Let us now bring in the differential structure. Assume for now that the algebraic variety X is defined over C_K the field of constants of K. Then it is easy to see that if $a \in X(\mathcal{U})$ then working in local coordinates with respect to a given covering of X by affine varieties over C_K, $(a, \partial(a)) \in T(X)$. If in addition $X = G$ is an algebraic group defined over C_K, then the map $\nabla : G(\mathcal{U}) \to T(G)(\mathcal{U})$ taking g to $(g, \partial(g))$ is a *group embedding*. Let $lD : G(\mathcal{U}) \to L(G)(\mathcal{U})$ be defined by $lD(g) = h_0(g, \partial(g))$. Then lD is a "differential rational" crossed homomorphism defined over K, which is precisely Kolchin's logarithmic derivative. The map lD depends on h_0, and clearly any other $h \in P_K(G)$ gives rise to another differential rational crossed homomorphism (over K) from $G(\mathcal{U})$ onto $L(G)(\mathcal{U})$.

Remark 2.2 *Suppose G is defined over C_K. Then the (standard) logarithmic derivative $lD : G(\mathcal{U}) \to L(G)(\mathcal{U})$ is given by $lD(g) = d(\rho^{g^{-1}})_g(\partial(g))$.*
(ii) lD is surjective.
(iii) $Ker(lD)$ is precisely $G(\mathcal{C})$.

Proof. (i) follows from Remark 2.1.
(ii) Let $a \in L(G)(\mathcal{U})$. Then a determines the right invariant vector field $r_a : G(\mathcal{U}) \to T(G)(\mathcal{U})$. As $\mathcal{U}$ is differentially closed, the main result of [7] gives $g \in G(\mathcal{U})$ such that $\partial(g) = r_a(g)$, hence (by (i)), $lD(g) = a$.
(iii) is obvious from (i).

Let us now work in a more general context, dropping our assumption that the variety X is defined over C_K. Then for $a \in X(\mathcal{U})$, $(a, \partial(a))$ may no longer be a point of $T(X)$ but rather a point of another bundle $\tau(X)$ over X, which we now describe. For $P(x_1, .., x_n)$ a polynomial over K, let P^∂ denote the polynomial obtained from P by applying ∂ to its coefficients. (So $P^\partial = 0$ if P is over C_K.) Then $\tau(X)$ is defined locally by equations: $\sum_i \partial P/\partial x_i(x_1, .., x_n)v_i + P^\partial(x_1, .., x_n) = 0$
where P again ranges over polynomials in the ideal of X over K. (That is, these affine pieces fit together to give an algebraic variety $\tau(X)$ over K, together with a canonical projection from $\tau(X)$ to X.) It is immediate that $(a, \partial a) \in \tau(X)$ for $a \in X(\mathcal{U})$. It is also immediate that for each $a \in G$, $\tau(X)_a$ is a principal homogeneous space for the tangent space $T(X)_a$, where the action is addition (with respect to local coordinates above). Moreover this happens uniformly, making $\tau(X)$ a *torsor* for the tangent bundle $T(X)$. In any case, if X is defined over C_K, then $\tau(X)$ coincides with $T(X)$.

Now assume again that $X = G$ is an algebraic group over K. Then $\tau(G)$ has the structure of an algebraic group over K such that the canonical projection $\tau(G) \to G$ is a homomorphism. In local coordinates, assuming multiplication in G is given by the sequence of polynomials $f = (f_i(x_1,..,x_n,y_1,..,y_n))_{i=1,..,n}$ over K, then for $(g,u),(h,v) \in \tau(G)$, the product of (g,u) and (h,v) in $\tau(G)$ is given by: $(g{\cdot}h, df_{(g,h)}(u,v)+(f_1^\partial(g,h),..,f_n^\partial(g,h)))$.

We again have the map $\nabla : G(\mathcal{U}) \to \tau(G)(\mathcal{U})$, given (in local coordinates) by $\nabla(g) = (g,\partial(g))$ and this is a *group embedding*.

So we have two K-algebraic groups $T(G)$ and $\tau(G)$. Although these need not be isomorphic as algebraic groups, they are "differential rationally" isomorphic. The following is left to the reader.

Lemma 2.3 *The map which takes (g,u) to $(g,\partial(g)-u)$ is an isomorphism of groups between $\tau(G)(\mathcal{U})$ and $T(G)(\mathcal{U})$. Although not necessarily rational, it IS rational when restricted to the fibres over G. In particular, the above map defines a K-rational isomorphism between the vector groups $\tau(G)_e$ and $T(G)_e = L(G)$.*

Note that we have again an exact sequence of algebraic groups over K: $0 \to \tau(G)_e \to \tau(G) \to G \to e$ which by virtue of the (canonical) isomorphism between $\tau(G)_e$ and $L(G)$ given by Lemma 2.3 can be rewritten as: $0 \to L(G) \to \tau(G) \to G \to e$.

Let us again write i for the (canonical) injection of $L(G)$ in $\tau(G)$, and π for the canonical surjection $\tau(G) \to G$. So if G is defined over C_K this agrees with our earlier notation: $i : L(G) \to T(G)$ and $\pi : T(G) \to G$.

We can consider splittings (as algebraic groups over K) of $\tau(G)$ as a semidirect product of G and $L(G)$. Again each such splitting is determined either by (a) a K-rational homomorphic section $s : G \to \tau(G)$, or (b) a K-rational crossed homomorphism $h : \tau(G) \to L(G)$ such that $h \circ i = id$ on $L(G)$. We will write h_s for the crossed homomorphism corresponding to the homomorphic section s, and give explicit formulas below.

In any case, we can now define an algebraic D-group.

Definition 2.4 *Let G an algebraic group over K. Then an algebraic D-group structure on G over K is precisely a K-rational homomorphic section $s : G \to \tau(G)$. We write the corresponding algebraic D-group as (G,s).*

Given an algebraic D-group (G,s) we obtain a generalized logarithmic derivative which we call lD_s, a crossed homomorphism (in the obvious sense) from $G(\mathcal{U})$ to $L(G)(\mathcal{U})$: $lD_s = h_s \circ \nabla$. Here is the analogue to Remarks 2.1 and 2.2.

Remark 2.5 *Let (G, s) be an algebraic D-group over K.*
(i) For $(g, u) \in \tau(G)$, $h_s(g, u) = d(\rho^{g^{-1}})_g(u - s(g))$.
(ii) For $g \in G(\mathcal{U})$, $lD_s(g) = d(\rho^{g^{-1}})_g(\partial(g) - s(g)) \in L(G)(\mathcal{U})$.
(iii) $lD_s : G(\mathcal{U}) \to L(G)(\mathcal{U})$ is surjective.
(iv) $Ker(lD_s)$ is precisely $\{g \in G(\mathcal{U}) : \partial(g) = s(g)\}$, and is a Zariski-dense subgroup of $G(\mathcal{U})$.

Proof. (i) follows from the formula for multiplication in $\tau(G)$, and (ii) is an immediate consequence of (i).
(iii) Let $a \in L(G)(\mathcal{U})$. Again we obtain the right invariant vector field $r_a : G(\mathcal{U}) \to T(G)(\mathcal{U})$. Then $r_a + s : G(\mathcal{U}) \to \tau(G)(\mathcal{U})$ is also a rational section of $\tau(G) \to G$. By [7] there is $g \in G(\mathcal{U})$ such that $\partial(g) = r_a(g) + s(g)$, hence $lD_s(g) = a$ by (i).
(iv) $Ker(lD_s)$ is a subgroup of $G(\mathcal{U})$ as lD_s is a crossed homomorphism. As $d(\rho^{g^{-1}})_g$ is an isomorphism between $T(G)_g$ and $T(G)_e$, we see that $Ker(lD_s)$ is as described in (iii). By [7] for any proper subvariety X of $G(\mathcal{U})$ there is $g \in G(\mathcal{U})$ such that $\partial(g) = s(g)$. Hence by (i) $Ker(lD_s)$ is Zariski dense in G.

In the context of Remark 2.5, we denote $Ker(lD_s)$ by $(G, s)^\sharp$. This is a finite-dimensional differential algebraic group, or ∂_0-group in the sense of [3], and is an object belonging to Kolchin's "differential algebraic geometry". Just for the record, here are some key properties: $(G, s)^\sharp(\mathcal{U})$ is Zariski-dense in G, the ∂_0-subvarieties of $(G, s)^\sharp$ are precisely of the form $X \cap (G, s)^\sharp$ for X a D-subvariety of (G, s), and for any other algebraic D-group (H, t), the ∂_0-homomorphisms between $(G, s)^\sharp$ and $(H, t)^\sharp$ are precisely those induced by algebraic D- group homomorphisms between (G, s) and (H, t).

In any case, we have seen that an algebraic D-group structure (G, s) on an algebraic group G over K determines a generalized logarithmic derivative LD_s on G. We point out now, just for completeness, that conversely any suitable differential rational crossed homomorphism map (over K) from $G(\mathcal{U})$ to $L(G)(\mathcal{U})$ determines an algebraic D-group structure on G.

Some notation: Let X and Y be algebraic varieties defined over K. By a *first order differential rational map* $h : X(\mathcal{U}) \to Y(\mathcal{U})$, defined over K, we mean a map h from $X(\mathcal{U})$ to $Y(\mathcal{U})$ such that for, for each $x \in X(\mathcal{U})$, $h(x) \in K(x, \partial(x))$.

Lemma 2.6 *Suppose G is a connected algebraic group over K. Let l :*

$G(\mathcal{U}) \to L(G)(\mathcal{U})$ be a first order differential rational crossed homomorphism, defined over K, which is surjective, and is such that $Ker(l)$ is Zariski-dense in $G(\mathcal{U})$. Then, there are a unique K-rational automorphism σ of $L(G)$, and a unique algebraic D-group structure (G, s) on G (defined over K), such that $l = \sigma \circ lD_s$.

Proof. By [7] for example, $\nabla(G(\mathcal{U}))$ is Zariski-dense in $\tau(G)(\mathcal{U})$. Define l_1 on $\nabla(G(\mathcal{U}))$ by $l_1(x, \partial(x)) = l(x)$. So by Zariski-denseness, and the properties of l, l_1 extends uniquely to a K-rational surjective crossed homomorphism f from $\tau(G)$ to $L(G)$.

By Zariski-denseness of $Ker(l)$ in G and the definition of f, $\pi(Ker(f)) = G$. On the other hand, $dim(\tau(G)) = 2dim(G)$ and $dim(G) = dim(L(G))$. Hence $dim(Ker(f)) = dim(G)$. Now $\pi|Ker(f) : Ker(f) \to G$ is a group homomorphism so it has finite kernel. But this finite kernel is a subgroup of the vector group $\tau(G)_e$ so has to be trivial. It follows that $f|\tau(G)_e$ is an isomorphism (over K) with $L(G)$. So there is a unique K-automorphism σ of $L(G)$ such that $(\sigma \circ f) \circ i$ is the identity on $L(G)$. Put $f_1 = \sigma \circ f$. Then f_1 gives an algebraic D-group structure (G, s) on G defined over K. For $g \in G(\mathcal{U})$, $\sigma \circ l(g) = f_1(g, \partial(g)) = lD_s(g)$. This completes the proof of the lemma.

There is a natural notion of a D-morphism between algebraic D-varieties (X, s) and (Y, t). First note that $\tau(-)$ is a functor, so if $f : X \to Y$ is a morphism between the algebraic varieties X and Y with everything defined over K then $\tau(f)$ is a morphism, over f, between $\tau(X)$ and $\tau(Y)$, again defined over f. (If X, Y, f are defined over the constants, then $\tau(f)$ is just the differential of f.) In any case, a morphism between algebraic D-varieties X and Y is by definition a morphism f of algebraic varieties, such that $t \circ f = \tau(f) \circ s$.

In particular we extract the notion of an algebraic D-subvariety of (X, s): so an algebraic subvariety Y of X will be the underlying variety of an algebraic D-subvariety of (X, s) if $s|Y$ maps Y to $\tau(Y)$.

A homomorphism of algebraic D-groups is a D-morphism which is also a homomorphism of algebraic D-groups.

We call an algebraic D-group (G, s) isotrivial if it is isomorphic over $\mathcal{U}$ to a trivial algebraic D-group (H, s_0) where H is defined over $\mathcal{C}$ and s_0 is the 0-section of $T(G)$.

The interest of the category of algebraic D-groups is that there exist non-isotrivial algebraic D-groups. If A is an abelian variety over $\mathcal{U}$ and $p: G \to A$ is the "universal extension of A by a vector group" then G has a D-group structure (defined over the field over which A is defined). Moreover any such D-group structure on G is nonisotrivial if A is not isomorphic (as an algebraic group) to an abelian variety defined over $\mathcal{C}$. Given such a D-group structure (G,s) on G, $p((G,s)^\sharp) < A$ is precisely the "Manin kernel" of A. This example is worked out in detail in [6].

Let us repeat from [9] the discussion of a nonisotrivial algebraic D-group structure on the commutative algebraic group $G_m \times G_a$. So let $G = G_m \times G_a$. Then $T(G) = \tau(G)$ can be identified with $\{(x,y,u,v) : x \neq 0\}$ with group structure $(x_1,y_1,u_1,v_1)\cdot(x_2,y_2,u_2,v_2) = (x_1x_2, y_1+y_2, u_1x_2+u_2x_1, v_1+v_2)$. Let $s: G \to T(G)$ be the homomorphic section $s(x,y) = (x,y,xy,0)$. Then (G,s) is an algebraic D-group which is known to be nonisotrivial.

Note that if $g = (x,y) \in G$, then the differential of multiplication by g^{-1} at g takes $(u,v) \in T(G)_g$ to $(u/x, v) \in L(G)$. Hence by 2.5, the generalized logarithic derivative lD_s corresponding to s is: $lD_s(x,y) = ((\partial(x) - xy)/x, \partial(y)) = (\partial(x)/x - y, \partial(y))$.

In particular $(G,s)^\sharp = Ker(lD_s) = \{(x,y) \in G : \partial(x)/x = y, \partial(y) = 0\}$, which can be identified with the subgroup of G_m defined by the second order equation $\partial(\partial(x)/x) = 0$.

3 Differential Galois theory

The conventions of the previous section are in force. In particular (K,∂) is a differential field of characteristic 0, and we may refer also to the universal differential field extension $(\mathcal{U},\partial)$ of K.

By a "logarithmic differential equation" over a differential field (K,∂) we mean something of the form
(*) $lD_s(x) = a$
where (G,s) is an algebraic D-group defined over K and $a \in L(G)(K)$. (So the indeterminate x ranges over G). As remarked in the introduction, a special case is the equation $\partial(X) = AX$, where X ranges over GL_n and A is an $n \times n$ matrix over K.

In order to give the right notion of a differential Galois extension of K for (*), we need to place a further restriction on K, G and s.

Definition 3.1 *Suppose (G,s) is an algebraic D-group over K. We say that (G,s) is K-large, if for every (maybe reducible) algebraic D-subvariety X of G, which is defined over K, $X(K)\cap(G,s)^\sharp$ is Zariski-dense in X.*

Remark 3.2 *(i) The intuitive meaning of (G,s) being K-large is that $(G,s)^\sharp$ has enough points with coordinates in K.*
(ii) Suppose that G is defined over C_K and that $s=s_0$, the 0-section of $T(G)$. Then, if C_K is algebraically closed, (G,s) is K-large.

In the next remark, we refer to *differential closures* of K. A differential closure of K is a differential field extension of K which embeds over K into any differentially closed field containing K. The differential closure of K is unique up to K-isomorphism, and is written as $\hat{K}$. Kolchin calls $\hat{K}$ the constrained closure of K.

Remark 3.3 *(G,s) is K-large if and only $(G,s)^\sharp(K)=(G,s)^\sharp(\hat{K})$ for some (any) differential closure $\hat{K}$ of K.*

Proof. Assume first that $(G,s)^\sharp(K)=(G,s)^\sharp(\hat{K})$. Let X be a D-subvariety of (G,s) defined over K. The irreducible components $X_1,..,X_r$ of X are defined over $\bar{K}$ so also $\hat{K}$ and are also D-subvarieties of (G,s). By [7] for example for any nonempty Zariski open subset of any X_i defined over $\hat{K}$, there is $a\in U(\hat{K}$ such that $\partial(a)=s(a)$. By our assumptions $a\in(G,s)^\sharp(K)$. So $X\cap(G,s)^\sharp(K)$ is Zariski-dense in X.

Conversely, suppose (G,s) is K-large. Let $a\in(G,s)^\sharp(\hat{K})$, and suppose for a contradiction that $a\notin G(K)$. Now a is *constrained* over K in the sense of section 10, Chapter III of [4]. This means that there is some differential polynomial $P(x)$ over K such that $P(a)\neq 0$, and whenever b is a differential specialization of a over K and $P(b)\neq 0$ then b is a "generic" specialization of a over K (in model-theoretic language $tp(b/K)=tp(a/K)$). Let X be the irreducible K-subvariety of G whose generic point is a. Then (as $\partial(a)=s(a)$), X is a D-subvariety of (G,s). Moreover the differential specializations of a over K are precisely those b such that $b\in X$ and $\partial(b)=s(b)$. Now subject to the conditions "$x\in X$ and $\partial(x)=s(x)$", the condition $P(x)\neq 0$ is clearly equivalent to $x \not{i}nY$ for Y some proper subvariety of X defined over K. By our assumptions there is $b\in G(K)$ such that $b\in X\setminus Y$ and $\partial(b)=s(b)$. This is a contradiction.

We call a differential ring (R, ∂) *simple* if it has no proper nontrivial differential ideals. We refer to [12] for a discussion of simple differential rings.

Lemma 3.4 *Suppose that R is a simple differential ring over K which is fintely generated over K. Then R embeds over K into some differential closure of K.*

Proof. As R has no zero-divisors, R embeds in $\mathcal{U}$ over K. Let $R = K[a]_\partial$ be differentially generated over K by the finite tuple a. Suppose that $\pi(a) = b$ is a differential specialization over K. Then π extends to a surjective ring homomorphism $\pi : R \to K[b]_\partial$. The kernel is a differential ideal, so must be trivial. Thus b is a generic specialization of a over K. It follows that a is constrained over K so lives in some differential closure of K, as does R.

We can now give the main definition.

Definition 3.5 *Let (G, s) be a K-large algebraic D-group defined over K, and $lD_s = a$ be a logarithmic differential equation over K for (G, s). By a differential Galois extension of K for the equation $lD_s = a$ we mean a differential field extension L of K of the form $K(\alpha)$ for some solution α of the equation such that $K[\alpha]$, the (differential) ring generated by K and the coordinates of α, is a simple differential ring.*

Lemma 3.6 *(Existence and uniqueness of differential Galois extensions.) If (G, s) is a K-large algebraic D-group defined over K, and $a \in L(G)(K)$, then there exists a differential Galois extension L of K for the equation $lD_s(x) = a$. Moreover, any two such extensions are isomorphic over K as differential fields.*

Proof. By Remark 2.5 (iii) there is a solution $\beta \in G(\mathcal{U})$ of $lD_s(x) = a$. Let α be a maximal differential specialization of β over K. Then $K[\alpha]$ is a simple differential ring and α is also a solution of $lD_s = a$, so we get existence. Let $L = K(\alpha)$.

Suppose L_1 is another differential Galois extension of K for the equation, generated by the solution γ say. By Lemma 3.4 we may assume that both L and L_1 are contained in a some differential closure $\hat{K}$ of K. By Remark 3.3, $(G, s)^\sharp(\hat{K}) = (G, s)^\sharp(K)$. As both α and γ are solutions of $lD_s = a$, it follows that $\alpha^{-1} \cdot \gamma \in (G, s)^\sharp(\hat{K}) = (G, s)^\sharp(K)$. Thus clearly $L = L_1$.

Here are some alternative characterizations of differential Galois extensions:

Lemma 3.7 *Let (G, s) be a K-large algebraic D-group over K, and $L = K(\alpha)$ a differential field extension of K where α is a solution of $lD_s = a$ (where $a \in L(G)(K)$). Then the following are equivalent:*
(i) L and α satisfy Definition 3.5.
(ii) L is contained in some differential closure of K,
(iii) $(G, s)^\sharp(K) = (G, s)^\sharp(L)$ and (G, s) is L-large.

Proof. (i) implies (ii) is given by Lemmas 3.4 and 3.6 and (ii) implies (iii) follows from Remark 3.3.
(iii) implies (i). Assume L satisfies (iii). Then using Remark 3.3, we have that $(G, s)^\sharp(\hat{L}) = (G, s)^\sharp(K)$. Now $\hat{K}$ embeds in $\hat{L}$ over K, hence by Lemmas 3.4 and 3.6, there is a solution $\beta \in G(\hat{L})$ of $lD_s(x) = a$, such that $K[\beta]$ is a simple differential ring. Now $\alpha = \beta \cdot g$ for some $g \in (G, s)^\sharp(K)$, so clearly $K[\alpha]$ is also a simple differential ring.

Condition (iii) is the analogue of "no new constants" in the strongly normal case.

Remark 3.8 *Suppose K is algebraically closed. Then the differential Galois extensions of K in the sense of Definition 3.5 coincide with the generalized strongly normal extensions of K in the sense of [8]. In particular, L is a strongly normal extension of K (in the sense of Kolchin) just if L is a differential Galois extension of K for an equation $lD_s(x) = a$, on an algebraic D-group (G, s) where G is defined over C_K and $s = s_0$.*

Proof. This follows from Proposition 3.4 of [8].

We now point out that given L as above, $Aut_\partial(L/K)$ has the structure of a differential algebraic group (over K) in two different ways. One corresponds to the usual differential Galois group in the linear case, and is simply of the form $(H, s)^\sharp(K)$ where H is an algebraic D-subgroup of (G, s). The other, corresponding to the "intrinsic" Galois group introduced by Katz, is of the form $(H_1, s_1)^\sharp(L)$ for s_1 another algebraic D-group structure on G (defined over K), and H_1 a D-subgroup of (G, s_1) defined over K. The algebraic D-groups (H, s) and (H_1, s_1) will be isomorphic, but not necessarily over K, unless they are commutative.

So fix $L = K(\alpha)$ as in Definition 3.5. By 3.7 we have $L < \hat{K}$ for some fixed copy of the differential closure of K. $Aut_\partial(L/K)$ denotes the group of differential field automorphisms of L over K. Note that for any $\sigma \in Aut_\partial(L/K)$, $\sigma(\alpha)$ is also a solution of $lD_s = a$, hence $\sigma(\alpha) = b \cdot c(\sigma)$ for a unique $c(\sigma) \in (G,s)^\sharp(L) = (G,s)^\sharp(K)$ $(= (G,s)^\sharp(\hat{K}))$.

Lemma 3.9 *The map c above is a group isomorphism between $Aut_\partial(L/K)$ and $(H,s)^\sharp(K)$ for some algebraic D-subgroup H of G defined over K.*

Proof. As an automorphism σ of L over K is determined by its action on α, clearly the map is a group isomorphism with its image. So all that we need is that the image is of the required form. The model-theoretic proof of this goes through showing that the image is a *definable* subgroup of $(G,s)^\sharp(\hat{K})$, and then using quantifier-elimination for differentially closed fields. We will give an algebraic proof, after first discussing the second incarnation of the Galois group.

First the equation $lD_s(x) = a$ equips G with another structure of an algebraic D-variety (but not in general an algebraic D-group). Let r_a be the right invariant vector field on G determined by a on G. So $s + r_a$ is a K-rational section of $\tau(G) \to G$, which we denote by s'. Note that the equation $lD_s(x) = a$ on G is equivalent to the equation $\partial(x) = s'(x)$.

Now G acts on itself by left translation. Let $S < G$ be the intersection of the stabilizers of the algebraic D-subvarieties of the algebraic D-variety (G,s') which are defined over K: namely (working in some algebraically closed field containing K), $S = \{g \in G : g \cdot X = X$ for all D-subvarieties X of (G,s') defined over $K\}$. S is an algebraic subgroup of G defined over K. S is precisely the analogue in our context of the intrinsic differential Galois group introduced by Katz in the Picard-Vessiot case and discussed by Bertrand in [1].

In fact G can be naturally equipped with another structure of an algebraic D-group, (G,s_1). Define the section $s_1 : G \to \tau(G)$ by $s_1(g) = s(g) + r_a(g) - l_a(g)$.

Lemma 3.10 *(G,s_1) is an algebraic D-group, and the action of G on itself by left multiplication is an action of the algebraic D-group (G,s_1) on the algebraic D-variety (G,s'). Moreover S is an algebraic D-subgroup of (G,s_1).*

Proof. An easy computation. Note that it follows that $(G,s_1)^\sharp(\mathcal{U})$ acts on $(G,s')^\sharp(\mathcal{U})$.

Let Z be the set of solutions of $lD_s(x) = a$ in $G(L)$. Note that Z is precisely $(G, s')^\sharp(L)$. In any case, Z is a principal homogeneous space for $(G, s)^\sharp(K)$ (acting on the right) As $(G, s)^\sharp(K) = (G, s)^\sharp(\hat{K})$, and $L < \hat{K}$, X is also the solution set of $lD_s(x) = a$ in $G(\hat{K})$. Clearly $Aut_\partial(L/K)$ acts on Z and any $\sigma \in Aut_\partial(L/K)$ is determined by its action on Z. In fact, as $L = K(\beta)$ for any $\beta \in Z$, $\sigma \in Aut_\partial(L/K)$ is determined by any pair $(\beta, \sigma(\beta))$ for $\beta \in Z$.

Lemma 3.11 *The action of $Aut_\partial(L/K)$ on Z is isomorphic to the action (by left multiplication) of $(S, s_1)^\sharp(L)$ on Z: The isomorphism d say takes σ to $\sigma(\beta) \cdot \beta^{-1}$ for some (any) $\beta \in Z$.*

Proof. We know that $(S, s_1)^\sharp$ acts on Z by left multiplication (by Lemma 3.10). Suppose $\sigma \in Aut_\partial(L/K)$. Let $d(\sigma) \in G(L)$ be such that $\sigma(\alpha) = d(\sigma) \cdot \alpha$. As $\sigma(\alpha) \in Z$, we have by Lemma 3.10 that $d(\sigma) \in (G, s_1)^\sharp(L)$. As any $\beta \in Z$ is of the form $\alpha \cdot c$ for $c \in (G, s)^\sharp(L) = (G, s)^\sharp(K)$ and σ fixes K pointwise, we see that
(*) $\sigma(\beta) = d(\sigma) \cdot \beta$ for all $\beta \in Z$.
We only have to see that $d(\sigma) \in S$. Let X be any D-subvariety of (G, s') defined over K. As $\hat{K}$ is differentially closed and $Z = (G, s')^\sharp(\hat{K})$, it follows that $Z \cap X$ is Zariski-dense in X. For $\beta \in Z \cap X$, $\sigma(\beta) \in Z \cap X$ too. By (*) and Zariski-denseness, $d(\sigma) \cdot X = X$. Thus $d(\sigma) \in S$.

Conversely, suppose that $g \in (S, s_1)^\sharp(L)$. Then $g \cdot \alpha \in Z$. Let X be the algebraic variety defined over K whose K-generic point is α. Then, X is a D-subvariety of (G, s'). Thus $g \cdot X = X$ and so $g \cdot \alpha \in X$, and $\partial(g \cdot \alpha) = s'(g \cdot \alpha)$. Hence $g \cdot \alpha$ is a differential specialization of α over K. By simplicity of $K[\alpha]$, α and $g \cdot \alpha$ satisfy exactly the same differential polyomial equations over K. As both α and $g \cdot \alpha$ generate L it follows that there is an automorphism σ of L over K such that $\sigma(\alpha) = g \cdot \alpha$. So $g = d(\sigma)$.

Conclusion of proof of Lemma 3.9.
Let H be the image of S under conjugation by α^{-1} in G ($g \to \alpha^{-1} \cdot g \cdot \alpha$). Then H is an algebraic D-subgroup of (G, s). By 3.11 and what we already know about 3.9, the image of the embedding $c : Aut_\partial(L/K) \to (G, s)^\sharp(K)$ is precisely $(H, s)^\sharp(K)$. As the latter is Zariski-dense in H, H is also defined over K.

Let us fix the isomorphism c between $Aut_\partial(L/K)$ and $(H, s)^\sharp(K) = (H, s)^\sharp(\hat{K})$.

Lemma 3.12 *There is a Galois correspondence between the set of differential fields in between K and L and the set of algebraic D-subgroups of (H,s) defined over K (equivalently defined over $\hat{K}$): given $K < F < L$ the corresponding group is H_F the Zariski closure of the set of $h \in (H,s)^\sharp(K)$ such that $c^{-1}(h)$ is the identity on F.*

Proof. This is Theorem 2.12 of [8] after making the translation between definable subgroups and D-subgroups.

Let us complete this section with an example. We will consider the non-isotrivial algebraic D-group structure (G,s) on $G_m \times G_a$ discussed at the end of section 2, and exhibit a (natural) differential Galois extension $K < L$ whose differential Galois group is (G,s) (or rather $(G,s)^\sharp(K)$). In fact there will be an intermediary differential field $K < F < L$ such that each of $K < F$ and $F < K$ are Picard-Vessiot extensions, but $K < L$ is NOT a Picard-Vessiot extension.

Recall that $s : G \to T(G)$ is given by $s(x,y) = (x,y,xy,0)$, and $lD_s : G \to L(G)$ is $lD_s(x,y) = (\partial(x)/x - y, \partial(y))$, and so a logarithmic differential equation on (G,s) has the form $\{\partial(x)/x - y = a_1, \partial(y) = a_2\}$. If we restrict our attention to equations where $a_1 = 0$, we obtain equations of the form $\partial(\partial(x)/x) = a$ on G_m.

Also $(G,s)^\sharp$ can be identified with $\{x \in G_m : \partial(\partial(x)/x)) = 0\}$.

Note that the embedding $x \to (x,0)$ of G_m in G and surjection $(x,y) \to y$ of G onto G_a induces an exact sequence
$0 \to (G_m, (s_0)_{G_m}) \to (G,s) \to (G_a, (s_0)_{G_a}) \to 0$
of algebraic D-groups where s_0 denotes the 0-sections for the corresponding groups. The important fact is that (G,s) is not a product (as a D-group) of the two groups, which is the reason that (G,s) is non isotrivial.

We will take the ground field of constants to be $\mathbf{C}$. Let $K = \mathbf{C}(e^{ct} : c \in \mathbf{C})$ and $L = K(t, e^{t^2})$. So L is a subfield of a fixed differential closure $\hat{\mathbf{C}} = \hat{K} = \hat{L}$ of $\mathbf{C}$.

Lemma 3.13 *(G,s) is K-large.*

Proof. It is enough to show that all solutions of $\partial(\partial(x)/x) = 0$ in $\hat{K}$ are in K, that is all solutions of $\partial(d)/d = c$ for $c \in \mathbf{C}$ which are in $\hat{K}$ are in K. But this is clear, because e^{ct} is one such such solution and the others are obtained by multiplying by a constant.

Lemma 3.14 *L is a differential Galois extension of K for the equation $\partial(\partial(x)) = 2$. The Galois group is $(G, s)^\sharp(K)$.*

Proof. Note that L is generated over K as a differential field by e^{t^2} which is a solution of $\partial(\partial(x)) = 2$. To show that the Galois group is as stated, it is enough to show that $tr.deg(L/K) = 2$, which is well-known.

Note that $K(t)$ is a Picard-Vessiot extension of K, and L is a Picard-Vessiot extension of $K(t)$, but L is not a Picard-Vessiot extension of K.

4 Isomorphisms of algebraic D-groups and algebraic D-varieties.

We will prove:

Proposition 4.1 *Suppose (K, ∂) is an algebraically closed differential field, (G, s) and (H, t) are connected algebraic D-groups over K and there is an isomorphism f between (G, s) and (H, t) defined over some differential field extension of K. Then there is such an isomorphism defined over a Picard-Vessiot extension of K.*

Here is a restatement of the theorem in the language of Kolchin's constrained cohomology (see [5]).

Corollary 4.2 *Suppose (K, ∂) has no proper Picard-Vessiot extensions. Then for any connected ∂_0-group G defined over K, $H^1_c(Aut_\partial(\hat{K}/K), G(\hat{K}))$ is trivial.*

In Proposition 3.11 of [9], Corollary 4.2 was stated in the special case that $H(K) = H(\hat{K})$ (which amounts to saying that the corresponding algebraic D-group is K-large).

Kolchin's differential tangent space and its properties (see Chapter 8 of [5]) play an important role in the proof of proposition 4.2. We will summarise the key properties (in the language of algebraic D-groups). Recall first that if V is a finite-dimensional vector space over $\mathcal{U}$, then a ∂-module structure on V is an additive map $D_V : V \to V$ such that $D_V(av) = \partial(a)v + aD_V(v)$ for all $a \in \mathcal{U}$ and $v \in V$. V^∂ denotes $\{v \in V : D_V(v) = 0\}$, a vector space over $\mathcal{C}$ with $\mathcal{C}$-dimension the same as the $\mathcal{U}$-dimension of V

Fact 4.3 *Suppose (G, s) is a connected algebraic D-group defined over K, and $V = L(G)$ is its Lie algebra (tangent space at the identity). Then*
(i) s equips V with a canonical ∂-module structure D_V, defined over K.
(ii) for any automorphism f of (G, s), $df_e \in GL(V)$ restricts to a $\mathcal{C}$-linear automorphism of V^∂, and moreover f is determined by $df_e|V^\partial$.

Proof of Proposition 4.2 First (G, s) and (H, t) will be isomorphic over $\hat{K}$. Let f be such an isomorphism. Let c be a finite tuple from $\hat{K}$ generating the smallest field of definition of f. Write f as f_c. Let (V, D_V) be as in Fact 4.3 for (G, s). Let d be a $\mathcal{C}$-basis for V^∂ which is contained in $\hat{K}$. Let $L = K < c, d >$ be the differential field generated by c and d over K.
Claim I. L is a strongly normal extension of K.
Proof. As $L < \hat{K}$, $C_L = C_K$. We have to show that for any automorphism σ of the differential field $\mathcal{U}$ fixing K pointwise, $\sigma(L)$ is contained in the differential field generated by L and $\mathcal{C}$. First $\sigma(d)$ is another basis of V^∂, so with respect to the basis d we may write $\sigma(d)$ as a $n \times n$ nonsingular matrix $B \in GL_n(\mathcal{C})$. On the other hand $f_{\sigma}(c) = f_c \cdot h$ for a unique automorphism h of (G, s). By Fact 4.3(ii), h is determined by $dh_e|V^\partial$, which by in terms of the basis d, is another nonsingular $n \times n$ matrix A over $\mathcal{C}$. It follows that $(\sigma(c), \sigma(d))$ is in the differential field generated by K, c, d, A and B. In particular $\sigma(L) \subseteq L < \mathcal{C} >$. This proves Claim I.

Claim II. L is a Picard-Vessiot extension of K.
Proof. For $\sigma \in Aut_\partial(L/K)$ write the matrices A, B (which will be in $GL_n(C_K)$) as A_σ, B_σ. Then it is easy to check that the map taking σ to (A_σ, B_σ) is an embedding of $Aut_\partial(L/K)$ into $GL_n(C_K) \times GL_n(C_K)$. From Kolchin's characterizations of Picard-Vessiot extensions, we get Claim II.

As f is defined over L, we have proved Proposition 4.1.

Finally let us give some additional results with a similar flavour. The first is really just a remark and is related to the themes of Buium's book [2]. Buium calls an algebraic D-variety (X, s) *split* if (X, s) is isomorphic (over $\mathcal{U}$) to a trivial algebraic D-variety, namely one of the form (Y, s_0) where Y is defined over $\mathcal{C}$ and s_0 is the 0-section. He proves that any D-variety (X, s) such that X is a projective variety is split. Moreover assuming Y defined over algebraically closed K, then (Y, s) is split over some strongly normal extension.

Remark 4.4 *Let K be a differential field with algebraically closed constant field. Suppose that (X, s) is an algebraic D-variety over K which is split. Then (X, s) is split over some strongly normal extension K_1 of K.*

Proof. We can find an isomorphism f of (X, s) with some trivial (Y, s_0) which is defined over $\hat{K}$. Let again $f = f_c$ with c the smallest field of definition of f. Let $K_1 = K < c >$. Then, as K_1 is contained in $\hat{K}$ and $C_{\hat{K}} = C_K$, we have that $C_{K_1} = C_K$. For any (differential) automorphism σ of $\mathcal{U}$, Then $f_{\sigma(c)} = f_c \circ g$ for some automorphism g of (Y, s_0). But then g must be defined over $\mathcal{C}$. Hence $\sigma(c)$ is rational over $K(c)(\mathcal{C})$, so $\sigma(K_1) \subseteq K_1 < \mathcal{C} >$. This shows that K_1 is a strongly normal extension of K.

The next result is a generalization of Proposition 4.1 in which the higher-dimensional versions of differential tangent spaces from [10] enter the picture. We use freely the results from that paper.

Proposition 4.5 *Suppose that K has algebraically closed constant field. Let (X, s) and (Y, t) be algebraic D-varieties defined over K. Suppose that $a \in (X, s)^\sharp(K)$, $b \in (Y, t)^\sharp(K)$ are nonsingular points on X, Y respectively, and that there is some isomorphism f between (X, s) and (Y, t) such that $f(a) = b$. Then there is such an isomorphism which is defined over a Picard-Vessiot extension of K.*

Proof. For each $m \geq 1$, let V_m the the $\mathcal{U}$-vector space $\mathcal{M}/\mathcal{M}^{m+1}$ where $\mathcal{M}$ is the maximal ideal of the local ring of X at a. V_m is defined over K. For any automorphism h of X such that $h(a) = a$, h induces a linear automorphism h_m say of V_m. Moreover if h' is another automorphism of X fixing a, then $h = h'$ iff $h_m = h'_m$ for all m. So far nothing has been said about the D-variety structure. As $a \in (X, s)^\sharp$, ∂ extends to a derivation ∂' on the local ring of X at a which preserves $\mathcal{M}$ and all its powers. This gives V_m the structure of a ∂ module (V_m, D_{V_m}) over $\mathcal{U}$, defined over K (for all m). If h is an automorphism of the algebraic D-variety (X, s) which fixes a then $h_m \in GL(V_m)$ restricts to a $\mathcal{C}$-linear automorphism h^∂_m of $GL(V^\partial_m)$ (where V^∂_m is the solution space of $D_{V_m} = 0$). Moreover h_m is determined by h'_m.

Now we may find an isomorphism f between (X, s) and (Y, t) such that $f(a) = b$ and f is defined over $\hat{K}$. Let c generate the smallest field of definition of f. So c is a finite tuple from $\hat{K}$. Write f as f_c. For any differential automorphism σ of $\mathcal{U}$ which fixes K pointwise, $\sigma(f_c) = f_{\sigma(c)}$ is

also an isomorphism of (X,s) with (Y,t) taking a to b. Hence $f_{\sigma(c)} \circ f_c^{-1}$ is an automorphism of (X,s) taking a to itself. Write h^σ for this map.
Claim I. There is m such that for all $\sigma, \tau \in Aut_\partial(\mathcal{U}/K)$, $h^\sigma = h^\tau$ iff $(h_m^\sigma)' = (h_m^\tau)'$.
Proof. This follows from compactness and the earlier remarks as the set of h^σ is a uniformly definable family of automorphisms of (X,s).

Now let m be as in Claim I and let d be a $\mathcal{C}$-basis for $(V_m)^\partial$.
Claim II. $K_1 = K < c, d >$ is a strongly normal extension of K.
Proof. Let $\sigma \in Aut_\partial(\mathcal{U}/K)$. As $\sigma(d)$ is also a basis for $(V_m)^\partial$, $\sigma(d) \in K < d >< \mathcal{C} >$. On the other hand, by virtue of d, $(V_m)^\partial$ can be identified with $\mathcal{C}^r$ for suitable r, and thus $(h_m^\sigma)^\partial$ can be identified with an element of $GL_r(\mathcal{C})$. By Claim I, it follows that $\sigma(c) \in K < c, d >< \mathcal{C} >$. Thus $\sigma(K_1) \subseteq K_1 < \mathcal{C} >$.

As in the proof of 4.1, we conclude that actually K_1 is a Picard-Vessiot extension of K. As $c \in K_1$, this gives the proposition.

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