O-MINIMAL PREPARATION THEOREMS

L. VAN DEN DRIES AND P. SPEISSEGGER

To Angus Macintyre, on his 60th birthday

1. Introduction

Macintyre, Marker and Van den Dries [6] use model theory to show that globally subanalytic functions are piecewise given by terms in a certain language. Adding symbols for the exponential and logarithm function to this language, they also show that functions definable from exponentiation and globally subanalytic functions are piecewise given by terms. This piecewise definability by terms inspired Lion and Rolin [9] to find geometric proofs for what they call *preparation theorems*, which are sometimes more useful in applications. (A *prepared* function of several variables depends in a piecewise simple way on any chosen variable. An earlier preparation theorem for *subanalytic* functions is due to Parusinski [11]. Still earlier, Denef [2],[3] obtained results of this kind in the p-adic *algebraic* setting; see also Cluckers [1] for a recent p-adic *analytic* version and a nice application.)

In Section 2 we prove a preparation theorem for functions belonging to a polynomially bounded o-minimal structure. In Section 3 we extend this to functions that are *logarithmic-exponential* over a polynomially bounded o-minimal structure.

For a survey of o-minimality emphasizing geometry we refer to [5]. The rest of the introduction updates aspects of this survey and fixes notation and terminology. Recall from [5] that a **structure** on $\mathbb{R}$ is a family $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ such that for each n:

(S1) $\mathcal{S}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$ with $\mathbb{R}^n \in \mathcal{S}_n$,
(S2) the graphs of addition and multiplication on $\mathbb{R}$ belong to $\mathcal{S}_3$,
(S3) $A \in \mathcal{S}_n$ implies $A \times \mathbb{R} \in \mathcal{S}_{n+1}$ and $\mathbb{R} \times A \in \mathcal{S}_{n+1}$,
(S4) $A \in \mathcal{S}_{n+1}$ implies $\pi(A) \in \mathcal{S}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection map defined by $\pi(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n)$.

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In this definition we view $\mathbb{R}$ as an *ordered field*, not just as an *ordered set*. Such an $\mathcal{S}$ is said to be **o-minimal** if in addition:

(O) the sets in $\mathcal{S}_1$ are exactly the subsets of $\mathbb{R} = \mathbb{R}^1$ that have only finitely many connected components, that is, the finite unions of intervals and points.

A key example of an o-minimal structure on $\mathbb{R}$ is the system of semialgebraic sets, another is the system of globally subanalytic sets (the subsets of the spaces $\mathbb{R}^n$, $n \in \mathbb{N}$, that are subanalytic in the ambient projective space $\mathbf{P}^n(\mathbb{R})$). For other examples, summarized in an inclusion diagram, see [5], p. 146. Since then, Rolin, Speissegger and Wilkie [12] established a new source of o-minimal structures, namely those generated by quasianalytic Denjoy-Carleman classes. They solve several open problems listed on p.147 of [5], namely Problem 2 (first part), Problem 3, and Problem 5.

Problem 6 of [5] asks whether Ecalle's *fonctions analysables* [8] generate an o-minimal structure on $\mathbb{R}$. This problem is still open, and a solution might throw light on the part of Hilbert's 16th problem that deals with limit cycles. The theory of o-minimal structures aims at a general framework for tameness results in real analytic geometry. This includes uniform finiteness phenomena such as conjectured by Hilbert in his 16th problem on limit cycles. Indeed, Roussarie's more precise conjectures [13] seem very much in the spirit of o-minimality.

Some terminological and notational conventions: l, m, n, p and q will range over $\mathbb{N} = \{0, 1, 2, 3, \dots\}$; given a structure $\mathcal{S}$ on $\mathbb{R}$, a set $S \subseteq \mathbb{R}^m$ is said to belong to $\mathcal{S}$ (or to be in $\mathcal{S}$) if $S \in \mathcal{S}_m$, and a map $F : S \to \mathbb{R}^n$ with $S \subseteq \mathbb{R}^m$ is said to belong to $\mathcal{S}$ (or to be in $\mathcal{S}$) if its graph (a subset of $\mathbb{R}^{m+n}$) belongs to $\mathcal{S}$. A **covering** of a set S is a collection $\mathcal{C}$ of subsets of S with union S.

A structure $\mathcal{S}$ on $\mathbb{R}$ is said to be **polynomially bounded** if for each function $f : \mathbb{R} \to \mathbb{R}$ in $\mathcal{S}$ there exists an $a > 0$ and an n such that $|f(x)| \leq x^n$ for all $x > a$. The system of semialgebraic sets, the system of globally subanalytic sets, the o-minimal structures generated by functions defined by series that are multisummable in the positive real direction (see [7]), and those generated by quasianalytic Denjoy-Carleman classes (see [12]) are all polynomially bounded. All known non-polynomially bounded o-minimal structures on $\mathbb{R}$ are obtained by performing certain o-minimality preserving operations on polynomially bounded o-minimal structures; we refer to [5] for details.

2. A preparation theorem for polynomially bounded o-minimal structures

We fix a polynomially bounded o-minimal structure $\mathcal{S}$ on $\mathbb{R}$. Let Λ be the set of all $\lambda \in \mathbb{R}$ such that the function $x \mapsto x^\lambda : (0,\infty) \to \mathbb{R}$ is in $\mathcal{S}$. Then Λ, a subfield of $\mathbb{R}$, is called the **field of exponents** of $\mathcal{S}$.

Theorem 2.1. *Let $f_1, \dots, f_l : \mathbb{R}^{n+1} \to \mathbb{R}$ be in $\mathcal{S}$. Then there is a finite covering $\mathcal{C}$ of $\mathbb{R}^{n+1}$ by sets in $\mathcal{S}$, and for each set $S \in \mathcal{C}$ there are exponents $\lambda_1, \dots, \lambda_l \in \Lambda$ and functions $\theta, a_1, \dots, a_l : \mathbb{R}^n \to \mathbb{R}$ and $u_1, \dots, u_l : \mathbb{R}^{n+1} \to \mathbb{R}$, all in $\mathcal{S}$, such that* graph θ *is disjoint from S and for $i = 1, \dots, l$ and all $(x, y) = (x_1, \dots, x_n, y) \in S$ we have*

$$f_i(x, y) = |y - \theta(x)|^{\lambda_i} a_i(x) u_i(x, y), \quad |u_i(x, y) - 1| < \frac{1}{2}.$$

Remarks.

(1) In the globally subanalytic case this holds with a more precise form for the units u_i, see [9]. This extra information has nice consequences for integrals of globally subanalytic functions in [10], similar to the p-adic integrations in [1], [2].

(2) We can replace the inequality $|u_i(x, y) - 1| < \frac{1}{2}$ for any given $\epsilon > 0$ by $|u_i(x, y) - 1| < \epsilon$, with a covering depending on ϵ. This follows from the proof of the theorem.

(3) The theorem can be seen as a geometric equivalent of valuation-theoretic facts established in [4] and [7], and the proof below consists of model-theoretic arguments deriving it from those facts. (Except for this proof we only use direct analytic and geometric arguments in the rest of the paper.)

We introduce the model-theoretic framework by setting $\widetilde{\mathbb{R}} = (\mathbb{R}, \mathcal{S})$ and viewing $\widetilde{\mathbb{R}}$ in the usual way as an $\mathcal{L}$-structure where $\mathcal{L}$ is the first-order language of ordered rings together with an n-ary function symbol for each function $\mathbb{R}^n \to \mathbb{R}$ in $\mathcal{S}$; "definable" will mean "definable with parameters". Thus by the properties (S1)–(S4) of $\mathcal{S}$, the sets and maps definable in $\widetilde{\mathbb{R}}$ are exactly the sets and maps in $\mathcal{S}$.

The valuation theoretic facts alluded to are first transformed into a result on one-variable functions defined in arbitrary models $\mathcal{R}$ of the theory $T := \mathrm{Th}(\widetilde{\mathbb{R}})$. Since $\widetilde{\mathbb{R}}$ has a unique elementary embedding into any such model $\mathcal{R}$, we identify $\widetilde{\mathbb{R}}$ via this embedding with an elementary submodel of $\mathcal{R}$. The underlying set of $\mathcal{R}$ is denoted by R.

Lemma 2.2. *Let $\mathcal{R} \models T$ and suppose $f_1, \dots, f_l : R \to R$ are definable in $\mathcal{R}$. Then there is a finite covering $\mathcal{C}$ of R by definable subsets, and* *for each $S \in \mathcal{C}$ there is $\theta \in R \setminus S$, and there are $\lambda_1, \dots, \lambda_l \in \Lambda$ and $a_1, ..., a_l \in R$ such that for $i = 1, \dots, l$ and all $y \in S$ we have*

$$f_i(y) = |y - \theta|^{\lambda_i} a_i u \quad \text{for some } u \in R \text{ with } |u - 1| < \frac{1}{2}.$$

Proof. Let $\mathcal{R}'$ be an elementary extension of $\mathcal{R}$ with underlying set R', and let $t \in R'$. The function $R' \to R'$ defined in $\mathcal{R}'$ by the same $\mathcal{L}_{\mathcal{R}}$-formula that defines f in $\mathcal{R}$ is also denoted by f. By model-theoretic compactness it suffices to show:

(∗) there exist $\lambda_1, \dots, \lambda_l \in \Lambda$, $\theta, a_1, \dots, a_l \in R$ and $u_1, \dots, u_l \in R'$ (all depending on t) such that $\theta \neq t$ and for $i = 1, \dots, l$ we have

$$f_i(t) = |t - \theta|^{\lambda_i} a_i u_i \quad \text{and} \quad |u_i - 1| < \frac{1}{2}.$$

Note that if i is such that $f_i(t) = 0$, then the equality and inequality above hold with $a_i = 0$ and $u_i = 1$ (and any $\theta \neq t$ and λ_i). Thus we may as well assume that $f_i(t) \neq 0$ for all i. Let $\mathcal{R}\langle t\rangle$ as in [4] denote the elementary submodel of $\mathcal{R}'$ generated by t over $\mathcal{R}$. We give $\mathcal{R}\langle t\rangle$ its natural valuation v, with valuation ring $\{b \in \mathcal{R}\langle t\rangle : |b| \leq r \text{ for some real } r > 0\}$. We identify the residue field of $\mathcal{R}\langle t\rangle$ in the usual way with the real field $\mathbb{R}$. By the Valuation Property [7], and by [4], we have $v(\mathcal{R}\langle t\rangle^\times) = v(\mathcal{R}^\times) + \Lambda v(t - \theta)$ for some $\theta \neq t$ in $\mathcal{R}$. With such a θ we have for $i = 1, \dots, l$ that $v(f_i(t)) = \lambda_i v(t - \theta) + v(a_i)$ with $\lambda_i \in \Lambda$ and $a_i \in \mathcal{R}$, hence $f_i(t) = |t - \theta|^{\lambda_i} a_i u_i$ as required in (∗), except that instead of $|u_i - 1| < 1/2$ we have $v(u_i) = 0$. But $v(u_i) = 0$ means that the residue class of u_i is a non-zero real number. Multiplying a_i by that real number and u_i by its inverse, we achieve that $v(u_i - 1) > 0$, in particular $|u_i - 1| < 1/2$. □

Since the lemma holds for all models of T, it implies a parametric version by a familiar model-theoretic compactness argument. With parameters in $\widetilde{\mathbb{R}}$ this gives the preparation theorem 2.1.

We can improve on this description as follows. This improvement is not used in the remainder of the paper, and for simplicity we only state it for a single function.

Theorem 2.3. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be in $\mathcal{S}$. Then there is a finite covering $\mathcal{C}$ of $\mathbb{R}^{n+1}$ by sets in $\mathcal{S}$, and for each set $S \in \mathcal{C}$ there are $\lambda, \mu \in \Lambda$ with $\mu > 0$, and functions $\theta, a, b : \mathbb{R}^n \to \mathbb{R}$ and $u : \mathbb{R}^{n+1} \to \mathbb{R}$,* *all in $\mathcal{S}$, such that* graph θ *is disjoint from S and for all $(x,y) \in S$,*

$$f(x,y) = |y-\theta(x)|^{\lambda} a(x) u(x,y), \quad |u(x,y)-1| < \frac{1}{2} \quad \textit{and}$$
$$|u(x,y)-1| \leq |y-\theta(x)|^{\mu} |b(x)|.$$

This follows by suitably modifying the lemma while keeping its proof, except for a corresponding change in $(*)$.

For use in later sections we actually need another variant, which mentions the function $\eta : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $\eta(x,y) = y$. Put

$$\mathbb{R}^{n+1}_{\eta\neq 0} := \{(x,y) \in \mathbb{R}^{n+1} : y \neq 0\}.$$

Theorem 2.4. *Let $f_1,\dots,f_l : \mathbb{R}^{n+1} \to \mathbb{R}$ be in $\mathcal{S}$. Then there is a finite covering $\mathcal{C}$ of $\mathbb{R}^{n+1}_{\eta\neq 0}$ by cells in $\mathcal{S}$, and for each $C \in \mathcal{C}$ there is a function $\theta : \mathbb{R}^n \to \mathbb{R}$ in $\mathcal{S}$ with either $\theta = 0$, or $0 < |\eta-\theta| \leq |\eta|/2$ on C, and there are exponents $\lambda_1,\dots,\lambda_l \in \Lambda$, functions $a_1,\dots,a_l : \mathbb{R}^n \to \mathbb{R}$ and $u_1,\dots,u_l : \mathbb{R}^{n+1} \to \mathbb{R}$, all in $\mathcal{S}$, such that*

$$f_i(x,y) = |y-\theta(x)|^{\lambda_i} a_i(x) u_i(x,y), \quad |u_i(x,y)-1| < \frac{1}{2}.$$

for $i = 1,\dots,l$ and all $(x,y) \in C$.

In this theorem we have *cells* in $\mathcal{S}$ rather than just *sets* in $\mathcal{S}$ as in Theorem 2.1. But this difference is only for later convenience: a covering by sets in $\mathcal{S}$ can always be refined to one by cells in $\mathcal{S}$ by cell decomposition. In the same way we obtained Theorem 2.1 from Lemma 2.2 we obtain Theorem 2.4 from a variant of that lemma:

Lemma 2.5. *Let $\mathcal{R} \models T$ and suppose $f_1,\dots,f_l : R \to R$ are definable in $\mathcal{R}$. Then there is a finite covering $\mathcal{C}$ of $R \setminus \{0\}$ by definable subsets, and for each $S \in \mathcal{C}$ there is $\theta \in R$ with either $\theta = 0$, or $0 < |y-\theta| \leq |y|/2$ for all $y \in S$, and there are $\lambda_1,\dots,\lambda_l \in \Lambda$ and $a_1,...,a_l \in R$ such that for $i = 1,\dots,l$ and all $y \in S$ we have*

$$f_i(y) = |y-\theta|^{\lambda_i} a_i u \quad \textit{for some } u \in R \textit{ with } |u-1| < \frac{1}{2}.$$

The proof follows closely that of Lemma 2.2 and we only indicate the few changes needed. In our situation we have $t \neq 0$, and in $(*)$ the condition $\theta \neq t$ is replaced by "$\theta = 0$ or $0 < |\theta - t| \leq |t|/2$". We have

$(**)$ $\qquad v(\mathcal{R}\langle t\rangle^{\times}) = v(\mathcal{R}^{\times}) + \Lambda v(t-\theta)$

for some $\theta \neq t$ in R. It remains to show that there is a $\theta \neq t$ in R satisfying $(**)$ with $\theta = 0$ or $0 < |\theta - t| \leq |t|/2$.

Indeed, if $v(\mathcal{R}\langle t\rangle^{\times}) = v(\mathcal{R}^{\times})$ or $v(t) \notin v(\mathcal{R}^{\times})$, then $\theta = 0$ works, and otherwise any $\theta \neq t$ in R satisfying $(**)$ will do.

3. Generalities on adjoining exponentiation

As in the previous section $\mathcal{S}$ is a polynomially bounded o-minimal structure on $\mathbb{R}$. In addition we assume that $\exp\big|[0,1]$ is in $\mathcal{S}$, as is the case for the polynomially bounded o-minimal structures mentioned in the introduction, except for the system of semialgebraic sets. (It follows that for any real numbers $a<b$ the function $\exp\big|[a,b]$ is in $\mathcal{S}$, as well as the function $\log\big|[a,b]$ if $a>0$.)

Our aim is to give an explicit description of the structure $\mathcal{S}(\exp)$ on $\mathbb{R}$ that is generated by the exponential function over $\mathcal{S}$, that is, the smallest structure on $\mathbb{R}$ that contains all sets in $\mathcal{S}$ and the exponential function. We achieve this aim in Theorem 3.2 and Corollary 3.3, following the treatment by Lion and Rolin [9] of the case $\mathcal{S} = \{\text{globally subanalytic sets}\}$. For efficient formulations of these results we need some terminology.

Consider a family $\mathcal{A} = (\mathcal{A}_n)_{n\in\mathbb{N}}$ such that for each n:

(i) $\mathcal{A}_n$ is a set of functions from $\mathbb{R}^n$ to $\mathbb{R}$;
(ii) whenever $f,g\in\mathcal{A}_n$, then $-f$, $f+g$ and fg belong to $\mathcal{A}_n$;
(iii) the function on $\mathbb{R}^n$ taking the constant value 1 belongs to $\mathcal{A}_n$;
(iv) for each $f\in\mathcal{A}_n$ the functions $(x_1,\dots,x_n,x_{n+1})\mapsto f(x_1,\dots,x_n)$ and $(x_1,\dots,x_n,x_{n+1})\mapsto f(x_2,\dots,x_{n+1})$ on $\mathbb{R}^{n+1}$ are in $\mathcal{A}_{n+1}$;
(v) the functions $(x,y)\mapsto x$ and $(x,y)\mapsto y$ on $\mathbb{R}^2$ belong to $\mathcal{A}_2$.

Note that $\mathcal{A}_0$ is a subring of $\mathbb{R}$ where we identify each function in $\mathcal{A}_0$ with its unique real value.

A function $f:\mathbb{R}^n\longrightarrow\mathbb{R}$ is said to be an **$\mathcal{A}$-function** if f belongs to $\mathcal{A}_n$. A **basic $\mathcal{A}$-set** in $\mathbb{R}^n$ is a set of the form

$$\{x\in\mathbb{R}^n:\ f(x)=0,\ g_1(x)>0,\dots,g_k(x)>0\},$$

where $f,g_1,\dots,g_k:\mathbb{R}^n\longrightarrow\mathbb{R}$ are $\mathcal{A}$-functions. An **$\mathcal{A}$-set** in $\mathbb{R}^n$ is a finite union of basic $\mathcal{A}$-sets in $\mathbb{R}^n$. Thus the $\mathcal{A}$-sets in $\mathbb{R}^n$ form a boolean algebra of subsets of $\mathbb{R}^n$.

An **$\mathcal{A}$-cylinder** in $\mathbb{R}^{n+1}$ is a subset of $\mathbb{R}^{n+1}$ of one of the following forms:

$$\begin{aligned}(f_1,f_2)_S &:= \big\{(x,y)\in\mathbb{R}^{n+1}:\ x\in S,\ f_1(x)<y<f_2(x)\big\},\\ (-\infty,f)_S &:= \big\{(x,y)\in\mathbb{R}^{n+1}:\ x\in S,\ y<f(x)\big\},\\ (f,+\infty)_S &:= \big\{(x,y)\in\mathbb{R}^{n+1}:\ x\in S,\ f(x)<y\big\},\\ \operatorname{graph}(f)_S &:= \big\{(x,y)\in\mathbb{R}^{n+1}:\ x\in S,\ y=f(x)\big\},\end{aligned}$$

where S is an $\mathcal{A}$-set in $\mathbb{R}^n$ and $f,f_1,f_2:\mathbb{R}^n\longrightarrow\mathbb{R}$ are $\mathcal{A}$-functions. For cylinders of the first type we can assume that $f_1<f_2$ on S, by shrinking S if necessary. Note that $\mathcal{A}$-cylinders in $\mathbb{R}^{n+1}$ are $\mathcal{A}$-sets in $\mathbb{R}^{n+1}$ of a

special form. The intersection of any two $\mathcal{A}$-cylinders in $\mathbb{R}^{n+1}$ is a finite union of $\mathcal{A}$-cylinders in $\mathbb{R}^{n+1}$, and the complement of any $\mathcal{A}$-cylinder in $\mathbb{R}^{n+1}$ is a finite union of $\mathcal{A}$-cylinders in $\mathbb{R}^{n+1}$; hence the finite unions of $\mathcal{A}$-cylinders in $\mathbb{R}^{n+1}$ form a boolean algebra of subsets of $\mathbb{R}^{n+1}$. In the expression "$\mathcal{A}$-cylinder in $\mathbb{R}^{n+1}$" we often omit the specification "in $\mathbb{R}^{n+1}$" if the ambient space is clear from the context.

We say that $\mathcal{A}$ admits **cylindrical decomposition** if for each n each $\mathcal{A}$-set in $\mathbb{R}^{n+1}$ is a finite union of $\mathcal{A}$-cylinders. Cylindrical decomposition is a very strong property, as shown by the next lemma, which follows easily from the definitions and remarks above.

Lemma 3.1. *Suppose $\mathcal{A}$ admits cylindrical decomposition. Then the family $(\{\mathcal{A}\text{-sets in } \mathbb{R}^n\})_{n\in\mathbb{N}}$ is a structure on $\mathbb{R}$. If in addition $\mathcal{A}_0 = \mathbb{R}$, then this structure is o-minimal.*

We are going to apply this to the family $\mathrm{LE}(\mathcal{S}) = (\mathrm{LE}(\mathcal{S})_n)_{n\in\mathbb{N}}$; each $\mathrm{LE}(\mathcal{S})_n$ is the ring of all functions on $\mathbb{R}^n$ obtained by superposition of functions in $\mathcal{S}$ with exp and log. Here exp is the usual exponential function on $\mathbb{R}$, and log its inverse, extended to all of $\mathbb{R}$ by setting $\log x := 0$ for $x \leq 0$. A precise definition of $\mathrm{LE}(\mathcal{S})_n$ is as follows. Let $\mathcal{L}_{\mathrm{LE}}$ be the language of the previous section augmented by the unary function symbols exp and log. Then each $\mathcal{L}_{\mathrm{LE}}$-term $t(v_1, \ldots, v_n)$ defines a function $(x_1, \ldots, x_n) \mapsto t(x_1, \ldots, x_n)$ on $\mathbb{R}^n$, and $\mathrm{LE}(\mathcal{S})_n$ is the set of all functions defined by such terms. Note that the conditions (i)—(v) imposed on $\mathcal{A}$ are satisfied by $\mathrm{LE}(\mathcal{S})$. Since the functions $(x, y) \mapsto \max(x, y)$ and $(x, y) \mapsto \min(x, y)$ on $\mathbb{R}^2$ belong to $\mathrm{LE}(\mathcal{S})_2$, the intersection of any two $\mathrm{LE}(\mathcal{S})$-cylinders in $\mathbb{R}^{n+1}$ is an $\mathrm{LE}(\mathcal{S})$-cylinder (not just a finite union of such cylinders).

Theorem 3.2. $\mathrm{LE}(\mathcal{S})$ *admits cylindrical decomposition.*

Taking into account the previous lemma, this theorem yields the promised explicit description of $\mathcal{S}(\exp)$ as well as o-minimality:

Corollary 3.3. $\mathcal{S}(\exp)_n = \{\mathrm{LE}(\mathcal{S})\text{-sets in } \mathbb{R}^n\}$ *for each* n.

Corollary 3.4. $\mathcal{S}(\exp)$ *is o-minimal.*

Theorem 3.2 follows easily from Theorem B in [7], obtained there by valuation theory and model-theory. Here we prefer to derive Theorem 3.2 from two more precise preparation theorems for $\mathrm{LE}(\mathcal{S})$-functions, namely Theorems 4.11 and 5.4, which are the main results of this paper.

4. Preparation for Logarithmic Functions

In this section we prepare the $\mathrm{LE}(\mathcal{S})$-functions in $n+1$ variables that are purely logarithmic in the last variable, Theorem 4.11. It is worth noting that no induction on n will be involved, so n will be fixed throughout. For the rest of the paper we keep the conventions of the previous section, and let $(x,y) = (x_1,\ldots,x_n,y)$ range over $\mathbb{R}^{n+1}$.

Lemma 4.1. *Let $E \subseteq \mathbb{R}^{n+1}$ be an $\mathrm{LE}(\mathcal{S})$-cylinder, and $g \in \mathrm{LE}(\mathcal{S})_{n+1}$ such that g has constant sign on $E \cap (\{x\} \times \mathbb{R})$, for each $x \in \mathbb{R}^n$. Then E is the disjoint union of $\mathrm{LE}(\mathcal{S})$-cylinders $E_<, E_=, E_>$ on each of which g has constant sign.*

The characteristic function $\chi_S : \mathbb{R}^n \to \mathbb{R}$ of any $\mathrm{LE}(\mathcal{S})$-set $S \subseteq \mathbb{R}^n$ belongs to $\mathrm{LE}(\mathcal{S})_n$.

Proof. Suppose $E = (f_1, f_2)_S$ with S an $\mathrm{LE}(\mathcal{S})$-set and $f_1, f_2 \in \mathrm{LE}(\mathcal{S})_n$. Then $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = \frac{1}{2}(f_1(x) + f_2(x))$ belongs to $\mathrm{LE}(\mathcal{S})_n$. Hence $S_* := \{x \in S : g(x, f(x)) * 0\}$ is an $\mathrm{LE}(\mathcal{S})$-set, for each $* \in \{<,=,>\}$. Thus the lemma holds with $E_* = (f_1, f_2)_{S_*}$, with $* \in \{<,=,>\}$. Other cases, such as $E = (f, +\infty)_S$, are handled similarly.

For the assertion on characteristic functions we just consider the case that $S = \{x \in \mathbb{R}^n : f(x) = 0 \text{ or } g(x) > 0\}$ with $f, g \in \mathrm{LE}(\mathcal{S})_n$. (The general case is handled similarly.) The function $\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $\alpha(s,t) = 1$ if $s = 0$ or $t > 0$, and $\alpha(s,t) = 0$ otherwise, belongs to $\mathcal{S}$, hence $\chi_S = \alpha(f,g)$ belongs to $\mathrm{LE}(\mathcal{S})_n$. ☐

Recall that $\eta : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is given by $\eta(x,y) = y$.

Definition 4.2. Let $E \subseteq \mathbb{R}^{n+1}$ be an $\mathrm{LE}(\mathcal{S})$-set. A **logarithmic scale on** E is a tuple $(\eta_0, \ldots, \eta_r)$ of functions $\eta_i : \mathbb{R}^{n+1} \to \mathbb{R}$ such that:

- (i) for each i, either $\eta_i > 0$ on E, or $\eta_i < 0$ on E;
- (ii) there is $\theta_0 \in \mathrm{LE}(\mathcal{S})_n$ and $0 < \epsilon_0 < 1$ such that $\eta_0 = \eta - \theta_0$, and either $0 < |\eta_0| \leq \epsilon_0 |\eta|$ on E, or $\theta_0 = 0$;
- (iii) for $i = 1, \ldots, r$ there is $\theta_i \in \mathrm{LE}(\mathcal{S})_n$ and $0 < \epsilon_i < 1$ such that $\eta_i = \log|\eta_{i-1}| - \theta_i$, and either $0 < |\eta_i| \leq \epsilon_i |\log|\eta_{i-1}||$ on E, or $\theta_i = 0$.

(Thus each $\eta_i \in \mathrm{LE}(\mathcal{S})_{n+1}$.)

Given such a logarithmic scale $(\eta_0, \ldots, \eta_r)$, we shall throughout write y_i instead of $\eta_i(x,y)$, in analogy with $\eta(x,y) = y$. Until Theorem 4.11 below we fix an $\mathrm{LE}(\mathcal{S})$-set E in $\mathbb{R}^{n+1}$ and a logarithmic scale $(\eta_0, \ldots, \eta_r)$ on E, with $r \geq 1$. We also let $\theta_0, \ldots, \theta_r \in \mathrm{LE}(\mathcal{S})_n$ be as in the definition above.

Here are some useful consequences of the definitions.

Lemma 4.3. *Let* $j \in \{1, \dots, r\}$ *and define* $\kappa_j : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ *by* $\kappa_j(x,y) = (x, y_j)$. *Then we have the following.*

(1) κ_j *is injective on* E.
(2) $\kappa_j(E)$ *is an* LE($\mathcal{S}$)*-set; if* E *is an* LE($\mathcal{S}$)*-cylinder, so is* $\kappa_j(E)$.
(3) *If* C *is an* LE($\mathcal{S}$)*-cylinder contained in* $\kappa_j(E)$, *then* $E \cap \kappa_j^{-1}(C)$ *is an* LE($\mathcal{S}$)*-cylinder contained in* E.
(4) *Define* $\zeta_0, \dots, \zeta_{r-j} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ *by* $\zeta_0(x,z) := z$ *and* $\zeta_i(x,z) := \log|\zeta_{i-1}(x,z)| - \theta_{i+j}(x)$ *for* $1 \le i \le r-j$. *Then* $\eta_i = \zeta_{i-j} \circ \kappa_j$ *for* $i = j, \dots, r$, *and* $(\zeta_0, \dots, \zeta_{r-j})$ *is a logarithmic scale on* $\kappa_j(E)$.
(5) *If* $(\overline{\zeta}_0, \dots, \overline{\zeta}_p)$ *is a logarithmic scale on* $\kappa_j(E)$ *and* $\overline{\eta}_{j+i} := \overline{\zeta}_i \circ \kappa_j$ *for* $0 \le i \le p$, *then* $(\eta_0, \dots, \eta_{j-1}, \overline{\eta}_j, \dots, \overline{\eta}_{j+p})$ *is a logarithmic scale on* E.

Proof. We leave most of this to the reader, and only sketch the case $j = 1$; the general case then follows by induction on j. In (2) and (3) we distinguish the cases $\eta_0 > 0$ on E and $\eta_0 < 0$ on E. If $\eta_0 > 0$ on E, then for $(x,z) \in \mathbb{R}^{n+1}$ we have

$$(x,z) \in \kappa_1(E) \iff (x, \theta_0(x) + e^{z+\theta_1(x)}) \in E,$$

and the inverse of the bijection $(x,y) \mapsto (x, y_1) : E \to \kappa_1(E)$ is given by $(x,z) \mapsto (x, \theta_0(x) + e^{z+\theta_1(x)})$. If $\eta_0 < 0$ on E, then we get similar explicit expressions. Now (2) and (3) follow by considering each type of cylinder separately. □

Lemma 4.4. *Let* $0 \le i \le r$ *and* $\theta \in \text{LE}(\mathcal{S})_n$, *and assume that* E *is an* LE($\mathcal{S}$)*-cylinder. Then the sets*

$$\{(x,y) \in E : y_i < \theta(x)\}, \quad \{(x,y) \in E : y_i = \theta(x)\},$$
$$\{(x,y) \in E : y_i > \theta(x)\}$$

are LE($\mathcal{S}$)*-cylinders.*

This follows easily by induction on i.

Given $M > 1$, we put

$$E_{>M} = E_{>M}(\eta_0, \dots, \eta_r) := \{(x,y) \in E : \ |y_i| > M \text{ for } i = 1, \dots, r\},$$
$$E_{i,M} = E_{i,M}(\eta_0, \dots, \eta_r) := \{(x,y) \in E : \ |y_i| \le M\}, \quad 1 \le i \le r.$$

By Lemma 4.4, if E is an LE($\mathcal{S}$)-cylinder, then $E_{>M}$ is a LE($\mathcal{S}$)-cylinder, and each $E_{i,M}$ is a union of two LE($\mathcal{S}$)-cylinders.

Let f and g be real valued functions and S a set contained in both their domains. We write $f \sim_S g$, if there exists ϵ with $0 < \epsilon < 1$ such that for all $x \in S$ we have $g(x) \neq 0$ and $\epsilon \le \frac{f(x)}{g(x)} \le 1/\epsilon$ (in particular, $\text{sign}(f(x)) = \text{sign}(g(x))$ for all $x \in S$).

Lemma 4.5. *Let $1 \le i \le r$.*

(1) *There is a constant $\alpha > 1$ such that if $(x, y) \in E$, and $|y_{i-1}| > 1$, then $|y_{i-1}| \ge \alpha^{|y_i|}$.*

(2) *Given any $M > 1$ there is a function $\theta \in \mathrm{LE}(\mathcal{S})_n$ such that $|\eta_{i-1}| \sim_{E_{i,M}} \theta$.*

Proof. Take a constant $a > 0$ such that $|\eta_i| \le a|\log|\eta_{i-1}||$ on E. Let $(x, y) \in E$ and $|y_{i-1}| > 1$. Then $|y_i|/a \le \log|y_{i-1}|$, hence $|y_{i-1}| \ge \alpha^{|y_i|}$, where $\alpha := \exp(1/a) > 1$. For the second assertion, let $(x, y) \in E_{i,M}$, so $-M \le \log|y_{i-1}| - \theta_i(x) \le M$, hence

$$\exp(-M + \theta_i(x)) \le |y_{i-1}| \le \exp(M + \theta_i(x)).$$

Thus $\theta := \exp(\theta_i)$ has the required property. □

In the next two lemmas we fix exponents $\lambda_1, \dots, \lambda_r \in \Lambda$.

Lemma 4.6. *Let $c \in \mathbb{R}$. Then there is $M > 1$ such that*

$$\left| c + \sum_{i=1}^{r} \lambda_i \log|y_i| \right| \le \frac{|y_1|}{2} \quad \text{for all } (x, y) \in E_{>M}.$$

This follows easily from the first part of the previous lemma.

Lemma 4.7. *Let $\psi \in \mathrm{LE}(\mathcal{S})_n$ and $M > 1$. Then*

(1) *If $1 \le i \le r$ and $|\eta_{i-1}| \sim_{E_{>M}} \psi$, then there is an $N > 1$ such that $|\eta_i| \sim_{E_{>N}} |\log\psi - \theta_i|$.*

(2) *If $|\eta_1| \sim_{E_{>M}} \psi$, then there are $N > 1$ and $\xi \in \mathrm{LE}(\mathcal{S})_n$ such that $|\eta_1|^{\lambda_1} \cdots |\eta_r|^{\lambda_r} \sim_{E_{>N}} \xi$.*

Proof. Let $0 < a < b$ be such that $a|y_{i-1}| \le \psi(x) \le b|y_{i-1}|$ for all $(x, y) \in E_{>M}$. Taking logarithms and subtracting $\theta_i(x)$ gives

$$\log a + y_i \le \log\psi(x) - \theta_i(x) \le \log b + y_i,$$

for all $(x, y) \in E_{>M}$; this, together with Lemma 4.5 implies (1) by taking absolute values. Part (2) follows easily from (1). □

Next we define the functions in $\mathrm{LE}(\mathcal{S})_{n+1}$ that are "purely logarithmic in the last variable".

Definition 4.8. The subring $\mathrm{L}(\mathcal{S})_{n+1,s}$ of $\mathrm{LE}(\mathcal{S})_{n+1}$ is obtained by recursion on $s \in \mathbb{N}$ as follows. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$. Then $f \in \mathrm{L}(\mathcal{S})_{n+1,0}$ iff there are $F : \mathbb{R}^{m+1} \to \mathbb{R}$ in $\mathcal{S}$ and $f_1, \dots, f_m \in \mathrm{LE}(\mathcal{S})_n$ such that $f(x, y) = F(f_1(x), \dots, f_m(x), y)$ for all (x, y); for $s > 0$ we declare that $f \in \mathrm{L}(\mathcal{S})_{n+1,s}$ iff there are $F : \mathbb{R}^{m+l} \longrightarrow \mathbb{R}$ in $\mathcal{S}$ and $f_1, \dots, f_{m+l} \in \mathrm{L}(\mathcal{S})_{n+1,s-1}$ such that for all (x, y) we have

$$f(x, y) = F(f_1(x, y), \dots, f_m(x, y), \log f_{m+1}(x, y), \dots, \log f_{m+l}(x, y)).$$

Note that by this definition, $\mathrm{L}(\mathcal{S})_{n+1,p} \subseteq \mathrm{L}(\mathcal{S})_{n+1,q}$ for $p \leq q$, and that $\eta_i \in \mathrm{L}(\mathcal{S})_{n+1,i}$ for $i = 0, \ldots, r$.

For the next two lemmas, let $\alpha : \mathbb{R}^{m+r+1} \to \mathbb{R}$ be in $\mathcal{S}$ and $\phi_1, \ldots, \phi_m$ in $\mathrm{LE}(\mathcal{S})_n$, and define $g : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ by

$$g(x, y) = \alpha(\phi_1(x), \ldots, \phi_m(x), y_0, y_1, \ldots, y_r).$$

(Thus $g \in \mathrm{L}(\mathcal{S})_{n+1,r}$.) There are $\eta_{ij} \in \mathrm{L}(\mathcal{S})_{n+1,j-i}$ for $0 \leq i < j \leq r$ such that $y_j = \eta_{ij}(x, y_i)$ for all (x, y), and below we fix such η_{ij}.

Lemma 4.9. *Suppose* $|\eta_0| \sim_E \xi$, *where* $\xi \in \mathrm{LE}(\mathcal{S})_n$. *Then there exists* $h \in \mathrm{L}(\mathcal{S})_{n+1,r-1}$ *such that* $g\big|E = h\big|E$.

Proof. On E we have

$$y_1 = \log|y_0| - \theta_1(x) = \log\frac{|y_0|}{\xi(x)} + \log\xi(x) - \theta_1(x).$$

Let $0 < \epsilon < 1$ be such that $\epsilon \leq \frac{|y_0|}{\xi(x)} \leq 1/\epsilon$ on E, and define $\log^* : \mathbb{R} \longrightarrow \mathbb{R}$ by $\log^*(t) = \log t$ if $\epsilon \leq t \leq 1/\epsilon$, and $\log^* t = 0$ otherwise. Then $\log^*$ belongs to $\mathcal{S}$. We modify η_1 to $\eta_1^* : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\eta_1^*(x, y) := \log^*\frac{|y_0|}{\xi(x)} + \log\xi(x) - \theta_1(x).$$

Then η_1 and η_1^* are equal on E, but $\eta_1^* \in \mathrm{L}(\mathcal{S})_{n+1,0}$. For $1 < j \leq r$ we define $\eta_j^* : \mathbb{R}^{n+1} \to \mathbb{R}$ by $\eta_j^*(x, y) := \eta_{1j}(x, \eta_1^*(x, y))$. Then the function $h : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$h(x, y) := \alpha(\phi_1(x), \ldots, \phi_m(x), \eta_0(x, y), \eta_1^*(x, y), \ldots, \eta_r^*(x, y))$$

has the desired properties. □

Let $\overline{\eta} = (\overline{\eta}_0, \ldots, \overline{\eta}_q)$ be a second logarithmic scale on E such that $\eta_0 = \overline{\eta}_0$ and $q \geq 1$. Let $\overline{\theta}_0, \ldots, \overline{\theta}_q$ be the associated functions. Put

$$\overline{E}_{>M} := E_{>M}(\overline{\eta}_0, \ldots, \overline{\eta}_q), \quad \overline{E}_{i,M} := E_{i,M}(\overline{\eta}_0, \ldots, \overline{\eta}_q).$$

Lemma 4.10. *Suppose* $|\eta_0| \sim_E |\overline{\eta}_1|^{\mu_1} \cdots |\overline{\eta}_q|^{\mu_q} B$, *where* $\mu_1, \ldots, \mu_q \in \Lambda$ *and* $B \in \mathrm{LE}(\mathcal{S})_n$. *Then there are* $M > 1$ *and* $h \in \mathrm{L}(\mathcal{S})_{n+1,r-1}$ *such that* $g = h$ *on* $\overline{E}_{>M}$.

Proof. Note that $B > 0$ on E. Let $0 < a < b$ be such that

$$a|\overline{y}_1|^{\mu_1} \cdots |\overline{y}_q|^{\mu_q} B(x) \leq |\overline{y}_0| \leq b|\overline{y}_1|^{\mu_1} \cdots |\overline{y}_q|^{\mu_q} B(x)$$

for all $(x, y) \in E$. Taking logarithms and subtracting $\overline{\theta}_1$ gives

$$\log a + L + \log B(x) - \overline{\theta}_1(x) \leq \overline{y}_1 \leq \log b + L + \log B(x) - \overline{\theta}_1(x)$$

for $(x,y) \in E$, where $L := \sum_{i=1}^{q} \mu_i \log|\overline{y}_i|$. By Lemma 4.6, we can choose $M > 1$ such that $|\log a + L| \le |\overline{y}_1|/2$ and $|\log b + L| \le |\overline{y}_1|/2$ for all $(x,y) \in \overline{E}_{>M}$. It follows that $1/2 \le |\log B(x) - \overline{\theta}_1(x)|/|\overline{y}_1| \le 3/2$ for $(x,y) \in \overline{E}_{>M}$, hence $|\overline{\eta}_1| \sim_{\overline{E}_{>M}} |\log B - \overline{\theta}_1|$. Increasing M if necessary, Lemma 4.7(2) and $\eta_0 = \overline{\eta}_0$ yield $|\eta_0| \sim_{\overline{E}_{>M}} \xi$ with $\xi \in \mathrm{LE}(\mathcal{S})_n$. Now Lemma 4.9 provides $h \in \mathrm{L}(\mathcal{S})_{n+1,r-1}$ such that $g = h$ on $\overline{E}_{>M}$. □

Call $U : \mathbb{R}^k \to \mathbb{R}$ a **unit** if $|U - 1| < 1/2$ on $\mathbb{R}^k$ (hence $U > 0$ on $\mathbb{R}^k$).

Theorem 4.11. *Let $f_1, \dots, f_l \in \mathrm{L}(\mathcal{S})_{n+1,s}$. Then there is a finite covering $\mathcal{C}$ of $\mathbb{R}^{n+1}_{\eta\neq 0}$ by $\mathrm{LE}(\mathcal{S})$-cylinders, and for each $C \in \mathcal{C}$ there are $r \le s$, a logarithmic scale $(\eta_0, \dots, \eta_r)$ on C, exponents $\lambda_{i0}, \dots, \lambda_{ir}$ in Λ, functions $A_i \in \mathrm{LE}(\mathcal{S})_n$ and units $U_i \in \mathrm{LE}(\mathcal{S})_{n+1}$ $(i = 1, \dots, l)$ such that for each such i*

$$f_i = |\eta_0|^{\lambda_{i0}} \cdots |\eta_r|^{\lambda_{ir}} \cdot A_i \cdot U_i \quad \text{on } C.$$

Corollary 4.12. *Let $f_1, \dots, f_l \in \mathrm{L}(\mathcal{S})_{n+1,s}$ and $\sigma_1, \dots, \sigma_l \in \{-1, 0, 1\}$. Then the set $\{(x,y) : y \neq 0, \operatorname{sign} f_1(x,y) = \sigma_1, \dots, \operatorname{sign} f_l(x,y) = \sigma_l\}$ is a finite union of $\mathrm{LE}(\mathcal{S})$-cylinders in $\mathbb{R}^{n+1}$.*

Proof. Take $\mathcal{C}$ and all associated data as in Theorem 4.11. Then for each $C \in \mathcal{C}$, $\operatorname{sign} f_i(x,y) = \operatorname{sign} A_i(x)$ for $i = 1, \dots, l$ and all $(x,y) \in C$, since the other factors on the right hand side in the display are positive. Hence we can always refine the covering $\mathcal{C}$ in the theorem to a finite covering of $\mathbb{R}^{n+1}_{\eta\neq 0}$ by $\mathrm{LE}(\mathcal{S})$-cylinders such that each f_i has constant sign on each cylinder of this refinement. □

Proof of Theorem 4.11. By induction on s; the case $s = 0$ follows from Theorem 2.4. So assume that $s > 0$ and that the theorem holds for lower values of s.

We first derive three claims from this inductive assumption. In all claims we are given $r \in \{1, \dots, s\}$, a logarithmic scale $(\eta_0, \dots, \eta_r)$ on an $\mathrm{LE}(\mathcal{S})$-cylinder $C \subseteq \mathbb{R}^{n+1}_{\eta\neq 0}$ and functions $\phi_1, \dots, \phi_m \in \mathrm{LE}(\mathcal{S})_n$. Put $\phi := (\phi_1, \dots, \phi_m) : \mathbb{R}^n \to \mathbb{R}^m$.

Claim 1. *Let $\alpha_1, \dots, \alpha_l : \mathbb{R}^{m+r} \longrightarrow \mathbb{R}$ be in $\mathcal{S}$. Then there is a finite covering $\mathcal{P}$ of C by $\mathrm{LE}(\mathcal{S})$-cylinders, and for each $P \in \mathcal{P}$ there are $q \in \{1, \dots, r\}$, a logarithmic scale $(\overline{\eta}_0, \dots, \overline{\eta}_q)$ on P with $\overline{\eta}_0 = \eta_0$, exponents $\mu_{i1}, \dots, \mu_{iq}$ in Λ, a function $B_i \in \mathrm{LE}(\mathcal{S})_n$ and a unit $U_i \in \mathrm{LE}(\mathcal{S})_{n+1}$ for $i = 1, \dots, l$ such that for each such i*

$$\alpha_i(\phi, \eta_1, \dots, \eta_r) = |\overline{\eta}_1|^{\mu_{i1}} \cdots |\overline{\eta}_q|^{\mu_{iq}} \cdot B_i \cdot U_i \quad \text{on } P.$$

Proof. With $(\zeta_0, \dots, \zeta_{r-1})$ as in Lemma 4.3, we note that $\kappa_1(C)$ is an $\mathrm{LE}(\mathcal{S})$-cylinder. The function $\alpha_i(\phi, \zeta_0, \dots, \zeta_{r-1})$ belongs to $\mathrm{L}(\mathcal{S})_{n+1,r-1}$

for each i, so the inductive hypothesis applies. This gives a finite covering $\mathcal{Q}$ of $\kappa_1(C)$ by $\mathrm{LE}(\mathcal{S})$-cylinders and for each $Q \in \mathcal{Q}$ a logarithmic scale $(\overline{\zeta}_0, \dots, \overline{\zeta}_{q-1})$ with $1 \leq q \leq r$, exponents $\mu_{i1}, \dots, \mu_{iq} \in \Lambda$, a function $B_i \in \mathrm{LE}(\mathcal{S})_n$ and a unit $U_i \in \mathrm{LE}(\mathcal{S})_{n+1}$ for $i = 1, \dots, l$, such that for each such i

$$\alpha_i(\phi, \zeta_0, \dots, \zeta_{r-1}) = |\overline{\zeta}_0|^{\mu_{i1}} \cdots |\overline{\zeta}_{q-1}|^{\mu_{iq}} B_i U_i \quad \text{on } Q$$

Claim 1 now follows from part (5) of Lemma 4.3. □

For $G \subseteq \mathbb{R}^{m+r+1}$ and $S \subseteq C$, put

$$S_G := \{(x,y) \in S : \ (\phi(x), y_1, \dots, y_r, y_0) \in G\}.$$

Claim 2. *Let $G \subseteq \mathbb{R}^{m+r+1}$ and $\alpha : \mathbb{R}^{m+r+1} \longrightarrow \mathbb{R}$ belong to $\mathcal{S}$ and put $g := \alpha(\phi, \eta_0, \dots, \eta_r)$. Moreover, let $S \subseteq C$ be an $\mathrm{LE}(\mathcal{S})$-cylinder, $q \in \{0, \dots, r\}$, $(\overline{\eta}_0, \dots, \overline{\eta}_q)$ a logarithmic scale on S with $\overline{\eta}_0 = \eta_0$, $\beta : \mathbb{R}^{k+q} \longrightarrow \mathbb{R}$ in $\mathcal{S}$ and $\psi_1, \dots, \psi_k \in \mathrm{LE}(\mathcal{S})_n$, and assume that*

$$|\eta_0| \sim_{S_G} \beta(\psi, \overline{\eta}_1, \dots, \overline{\eta}_q),$$

where $\psi := (\psi_1, \dots, \psi_k) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$. Then there is a finite covering $\mathcal{T}$ of S by $\mathrm{LE}(\mathcal{S})$-cylinders such that for each $T \in \mathcal{T}$ we have $g|T_G = h|T_G$ for some $h \in \mathrm{L}(\mathcal{S})_{n+1,r-1}$.

Proof. Note that $g \in \mathrm{L}(\mathcal{S})_{n+1,r}$ and $\beta(\psi, \overline{\eta}_1, \dots, \overline{\eta}_q) \in \mathrm{L}(\mathcal{S})_{n+1,q}$. We proceed by induction on q. If $q = 0$, Claim 2 follows from Lemma 4.9, so we assume that $q > 0$ and Claim 2 holds for lower values of q. By Claim 1 with β in place of $\alpha_1, \dots, \alpha_k$, and ψ and $\overline{\eta}_0, \dots, \overline{\eta}_q$ in place of ϕ and $\eta_0, \dots, \eta_r$ we reduce to the situation that we have exponents $\mu_1, \dots, \mu_q \in \Lambda$ and a function $B \in \mathrm{LE}(\mathcal{S})_n$ such that

$$|\eta_0| \sim_{S_G} |\overline{\eta}_1|^{\mu_1} \cdots |\overline{\eta}_q|^{\mu_q} \cdot B. \tag{4.1}$$

(This reduction involves replacing S by each cylinder of some finite covering of S by $\mathrm{LE}(\mathcal{S})$-cylinders, and $(\overline{\eta}_0, \dots, \overline{\eta}_q)$ by a logarithmic scale on each of those cylinders, without increasing q.)

Hence Lemma 4.10 yields $M > 1$ such that $g = h$ on $(\overline{S}_{>M})_G$ for some $h \in \mathrm{L}(\mathcal{S})_{n+1,r-1}$. Since $S = \overline{S}_{>M} \cup \overline{S}_{1,M} \cup \cdots \cup \overline{S}_{q,M}$, it remains to show that for each $j \in \{1, \dots, q\}$ the claim holds with $\overline{S}_{j,M}$ in place of S. So after renaming we have $j \in \{1, \dots, q\}$ such that $|\overline{\eta}_j| \leq M$ on S. Then Lemma 4.5(2) gives $\xi \in \mathrm{LE}(\mathcal{S})_n$ such that $|\overline{\eta}_{j-1}| \sim_S \xi$. If $j = 1$, Claim 2 follows from Lemma 4.9 (because $\eta_0 = \overline{\eta}_0$), so we assume below that $j \geq 2$.

Define $\kappa : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ by $\kappa(x,y) := (x, \overline{y}_{j-1})$ and $\overline{\zeta}_0, \dots, \overline{\zeta}_{q-j+1} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ by $\overline{\zeta}_0(x,z) := z$ and $\overline{\zeta}_i(x,z) := \log|\overline{\zeta}_{i-1}(x,z)| - \overline{\theta}_{i-j+1}(x)$ for $1 \leq i \leq q-j+1$. So $\overline{\eta}_i = \overline{\zeta}_{i-j+1} \circ \kappa$ for $i = j-1, \dots, q$ and

$(\overline{\zeta}_0,\dots,\overline{\zeta}_{q-j+1})$ is a logarithmic scale on $\kappa(S)$ by part (4) of Lemma 4.3. Then $|\overline{\zeta}_0| \sim_{\kappa(S)} \xi$, so Lemma 4.9 gives $h \in \mathrm{L}(\mathcal{S})_{n+1,q-j}$ such that

$$|\overline{\zeta}_0|^{\mu_{j-1}} \cdots |\overline{\zeta}_{q-j+1}|^{\mu_q} B = h \quad \text{on } \kappa(S). \tag{4.2}$$

The inductive hypothesis of the theorem applied to h yields a reduction to the case where we have a $p \le q-j$, a logarithmic scale $(\widehat{\zeta}_0,\dots,\widehat{\zeta}_p)$ on $\kappa(S)$, exponents $\nu_0,\dots,\nu_p \in \Lambda$, a function $\widehat{B} \in \mathrm{LE}(\mathcal{S})_n$ and a unit $\widehat{U} \in \mathrm{LE}(\mathcal{S})_{n+1}$ such that

$$h = |\widehat{\zeta}_0|^{\nu_0} \cdots |\widehat{\zeta}_p|^{\nu_p} \widehat{B}\widehat{U} \quad \text{on } \kappa(S). \tag{4.3}$$

(This reduction uses part(3) of Lemma 4.3 and involves replacing S by each cylinder of some finite covering of S by LE($\mathcal{S}$)-cylinders.) Put $\widehat{\eta}_i := \widehat{\zeta}_{i-j+1}$ for $j-1 \le i \le j-1+p$. Then $(\overline{\eta}_0,\dots,\overline{\eta}_{j-2},\widehat{\eta}_{j-1}.\dots,\widehat{\eta}_{j-1+p})$ is a logarithmic scale on S by part (5) of Lemma 4.3. By (4.1), (4.2) and (4.3) we have

$$|\eta_0| \sim_{S_G} |\overline{\eta}_1|^{\mu_1} \cdots |\overline{\eta}_{j-2}|^{\mu_q} |\widehat{\eta}_{j-1}|^{\nu_0} \cdots |\widehat{\eta}_{j-1+p}|^{\nu_p} \widehat{B}.$$

The function on the right hand side has the form

$$\widehat{\beta}(\widehat{B},\overline{\eta}_1,\dots,\overline{\eta}_{j-2},\widehat{\eta}_{j-1},\dots,\widehat{\eta}_{j-1+p})$$

with $\widehat{\beta} : \mathbb{R}^{1+(j-1+p)} \longrightarrow \mathbb{R}$ in $\mathcal{S}$ and $j-1+p<q$. Now the inductive assumption on q yields the desired conclusion. ☐

Claim 3. Let $D \subseteq \mathbb{R}^{m+r+1}$ be a cell in $\mathcal{S}$. Then C_D is a finite union of LE($\mathcal{S}$)-cylinders.

Proof. Below, $u = (u_1,\dots,u_m)$ ranges over $\mathbb{R}^m$ and $v = (v_1,\dots,v_r,v_0)$ over $\mathbb{R}^{r+1}$; we set $v' = (v_1,\dots,v_r)$. Put $D' = \pi(D)$, where $\pi : \mathbb{R}^{m+r+1} \longrightarrow \mathbb{R}^{m+r}$ is given by $\pi(u,v) = (u,v')$, and put

$$C_{D'} = \{(x,y) \in C : \ (\phi(x),y_1,\dots,y_r) \in D'\}.$$

Let $\kappa := \kappa_1$ and $(\zeta_0,\dots,\zeta_{r-1})$ be as in Lemma 4.3, with C in place of E. Define $a : \mathbb{R}^{n+1} \to \mathbb{R}$ by $a = \chi_{D'}(\phi,\zeta_0,\dots,\zeta_{r-1})$, so $a \in \mathrm{L}(\mathcal{S})_{n+1,r-1}$. By the inductive hypothesis and (the proof of) Corollary 4.12 the set

$$S := \{(x,z) : a(x,z) > 0\} = \{(x,z) : a(x,z) = 1\}$$

is a finite union of LE($\mathcal{S}$)-cylinders. Since $C_{D'} = C \cap \kappa^{-1}(S)$, it follows by Lemma 4.3 that $C_{D'}$ is a finite union of LE($\mathcal{S}$)-cylinders. Replacing C by each of these cylinders we may, for the purpose of proving the claim, assume that $C_{D'} = C$. (Further reductions of this kind are made later in the proof.) We distinguish several cases.

Case a: $D = \mathrm{graph}(\alpha|D')$ where $\alpha : \mathbb{R}^{m+r} \to \mathbb{R}$ belongs to $\mathcal{S}$ and $\alpha|D'$ is continuous. Put $g = g_\alpha := \eta_0 - \alpha(\phi,\eta_1,\dots,\eta_r) : \mathbb{R}^{n+1} \to \mathbb{R}$. We will show that there is a finite covering $\mathcal{P}_\alpha$ of C by $\mathrm{LE}(\mathcal{S})$-cylinders such that g has constant sign on each $P \in \mathcal{P}_\alpha$. (Then the claim follows by noting that C_D is the union of those $P \in \mathcal{P}_\alpha$ on which g vanishes.) We can further reduce to the case that α has constant sign on D'. If α vanishes identically on D', then $g = \eta_0$ on C, so g has constant sign on C. Suppose $\alpha > 0$ on D'. (The case $\alpha < 0$ on D' is handled similarly.) Let $G := (\frac{1}{2}\alpha|D', 2\alpha|D')$, so $D \subseteq G$, $C_D \subseteq C_G$, and $\frac{1}{2}\alpha < \eta_0 < 2\alpha$ on C_G; in particular

$$|\eta_0| \sim_{C_G} \alpha(\phi, \eta_1, \dots, \eta_r).$$

By Claim 2 we can reduce to the situation where $g|C_G = h|C_G$ for some $h \in \mathrm{L}(\mathcal{S})_{n+1,r-1}$. Now the inductive hypothesis of the theorem and (the proof of) Corollary 4.12 produce a finite covering $\mathcal{Q}$ of C by $\mathrm{LE}(\mathcal{S})$-cylinders such that h has constant sign on each $Q \in \mathcal{Q}$. The continuity of $\alpha|D'$ and $C_{D'} = C$ yield that $g|C \cap (\{x\} \times \mathbb{R})$ is continuous for each x. Since

$$C_D = \{(x,y) \in C : g(x,y) = 0\} = \{(x,y) \in C_G :\ h(x,y) = 0\},$$

g has constant sign on $Q \cap (\{x\} \times \mathbb{R})$ for each $Q \in \mathcal{Q}$ and each x. Lemma 4.1 applied to g and each $Q \in \mathcal{Q}$ thus produces a refinement $\mathcal{P}$ of $\mathcal{Q}$ such that g has constant sign on each $P \in \mathcal{P}$. This finishes the proof of Claim 3 for Case a.

Case b: $D = (\alpha_1|D', \alpha_2|D')$ where $\alpha_1, \alpha_2 : \mathbb{R}^{m+r} \longrightarrow \mathbb{R}$ belong to $\mathcal{S}$, have continuous restrictions to D', and $\alpha_1 < \alpha_2$ on D'. By Case a, there are finite coverings $\mathcal{P}_{\alpha_1}$ and $\mathcal{P}_{\alpha_2}$ of C by $\mathrm{LE}(\mathcal{S})$-cylinders such that g_{α_1} has constant sign on each $P \in \mathcal{P}_{\alpha_1}$ and g_{α_2} has constant sign on each $P \in \mathcal{P}_{\alpha_2}$. Let $\mathcal{P}$ be any finite covering of C by $\mathrm{LE}(\mathcal{S})$-cylinders that is compatible with each element of $\mathcal{P}_{\alpha_1} \cup \mathcal{P}_{\alpha_2}$; then C_D is a finite union of elements of $\mathcal{P}$.

The other cases are treated similarly to Case b. This finishes the proof of Claim 3. □

We now return to the proof of Theorem 4.11. Applying the inductive hypothesis to the functions in $\mathrm{L}(\mathcal{S})_{n+1,s-1}$ involved in expressing the functions f_i as elements of $\mathrm{L}(\mathcal{S})_{n+1,s}$, we obtain a finite covering $\mathcal{C}$ of $\mathbb{R}^{n+1}_{\eta\neq 0}$ by $\mathrm{LE}(\mathcal{S})$-cylinders, and for each cylinder $C \in \mathcal{C}$ a number $r \in \{1, \dots, s\}$, a logarithmic scale $(\eta_0, \dots, \eta_{r-1})$ on C, an m, functions $F_i : \mathbb{R}^{m+r+1} \longrightarrow \mathbb{R}$ in $\mathcal{S}$ for $i = 1, \dots, l$ and a tuple $\phi = (\phi_1, \dots, \phi_m) \in (\mathrm{LE}(\mathcal{S})_n)^m$ such that for $i = 1, \dots, l$ and $(x,y) \in C$,

$$f_i(x,y) = F_i(\phi(x), y_1, \dots, y_{r-1}, y_r, y_0), \qquad \text{where } y_r = \log|y_{r-1}|.$$

We now focus attention on one particular $C \in \mathcal{C}$ with corresponding $\eta_0, \dots, \eta_{r-1}$, m and $F_1, \dots, F_l, \phi$ as above, and we write $y_r = \log|y_{r-1}|$

and $\eta_r = \log|\eta_{r-1}|$. If $r < s$, then the above identities imply $f_i|C = g_i|C$ with $g_1, \dots, g_l \in \mathrm{L}(\mathcal{S})_{n+1,s-1}$, and the inductive assumption applies to $g_1, \dots, g_l$. Thus from now on we may and shall assume that $r = s$. As before $u = (u_1, \dots, u_m)$ ranges over $\mathbb{R}^m$, $v = (v_1, \dots, v_r, v_0)$ over $\mathbb{R}^{r+1}$, and $v' := (v_1, \dots, v_r)$. By Lemma 4.4 and the inductive hypothesis, we may also assume that either $\eta_r > 0$ on C, or $\eta_r < 0$ on C; in particular, $(\eta_0, \dots, \eta_r)$ is a logarithmic scale on C.

Theorem 2.4 for $F_1, \dots, F_l$ produces a finite partition $\mathcal{D}$ of the set

$$\{(u,v) \in \mathbb{R}^{m+r+1} : \ v_i \neq 0, \ i = 0, \dots, r\}$$

by cells in $\mathcal{S}$ such that for each $D \in \mathcal{D}$ there is $\theta : \mathbb{R}^{m+r} \longrightarrow \mathbb{R}$ in $\mathcal{S}$ with either $\theta = 0$, or $0 < |v_0 - \theta(u,v')| \leq |v_0|/2$ for all $(u,v) \in D$, and there are exponents $\mu_1, \dots, \mu_l \in \Lambda$, functions $\alpha_1, \dots, \alpha_l : \mathbb{R}^{m+r} \longrightarrow \mathbb{R}$ and $U_1, \dots, U_l : \mathbb{R}^{m+r+1} \longrightarrow \mathbb{R}$, all in $\mathcal{S}$, such that

$$F_i(u,v) = |v_0 - \theta(u,v')|^{\mu_i} \alpha_i(u,v') U_i(u,v), \quad |U_i(u,v) - 1| < \frac{1}{2}$$

for $i = 1, \dots, l$ and all $(u,v) \in D$. By Claim 3 we reduce to the case that $C = C_D$, where $D \in \mathcal{D}$. Thus it remains to examine the following two cases:

Case 1: $0 < |v_0 - \theta(u,v')| \leq |v_0|/2$ for all $(u,v) \in D$. Since $C = C_D$, it follows that

$$|\eta_0| \sim_C \theta(\phi, \eta_1, \dots, \eta_r).$$

Applying Claim 2 to each f_i in place of g and taking a common refinement of the coverings obtained, we reduce to the case that $f_i\big|C = h_i\big|C$ with $h_i \in \mathrm{L}(\mathcal{S})_{n+1,r-1}$, for $i = 1, \dots, l$. Applying the inductive hypothesis once more to $h_1, \dots, h_l$ yields the desired conclusion.

Case 2: $\theta = 0$; so for $i = 1, \dots, l$ we have $\lambda_{i0} \in \Lambda$, $\alpha_i : \mathbb{R}^{m+r} \longrightarrow \mathbb{R}$ in $\mathcal{S}$ and a unit $U_i : \mathbb{R}^{m+r+1} \longrightarrow \mathbb{R}$ in $\mathcal{S}$ such that $F_i(u,v) = |v_0|^{\lambda_{i0}} \alpha_i(u,v') U_i(u,v)$ for all $(u,v) \in D$. Since $C = C_D$, it follows that for each i and all $(x,y) \in C$,

$$f_i(x,y) = |y_0|^{\lambda_{i0}} \alpha_i(\phi(x), y_1, \dots, y_r) U_i(\phi(x), y_0, \dots, y_r).$$

The theorem now follows from Claim 1 applied to $\alpha_1, \dots, \alpha_l$. □

5. Preparation for Functions in $\mathrm{LE}(\mathcal{S})$

The notations and conventions of the previous section remain in force, and n is fixed throughout. We need to consider the level of nestedness of exp in functions in $\mathrm{LE}(\mathcal{S})_{n+1}$ with respect to the last variable. To define such a level (5.3 below), we fix distinct formal variables $v_1, \dots, v_{n+1}$.

In this section we heavily use the syntax of terms. Throughout, "term" means "$\mathcal{L}_{\mathrm{LE}}$-term in which no other variables than $v_1, \dots, v_{n+1}$ occur", and t denotes a term. We assume familiarity with the notion (for terms s) of "occurrence of s in t", with the fact that different occurrences of s in t do not overlap, and with the operation of substituting a term s' for all occurrences of s in t; if s occurs in t we call s a *subterm* of t.

Note that t defines a function $(x, y) \mapsto t(x, y)$ in $\mathrm{LE}(\mathcal{S})_{n+1}$, which we also denote by t when the context demands that t be a function. Similarly, when dealing with a function $f : \mathbb{R}^p \to \mathbb{R}$ in $\mathcal{S}$, the p-ary function symbol of $\mathcal{L}$ that corresponds to the function f is denoted by f as well.

Definition 5.1. The **exponential level** of t is the number $\mathrm{e}(t) \in \mathbb{N}$ defined inductively as follows:

(i) if t is a variable v_i or a constant symbol, then $\mathrm{e}(t) = 0$;

(ii) if $t = g(t_1, \dots, t_m)$ where g is an m-ary function symbol of $\mathcal{L}$, $m \geq 1$, and $t_1, \dots, t_m$ are terms, then

$$\mathrm{e}(t) = \max \{\mathrm{e}(t_1), \dots, \mathrm{e}(t_m)\};$$

(iii) if $t = \log(s)$, where s is a term, then $\mathrm{e}(t) = \mathrm{e}(s)$;

(iv) if $t = \exp(s)$, where s is a term, then

$$\mathrm{e}(t) = \begin{cases} \mathrm{e}(s) + 1 & \text{if } v_{n+1} \text{ occurs in } s, \\ \mathrm{e}(s) & \text{otherwise.} \end{cases}$$

Remark. If $\mathrm{e}(t) = 0$, then the function t belongs to the ring

$$\mathrm{L}(\mathcal{S})_{n+1} := \bigcup_s \mathrm{L}(\mathcal{S})_{n+1,s}.$$

Every function in $\mathrm{L}(\mathcal{S})_{n+1}$ arises in this way from some t with $\mathrm{e}(t) = 0$.

We call t **exponential** if $t = \exp(s)$ for some subterm s of t in which v_{n+1} occurs. It follows by induction on $\mathrm{e}(t)$ that t has an exponential subterm if and only if $\mathrm{e}(t) > 0$, in which case t has an exponential subterm s with $\mathrm{e}(s) = \mathrm{e}(t)$. Put

$$\mathcal{E}(t) := \{s : \ s \text{ is an exponential subterm of } t \text{ and } \ \mathrm{e}(s) = \mathrm{e}(t)\}.$$

Thus $|\mathcal{E}(t)|$ counts, without multiplicity of occurrence, the number of exponential subterms of t of maximal exponential level.

Lemma 5.2. *Suppose* $\mathrm{e}(t) > 0$, $s_1, \dots, s_q$ *are distinct subterms of* t, $q \geq 1$, $r_1, \dots, r_p$ *are terms,* $C \subseteq \mathbb{R}^{n+1}$ *is an* $\mathrm{LE}(\mathcal{S})$*-cylinder, and* $\theta : \mathbb{R}^{p+q-1} \longrightarrow \mathbb{R}$ *belongs to* $\mathcal{S}$*, such that*

(i) $\exp s_i \in \mathcal{E}(t)$ *for* $i = 1, \dots, q$,
(ii) $\mathrm{e}(r_j) < \mathrm{e}(t)$ *for* $j = 1, \dots, p$, *and*
(iii) $\exp s_q \sim_C \theta(r_1, \dots, r_p, \exp s_1, \dots, \exp s_{q-1})$.

Then there is a term t' *such that* $t\big|C = t'\big|C$, *and either* $\mathrm{e}(t') < \mathrm{e}(t)$, *or* $\mathrm{e}(t') = \mathrm{e}(t)$ *and* $\mathcal{E}(t') \subseteq \mathcal{E}(t) \setminus \{\exp s_q\}$.

Proof. Let σ be the term $\theta(r_1, \dots, r_p, \exp s_1, \dots, \exp s_{q-1})$, and let $0 < a < b$ be such that $\log a \le s_q(x,y) - \log \sigma(x,y) \le \log b$ for all $(x,y) \in C$. Let $\exp^* : \mathbb{R} \to \mathbb{R}$ be defined by $\exp^*(z) := \exp(z)$ if $\log a \le z \le \log b$ and $\exp^*(z) := 0$ otherwise. Then $t\big|C = t'\big|C$ where t' is the term obtained from t by replacing all occurrences of $\exp s_q$ in t by $\sigma \cdot \exp^*\big(s_q - \log \sigma\big)$. Clearly $\mathrm{e}(\sigma \cdot \exp^*(s_q - \log \sigma)) < \mathrm{e}(\exp s_q)$. □

We define the **logarithmic depth** $\mathrm{ld}_t(s) \in \mathbb{N}$ **of a term** s **in** t inductively as follows (where $t_1, ..., t_m, t'$ denote terms):

(i) if t is a variable v_i or a constant symbol, then $\mathrm{ld}_t(s) = 0$;
(ii) if $t = g(t_1, \dots, t_m)$ where g is an m-ary function symbol of $\mathcal{L}$, $m \ge 1$, then $\mathrm{ld}_t(s) = \max\{\mathrm{ld}_{t_1}(s), \dots, \mathrm{ld}_{t_m}(s)\}$;
(iii) if $t = \exp(t')$, then $\mathrm{ld}_t(s) = \mathrm{ld}_{t'}(s)$;
(iv) if $t = \log(t')$, then $\mathrm{ld}_t(s) = \mathrm{ld}_{t'}(s) + 1$ if s occurs as a subterm in t', and $\mathrm{ld}_t(s) = 0$ otherwise.

It follows easily that $\mathrm{ld}_t(s) = 0$ if s is not a proper subterm in t, in particular, $\mathrm{ld}_t(t) = 0$.

We need to measure the logarithmic depth of the exponential subterms of t of maximal exponential level; so we put

$$\mathrm{l}(t) := \max\{\mathrm{ld}_t(s) : \ s \in \mathcal{E}(t)\} \ \text{if} \ \mathrm{e}(t) > 0, \quad \mathrm{l}(t) := 0 \ \text{if} \ \mathrm{e}(t) = 0$$

and

$$\mathcal{LE}(t) := \{s \in \mathcal{E}(t) : \ \mathrm{ld}_t(s) = \mathrm{l}(t)\}.$$

Definition 5.3. We define the **logarithmic-exponential nestedness** of t with respect to v_{n+1} to be the quadruple

$$\mathrm{n}(t) := \big(\mathrm{e}(t), \ |\mathcal{E}(t)|, \ \mathrm{l}(t), \ |\mathcal{LE}(t)|\big) \in \mathbb{N}^4.$$

We can now prove the main result of this section.

Theorem 5.4. *Suppose* $\mathrm{e}(t) > 0$. *Then there is a finite covering* $\mathcal{C}$ *of* $\mathbb{R}^{n+1}_{\eta \ne 0}$ *by* $\mathrm{LE}(\mathcal{S})$*-cylinders such that*

$(*)_{t,\mathcal{C}}$ *for each* $C \in \mathcal{C}$ *there exist terms* σ, τ *and* u *such that* $\mathrm{e}(\sigma) = 0$, $\mathrm{e}(\tau) < \mathrm{e}(t)$ *and for all* $(x,y) \in C$,

$$t(x,y) = \sigma(x,y) \cdot \exp \tau(x,y) \cdot u(x,y), \quad |u(x,y) - 1| < \frac{1}{2}.$$

Proof. Let $\mathrm{n}(t) = (e, m, l, q) \in \mathbb{N}^4$, where $\mathbb{N}^4$ is lexicographically ordered. We assume inductively that the proposition holds for values of $\mathrm{n}(t)$ lower than (e, m, l, q). Since $e > 0$ we have $m, q > 0$.

Case 1: $l = 0$. Then $m = q$, and there is $F : \mathbb{R}^{p+q} \longrightarrow \mathbb{R}$ in $\mathcal{S}$ and there are subterms $r_1, \ldots, r_p$ and distinct subterms $s_1, \ldots, s_q$ of t, all of exponential level strictly less than e, such that $\mathcal{LE}(t) = \mathcal{E}(t) = \{\exp s_1, \ldots, \exp s_q\}$ and

$$t(x,y) = F\big(r_1(x,y), \ldots, r_p(x,y), \exp s_1(x,y), \ldots, \exp s_q(x,y)\big)$$

for all $(x, y) \in \mathbb{R}^{n+1}$. Since $\mathrm{n}(t) = \mathrm{n}(F(r_1, \ldots, r_p, \exp s_1, \ldots, \exp s_q))$, we assume below that t is the term $F\big(r_1, \ldots, r_p, \exp s_1, \ldots, \exp s_q\big)$. We also write $r = (r_1, \ldots, r_p)$, and we let $z = (z_1, \ldots, z_p)$ range over $\mathbb{R}^p$, $w = (w_1, \ldots, w_q)$ range over $\mathbb{R}^q$ and put $w' = (w_1, \ldots, w_{q-1})$.

Using Theorem 2.4 we take a finite covering of $\mathbb{R}^{p+q}$ by cells A in $\mathcal{S}$ each of which satisfies one of the following three conditions:

(A1) $w_q = 0$ for all $(z, w) \in A$;
(A2) there is a function $\theta : \mathbb{R}^{p+q-1} \longrightarrow \mathbb{R}$ in $\mathcal{S}$ such that $0 < |w_q - \theta(z, w')| \le |w_q|/2$ for all $(z, w) \in A$;
(A3) $w_q \neq 0$ for all $(z, w) \in A$, and there are $\lambda \in \Lambda$ and functions $a : \mathbb{R}^{p+q-1} \longrightarrow \mathbb{R}$ and $U : \mathbb{R}^{p+q} \longrightarrow \mathbb{R}$, both in $\mathcal{S}$, such that $F(z, w) = |w_q|^\lambda a(z, w')U(z, w)$ and $|U(z, w) - 1| < \frac{1}{2}$ for all $(z, w) \in A$.

For each A in this finite covering we put

$$B := \left\{(x,y) \in \mathbb{R}^{n+1}_{\eta\neq 0} : \ \big(r(x,y), \exp s_1(x,y), \ldots, \exp s_q(x,y)\big) \in A\right\}.$$

It clearly suffices to show that for each A in this finite covering the corresponding B has a finite covering $\mathcal{C}_B$ by LE($\mathcal{S}$)-cylinders such that $(*)_{t,\mathcal{C}_B}$ holds. We now fix one such A and its corresponding B.

Claim. *B is a finite union of* LE($\mathcal{S}$)*-cylinders.*

Proof. Assume that $A = \operatorname{graph}(g\big|A')$, where $A' \subseteq \mathbb{R}^{p+q-1}$ is a cell in $\mathcal{S}$ and $g : \mathbb{R}^{p+q-1} \longrightarrow \mathbb{R}$ is a function in $\mathcal{S}$ such that $g\big|A'$ is continuous; other possibilities for A are handled similarly.

Let $h : \mathbb{R}^{p+q-1} \longrightarrow \mathbb{R}$ be the characteristic function of A', so h is in $\mathcal{S}$ and $\mathrm{n}(h(r, \exp s_1, \ldots, \exp s_{q-1})) < \mathrm{n}(t)$. Using the equivalence $(z, w') \in A' \iff h(z, w') > 0$, the inductive hypothesis and Corollary 4.12 yield a finite covering $\mathcal{P}$ of $\mathbb{R}^{n+1}_{\eta\neq 0}$ by LE($\mathcal{S}$)-cylinders such that

$$B' := \left\{(x,y) \in \mathbb{R}^{n+1}_{\eta\neq 0} : \ \big(r(x,y), \exp s_1(x,y), \ldots, \exp s_{q-1}(x,y)\big) \in A'\right\}$$

is a union of cylinders in $\mathcal{P}$. Since $B \subseteq B'$, it is enough to show that for each $P \in \mathcal{P}$, the set $B \cap P$ is a finite union of LE($\mathcal{S}$)-cylinders. We therefore fix a $P \in \mathcal{P}$ with $P \subseteq B'$. Then

$$B \cap P = \{(x,y) \in P : \ t_1(x,y) > 0, \ t_2(x,y) = 0\},$$

where $t_1 := g(r, \exp s_1, \ldots, \exp s_{q-1})$ and

$$t_2 := \log g(r, \exp s_1, \ldots, \exp s_{q-1}) - s_q.$$

Clearly $\mathrm{n}(t_i) < \mathrm{n}(t)$ for $i = 1, 2$. By the inductive hypothesis and after replacing P by each of the cylinders of a suitable finite covering of P by $\mathrm{LE}(\mathcal{S})$-cylinders, there are terms σ_1 and σ_2 such that $\mathrm{e}(\sigma_1) = \mathrm{e}(\sigma_2) = 0$ and

$$B \cap P = \{(x,y) \in P : \ \sigma_1(x,y) > 0, \ \sigma_2(x,y) = 0\}.$$

Since $\sigma_1, \sigma_2 \in \mathrm{L}(\mathcal{S})_{n+1}$, Corollary 4.12 implies that $B \cap P$ is a finite union of $\mathrm{LE}(\mathcal{S})$-cylinders. □

To finish the treatment of Case 1, we use the claim and take a finite covering of B by nonempty $\mathrm{LE}(\mathcal{S})$-cylinders; it suffices to show that each cylinder C in this covering has a finite covering $\mathcal{C}$ by $\mathrm{LE}(\mathcal{S})$-cylinders such that $(*)_{t,\mathcal{C}}$ holds. Fix such a cylinder C. Since C is not empty, condition (A1) is not satisfied.

Suppose (A2) holds. Then by Lemma 5.2 we have $t\big|C = t'\big|C$ for some term $t'(v)$ such that $\mathrm{n}(t') < \mathrm{n}(t)$. The inductive hypothesis yields a finite covering $\mathcal{C}'$ of $\mathbb{R}^{n+1}_{\eta \neq 0}$ by $\mathrm{LE}(\mathcal{S})$-cylinders such that $(*)_{t',\mathcal{C}'}$ holds. Since C is itself an $\mathrm{LE}(\mathcal{S})$-cylinder, $\mathcal{C}'$ induces a finite covering $\mathcal{C}$ of C by $\mathrm{LE}(\mathcal{S})$-cylinders such that $(*)_{t,\mathcal{C}}$ holds, as required.

Suppose (A3) holds. Then $t\big|C = t'\big|C$, where

$$t' := \exp(\lambda s_q) \cdot a(r, \exp s_1, \ldots, \exp s_{q-1}) \cdot U(r, \exp s_1, \ldots, \exp s_q).$$

Since $t'' := a(r, \exp s_1, \ldots, \exp s_{q-1})$ satisfies $\mathrm{n}(t'') < \mathrm{n}(t)$, the inductive hypothesis gives a finite covering $\mathcal{C}$ of C such that $(*)_{t'',\mathcal{C}}$ holds. It follows immediately that $(*)_{t,\mathcal{C}}$ holds as well. This concludes the treatment of Case 1.

Case 2: $l > 0$. Let $s_1, \ldots, s_q$ be the distinct subterms of t such that $\mathcal{LE}(t) = \{\exp s_1, \ldots, \exp s_q\}$, and let $\mathrm{o}(t)$ be the number of occurrences of $\exp s_q$ in t; we proceed by an auxiliary induction on $\mathrm{o}(t)$. There are a subterm s of t, a function $F : \mathbb{R}^{p+q} \longrightarrow \mathbb{R}$ in $\mathcal{S}$ and terms $r_1, \ldots, r_p$ such that

(i) $\mathrm{e}(r_j) < e$ for $j = 1, \ldots, p$, and $\mathrm{ld}_t(s) = l$,
(ii) $\exp s_q$ is a subterm of s and $\log s$ is a subterm of t, and
(iii) $s(x,y) = F(r, \exp s_1, \ldots, \exp s_q)(x,y)$ for all $(x,y) \in \mathbb{R}^{n+1}$.

Replacing all occurrences of s in t by $F(r, \exp s_1, \ldots, \exp s_q)$ does not change $\mathrm{n}(t)$ or $o(t)$, so we may and shall assume that s is the term $F(r, \exp s_1, \ldots, \exp s_q)$.

Using Theorem 2.4 we take a finite covering of $\mathbb{R}^{p+q}$ by cells A in $\mathcal{S}$ each of which satisfies one of the conditions (A1), (A2), (A3) as in Case 1. To each such A we associate a set $B \subseteq \mathbb{R}^{n+1}$ as in Case 1. It suffices to show that for each A in this finite covering the corresponding B has a finite covering $\mathcal{C}_B$ by LE($\mathcal{S}$)-cylinders such that $(*)_{t,\mathcal{C}_B}$ holds. We now fix one such A and its corresponding B. As in Case 1 we obtain a finite covering of B by nonempty LE($\mathcal{S}$)-cylinders; it suffices to show that each cylinder C in this covering has a finite covering $\mathcal{C}$ by LE($\mathcal{S}$)-cylinders such that $(*)_{t,\mathcal{C}}$ holds. Fix such a cylinder C. Since C is not empty, condition (A1) is not satisfied.

Suppose (A2) holds. Then by Lemma 5.2 we have $s\big|C = s'\big|C$ where s' is a term such that $\mathrm{e}(s') \leq \mathrm{e}(s)$ and $\mathcal{E}(s') \subseteq \mathcal{E}(s) \setminus \{\exp s_q\}$. Let t' be the term obtained by replacing every occurrence of s in t by s'. Then $t\big|C = t'\big|C$, and by the above, either $\exp s_q \in \mathcal{LE}(t')$, $\mathrm{n}(t') \leq \mathrm{n}(t)$ and $\mathrm{o}(t') < \mathrm{o}(t)$, or $\exp s_q \notin \mathcal{LE}(t')$ and $\mathrm{n}(t') < \mathrm{n}(t)$. In both cases, we are done by the inductive hypothesis.

Assume (A3) holds. Then $\log s\big|C = s'\big|C$ where

$$s' := \lambda s_q + \log a(r, \exp s_1, \ldots, \exp s_{q-1}) + \log U(r, \exp s_1, \ldots, \exp s_q).$$

We let t' be the term obtained from t by replacing every occurrence of $\log s$ in t by s'. Since $\log U$ is in $\mathcal{S}$, we are in a similar situation as in the previous paragraph: either $\exp s_q \in \mathcal{LE}(t')$, $\mathrm{n}(t') \leq \mathrm{n}(t)$ and $\mathrm{o}(t') < \mathrm{o}(t)$, or $\exp s_q \notin \mathcal{LE}(t')$ and $\mathrm{n}(t') < \mathrm{n}(t)$. Either way, we are done by the inductive hypothesis. □

Combining Theorems 4.11 and 5.4 yields:

Corollary 5.5. *Let $t_1, \ldots, t_l$ be terms. Then there is a finite covering $\mathcal{C}$ of $\mathbb{R}^{n+1}_{\eta \neq 0}$ by LE($\mathcal{S}$)-cylinders, and for each $C \in \mathcal{C}$ there are a logarithmic scale $(\eta_0, \ldots, \eta_m)$ on C, exponents $\lambda_{i0}, \ldots, \lambda_{im}$ in Λ, a term s_i, a function $A_i \in \mathrm{LE}(\mathcal{S})_n$ and a unit $u_i \in \mathrm{LE}(\mathcal{S})_{n+1}$ for $i = 1, \ldots, l$ such that for each such i*

$$t_i(x,y) = |y_0|^{\lambda_{i0}} \cdots |y_m|^{\lambda_{im}} A_i(x) \exp(s_i(x,y)) u_i(x,y)$$

for all $(x,y) \in C$, and $\mathrm{e}(s_i) < \mathrm{e}(t_i)$ if $\mathrm{e}(t_i) > 0$, and $s_i = 0$ if $\mathrm{e}(t_i) = 0$.

6. Proof of Theorem 3.2, and concluding remarks

Theorem 3.2 asserts that each LE($\mathcal{S}$)-set in $\mathbb{R}^n$ is a finite union of LE($\mathcal{S}$)-cylinders in $\mathbb{R}^n$. To prove this assertion we proceed by induction on n. The case $n = 0$ is trivial. Assume the assertion holds for a certain n. To derive the assertion for n replaced by $n+1$ we consider any functions $f_1, \dots, f_l \in \mathrm{LE}(\mathcal{S})_{n+1}$ and signs $\sigma_1, \dots, \sigma_l \in \{-1, 0, 1\}$; it suffices to show that then the set

$$\{(x, y) : \operatorname{sign} f_1(x, y) = \sigma_1, \dots, \operatorname{sign} f_l(x, y) = \sigma_l\},$$

where $(x, y) = (x_1, \dots, x_n, y)$ ranges over $\mathbb{R}^{n+1}$, is a finite union of $\mathrm{LE}(\mathcal{S})$-cylinders in $\mathbb{R}^{n+1}$. Representing $f_1, \dots, f_l$ by terms $t_1, \dots, t_l$ in $\mathcal{L}_{\mathrm{LE}}$, Corollary 5.5 yields a finite covering $\mathcal{C}$ of $\mathbb{R}^{n+1}_{\eta \neq 0}$ by $\mathrm{LE}(\mathcal{S})$-cylinders, such that for each $C \in \mathcal{C}$ we have $A_1, \dots, A_l \in \mathrm{LE}(\mathcal{S})_n$ with $\operatorname{sign} f_i(x, y) = \operatorname{sign} A_i(x)$ on C for $i = 1, \dots, l$. Thus by the inductive hypothesis, the set

$$\{(x, y) \in \mathbb{R}^{n+1}_{\eta \neq 0} : \operatorname{sign} f_1(x, y) = \sigma_1, \dots, \operatorname{sign} f_l(x, y) = \sigma_l\}$$

is a finite union of $\mathrm{LE}(\mathcal{S})$-cylinders in $\mathbb{R}^{n+1}$. The inductive hypothesis also yields that the remaining part

$$\{(x, 0) : x \in \mathbb{R}^n, \operatorname{sign} f_1(x, 0) = \sigma_1, \dots, \operatorname{sign} f_l(x, 0) = \sigma_l\}$$

is a finite union of $\mathrm{LE}(\mathcal{S})$-cylinders in $\mathbb{R}^{n+1}$. This finishes the proof of Theorem 3.2.

Comparison with the proof by Lion and Rolin. Our original intent for Section 4 was to simply "o-minimalize" the proof of Theorem II in [9], while removing some ambiguities. In the course of doing so we found a gap in [9]: in part a) of the proof of $\mathbf{D_r}$, it is assumed without justification that one can reduce to working on cylinders. In discussing this issue with Lion, he sketched a way to remedy the situation. The extra arguments needed led to the three claims in the proof of Theorem 4.11 and to various adjustments in the lemmas preceding the theorem. This explains the length of our treatment compared to [9].

The Valuation Property. In one sense the proof of Theorem 3.2 is not so different from that of the closely related Theorem B in [7]: both depend on the Valuation Property in a crucial way, here via its consequence Theorem 2.4. This Valuation Property was conjectured for power bounded o-minimal theories in [4], proved for the polynomially bounded case in [7], and recently established in full (power bounded) generality by James Tyne [14].

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University of Illinois, Department of Mathematics, 1409 W. Green Street, Urbana, IL 61801

E-mail address: `vddries@math.uiuc.edu`

University of Wisconsin, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706

E-mail address: `speisseg@math.wisc.edu`