

**THE SHARPNESS OF A CRITERION
OF MACLANE FOR THE CLASS \mathcal{A}**

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1. Introduction. A holomorphic function defined in the unit disk $\Delta = \{z : |z| < 1\}$ belongs to the MacLane class \mathcal{A} if each point ζ of a dense subset of $\partial\Delta$ is the endpoint of a curve γ_ζ (with $\gamma_\zeta \setminus \zeta \subset \Delta$) such that $f(z)$ tends to a limit (perhaps ∞) as $z \rightarrow \zeta$ on γ_ζ . The classical Fatou theorem ensures that $f \in \mathcal{A}$ when f is bounded. G. R. MacLane introduced \mathcal{A} in [5], where he proved that $f \in \mathcal{A}$ if there is a set E dense in $\partial\Delta$ with

$$(1.1) \quad \int_0^1 (1-r) \log^+ |f(re^{i\theta})| dr < \infty \quad (\theta \in E).$$

For example, if f is the modular function and $M(r) = \max_{|z|=r} |f(z)|$ its maximum modulus, then

$$\log M(r) \leq \log \frac{1}{1-r} + O(1),$$

so that (1.1) applies. An ample discussion of \mathcal{A} is in [4, Chapter 10].

MacLane's theorem concerns only local growth of f along certain rays, and is compatible with arbitrarily large growth. We show here that it is precise.

Theorem 1.2. *Let $I \subset \partial\Delta$ be an arc, and let $\{\phi_n(r); n \in \mathbb{Z}\}$ be a sequence of non-negative continuous functions with*

$$(1.3) \quad \int_0^1 (1-r)\phi_n(r) dr = \infty \quad (n \in \mathbb{Z}).$$

Let $E = \{\theta_n\}$ be dense in I . Then there exists a holomorphic function f with

$$(1.4) \quad \log |f(re^{i\theta_n})| \leq \phi_n(r) \quad (r_n \leq r \leq 1, n \in \mathbb{Z}),$$

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having no asymptotic paths terminating at points of I .

Theorem 1.2 improves earlier work [2] which produced a subharmonic function $u \notin \mathcal{A}$ whose growth along the rays $\{re^{i\theta}; \theta \in E\}$ satisfies (1.4). Our argument follows the general outline of [2], but exploits the approximation methods of R. Yulmukhametov [7] which transforms a subharmonic function u to one of the form $v(z) = \log|f(z)|$, where f is holomorphic, with good control of error. The method is based on approximating the Riesz mass $\mu = \Delta u$ by a measure ν consisting of point masses. Yulmukhametov's insight to place these masses where certain first moments are zero (cf. (4.1) below) is decisive.

It is more natural to work in the unit square $Q = (0, 1] \times [0, 1]$. Our function f will be holomorphic in Q° and continuous on Q , and we analyze the behavior of f as z approaches $\partial Q \cap \{\Re z = 0\}$. Let $E = \{Y_n; n \in \mathbb{Z}\}$ be dense in $(0, 1)$. Corresponding to (1.3), suppose a family $\Phi = \{\phi_n(x)\}$ of nonnegative continuous functions on $(0, 1]$ is given, with

$$(1.5) \quad \int_0^1 x\phi_n(x) dx = \infty \quad (\phi_n \in \Phi).$$

(Condition (1.5) is the analogue of (1.3) when Δ is replaced by Q .) We then produce systems of curves Γ'_N and γ'_N whose Hausdorff distances to $\partial Q \cap \{x = 0\}$ tend to zero as $N \rightarrow \infty$ such that

$$(1.6) \quad \inf_{\Gamma'_N} |f| \uparrow \infty, \quad \sup_{\gamma'_N} |f(z)| = O(1) \quad (N \rightarrow \infty),$$

such that for some positive sequence $\{x_n\}$,

$$(1.7) \quad \log|f(x + iY_n)| < \phi_n(x) \quad (0 < x \leq x_n, n \in \mathbb{Z}^+).$$

MacLane showed that (1.6) is equivalent to f having no asymptotic curves which terminate at points of $I_0 = \{(0, y); 0 < y < 1\} \subset \partial Q$. We sketch the proof. It is obvious that (1.6) prevents asymptotic curves which terminate at any point of I_0 . On the other hand, if f has no asymptotic curves which terminate at points of I_0 , then ([5, Theorem 1]) there must be sequence of level-sets (Koebe arcs) $\{|f| = \lambda\}$ which coverge to I_0 . These play the role of the γ'_N in (1.6), and if there is no corresponding sequence Γ'_N as in (1.6), then it is easy to see that f must be

bounded in some subregion $\{(x, y); 0 < x < \varepsilon, y \in I_1\}$, where I_1 is some subarc of I_0 . Fatou's theorem then produces a set of asymptotic curves (lines) in Q which terminate at a dense set of points of I_1 .

Another well-known fact concerning \mathcal{A} was obtained by R. Hornblower: if

$$\int^1 \log^+ \log^+ M(r) dr < \infty,$$

then $f \in \mathcal{A}$. All efforts to show this condition sharp require supplementary hypotheses (see [4]); to identify the sharp growth condition of $M(r)$ to guarantee that $f \in \mathcal{A}$ remains open.

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2. Notation and Lemmas.

The methods of this section are standard. A rectangle Q is called an n -square (near-square) if its sides are parallel to the coordinate axes, with ratio of sides are in the interval $(0.9, 1.1)$. Write $\partial Q = Q^t \cup Q^b \cup Q^\ell \cup Q^r$ using obvious abbreviations for top, bottom, left and right.

For a function u defined on a closed rectangle Q , let $\partial u / \partial n$ be the inward normal on ∂Q ; i.e. $-\partial u / \partial y^-$ on Q^t , $\partial u / \partial y^+$ on Q^b , $\partial u / \partial x^+$ on Q^ℓ and $-\partial u / \partial x^-$ on Q^r . This distinction is important, since our subharmonic function u will be constructed as Q is exhausted by a sequence of closed n -squares.

If Q is an n -square and $z \in Q$, then $\rho(z)$ is the distance from z to the nearest vertex of Q .

Write $A \cong B$ provided that A/B is bounded above and below by positive constants, and $A \lesssim B$ when A/B is bounded above by a positive constant. These will usually be absolute constants in the context of our normalizations, but should a constant C depend on a parameter α , we write $C(\alpha)$.

Lemma 2.1. *Let U be positive harmonic in the first quadrant Ω with boundary values 0 except on the segment $\{(x, 0) : 0 < \epsilon < x < \delta < 1\}$. Then*

$$U(x, x) \cong x^{-2} \int_{\epsilon}^{\delta} U(t, 0) t dt \quad \text{for } x > \delta/3.$$

Proof. The transformation $w = z^2$ maps the first quadrant onto the upper half plane where we may use the Poisson integral. On taking account of boundary values, we have

$$(2.2) \quad U(x, y) = \frac{4}{\pi} \int_{\epsilon}^{\delta} \frac{xy}{(x^2 - y^2 - t^2)^2 + 4x^2y^2} U(t, 0) t dt,$$

which yields the Lemma.

Lemma 2.3. *Let $Q = [0, a] \times [0, b]$ be an n -square and v a positive harmonic function on Q° with boundary values zero except on $Q^b \cap \{0 < \epsilon < x < \delta\}$, where ($0 < \delta < a/3$). Then*

$$v(z) = v(x, y) \cong x(b - y)a^{-4} \int_{\epsilon}^{\delta} v(t, 0) t dt$$

on $[0, a/2] \times [b/2, b]$. In addition, with $\rho(z)$ the distance from z to the nearest vertex of Q ,

$$\frac{\partial v}{\partial n}(z) \cong \rho(z)a^{-4} \int_{\epsilon}^{\delta} v(t, 0) t dt \quad (z \in \partial Q \setminus [0, a/2] \times [0, b/2]).$$

Proof. Let Ω be as in Lemma 2.1 and let U_1 be harmonic in Ω with $U_1 = v$ on $\partial Q \cap \partial\Omega$, $U_1 = 0$ on the rest of $\partial\Omega^*$. Let $z_Q = (a/2, a/2)$ be the ‘center’ of Q . By the maximum principle, $v(z_Q) \leq U_1(z_Q)$. On the other hand,

$$U_1(z_Q) \leq v(z_Q) + \max\{U_1(z); z \in Q^r \cup Q^t\}.$$

Now the Poisson formula (2.2) and the hypothesis $3\delta < a$ yield that

$$U_1(z) \leq \frac{3}{4}U_1(z_Q) \quad (z \in Q^r \cup Q^t),$$

and so

$$v(z_Q) \cong U_1(z_Q) \cong a^{-2} \int_{\epsilon}^{\delta} v(t, 0) t dt \quad (z \in Q^r).$$

We now apply the Harnack and boundary Harnack principles ([1], [3, Ch. 4]) to the positive harmonic functions $v(z)$ and $x(b - y)a^{-4} \int_{\epsilon}^{\delta} v(t, 0) t dt$ (whose values at z_Q are comparable), and deduce both assertions of the Lemma when $z \in \partial Q \cap [0, a/2] \times [b/2, b]$. The same argument, now comparing v and $(a - x)(b - y)a^{-4} \int_{\epsilon}^{\delta} v(t, 0) t dt$, yields the second inequality for $z \in \partial Q \setminus ([0, a/2] \times [0, b/2])$.

This lemma will be used in a slightly different formulation:

Corollary 2.4. *Let $Q = [0, a] \times [0, b]$ be an n -square and v a positive harmonic function on Q° with boundary value zero except on $Q^t \cap \{0 < \epsilon < x < \delta\}$ ($0 < \delta < a/3$). Then*

$$(2.5) \quad v(z) = v(x, y) \cong xy a^{-4} \int_{\epsilon}^{\delta} v(t, b) t dt$$

on $[0, a/2] \times [0, b/2]$, and (with $\partial/\partial n$ as defined at the beginning of this section),

$$(2.6) \quad \frac{\partial v}{\partial n}(z) \cong \rho(z) a^{-4} \int_{\epsilon}^{\delta} v(t, b) t dt \quad (z \in \partial Q \setminus [0, a/2] \times [b/2, b]).$$

Lemma 2.7. *Let $Q = [0, a] \times [0, b]$ be an n -square and u be a function harmonic in Q° having C^2 boundary values ϕ . Then*

$$\left| \frac{\partial u}{\partial n} \right| < K(a, \|\phi\|_{\infty}, \|D\phi\|_{\infty}, \|D^2\phi\|_{\infty}).$$

If in addition ϕ is constant in an ϵ -neighborhood of $(0, 0)$ then

$$\left| \frac{\partial u}{\partial n}(z) \right| \leq M|z| \quad \text{on} \quad \partial Q \cap \{|z| < 2a/3\}$$

for some M which depends on ϵ , a and the above data of ϕ .

The Lemma follows from [6, p. 143] and the fact that xy is harmonic in Q .

Lemma 2.8. *Let I be a connected set which is a line segment or the union of two perpendicular line segments which meet at a common endpoint. Assume that $|I| < 1/10$ and that μ is a probability measure on I absolutely continuous with respect to arc length. Let I^* be the smallest rectangle which contains I and ζ_I be the point in I^* such that*

$$(2.9) \quad \int_I (\zeta - \zeta_I) d\mu(\zeta) = 0.$$

(1) *If $\text{dist}(z, I^*) > \alpha|I|$, then*

$$\left| \int_I (\log|z - \zeta_I| - \log|z - \zeta|) d\mu(\zeta) \right| \leq C(\alpha)|I|^2 \text{dist}(z, I)^{-2};$$

(2) *if $\text{dist}(z, \zeta_I) > \delta|I|$, then*

$$\left| \int_I \log|z - \zeta_I| - \log|z - \zeta| d\mu(\zeta) \right| \leq C(\delta);$$

(3) if $d\mu/ds \leq K|I|^{-1}$ on I and $\zeta_0 \in I$, then

$$\int_I (\log|z - \zeta_0| - \log|z - \zeta|) d\mu(\zeta) \leq 4K \quad (z \in \mathbb{C}).$$

Proof of (1). Let $\zeta \in I$. We use the first-order Taylor expansion of $\log(z - \zeta)$ about ζ_I , together with an elementary estimate of the remainder. It follows that for some $\zeta' \in I^*$ we have that

$$\begin{aligned} \left| \int_I (\log|z - \zeta_I| - \log|z - \zeta|) d\mu(\zeta) \right| &\leq \left| \int_I (\log(z - \zeta) - \log(z - \zeta_I)) d\mu(\zeta) \right| \\ &= \frac{1}{2} \left| \int_I \frac{1}{(z - \zeta_I)^2} (\zeta' - \zeta_I)^2 d\mu(\zeta) \right| \leq C(\alpha) \frac{|I|^2}{\text{dist}(z, I)^2}. \end{aligned}$$

Proof of (2). This is similar to that of (1); we use that $|z - \zeta_I| \geq C(\delta)|I|$ in estimating the last integral.

Proof of (3). Denote the left side of the inequality by $h(z)$. Since $h \rightarrow 0$ as $z \rightarrow \infty$, it suffices to consider $z \in I$. I is either an interval or composed of two perpendicular intervals which meet at a point p such that $I \setminus \{p\}$ has two components. In the first case, we may suppose that $I = [-a, b]$, $\zeta_0 = 0$ and $z = x > 0$. Then, on taking account of positivity, we have that

$$\begin{aligned} h(x) &\leq \int_0^x (\log x - \log(x - t)) d\mu(t) + \int_x^{2x} (\log x - \log(t - x)) d\mu(t) \\ &\leq \frac{K}{|I|} \left[\int_0^x (\log x - \log(x - t)) dt + \int_x^{2x} (\log x - \log(t - x)) dt \right] \\ &\leq \frac{K}{|I|} 2x \leq 2K. \end{aligned}$$

When I consists of two orthogonal segments we apply this estimate to each of the pieces.

3. Construction.

In our construction, $Q = (0, 1] \times [0, 1] = \cup Q_k$ and the $\{Q_k\}$ are n -squares with mutually disjoint interiors; u is continuous and subharmonic in all of Q° , and harmonic in each Q_k° . Then the Riesz measure μ of u is supported on $\cup \partial Q_k \cap Q^\circ$ and will have the form

$$\Delta u \, dx dy = \frac{\partial u}{\partial \eta} dx + \frac{\partial u}{\partial \eta} dy \quad \text{on} \quad Q^0 \cap (\cup \partial Q_k),$$

where

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x^+} - \frac{\partial u}{\partial x^-}$$

on vertical line segments, and

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial y^+} - \frac{\partial u}{\partial y^-}$$

on horizontal line segments. At a point z common to an arc σ of $\partial Q_k \cup \partial Q_j$, $\partial/\partial \eta$ is the sum of the corresponding two inward normals $\partial/\partial n$, which have been considered in Lemmas 2.3 and 2.7 and Corollary 2.4. Thus, $\partial/\partial \eta$ measures the jump of the normal derivative across the two sides of σ .

The $\{Q_k\}$ and $\{u_k\}$ shall be constructed in order, each Q_k belonging to some stage $N \in [1, \infty)$. At stage N , the $\{Q_k\}$ ($k(N) \leq k \leq k(N+1) - 1$) are constructed from bottom up, Thus $R_N \equiv \cup_{k(N)}^{k(N+1)-1} Q_k$ is the rectangle $\{a(N) \leq x \leq a(N+1), 0 \leq y \leq 1\}$ with $\cup R_N = Q$ for a sequence $a(N) \downarrow 0$ to be determined. The assignment of boundary values on the $\{Q_k\} \subset R_N$ will depend on the family Φ of (1.5), the values on $\cup_{j < k} Q_j$ and a number A_N , which is produced once R_{N-1} has been determined; the $\{A_N\}$ rapidly tend to infinity.

For the first stage, let $Q_1 = [1/10, 1] \times [0, 1]$, $R_1 = Q_1$ and $u_1 \equiv 0$ on Q_1 . Thus we may take $a(0) = 1$, $a(1) = 1/10$, and let $A_1 = 0$, and $A_2 = 1$.

Suppose u has been constructed through stage $N - 1$, and that A_N has been chosen. Thus we have constructed n -squares Q_k , $k \leq k(N) - 1$, and assigned boundary values on each Q_k ; inside each Q_k , u is the Poisson integral of its boundary values. We now choose a set $\mathcal{Y}(N) = \{y_k; k(N) \leq k \leq k(N+1) - 2\} \subset E$ and, slightly abusing notation, set $y_{k(N)-1} = 0$ and $y_{k(N+1)-1} = 1$, so that for $k(N) \leq k \leq k(N+1) - 1$

$$(3.1a) \quad .99 \leq (y_k - y_{k-1})/a(N-1) \leq 1.01,$$

$$(3.1b) \quad \mathcal{Y}(N) \supset \mathcal{Y}(N-1),$$

and

$$(3.1c) \quad y_k < y_{k'} \text{ when } k < k' \text{ and } y_k, y_{k'} \in \mathcal{Y}[N].$$

This may be arranged so that

$$(3.1d) \quad \cup_N \mathcal{Y}(N) = E;$$

for example, given the $\{y_j\}(j < k, k(N) - 1 \leq j < k(N + 1) - 2)$ we take $y_k = Y_\ell$, where $Y_\ell \in E$: if (3.1a) has a solution with $y_k = y_s$ for some $s < k(N - 1) - 1$, let $Y_\ell = y_k$; otherwise take ℓ the least index compatible with (3.1a) with $Y_\ell \notin \cup_{j < k} y_j$. We will have $Q_k = [a(N), a(N - 1)] \times [y_{k-1}, y_k]$.

The precise value of $a(N)$ cannot be fixed until boundary conditions for the last Q_k in stage N are finalized. For the time being, assume that $a(N)/a(N - 1)$ is a small but positive number (see the discussion concerning (3.4)) so that the $\{Q_k\}$ are n -squares. Note from (3.1a) that no matter how small $a(N) > 0$ is taken, each $Q_k \in R_N$ will be an n -square. In extending u to the n -squares of stage N , we often ignore dependence on N of the various constants.

Recall that A_N is known when we arrive at stage N . Of course, u is defined already on

$$\gamma_{N-1} \equiv \bigcup_{Q_k \subset R_N} Q_k^r \equiv \bigcup_{Q_j \subset R_{N-1}} Q_j^\ell \equiv R_{N-1} \cap \{x = a(N - 1)\}.$$

The construction for stage N will ensure that

$$(3.2a) \quad u(z) \geq -A_N \quad (z \in R_N)$$

as well as

$$(3.2b) \quad u(z) = -A_N \quad (z \in \gamma_N \equiv R_N \cap \{x = a(N)\}).$$

The key problem is to choose choose boundary values on $Q_k^t \cup Q_k^b$ to achieve two conditions. the first is that u be subharmonic in Q ; the second is that

$$(3.3) \quad u(z) \geq 2A_N \quad (z \in \Gamma_N)$$

on a continuum $\Gamma_N \subset R_N^\circ$ such that as $N \rightarrow \infty$, $\Gamma_N \rightarrow \partial Q \cap \{x = 0\}$ with respect to Hausdorff topology, in a manner that (3.2b) and (3.3) transfer appropriately to the analytic function f such that $\log |f|$ approximates u . The geometry of Q_k (in

particular the final choice of Q_k^ℓ) will depend on certain positive numbers $\{a_k\}$, these chosen in succession such that

$$(3.4) \quad 0 < 9a_{k+1} < a_k < a(N-1)/9 \quad (k(N) \leq k < k(N+1) - 2).$$

At the end of the construction in stage N , we define $a(N) = a_{k(N+1)-1}$, and conditions (3.5) will ensure that (3.2) and (3.3) hold. For this reason, we will require that

$$(3.5) \quad \begin{aligned} u(z) &= -A_N & (z \in \partial Q_k, a(N) \leq \Re z \leq 2a_k) \\ u(z) &= -A_{N-1} & (z \in \partial Q_k, a(N-1)/2 \leq \Re z \leq a(N-1)), \end{aligned}$$

so that (3.2b) follows at once. This, together with (3.2), ensures that the estimates of Corollary 2.4 and Lemmas 2.7 and 2.8 may be applied to each $Q_k \subset R_N$ independent of the ultimate choice of $a(N)$.

We start the construction in stage N by taking $u(x, 0)$ to be a C^2 function on $[a(N), a(N-1)]$ subject to (3.2a), with

$$(3.6a) \quad u(x, 0) = 5A_N \quad (a(N-1)/4 \leq x \leq a(N-1)/3),$$

$$(3.6b) \quad \begin{aligned} u(x, 0) &= -A_N & (a(N) \leq x \leq 2a(N-1)/9) \\ &= -A_{N-1} & (a(N-1)/2 \leq x \leq a(N-1)). \end{aligned}$$

From this point on, the procedure used to define u on each $Q_k \subset R_N$ is the same: u is known on $\partial Q_k \setminus Q_k^t$ and on all n -squares Q_j , $j < k$, subject to the final choice of $a(N)$.

Note that, independent of the specific choice of a_k , there is a point $(t_{k-1}, y_{k-1}) \in Q_k^b$ and $\eta_{k-1} > 0$ such that

$$u(t, y_{k-1}) > 4A_N \quad (|t - t_{k-1}| < \eta_{k-1}).$$

When $k = k(N)$ this is the content of (3.6a), and for $k(N) + 1 \leq k \leq k(N+1) - 1$ it will follow from our assignment of u on Q_{k-1}^t in (3.8d) and (3.12), together with the fact that $Q_k^b = Q_{k-1}^t$. Recall the first inequality of (3.5); by decreasing η_{k-1} if necessary, we may assume that $t_{k-1} - \eta_{k-1} > 2a_{k-1}$ and that, independent of values assigned to u on Q_k^t , we have in addition that

$$(3.7) \quad u(x, y) > 3A_N \quad (|x - t_{k-1}| < \eta_{k-1}, 0 \leq y - y_{k-1} \leq \eta_{k-1}).$$

To determine a_k (consistent with (3.4)) and u on Q_k^t requires several steps, culminating in (3.12) and (3.13). Recall that $y_k = Y_{n(k)}$ for some $n(k) \in \mathbb{Z}$. We will define $u(x, y_k) = \phi^k(x)$ where ϕ^k is C^2 , subject to

$$(3.8a) \quad \phi^k(x) = \begin{cases} -A_N & \text{on } a(N) \leq x \leq 2a_k, \\ -A_{N-1} & \text{on } a(N-1)/2 \leq x \leq a(N-1) \end{cases}$$

(consistent with (3.5)), as well as

$$(3.8b) \quad -A_N \leq \phi^k(x) \leq \phi_{n(k)}(x) - A_{N-1} \quad (2a_k \leq x \leq 3a_k),$$

$$(3.8c) \quad -A_{N-1} \leq \phi^k(x) \leq \phi_{n(k)}(x) - A_{N-1} \quad (3a_k \leq x \leq a(N-1)/2),$$

and finally

$$(3.8d) \quad \int_{3a_k}^{a(N-1)/3} (x - a_k) \phi^k(x) dx > L_k,$$

where L_k is chosen below in (3.12) and (3.13). Note from the essential condition (1.5) as well as (3.8c) that L_k may be made as large as desired, on choosing a_k appropriately small. When $k = k(N+1) - 1$, so that Q_k is the top n -square of R_N , we may choose ϕ^k consistent with (3.8) such that the integral in (3.8d) is as large as desired, since the hypotheses of Theorem 1.2 impose no restriction on the behavior of u on $\{y = 1\}$. We then define u on Q_k^t by

$$(3.9) \quad u(x, y_k) = \phi^k(x) \quad (a(N) \leq x \leq a(N-1)).$$

Recall (3.7) and decrease a_k if necessary so that there exist $(t_k, y_k) \in Q_k^t$ and $\eta_k > 0$ with $u(t_k, y_k) > 4A_N$, and

$$(3.10a) \quad u(z) > 3A_N \quad (z \in D_k),$$

where

$$(3.10b) \quad \begin{aligned} D_k &= D'_k \cup D''_k \\ &\equiv \{|x - t_{k-1}| < \eta_{k-1}, 0 \leq y - y_{k-1} \leq \eta_{k-1}\} \\ &\quad \cup \{|x - t_k| < \eta_k, 0 \leq y_k - y \leq \eta_k\}. \end{aligned}$$

Conditions (1.5) and (3.8b) and the reasoning which gave (3.7) show this is possible.

Thus u is large in a relative Q_k -neighborhood of (t_{k-1}, y_{k-1}) and (t_k, y_k) .

Now recall the bounds of Corollary 2.4 as well as (3.8d) and (3.9), and define

$$(3.11) \quad \Lambda_k = [a(N-1)]^{-4} \int_{3a_k}^{a(N-1)/3} (x - a_k) \phi^k(x) dx,$$

so that $\Lambda_k > [a(N-1)]^{-4} L_k$. We require four conditions on L_k (or Λ_k):

$$(3.12) \quad \begin{aligned} L_k &> kA_N [a(N-1)]^{-4} \min\{(t_k \eta_k)^{-1}, (t_{k-1} \eta_{k-1})^{-1}\}; \\ L_k &> 4^k; \\ \Lambda_k &> k\eta_k^{-2}, \end{aligned}$$

as well as the final conditions (3.13) below. Inequality (3.3) (at least for large k) will then follow from (2.5) and the first condition on L_k .

To describe the final restriction of L_k , let $\partial_k = \partial_k^r \cup \partial_k^b$ be the ‘initial’ portion of ∂Q_k : $\partial_k^r = Q_k^r$, $\partial_k^b = Q_k^b$ (since boundary are assigned *a priori* on the initial portion of ∂Q_k ; when Q_k is the bottom of R_N , let $\partial_k = Q_k^r$ so that $\partial_k^b = \emptyset$). We need only check that $\Delta u \geq 0$ on $U_k \partial_k$. Let Q_ℓ be the n -square of stage $N-1$ such that $Q_\ell^t \supset Q_k^r$, and define $\lambda_k = \sup_{\partial_k^r} \lambda_k(z)$, where $\lambda_k(z) = \partial u_\ell / \partial n(z)$ ($z \in \partial_k^r$) is the inner normal derivative with respect to Q_ℓ . Note that $\lambda_k(z) \geq 0$ and is continuous on ∂_k^r . Next, let $h(x)$ be any C^2 function such that $h(x) = -A_N$ for $(a(N) \leq x \leq a(N-1)/4)$, $h(x) = -A_{N-1}$ on $(a(N-1)/2, a(N-1))$ and h increases on $[A(N-1)/4, a(N-1)/2]$. Let w_k be the harmonic function in Q_k with $w_k = u$ on $\partial Q_k \setminus Q_k^t$ and $w_k = h$ on Q_k^t , and let v_k be harmonic in Q_k with $v_k = 0$ on $\partial Q \setminus Q_k^t$ and $v_k = \phi^k - h$ on Q_k^t . Lemma 2.7 implies that if $\rho_k(z)$ is the distance of $z \in Q_k$ to the nearest vertex of Q_k , then

$$\left| \frac{\partial w_k}{\partial n}(z) \right| \leq M'_k \rho_k(z) \quad (z \in \partial_k),$$

and when Q_k is not the bottom n -square of R_N ,

$$\left| \frac{\partial u_{k-1}}{\partial n}(z) \right| \leq M'_k \rho_k(z) \quad (z \in \partial_k^b),$$

where M'_k is independent of the specific choices of $a(N)$, a_k and ϕ^k . Now

$$u(z) = w_k(z) + v_k(z) \quad (z \in Q_k),$$

so that the jump of the normal derivatives along ∂_k is given by

$$\frac{\partial u}{\partial \eta}(z) = \begin{cases} \lambda_k(z) + \partial w_k / \partial n(z) + \partial v_k / \partial n(z) & \text{on } \partial_k^r \\ \partial u_{k-1} / \partial n(z) + \partial w_k / \partial n(z) + \partial v_k / \partial n(z) & \text{on } \partial_k^b, \end{cases}$$

(suitably interpreted when Q_k is the bottom n -square). Thus by taking Λ_k in (3.11) sufficiently large (possible by decreasing a_k), we may now record our final conditions:

$$(3.13) \quad \Lambda_k > \lambda_k^2 + M'_k; \quad \frac{\partial u}{\partial \eta}(z) \cong \Lambda_k \rho_k(z) + \lambda_k(z) \quad (z \in \partial_k).$$

That the first of these is possible is the content of (1.5) and (3.11), and the second follows from (2.6) and (3.11).

To see that u is subharmonic in Q observe that (3.13) ensures that $\partial u / \partial \eta \geq 0$ on each ∂_k .

By minor changes in each ϕ^k , as k increases, we can arrange that

$$(3.14) \quad \mu(\partial_k) = \int_{\partial_k} \frac{\partial u}{\partial \eta} dx + \frac{\partial u}{\partial \eta} dy$$

always be a positive integer. (Recall that when Q_k is at the bottom of R_N , we take $\partial_k = Q_k^r$). Once all a_k and ϕ^k have been selected in stage N , fix $a(N) = a_{k(N+1)-1}$ and define

$$(3.15) \quad A_{N+1} = 100a(N)^{-2} \mu(\cup_1^{k(N+1)-1} \partial_k),$$

and begin the next stage.

Recall the components D'_k and D''_k of D_k from (3.10b). Let β'_k be the convex hull of $(D'_k)^t$ and $(D''_k)^b \subset Q_k$ and $\beta_k = \beta'_k \cup D_k$. It is clear that β_k is a continuum which joins Q_k^t to Q_k^b within Q , and (2.5) and the first condition of (3.12) ensure that

$$(3.16) \quad u \geq 2A_N \quad \text{on } \beta_k.$$

Thus if $\Gamma_N = \cup_{k(N)}^{k(N+1)-1} \beta_k$, condition (3.3) holds. In addition, (3.2b) implies that $H_N \equiv \sup_{\gamma_N} u(z) \rightarrow -\infty$ ($N \rightarrow \infty$).

This completes the construction of stage N .

It is obvious from (3.8) and (3.9) that if $y_k = Y_{n(k)} \in E$, then

$$(3.17) \quad u(x, y) \leq \phi_{n(k)}(x) - A_{N-1} \quad (y = Y_{n(k)} \in E, 0 < x < a(N-1)).$$

Thus the growth of u on the lines $\{x + iY_n; Y_n \in E\}$ fulfills (1.7), and this, (3.1d), (3.2b) and (3.16) imply that $u \notin \mathcal{A}$.

The remainder of the paper shows that u may be approximated by a subharmonic function $v(z)$ with $v(z) = \log |f(z)|$, where f is holomorphic in Q , so that f inherits these properties .

4. The Approximation of Subharmonic Functions.

Divide each initial boundary $\partial_k \equiv Q_k^b \cup Q_k^r \subset \partial Q_k$ into closed arcs $\{I\} = \mathcal{I}_k$ each of μ -measure 1; this is possible from (3.14)). We denote the entire collection $\cup \mathcal{I}_k$ by \mathcal{I} . If Q_k has stage N , then for $I \subset \partial_k$ we write $s(I) = s(\partial_k) = N$.

Following [7], to each $I \in \mathcal{I}$, we choose ζ_I (cf. (2.9)) with

$$(4.1) \quad \int_I (z - \zeta_I) d\mu(z) = 0$$

and let ν be the Borel measure with unit mass at each ζ_I .

For any Borel measure τ , we consider the potential

$$\pi(\tau, z) = \int \log |z - \zeta| d\tau(\zeta),$$

and now estimate

$$V(z) = \pi(\nu, z) - \pi(\mu, z) \quad (z \in Q).$$

Note that nothing is asserted about the individual terms $\pi(\nu, z)$ and $\pi(\mu, z)$.

Proposition 4.2. *If Q_k is of stage N and its closure is contained in $(0, 1) \times (0, 1)$, then for all large N ,*

$$(1) \quad V(z) < A_{N-1}/5 \quad (z \in Q_k).$$

In addition, there exists a system of curves $\{\Gamma'_N\}$, with $\Gamma'_N \subset \Gamma_N$ and the Hausdorff distance from Γ'_N to $\partial Q \cap \{x = 0\}$ tending to zero, such that

$$(2) \quad |V(z)| \leq A_N/5 \quad (z \in \Gamma'_N).$$

Granted the Proposition, note that $v(z) \equiv u(z) + V(z)$ is subharmonic in Q , and that its Laplacian consists of unit masses: $\Delta v = \{\delta_{\zeta_I}\}$. Hence there is a function

$f(z)$, analytic in Q , with $v(z) = \log|f(z)|$. We have $\gamma_N = \{x = a(N)\} \cap Q$ and now let γ'_N be the portion of γ_N that is contained in the union of all Q_k with $Q_k \subseteq (0, 1) \times (0, 1)$. Proposition 4.2(1) and (3.2b) imply that f is bounded (indeed, tends to zero) on the $\{\gamma'_N\}$. Assertion (2) of Proposition 4.2 and (3.3) show that $\inf_{\Gamma'_N} |f| \rightarrow \infty$, and thus (1.6) holds. Finally, (1.7) follows from conclusion (1) of Proposition 4.2 and the conditions (3.8) and (3.9) for u . Thus Theorem 1.2 follows from this Proposition. The remainder of the paper is devoted to establish Proposition 4.2. (Note that with more care, the bound in assertion (2) can be improved to $A_{N-1}/5$, but this refinement is not needed here.)

For $I \in \mathcal{I}_k$, let \tilde{I}_k be I and its neighbors from \mathcal{I}_k , and \tilde{I} be I and its neighbors from \mathcal{I} ; for $z \in \partial_k$, let $\tilde{I}_k(z)$ consist of all $J \in \mathcal{I}_k$ such that \tilde{J}_k contains z , and $\tilde{I}(z)$ consist of all $J \in \mathcal{I}$ such that \tilde{J} contains z . Recall that all arcs are closed, and a neighbor of I is an arc of measure 1, chosen as at the beginning of §4, that meets I . To avoid excessive notations, we use \tilde{I}_k , \tilde{I} , $\tilde{I}_k(z)$ and $\tilde{I}(z)$ to denote collections of arcs or unions of arcs depending on the situation, and when considering a fixed Q_k we often ignore the subscript in our notations. Note that there are at most eight arcs in any one of them.

Lemma 4.3. *Let $I \in \mathcal{I}_k$. If \tilde{I}_k contains a vertex of Q_k , then*

$$(4.4) \quad \frac{\partial u}{\partial \eta}(z) \lesssim \Lambda_k^{1/2} \cong |I|^{-1} \quad \text{on } I,$$

and otherwise

$$(4.5) \quad \frac{\partial u}{\partial \eta}(z) \cong \Lambda_k \rho_k(z) \cong |I|^{-1} \quad \text{on } I.$$

Proof. We first consider (4.4), and note that it suffices to consider line segments $J \subset \partial_k$ which contain a corner of Q and have $1/2 \leq \mu(J) \leq 3$. Let us assume J is a segment of ∂_k^r which contains the bottom right corner of Q_k . To simplify notation, write $J = [0, \tau]$ for an appropriate τ ; the remaining cases are left to the reader. Note that Λ is associated to Q_k by (3.11), and (3.13) implies that $\int_0^\tau \Lambda t + \lambda dt \cong 1$ and $\lambda < \Lambda^{1/2}$. Hence

$$\Lambda \tau^2 \lesssim \int_0^\tau \Lambda t + \lambda dt \cong 1 \leq \int_0^\tau \Lambda t + \Lambda^{1/2} dt \lesssim \Lambda \tau^2 + \Lambda^{1/2} \tau;$$

thus $\tau = |J| \cong \Lambda^{-1/2}$. Moreover, (3.13) shows that on J we have $(\partial u/\partial \eta) \cong \Lambda t + \lambda \leq \Lambda \tau + \Lambda^{1/2} \cong \Lambda^{1/2}$.

Now we prove (4.5). Write $I = [\alpha, \alpha + \tau]$ where, by (4.4), $\tau \gtrsim \Lambda^{-1/2}$ and $\alpha \gtrsim \Lambda^{-1/2}$. Again, $(\partial u/\partial \eta) = \Lambda t + \lambda$, and so, using (3.13),

$$\Lambda t \lesssim \frac{\partial u}{\partial \eta} \leq \Lambda t + \Lambda^{1/2} \lesssim \Lambda t.$$

But $\int_{\alpha}^{\alpha+\tau} \Lambda t dt \cong 1$, so that $\tau \lesssim \Lambda^{-1/2}$; hence $\Lambda \alpha \tau \cong 1$. This means that

$$|I| \cong \tau \cong (\Lambda \alpha)^{-1} \cong (\Lambda t)^{-1}$$

for any $t \in I$, since $\tau \cong (\Lambda \alpha)^{-1} < \alpha$.

Corollary 4.6. *Let $I \in \mathcal{I}_k$. Then $|I| \lesssim \Lambda_k^{-1/2}$.*

Proof. We have that $|I| \cong \Lambda^{-1/2}$ when (4.4) applies, and $|I| = \tau \cong (\Lambda \alpha)^{-1} \leq \Lambda^{-1/2}$ otherwise, since $\alpha \gtrsim \Lambda^{-1/2}$.

Remark. It now follows from Corollary 4.6 and the conditions (3.12) that the μ -measure of the interval $[t_k - \eta_k, t_k + \eta_k] \times \{y_k\}$ tends to infinity with k .

Note also that if $z \in \partial_k$ and I is an arc in \mathcal{I}_k which is not contained in $\tilde{I}_k(z)$, then

$$(4.7) \quad \text{dist}(z, I) \gtrsim |I|.$$

A simple calculation also gives

Lemma 4.8. *For each ∂_k in stage N we have that*

$$\sum_{\mathcal{I}_k} |I|^2 \lesssim \Lambda_k^{-1} \log \Lambda_k \lesssim a(N-1)^3 2^{-k}.$$

Proof. Identify ∂_k^b with the real interval $[0, c]$, where $c = a(N-1) - a(N)$, and identify $z \in \{\Re z = a(N) + t\} \cap \partial_k^b$ with t .

We first sum over all I with $I \subset \partial_k^b$. If I or its neighbor in ∂_k meets a corner of Q_k , then (4.4) yields that $|I|^2 \cong |\Lambda|^{-1}$; otherwise, we have from (4.5) that $|I| \cong \min\{(t\Lambda)^{-1}, ((c-t)\Lambda)^{-1}\}$.

Let \sum^* be the sum over those I with $I \subset [0, 2c/3]$. Recall from (3.8d) and (3.11) that $\Lambda_k > a(N-1)^{-4}L_k$. Then it follows from (3.12), Lemma 4.3 and the estimate $\Lambda^{-1} \log \Lambda \lesssim \Lambda^{-3/4}$ that

$$\begin{aligned} \sum^* |I|^2 &\cong \Lambda^{-1} + \int_{\Lambda^{-1/2}}^{2c/3} (\Lambda t)^{-1} dt \\ &\cong \Lambda^{-1} + \Lambda^{-1}(\log c + \log \Lambda) \lesssim \Lambda^{-1} \log \Lambda \lesssim a(N-1)^3 2^{-k}. \end{aligned}$$

The same estimate, with $t\Lambda$ replaced by $(c-t)\Lambda$ and the range of integration over $[c/3, c - \Lambda^{-1/2}]$, shows that the sum over those I contained in the right portion of ∂_k^b also may be absorbed in this sum. We make the same estimates over the top and bottom halves of ∂_k^r to complete the proof.

Lemma 4.9. *Let $z \in \partial_k^i$ and \mathcal{I}_k^i consist of all $I \in \mathcal{I}_k$ that meet ∂_k^i ($i = r$ or b). Then*

$$\sum_{\mathcal{I}_k^i \setminus \tilde{\mathcal{I}}_k(z)} \frac{|I|^2}{(\text{dist}(z, I))^2} \lesssim 1.$$

Proof. We consider the case when z is in ∂_k^r , the other case being similar, and set $\sigma(z, I) = |I|^2 / (\text{dist}(z, I))^2$.

Identify $\partial_k^r = \{a(N) + i(y_{k-1} + t) : 0 \leq t \leq b\}$ with the interval $0 \leq t \leq b$, and z with some $s \in [0, b]$. By symmetry we assume $0 \leq s \leq b/2$ and estimate the sum over those I 's that are closer to 0 than to b .

Consider an $I \in \mathcal{I}_k^i \setminus \tilde{\mathcal{I}}_k(z)$ which is closer to 0 than to b . If $I \in \tilde{\mathcal{I}}_k(0)$, then by (4.4), $|I| \cong \Lambda^{-1/2}$ and $|I|/\text{dist}(z, I) \lesssim 1$; if $I \notin \tilde{\mathcal{I}}_k(0)$ then $|I| \cong (\Lambda t)^{-1}$ for any $t \in I$, and since $\text{dist}(z, I) \cong |t - s|$, we have

$$\sigma(z, I) \cong \int_I \frac{1}{\Lambda t |t - s|^2} dt.$$

Summing over I 's, we have

$$\sum_{\mathcal{I}_k^i \setminus \tilde{\mathcal{I}}_k(z)} \sigma(z, I) \lesssim 1 + \int_J \frac{1}{\Lambda t |t - s|^2} dt$$

where $J = [0, 2b/3] \setminus (\tilde{\mathcal{I}}_k(0) \cup \tilde{\mathcal{I}}_k(s))$. Recall that $|\tilde{\mathcal{I}}_k(0)| \cong \Lambda^{-1/2}$ and consider two possibilities.

First, suppose that $\tilde{I}_k(0) \cap \tilde{I}_k(z) \neq \emptyset$. Then $|\tilde{I}_k(z)| \cong \Lambda^{-1/2}$, $|t - s| \cong t$, and so

$$\int_J \frac{1}{\Lambda t |t - s|^2} dt \lesssim \int_{\Lambda^{-1/2}}^{\Lambda^{2b/3}} \frac{1}{\Lambda t^3} dt \cong 1;$$

On the other hand, suppose that $\tilde{I}_k(0) \cap \tilde{I}_k(z) = \emptyset$. Then $s \gtrsim \Lambda^{-1/2}$, $|\tilde{I}_k(s)| \cong (\Lambda s)^{-1}$ and there exist $\alpha \cong \beta \cong \gamma \cong 1$ so that $J = [0, b] \setminus ([0, \alpha \Lambda^{-1/2}] \cup [s - \beta(\Lambda s)^{-1}, s + \gamma(\Lambda s)^{-1}])$. We then calculate the integral $\int (\Lambda t |t - s|^2)^{-1} dt$ over $J \cap [0, s/2]$, $J \cap [s/2, 3s/2]$ and $J \cap [3s/2, 2b/3]$ separately, and conclude that $\int_J (\Lambda t |t - s|^2)^{-1} dt \lesssim 1$.

This completes the proof.

5. Approximation on the grid.

In this section we prove (1) of Proposition 4.2. Since V is harmonic in each Q_k , we need only check boundary values. Thus we consider a $Q_k \subset R_N$ whose boundary does not meet ∂Q , and recall that ∂_k is the initial portion of its boundary (for the bottom $Q_k \subset R_N$ see the discussion after (3.14)).

We will divide $\mathcal{I} = \cup \mathcal{I}_k$ introduced at the beginning of §4 into four portions $\{\mathcal{I}^\ell : \ell = 1, 2, 3, 4\}$, and let the restrictions of μ and ν to each subset be μ_ℓ and ν_ℓ , a convention to be used from now on.

Let Q_j be the n -square from stage $N - 1$ adjacent to Q_k . With $z \in Q_k$, let

$$\mathcal{S}(z) = Q_k^t \cup Q_j^b \cup \{\partial_m^i : \partial_m^i \cap \partial_k \neq \emptyset, i = b \text{ or } r\}.$$

Then $\mathcal{S}(z)$ contains at most two segments from stage $N - 1$, five from stage N (three vertical and two horizontal) and three from stage $N + 1$ (two vertical and one horizontal). We omit Q_k^t (or Q_j^b) in the definition of $\mathcal{S}(z)$ when it lies on $y = 1$ (or $y = 0$).

Note that if the stage of Q_n (which we call $s(Q_N)$) is less than $N - 1$ and ∂_n^i is not in $\mathcal{S}(z)$, then

$$(5.1) \quad \text{dist}(z, \partial_n^i) \geq a(N - 2)/2,$$

while if $s(\partial_n^i) \geq N$ and ∂_n^i is not in $\mathcal{S}(z)$, then

$$(5.2) \quad \text{dist}(z, \partial_n^i) \gtrsim \text{diam } Q_n.$$

Write $s(I)$ for the stage of I (i. e., the stage of the Q_j with $I \subset \partial_j$). First, let $\mathcal{I}^1(z) = \{I \in \mathcal{I} : s(I) \leq N-1, I \cap \mathcal{S}(z) = \emptyset\}$, and $\mathcal{I}^2(z) = \{I \in \mathcal{I} : s(I) \geq N, I \cap \mathcal{S}(z) = \emptyset\}$.

Using the obvious estimates of for the individual potentials and (3.15) and (5.1), we have that

$$(5.3) \quad |\pi(\mu_1, z) - \pi(\nu_1, z)| \leq 3 \log \frac{1}{a(N-2)} \mu(\cup_1^{k(N-1)-1} \partial_k) \leq \frac{A_N - 1}{10}.$$

We now use Lemma 2.8(1), Lemma 4.8, (5.1) and (5.2) to control the sum over \mathcal{I}^2 :

$$(5.4) \quad \begin{aligned} |\pi(\mu_2, z) - \pi(\nu_2, z)| &\leq \sum_{\substack{I \in \mathcal{I}^2(z) \\ s(I)=N}} \frac{|I|^2}{\text{dist}(z, I)^2} + \sum_{\substack{\mathcal{I}^2(z) \\ s(I) \geq N+1}} \frac{|I|^2}{\text{dist}(z, I)^2} \\ &\lesssim a(N-1)^{-2} a(N-1)^3 \sum 2^{-k} + a(N)^{-2} a(N)^3 \sum 2^{-k} \\ &= o(1). \end{aligned}$$

We next confront the contribution from the arcs of $\mathcal{S}(z)$. Recall that $\mathcal{S}(z)$ consists of at most ten segments J , and the endpoints of these segments form a set of at most fourteen points: four as endpoints of segments from ∂_n with $s(\partial_n) = N+1$, six from from endpoints of ∂_ℓ , $s(\partial_\ell) = N$, and at most four from intervals from stage $N-1$ (this is an inefficient counting, as some points appear more than once). Let $\mathcal{P}(z)$ be the set consisting of z and these endpoints. Let

$$\mathcal{I}^3(z) = \{I \in \mathcal{I} : I \subset J \text{ with } J \text{ from } \mathcal{S}(z) \text{ but } I \notin \tilde{I}(\eta) \text{ for any } \eta \in \mathcal{P}(z)\},$$

and, finally, take $\mathcal{I}^4(z) = \{I \in \tilde{I}(\eta); \eta \in \mathcal{P}(z)\}$. Clearly $\mathcal{I} = \cup_1^4 \mathcal{I}^j(z)$.

We study the contribution to V which arises from $\mathcal{I}^3(z)$. If J is a segment in $\mathcal{S}(z)$ that contains z , Lemma 4.9 shows that

$$(5.5) \quad \sum_{\mathcal{I}^3(z) \cap J} \sigma(z, I) \lesssim 1.$$

If J is a segment in $\mathcal{S}(z)$ that does not contain z , choose $p \in \mathcal{P}(z)$ so that

$$(5.6) \quad \text{dist}(z, w) \geq \text{dist}(p, w) \quad (w \in I).$$

Then $\sigma(z, I) \leq \sigma(p, I)$ for each $I \subset J$ and again (5.5) holds. Hence by Lemma 2.8(1),

$$(5.7) \quad |\pi(\nu_3, z) - \pi(\mu_3, z)| \lesssim \sum_{\mathcal{I}^3(z)} \sigma(z, I) \lesssim 1.$$

Recall from Lemma 4.3 that $\partial u / \partial \eta \lesssim |I|^{-1}$ for every I and that there are at most 40 arcs in $\mathcal{I}^4(z)$ (again not a precise count). It then follows from Lemma 2.8(3) that

$$(5.8) \quad \pi(\nu_4, z) - \pi(\mu_4, z) \lesssim 1.$$

Since $\pi(\nu_4, z)$ can be $-\infty$, this estimate is only one-sided. Estimates (5.3)-(5.8) and the maximum principle yield the first estimate of Proposition 4.2.

6. Interior approximation from below.

Two-sided estimates of V are needed to guarantee that $\log |f|$ remain large on a suitable curve $\Gamma'_N \subset \Gamma_N$; recall from (3.16) that $u \geq 2A_N$ on Γ_N . This requires steps similar to those of the previous section, except we avoid (5.8).

Suppose that $z = x + iy \in \Gamma_N \cap Q_k \equiv \beta_k$, and that $y_k - y \leq y - y_{k-1}$ (the other case can be treated similarly). Recall from (3.10b) that $\xi_k \equiv (t_k, y_k)$ is in $Q_k^t \cap \beta_k$. Let $\mathcal{I}^5(z) = \{I \in \mathcal{I}, s(I) \leq N - 1\}$, $\mathcal{I}^6(z) = \{I \in \mathcal{I} : s(I) \geq N, I \not\subseteq Q_k^t\}$, $\mathcal{I}^7(z) = \{I \subseteq Q_k^t\} \setminus \tilde{I}(\xi_k)$ and $\mathcal{I}^8(z) = \tilde{I}(\xi_k)$. Clearly $\mathcal{I} = \mathcal{I}^5(z) \cup \mathcal{I}^6(z) \cup \mathcal{I}^7(z) \cup \mathcal{I}^8(z)$. Note that Γ_N lies in the left half of R_N ; also (3.8) and (3.10) and the definition of β_k in (3.16) show that $\text{dist}(z, Q_k^t) \geq a_k > a(N)$.

Since $\text{dist}(\Gamma_N, R_{N-1}) \gtrsim a(N - 1)$, the choice (3.15) now yields (cf. (5.3)) that

$$|\pi(\nu_5, z) - \pi(\mu_5, z)| \leq \frac{A_N}{10}.$$

However, $\text{dist}(z, \partial_n^i) \gtrsim \text{diam} Q_n$ when $\partial_n^i (\neq Q_k^t)$ is of stage N , and $\text{dist}(z, Q_n) \geq a(N) \gtrsim \text{diam} Q_n$ for any Q_n with $s(Q_n) \geq N + 1$. Thus a calculation similar to (5.4) shows that

$$|\pi(\nu_6, z) - \pi(\mu_6, z)| \lesssim 1.$$

If $I \in \mathcal{I}^7(z)$ then $\text{dist}(z, I) \gtrsim \text{dist}(\xi_k, I)$ and $\sigma(z, I) \lesssim \sigma(\xi_k, I)$; again by Lemma 2.8(1) and Lemma 4.9,

$$|\pi(\nu_7, z) - \pi(\mu_7, z)| \lesssim 1.$$

Finally, let $\{\zeta_k^j\}$ be the points on $Q_k^t \cap \beta_k$ such that $\nu(\zeta_k^j) = 1$ and $I(\zeta_k^j) \subset \tilde{I}(\xi_k)$. There are at most four such points for each k . According to (4.5), there exists $\delta \in (0, 1/10)$ so that the $\{B(\zeta_k^j, \delta|I(\zeta_k^j)|)\}$ are mutually disjoint.

Suppose now that $z \notin \cup B(\zeta_k^j, \delta|I(\zeta_k^j)|)$, $j = 1, 2, 3, 4$. Then by Lemma 2.8(2),

$$|\pi(\nu_8, z) - \pi(\mu_8, z)| \leq \sum_1^4 \left| \int_{I(\zeta_k^j)} \log |z - \zeta_k^j| - \log |z - \zeta| d\mu(\zeta) \right| \leq C(\delta) \lesssim 1.$$

Since $\{A_N\} \uparrow \infty$, we deduce for large N that

$$|\pi(\nu, z) - \pi(\mu, z)| \leq \frac{A_N}{5}$$

on

$$\Gamma_N \setminus \bigcup_{k(N)}^{k(N+1)-1} \{\cup_k \cup_j B(\zeta_k^j, \delta|I(\zeta_k^j)|)\}.$$

From this we readily obtain arcs $\Gamma'_n \subset \Gamma_N$ joining $y = y_{k(N)}$ to $y = y_{k(N+1)-2}$ in R_N so that property (2) of Proposition 4.2 holds. In view of (3.3), we have that $\inf_{\Gamma'_N} |f| \uparrow \infty$ as $N \rightarrow \infty$, and the proof of Proposition 4.2 is complete.

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