DOOB'S CONTRIBUTION TO FUNCTION THEORY

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ABSTRACT. A brief survey of Doob's work on cluster sets and fine limits.

Joseph Doob's early work, including his thesis in 1932 and four published papers [7], [8], [9], [11] between 1932 and 1935, focused on function theory, in particular on cluster sets. At that time, this subject remained where Lindelöf, Gross and Iversen had left it a decade earlier. Some of Gross and Iversen's work was considered difficult to follow, and was described many years later by Doob as "obscure".

In a space of two or three years, Doob grasped the nature of the subject and reduced it to its essentials. He untangled the arguments of Gross and Iversen, dispensed with extraneous matters and observed the connection among cluster sets, omitted values and a theorem of Lindelöf on asymptotic values. He then gave a clarified proof of the Gross-Iversen Theorem.

He went on to prove many nuts-and-bolts theorems on cluster sets. From this there emerged an intriguing covering property of analytic functions. Suppose that f is an analytic function on the unit disk with $f(0) = 0$, whose cluster values on a boundary arc of length $\rho > 0$ have modulus at least 1, and that the part of $|w| < 1$ covered by f at least q times has area less than $\sigma < \pi$. Then the image of f must cover a disk $|z| < d$, with d depending only on ρ, q and σ.

Because of the Great Depression few academic positions were open, so Hotelling suggested that Doob study statistics, an area with more suitable positions available. In short order Doob wrote *Probability and Statistics* [10] in 1934, in which he sharpened then-current work of Kolmogorov and gave rigorous proofs of theorems of Fisher and Hotelling in statistics. From there he started a life-long journey in stochastic processes. Years later Doob would discover profound connections between probability and classical potential theory.

1. CLUSTER SETS [1]

Early developments on boundary behavior of analytic functions were centered around three types of sets: cluster sets, range sets and asymptotic sets. Let f be a nonconstant meromorphic function on a domain D in $\hat{\mathbb{C}}$.

1991 *Mathematics Subject Classification.* Primary:30D40; Secondary 31C40.
partially supported by the National Science Foundation under Grant No. DMS-0400810.

[1] partially based on two classical texts, Collingwood and Lohwater [6] and Noshiro [36].

Suppose D has an isolated boundary point x_0. Then the cluster set of f at x_0 is either a single point or $\hat{\mathbb{C}}$ (Casorati 1868 [5], Weierstrass 1876 [42]); if the cluster set at x_0 is $\hat{\mathbb{C}}$, then the complement of the range set at x_0 consists of at most two points (Picard 1879 [38]), and points not in the range set at x_0 are asymptotic values (Iversen 1914 [28]).

Let z_0 be any boundary point of D. The *cluster set* $\mathcal{C}(z_0)$ of f at z_0 consists of those values α such that there is a sequence z_n in D converging to z_0 with $\lim_{n\to\infty} f(z_n) = \alpha$. $\mathcal{C}(z_0)$ is connected if D is locally connected at z_0. The *boundary cluster set* $\mathcal{B}(z_0)$ at z_0 consists of those values α such that there is a sequence z_n in $\partial D \backslash z_0$ converging to z_0, and a corresponding sequence w_n, in $\mathcal{C}(z_n)$, converging to α. The *range set* $\mathcal{R}(z_0)$ of f at z_0 is the set of values α such that there is a sequence z_n in D converging to z_0 with $f(z_n) = \alpha$ for all n, i.e., the set of values taken by f in every neighborhood of z_0. $\mathcal{R}(z_0)$ is a G_δ set. A number α is an *asymptotic value* at z_0 if it is the limit of f along some continuous path in D ending at z_0.

The notion of cluster set was introduced by Painlevé in 1895 ([37]). In the beginning, the focus was on the behavior of an analytic function in a neighborhood of a part of the boundary that has linear measure zero, notably in work due to Painlevé, Besicovitch and Cartwright.

Cluster sets at the boundary of a Jordan domain were systematically studied independently by Gross and Iversen between 1914 and 1922. They obtained some of most powerful theorems in the theory [23],[24],[25],[26], [28],[29], [30].

Gross-Iversen Theorem. *Let f be a meromorphic function on the unit disk Δ and z_0 be a point on the boundary $\partial\Delta$. Then*

(i) $\partial\mathcal{C}(z_0) \subseteq \mathcal{B}(z_0)$; and

(ii) every value of $\mathcal{C}(z_0)\backslash\mathcal{B}(z_0)$, if one exists, belongs to the range set $\mathcal{R}(z_0)$ with at most two possible exceptions; any exception, if one exists, is an asymptotic value of f at z_0; if there are two exceptions, then the range set $\mathcal{R}(z_0)$ is $\hat{\mathbb{C}}$ less these two points.

To illustrate, note that the function $e^{(z+1)/(z-1)}$ is bounded analytic in Δ and has radial limit of modulus 1 at every point of $\partial\Delta$ except at the point 1, where f has radial limit 0 and has cluster sets $\mathcal{C}(1) = \{|w| \leq 1\}$ and $\mathcal{B}(1) = \{|w| = 1\}$. As for a Blaschke product, if the point 1 is a limit point of the zeros, then $\mathcal{C}(1) = \{|w| \leq 1\}$; if every point in $\partial\Delta$ is a limit point of the zeros, then $\mathcal{C}(z_0) = \mathcal{B}(z_0) = \{|w| \leq 1\}$. See [6].

The proofs of Gross and Iversen depend on uniformization of Riemann surfaces and are difficult. Seidel (1932 [40]) gave a simpler proof of part (i) based on the maximum modulus principle for analytic functions.

The identification of the range set is always more complicated than the identification of cluster sets, as one sees by comparing the theorem of Picard with that of Casorati-Weierstrass.

In Doob's first published work *On a Theorem of Gross and Iversen* [7], he gave a simple, transparent proof of part (ii) of the Gross-Iversen Theorem.

He showed that every value of $\mathcal{C}(z_0)$ not belonging to $\mathcal{B}(z_0)$ and omitted by f in some neighborhood of z_0 is an asymptotic value of f at z_0. The conclusion then follows from a theorem of Lindelöf [32], which states that if f admits two distinct asymptotic values at some boundary point z_0 then f assumes infinitely often in any neighborhood of z_0 all values of $\hat{\mathbb{C}}$ with at most two possible exceptions. Doob's construction of asymptotic paths uses only simple ideas—the maximum principle and the monodromy theorem.

In another paper [8] in 1932, Doob introduced *metric cluster values*, a notion between the limit along an entire path and the limit along a subset of a path of prescribed density. Let f be a bounded analytic function in Δ and F be its Fatou boundary function on $\partial\Delta$. He proved a connection between cluster values of f along curves of various orders of tangency to the unit circle at the point 1, and the metric cluster values of F at 1 for various degrees of density. In modern language, there were many harmonic measure estimates.

Let $f_1, f_2, ...$ be a uniformly bounded sequence of functions analytic in Δ, and $\mathcal{C}_1, \mathcal{C}_2, ...$ be the corresponding (cluster) set functions on $\partial\Delta$. In [9] Doob defined *strong cluster sets* from the interior and from the boundary for such sequences; and proved several Gross-Iversen type theorems for strong cluster sets and omitted values, which are pivotal in his Bloch-type theorems.

Years later (1963), after he had made major contribution in probability theory and fine-limit theorems, Doob returned to study cluster sets [20]. This time he proved an elegant and technically demanding one-sided generalization of the Gross-Iversen Theorem. The concept of boundary cluster set from one side originates in his work on tangential limits [8]. In the proof, ideas from [9] on strong cluster values for sequences of function-arc pairs and normal family argument are used, and harmonic measure estimates are applied to obtain one-sided limits.

To show the uncompromising style that is common in Doob's papers, we state this theorem in its original form [20], aside from changes of notation.

Let f be a meromorphic function on the unit disk Δ and let $z_0 = 1$. The *boundary cluster set* $\mathcal{B}_s$ *(at* $z_0 = 1$*) from the south*, is the intersection $\bigcap_{n\geq 1}$ {closure of the set of cluster values of f at points z on the unit circle satisfying $-\frac{1}{n} < \arg z < 0$}. The boundary cluster set $\mathcal{B}_n$ from the north is defined similarly. The boundary cluster set $\mathcal{B}(1)$ defined earlier is then $\mathcal{B}_n \cup \mathcal{B}_s$. If $0 < \omega \leq \pi$, the set $\{z : |z| < 1,\ 3\pi/2 - \arg(z-1) \leq \omega\}$ is called the ω-sector at 1. Let $\mathcal{C}_\omega$ be the cluster set of f at 1 from the ω-sector, and let $\mathcal{C}^\omega = \bigcup_{\omega'<\omega} \mathcal{C}_{\omega'}$, and $\mathcal{T}_s = \bigcap_{\omega>0} \mathcal{C}^\omega = \bigcap_{\omega>0} \mathcal{C}_\omega$. Then $\mathcal{T}_s$ is the set of *tangential-from-south cluster values* of f at 1; both $\mathcal{T}_s$ and $\mathcal{C}_\omega$ are closed.

If $\omega > 0$, let $\mathcal{R}^\omega$ be the set of those values taken by f arbitrarily near 1 in some ω'-sector with $\omega' < \omega$, and define $\mathcal{R}_s = \bigcap_{\omega>0} \mathcal{R}^\omega$. It is trivial that $\mathcal{R}^\omega \subseteq \mathcal{C}^\omega$ for $\omega \geq 0$.

Theorem(Doob). *Let f be meromorphic function on Δ and $z_0 = 1$.*

(i) Then $\mathcal{B}_s$ *contains the boundary of* $\mathcal{T}_s$.

(ii) If $\omega > 0$, *then* $\mathcal{C}^\omega \cap (\hat{\mathbb{C}}\backslash\mathcal{B}_s)$ *is open, and each component of this set has a boundary point on* $\mathcal{B}_s$.

(iii) If $\alpha \in \mathcal{T}_s \cap (\hat{\mathbb{C}}\backslash\mathcal{B}_s)$, *then* $\mathcal{R}_s \supseteq D(\alpha, \mathcal{B}_s)$ *aside from at most two exceptional points. (Here* $D(\alpha, \mathcal{B}_s)$ *is the component of* $\hat{\mathbb{C}}\backslash\mathcal{B}_s$ *containing* α.*) If there are two exceptional points,* $\mathcal{R}_s$ *is* $\hat{\mathbb{C}}$ *less these two points.*

(iv) If $\omega > 0$, $\alpha \in \mathcal{C}^\omega \cap (\hat{\mathbb{C}}\backslash\mathcal{B}_s)$ *and* $\mathcal{T}_s \cap D(\alpha, \mathcal{B}_s) = \emptyset$, *then either* $(\mathcal{C}^\omega\backslash\mathcal{R}^\omega) \cap D(\alpha, \mathcal{B}_s) = \emptyset$ *or* $\mathcal{R}^\omega$ *is* $\hat{\mathbb{C}}$ *less at most two points.*

Whether it is the motivation for, or a consequence of, the one-sided theorem, a simpler proof is deduced replacing the original "obscure" proof, according to Doob ([20], p.462), of another deep theorem of Gross [26]; it states that every angular cluster value of a meromorphic f at 1 that is also a limit point of $\hat{\mathbb{C}}\backslash\mathcal{R}(1)$ must be a principal value at 1, in other words, a cluster value along every path in Δ ending at 1.

Caretheódory's theory of prime ends can also be considered a theory of cluster sets. Doob's one-sided cluster value theorem is important in studying prime ends, and has been used to give an alternative proof of Lindelöf's theorem relating angular cluster sets of a univalent function f on Δ to the principal points of prime ends of $f(\Delta)$. (See [6] pp.113, 178).

2. Bloch-Type Theorems

A theorem of Bloch [1] in 1926 states that there is an absolute constant $b > 1/6$ such that if f is analytic on the unit disk Δ with $f'(0) = 1$, then $f(\Delta)$ contains an open disk of radius b, and that there exists b', $1/72 < b' < b$, such that f maps some domain in Δ conformally onto a disk of radius b'.

A function f analytic in the unit disk Δ with $f(0) = 0$, is said to have property $K(\rho)$, if there is an open arc $\mathcal{A}$ of length at least $\rho > 0$ on $\partial\Delta$ such that $\liminf_{n\to\infty} |f(z_n)| \geq 1$ for every sequence z_n in Δ converging to a point on $\mathcal{A}$. In fact, the origin is the only point common to all images $f(\Delta)$, $f \in K(\rho)$, for a fixed $\rho < 2\pi$ ([11]).

In the spirit of Bloch's theorem, Doob proved the following in 1935 [11].

Theorem(Doob). *Let* f *be analytic on* Δ *and* $f(0) = 0$.

(i) Given an arbitrary closed set D_1 *in* $|w| < 1$ *not containing* 0 *and an arbitrary integer* q, *there exists a neighborhood* D_0 *of* 0 *depending only on* D_1, ρ *and* q *such that if* f *has property* $K(\rho)$, *then* $f(\Delta)$ *covers* D_1 *at least* q *times if it does not cover* D_0.

(ii) There exists $k(\rho) > 0$, *such that if* f *has property* $K(\rho)$ *then* $f(\Delta)$ *contains a disk of radius* $k(\rho)$.

The disk in (ii) is generally not centered at the origin, because 0 is the only point in all $f(\Delta)$, $f \in K(\rho)$.

The proof is based on a theorem on strong cluster sets of sequences of function-arc pairs introduced in [9], whose proof in turn uses elliptic modular functions, the monodromy theorem and a normal family argument.

Bloch's theorem can be deduced from the above; Doob asked whether the opposite deduction is possible, and whether the conformal part of Bloch's theorem can be extended to functions having property $K(\rho)$.

There are many other interesting theorems of this nature in [11]; we state one on covering properties of analytic functions. A function with the property $K(\rho)$ is said to have property $K_\sigma^q(\rho)$, $0 < \sigma < \pi$, $q \geq 0$, if the set of points in $|w| < 1$ assumed at least q times by f has measure at most σ.

Theorem(Doob). *Let f be analytic on Δ and $f(0) = 0$.*

(i) If $0 < \sigma < \pi$, there exists a neighborhood of the origin $E_\sigma^q(\rho)$, depending only on ρ, σ and q, such that if f has property $K_\sigma^q(\rho)$ then $f(\Delta) \supseteq E_\sigma^q(\rho)$.

(ii) If $0 < \sigma < \pi$, the family of functions with property $K_\sigma^1(\rho)$ forms a normal family and is compact.

(iii) Let σ_0 be the infimum of σ for which the family in (ii) is nonempty. Then $\sigma_0 > 0$ and there is a function which has property $K_{\sigma_0}^1(\rho)$.

Part (i) follows from the previous theorem; parts (ii) and (iii) are in the spirit of Montel's theorem and the proof uses Montel's theorem and Hurwitz' theorem.

3. Relative Fatou Theorems

Bounded harmonic functions defined on the unit ball B in $\mathbb{R}^n$ have nontangential limit almost everywhere on ∂B (Fatou 1906 [22]). A Green potential in B has radial limit zero at almost every point on ∂B (Littlewood 1928 for $n = 2$ [33], and Privalov 1938 for $n \geq 3$ [39]); the nontangential limits of a Green potential need not exist at any point on the boundary (Zygmund 1950, see [41]).

In 1954 Doob [12] envisioned that Brownian motion provides a suitable path system in any domain, replacing the set of radii to the boundary of a ball, for both harmonic and superharmonic functions. He also observed that if u is superharmonic in $\mathbb{R}^n$, and $X(t), 0 \leq t < \infty$, is a Brownian motion process, then the stochastic process $u(X(t)), 0 \leq t < \infty$, is a supermartingale; and that any superharmonic function is continuous on almost all Brownian paths starting at any point in its domain of definition.

In 1945 Cartan [4] introduced the fine topology, the coarsest topology making all subharmonic functions continuous. Brelot [2, 3] had earlier introduced "thinness" of a set. In the language of fine topology, a set E is *thin* at a point x if x is not a fine limit point of E. In 1941, Martin [34] constructed a compactification D^M for any domain D in $\mathbb{R}^n$ such that there is a one-to-one correspondence between points on the boundary $D^M \backslash D$ and normalized minimal harmonic functions. He also showed that a strictly positive superharmonic function h in D has a canonical integral representation, involving a uniquely determined measure μ^h on D^M. In 1957, Naïm [35] extended Cartan's fine topology from D to D^M.

Doob showed that if E is thin at x, then almost no Brownian path from x meets E for arbitrarily small strictly positive time, and regarded fine limits as rivals of nontengential limits for boundary behavior of harmonic functions [12].

Meanwhile, Kakutani [31] observed in 1944 the connection between harmonic measure and exit distribution of Brownian motion from a domain. Hunt [27] noticed in 1956 the relation between the occupation time and the Green function.

The cumulative result in this theory is the following relative Fatou theorem. Doob gave a probabilistic proof using conditional Brownian motion and the h-processes introduced for this purpose [14], and a nonprobabilistic proof two years later [16].

Brelot-Naïm-Doob Theorem. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, whose complement has positive capacity, and let u and h be strictly positive superharmonic functions on D. Then u/h has a fine limit at μ^h-almost every point of D^M.*

Although it is natural in the case of a general D to consider boundary behavior via Brownian paths or fine topology, Doob thought it is also natural to consider classical nontangential approach if D is a ball or the half space [17].

Theorem(Doob). *Let u and h be positive harmonic functions on the unit ball B in $\mathbb{R}^n$, $n \geq 2$, corresponding to the Poisson-Stieltjes measures ν^u and ν^h respectively on the ∂B. Then u/h has a finite nontangential limit at ν^h-almost every point on the boundary. The limit is ν^h-almost everywhere the Radon-Nikodym derivative of the absolutely continuous part of ν^h with respect to ν^u.*

Doob's analytical proof of relative Fatou-type theorems [17] has inspired at least one of the three original proofs of the Boundary Harnack Principle for harmonic functions on Lipschitz domains – a subject still actively pursued for some other operators on nonsmooth domains.

Doob's vision in relating the Brownian trajectories to the heat equation [13] in 1955 led eventually to the solution of the Dirichlet problem for the heat equation on noncylindrical domains with mixed time and space variables by means of the parabolic measure defined probabilistically. He later proved these theorems by analytical methods. These led to prolonged search by many for the absolute continuity of parabolic measure and for smoothness of solutions of the heat equation near the boundary of very general domains.

For all this and much more in 846 pages, see the encyclopedic *Classical Potential Theory and Its Probabilistic Counterpart* [21].

4. Random Notes

Doob's early papers reveal an amazing acquaintance with the great works of complex function theory and probability theory, reflected in a quick attack on the central problems. He insisted that probability is rational, not mystical, a new idea in 1934.

His favorite subjects in analysis were boundary value problems: along sequences, rays, curves and sectors, by fine topology and of course by Brownian paths.

During analysis seminars, his favorite place in Altgeld Hall was near a window, rocking his chair back and forth and watching the activities in the Quad, but he was listening as well. Many speakers could not escape his challenge "Can this be done by Brownian motion? by fine limit?" at the end. He liked to debate on the best "approach" to the boundary.

Once or twice he doubted whether Brownian motion solves all problems. After proving that u/h has finite limit along almost all h-paths from any point x in the domain of definition for positive harmonic h and positive superharmonic u, he was not satisfied. He wrote in [14], "This theorem suffers from the unsatisfactory feature that it stresses the paths from x rather than the path endpoints. It is not clear how the limits along paths to the same boundary point are related. The corresponding advantage is that the theorem does not even involve a boundary. All it really states is that u/h has a finite limit on almost all h-paths, up to the first hitting time." He was equal to the challenge and went on to prove the fine limit version.

The style in his first papers on cluster sets is incisive, unpolished and straightforward, not unlike Doob himself. Reading his early work is like mining—requiring concentration and digging; hidden gems will be the reward.

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