

# HARMONIC MEASURES FOR SYMMETRIC STABLE PROCESSES

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ABSTRACT. Let  $D$  be an open set in  $\mathbb{R}^n$  ( $n \geq 2$ ) and  $\omega(\cdot, D)$  be the harmonic measure on  $D^c$  with respect to the symmetric  $\alpha$ -stable process ( $0 < \alpha < 2$ ) killed upon leaving  $D$ . We study inequalities on volumes or capacities which imply that a set  $S$  on  $\partial D$  has zero harmonic measure and others which imply that  $S$  has positive harmonic measure. In general, it is the relative size of a set  $S$  to  $D^c \setminus S$  that determines whether  $\omega(S, D)$  is zero or positive.

We study null sets of harmonic measures for the symmetric  $\alpha$ -stable process.

A symmetric  $\alpha$ -stable process  $X$  on  $\mathbb{R}^n$  is a Lévy process whose transition density  $p(t, x - y)$  relative to Lebesgue measure is uniquely determined by its Fourier transform

$$\int_{\mathbb{R}^n} e^{ix \cdot \xi} p(t, x) dx = e^{-t|\xi|^\alpha}.$$

Here  $\alpha$  must be in the interval  $(0, 2]$ . When  $\alpha = 2$ , it is a Brownian motion running with a time clock twice as fast as the standard one. From now on, we assume  $0 < \alpha < 2$ , when referring to symmetric stable processes. Unlike the Brownian motion, the generator of a symmetric  $\alpha$ -stable process is nonlocal, as it is the fractional Laplacian  $-(-\Delta)^{\alpha/2}$ , an integro-differential operator. A symmetric stable process has discontinuous sample paths and heavy tails, while Brownian motion has continuous sample paths and exponentially decaying tails. Basic properties of symmetric stable processes and their potential theoretic formulations in terms of Riesz kernels can be found in [BG] and [L].

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From now on,  $D$  is an open set in  $\mathbb{R}^n$  ( $n \geq 2$ ),  $0 < \alpha < 2$ , and  $X^D$  is the symmetric  $\alpha$ -stable process  $X$  killed upon leaving  $D$  and  $\tau_D$  is the first exit time.

An  $\alpha$ -harmonic function  $u$  in  $D$  is a locally integrable function in  $\mathbb{R}^n$  continuous in  $D$  satisfying  $\int_{|x|>1} |u(x)| |x|^{-n-\alpha} dx < \infty$  and

$$(0.1) \quad u(x) = E^x u(X_{\tau_{B(x,r)}})$$

for all balls  $B(x, r) \subseteq \overline{B(x, r)} \subseteq D$ ;  $u$  is said to be *regular  $\alpha$ -harmonic* in  $D$  if, in addition,

$$(0.2) \quad u(x) = E^x u(X_{\tau_D}) \quad \forall x \in D,$$

and  $u$  is said to be *singular  $\alpha$ -harmonic* if it is  $\alpha$ -harmonic in  $D$  and  $u \equiv 0$  in  $D^c$ . Strong Markov property yields that (0.2) implies (0.1); and if  $u$  is  $\alpha$ -harmonic in  $D$  then  $u$  is regular  $\alpha$ -harmonic in  $B(x, r)$  whenever  $\overline{B(x, r)} \subseteq D$ .

For  $x \in D$ , the  $\alpha$ -harmonic measure  $\omega^x(\cdot, D)$  is a measure on  $D^c$  given by

$$\omega^x(A, D) = P^x(X_{\tau_D} \in A), \quad A \subseteq D^c.$$

Fix  $A \subseteq D^c$ , let  $\omega^x(A, D)$  be 1 for  $x \in A$ , and 0 for  $x \in D^c \setminus A$ , then  $\omega^x(A, D)$  is regular  $\alpha$ -harmonic in  $D$ .

We say  $A$  has *zero harmonic measure* if  $\omega^x(A, D) = 0$  for some  $x \in D$ , equivalently for all  $x \in D$  by the Harnack inequality; and we write  $\omega(A, D) = 0$ .

If  $u$  is regular  $\alpha$ -harmonic in  $D$ , then

$$(0.3) \quad u(x) = \int_{D^c} u(y) d\omega^x(y, D) \quad \forall x \in D.$$

In the case of a ball  $B = B(0, r)$ , it was shown by M. Riesz that

$$(0.4) \quad d\omega^x(y, B) = k_B(x, y) dy$$

where

$$(0.5) \quad k_B(x, y) = \begin{cases} C_1(n, \alpha) \left[ \frac{r^2 - |x|^2}{|y|^2 - r^2} \right]^{\alpha/2} |x - y|^{-n}, & |y| > r, \\ 0, & |y| \leq r. \end{cases}$$

Denote by  $G$  the *Green function* of  $X$ , then

$$G(x, y) = \int_0^\infty p(t, x - y) dt = C_2(n, \alpha) |x - y|^{-n+\alpha}.$$

and  $G(\cdot, y)$  is  $\alpha$ -harmonic in  $\mathbb{R}^n \setminus \{y\}$ . The *Green function*  $G_D$  of  $X^D$  is defined by

$$G_D(x, y) = C_2(n, \alpha) \left[ |x - y|^{-n+\alpha} - \int_{D^c} |y - z|^{-n+\alpha} d\omega^x(z, D) \right];$$

note that  $G_D(x, x) = \infty$  if  $x \in D$ ,  $G_D(x, y) = 0$  in  $(D \times D)^c$ ,  $G_D(x, y) = G_D(y, x)$  and that  $G_D(\cdot, y)$  is  $\alpha$ -harmonic in  $D \setminus \{y\}$ . It is known that for any measurable  $f \geq 0$  on  $D$ ,

$$E^x \left[ \int_0^{\tau_D} f(X_s) ds \right] = \int_D G_D(x, y) f(y) dy \quad \forall x \in D.$$

For any bounded measurable  $\phi \geq 0$  on  $D^c$ ,

$$E^x[\phi(X_{\tau_D}) : X_{\tau_D} \neq X_{\tau_{D^-}}] = C_3(n, \alpha) \int_{D^c} \int_D \frac{G_D(x, y)}{|y - z|^{n+\alpha}} dy \phi(z) dz,$$

where  $X_{\tau_{D^-}} = \lim_{t \uparrow \tau_D} X(t)$  exists a.s. ([IW]). Note from  $X_{\tau_{D^-}} \in \bar{D}$  that

$$d\omega^x(z, D) = C_3(n, \alpha) \left( \int_D \frac{G_D(x, y)}{|y - z|^{n+\alpha}} dy \right) dz \quad \forall x \in D \text{ and } z \in \bar{D}^c.$$

From this it follows that  $\omega^x(\cdot, D)$  and the Lebesgue measure are mutually absolutely continuous in  $\bar{D}^c$ .

Consider  $S \subseteq \partial D$ . If  $S$  has positive Lebesgue measure then by (0.4),

$\omega^x(S, D) > \omega^x(S, B(x, \frac{1}{2} d(x, \partial D)))$ ; and if  $S$  has zero  $\alpha$ -capacity then  $\omega^x(S, D) = 0$  ([L]).

*Therefore in determining null sets for harmonic measures, it remains to study subsets of the boundary which have positive  $\alpha$ -capacity and zero volume. (Definition of  $\alpha$ -capacity shall be given in §1.)*

Potential theoretic problems of  $\alpha$ -processes for various domains have been studied by Byczkowski, Bogdan, Chen, Kulczycki, Song, and Wu. Bogdan [B1, B2] proved that for a Lipschitz domain  $\Omega$ , the  $\alpha$ -harmonic measure of  $\partial\Omega$  is zero, the boundary Harnack principle holds and the Martin boundary coincides with the Euclidean boundary. Subsequently, Song and Wu [SW] extended the boundary Harnack principle to all bounded open sets and showed that the Martin boundary and the Euclidean boundary are the same for slit domains. The discontinuity of the processes and the nonlocal definition of the harmonicity have imposed many technical complications; on the other hand, the results of Song and Wu suggest that the jumps of the processes make the roughness of  $\partial D$  and the disconnectedness of  $D$  harmless.

Bogdan's result on harmonic measure says that in a Lipschitz domain  $\Omega$ , the  $\alpha$ -processes skip the boundary and jump directly to  $\bar{\Omega}^c$  almost surely. It would be interesting to know which other open sets have this property; we have no answer to this question.

In this note, we study inequalities on volumes or capacities which imply that a set  $S$  on  $\partial D$  has zero harmonic measure, and others which imply a set has positive harmonic measure. In general, it is the relative size of a set  $S$  to  $D^c \setminus S$  that determines whether  $\omega(S, D)$  is positive or zero.

In the following, we use  $c$  to denote positive constants depending at most on  $n$  and  $\alpha$ , use  $a \gtrsim b$  when  $a/b \geq c$  for some  $c > 0$ , and use  $a \cong b$  when  $a \gtrsim b$  and  $b \gtrsim a$ . We use  $d$  to

denote the Euclidean distance,  $|S|$  to denote the Lebesgue measure (or volume) of a set  $S$  and  $\dim S$  to denote the Hausdorff dimension of  $S$ .

The author thanks R. Song for discussions related to Theorem 1.

## 1. Zero Harmonic Measure.

Let  $D$  be an open set, we say  $D^c$  satisfies the *volume density condition* (VDC) provided that there exists  $c > 0$  so that

$$(1.1) \quad |D^c \cap B(x, 2d(x, \partial D))| > c d(x, \partial D)^n \quad \forall x \in D.$$

We note that in (1.1), balls are centered in the interior of  $D$ . There is a more commonly used volume density condition ( $VDC_b$ ) in analysis which states that there exists  $c > 0$  so that

$$|D^c \cap B(x, r)| > cr^n \quad \forall x \in \partial D \quad \text{and} \quad r > 0.$$

It is easy to check that  $VDC_b$  implies  $VDC$  and that  $VDC_b$  implies  $|\partial D| = 0$ . On the other hand, there exist domains  $D$  such that  $D^c$  satisfy  $VDC$  and yet  $|\partial D| > 0$ ; for example, let  $D$  be a domain having the shape of an infinite branching tree and with branches accumulating at a Cantor set  $S$  of positive volume, and having branches chosen so that  $D^c$  satisfies  $VDC$ .

Our first theorem generalizes Bogdan's result on harmonic measures for Lipschitz domains and possibly has been known to him; however the analytic approach here is more direct.

**Theorem 1.** *Let  $0 < \alpha < 2$  and  $D$  be an open set. If  $D^c$  satisfies the volume density condition (VDC) and  $\partial D$  has zero volume then  $\omega(\partial D, D) = 0$ .*

Theorem 1 follows from the more technical result below.

**Theorem 1'.** *Let  $0 < \alpha < 2$ ,  $D$  be an open set and  $S$  be a measurable subset of  $\partial D$  of zero volume. Suppose that there exists  $c' > 0$  so that*

$$(1.2) \quad |D^c \cap B(x, 2d(x, S))| > c' d(x, S)^n \quad \forall x \in D.$$

Then  $\omega(S, D) = 0$ .

*Proof.* Fix  $c_0 \in (0, \frac{1}{2})$ , depending only on  $n$  and  $c'$ , so that

$$(1.3) \quad |D^c \cap B(x, 2d(x, S)) \setminus B(x, c_0 d(x, S))| > \frac{c'}{2} d(x, S)^n \quad \forall x \in D.$$

For  $x \in D$ , let  $\Delta_x = B(x, c_0 \text{dist}(x, S))$ . Then by the Markov property and the maximum principle,

$$(1.4) \quad \begin{aligned} \omega^x(S, D) &= P^x\{X_{\tau_{\Delta_x \cap D}} \in S\} + P\{X_{\tau_{\Delta_x \cap D}} \in D \setminus \Delta_x, X_{\tau_D} \in S\} \\ &= \omega^x(S, \Delta_x \cap D) + \int_{D \setminus \Delta_x} \omega^y(S, D) d\omega^x(y, \Delta_x \cap D) \\ &\leq \omega^x(S, \Delta_x) + \sup_{y \in D} \omega^y(S, D) \omega^x(D \setminus \Delta_x, \Delta_x \cap D) \\ &\leq \sup_{y \in D} \omega^y(S, D) \omega^x(D \setminus \Delta_x, \Delta_x). \end{aligned}$$

Note that  $\omega^x(S, \Delta_x) = 0$  because  $S$  is contained in  $\bar{\Delta}_x^c$  and has zero volume. It follows from the integral representation for  $\alpha$ -harmonic functions in the ball  $\Delta_x$  (0.3)  $\sim$  (0.5) and (1.3) that

$$\begin{aligned} \omega^x(D^c \setminus \Delta_x, \Delta_x) &\geq \omega^x(D^c \cap B(x, 2d(x, S)) \setminus \Delta_x, \Delta_x) \\ &\gtrsim |D^c \cap B(x, 2d(x, S)) \setminus \Delta_x| d(x, S)^{-n} > c_0 > 0. \end{aligned}$$

Therefore

$$(1.5) \quad \sup_{x \in D} \omega^x(D \setminus \Delta_x, \Delta_x) < 1 - c_0 < 1,$$

and it follows from (1.4) that

$$(1.6) \quad \sup_{x \in D} \omega^x(S, D) < 1 - c_0.$$

Apply (1.5) and (1.6) to (1.4), we obtain  $\omega^x(S, D) < (1 - c_0)^2$  for all  $x \in D$ . Iterating this process, we obtain  $\sup_D \omega^x(S, D) < (1 - c_0)^k$  for all  $k \geq 1$ . Therefore  $\omega(S, D) = 0$ .

In examining the sharpness of (1.2), we arrive at the following example.

**Example 1.** Let  $0 < \alpha < 2$ ,  $\lambda > \frac{n+\alpha}{n-\alpha}$  and  $h(t) = (\log^+ \frac{1}{t})^{-\lambda}$ . Then there exist an open set  $D$  in  $\mathbb{R}^n$  and a Cantor set  $S \subseteq \partial D$  of dimension  $n - \alpha$  such that

$$(1.7) \quad |D^c \cap B(x, 2d(x, S))| \gtrsim d(x, S)^n h(d(x, S)) \quad \forall x \in D;$$

however  $\omega(S, D) > 0$ . Moreover  $\omega(\partial D, \bar{D}^c) = 0$ .

When  $n \geq 3$  or when  $n = 2$  and  $0 < \alpha \leq 1$ ,  $\partial D$  can be chosen to be the image of a sphere under a homeomorphism of  $\mathbb{R}^n$ .

Detailed construction shall be given in §3.

We believe that the sharp condition to replace (1.2) in Theorem 1' lies strictly between (1.2) and (1.7).

Next, we give a sufficient condition for a set  $S$  to have zero  $\alpha$ -harmonic measure by comparing capacities of  $S$  and  $D^c \setminus S$ .

For  $0 < \alpha < n$  and  $S$  compact in  $\mathbb{R}^n$ , the  $\alpha$ -capacity of  $S$ ,  $C_\alpha(S)$ , is

$$\inf \left\{ \mu(S) : \int_S |x - y|^{-n+\alpha} d\mu(y) \geq 1 \text{ on } S \right\}.$$

There exists a measure  $\mu$  ( $\alpha$ -capacitory measure), for which the infimum is attained; furthermore  $\mu(S) = C_\alpha(S)$  and  $\int_S |x - y|^{-n+\alpha} d\mu(y) = 1$  on  $S$  except on a set of zero  $\alpha$ -capacity.

For a Borel set  $T$ , define

$$C_\alpha(T) = \sup\{C_\alpha(S) : S \text{ compact, } S \subseteq T\},$$

and note that

$$C_\alpha(B(0, r)) \cong r^{n-\alpha}.$$

Let  $D$  be an open set in  $\mathbb{R}^n$  ( $n \geq 2$ ). We say  $D^c$  satisfies the  $\alpha$ -capacity density condition ( $\alpha - CDC$ ) provided that there exists  $\tau > 0$  so that

$$(1.8) \quad C_\alpha(D^c \cap \overline{B(x, 2d(x, \partial D))}) > \tau d(x, \partial D)^{n-\alpha} \quad \forall x \in D.$$

Here again our condition is weaker than the usual one, namely, there exists  $\tau > 0$  so that

$$C_\alpha(D^c \cap B(x, r)) > \tau r^{n-\alpha} \quad \forall x \in \partial D, r > 0.$$

**Theorem 2.** *Let  $0 < \alpha < 2$ ,  $D$  an open set and  $S$  a compact subset of  $\partial D$ . Suppose that  $D^c$  satisfies the  $\alpha$ -capacity density condition (1.8) and  $S$  satisfies*

$$(1.9) \quad C_\alpha(S \cap \overline{B(x, 5d(x, \partial D))}) < \frac{\tau}{4}(2^{-n+\alpha} - 3^{-n+\alpha})(d(x, \partial D))^{n-\alpha} \quad \forall x \in D.$$

*Then  $\omega(S, D) = 0$ .*

The theorem says that if  $\alpha$ -capacity of  $D^c$  is large and  $\alpha$ -capacity of  $S$  is small in comparison at all scales then  $\omega(S, D) = 0$ .

*Proof.* Let  $T = D^c \setminus S$ ; and for a given  $x \in D$ , let  $cB = B(x, cd(x, \partial D))$ . Note from (1.8) and (1.9) that  $C_\alpha(T \cap \overline{2B}) > \frac{3}{4}\tau d(x, \partial D)^{n-\alpha}$ . Choose  $\tilde{T}$  a compact subset of  $T \cap \overline{2B}$  so that  $C_\alpha(\tilde{T}) > \frac{\tau}{2} d(x, \partial D)^{n-\alpha}$  and  $\mu$  its  $\alpha$ -capacitory measure.

Let  $\Omega = (D \cap 5B) \cup ((5B \setminus \overline{2B}) \setminus S)$  and note that  $\Omega^c = (T \cap \overline{2B}) \cup (S \cap 5B) \cup (5B)^c$ . It follows from the maximum principle that

$$\omega^x(T, D) \geq \omega^x(T \cap \overline{2B}, \Omega).$$

Since  $\omega(T \cap \overline{2B}, \Omega) + \omega(S \cap 5B, 5B \setminus S)$  and  $\omega(\tilde{T}, 5B \setminus \tilde{T})$  are  $\alpha$ -harmonic in  $\Omega$  and

$$(1.10) \quad \omega(T \cap \overline{2B}, \Omega) + \omega(S \cap 5B, 5B \setminus S) \geq \omega(\tilde{T}, 5B \setminus \tilde{T})$$

in  $\Omega^c$ , the inequality (1.10) holds at  $x$  also.

Then, by the conditions imposed on  $\tilde{T}$ ,

$$(1.11) \quad \begin{aligned} \omega^x(\tilde{T}, 5B \setminus \tilde{T}) &\geq \int_{\tilde{T}} |x - y|^{-n+\alpha} d\mu(y) - \int_{(5B)^c} \int_{\tilde{T}} |z - y|^{-n+\alpha} d\mu(y) d\omega^x(z, 5B) \\ &\geq \int_{\tilde{T}} |x - y|^{-n+\alpha} d\mu(y) - (3 \operatorname{dist}(x, \partial D))^{-n+\alpha} \mu(\tilde{T}) \\ &\geq (2^{-n+\alpha} - 3^{-n+\alpha}) d(x, \partial D)^{-n+\alpha} \mu(\tilde{T}) \\ &\geq (2^{-n+\alpha} - 3^{-n+\alpha}) \tau / 2. \end{aligned}$$

Let  $\nu$  be the  $\alpha$ -capacity measure for  $S \cap \overline{5B}$ , then

$$(1.12) \quad \begin{aligned} \omega^x(S \cap \overline{5B}, 5B \setminus S) &\leq \int_{S \cap 5B} |x - y|^{-n+\alpha} d\mu(y) \\ &\leq \mu(S \cap \overline{5B}) d(x, \partial D)^{-n+\alpha}. \end{aligned}$$

From (1.9)  $\sim$  (1.12) and the subadditivity of capacity it follows that

$$\omega^x(T, D) \geq \frac{\tau}{4} (2^{-n+\alpha} - 3^{-n+\alpha}) \quad \forall x \in D,$$

or

$$\omega^x(S, D) < c(n, \alpha, \tau) < 1 \quad \forall x \in D.$$

Therefore  $\omega(S, D) = 0$  by Lemma 1 below.

Let  $\mathcal{H}_D$  be the class of functions continuous in  $\mathbb{R}^n$  and  $\alpha$ -harmonic in  $D$ ,  $C(D^c)$  be the class of functions  $f$  continuous in  $D^c$  satisfying  $\int_{D^c \cap \{|x|>1\}} |f(x)||x|^{-n-\alpha} < \infty$  and  $H(D)$  be the class of functions on  $\mathbb{R}^n$ ,  $\alpha$ -harmonic in  $D$ . Following Landkof ([L], Chap. IV, 15 and 21), there exists a unique positive linear operator  $A$  from  $C(D^c)$  into  $H(D)$  so that for  $f \in \mathcal{H}_D$ ,  $A(f|D^c) = f$  in  $\mathbb{R}^n$ ; moreover

$$Af(x) = \int_{D^c} f(y) d\omega^x(y, D) \quad \text{on } D,$$

and  $A(f(x))$  is the unique (up to a set of  $\alpha$ -capacity zero)  $\alpha$ -harmonic function in  $D$  which has limit  $f(z)$  at each regular boundary point  $z$ . The set of irregular boundary points has zero  $\alpha$ -capacity hence zero  $\alpha$ -harmonic measure; and  $Af$  is called the solution of the Dirichlet problem for the given  $f$ .

Given  $f \in C(D^c)$ , let  $\mathcal{P}_f$  (Perron family) be the class of functions  $u$ ,  $\alpha$ -subharmonic in  $D$ , with  $u \leq f$  in  $\bar{D}^c$  and boundary values  $u^* \leq f$  on  $\partial D$ . (An  $\alpha$ -subharmonic function in  $D$  is a function locally integrable in  $\mathbb{R}^n$ , upper semi-continuous in  $D$  satisfying  $\int_{|x|>1} |u(x)||x|^{-n-\alpha} dx < \infty$  and  $u(x) \leq E^x u(X_{\tau_{B(x,r)}})$  for all balls  $B(x, r) \subseteq \overline{B(x, r)} \subseteq D$ ; and  $u^*(z) \equiv \limsup u(x)$  as  $x \rightarrow z$  in  $D$  for  $z \in \partial D$ .) Define

$$Lf(x) = \sup\{u(x) : u \in \mathcal{P}_f\} \quad \text{in } \mathbb{R}^n.$$

Following the argument for Brownian motion ( $\alpha = 2$ ) as in [F] or [H], we may prove that

$Lf$  is  $\alpha$ -harmonic in  $D$ , and  $L$  is a positive linear operator from  $C(D^c)$  into  $H(D)$  with  $A(f|D^c) = f$  for  $f \in \mathcal{H}_D$ . By the uniqueness,  $L = A$ .

The lemma below is known for the Brownian motion ( $\alpha = 2$ ). When  $0 < \alpha < 2$ , it is less apparent and is false if the assumption  $S \subseteq \partial D$  is removed; note for example, if  $D = B(0, 1)$  and  $S = \{2 \leq |x| \leq 3\}$ , then  $\sup_{x \in D} \omega^x(S, D) < 1$  and  $\omega^0(S, D) > 0$ .

**Lemma 1.** *Let  $0 < \alpha < 2$ ,  $D$  be an open set and  $S$  be a compact subset of  $\partial D$ . Suppose that  $\sup_{x \in D} \omega^x(S, D) < 1$ , then  $\omega(S, D) = 0$ .*

*Proof.* Let  $f_k$  be a decreasing sequence of continuous functions on  $D^c$  with  $0 \leq f_k \leq 1$ , support  $f_k \downarrow S$  and  $f_k \downarrow \chi_S$ . Then

$$Af_k(x) = \int_{D^c} f_k(y) d\omega^x(y, D) \downarrow \omega^x(S, D) \quad \text{on } D.$$

Let  $a = \sup_{x \in D} \omega^x(S, D) < 1$ . We claim that for each  $k$ ,

$$(1.13) \quad \omega^x(S, D) \leq aAf_k(x) \quad \text{on } D.$$

This shows that  $\omega^x(S, D) \leq a\omega^x(S, D)$  on  $D$ , which implies that  $\omega(S, D) = 0$ .

To prove (1.13), we fix  $k$  and let

$$v(x) = \omega^x(S, D) - aAf_k(x) \quad \text{in } \mathbb{R}^n.$$

Note that  $v \leq 1$  and that  $v^*(z) \leq 0$  at every regular boundary point  $z$ . Take  $\epsilon > 0$  and  $F_\epsilon$  the set of boundary points  $z$  where  $v^*(z) \geq \epsilon$ . The set  $F_\epsilon$  is closed and consists of irregular points only, hence  $\omega(F_\epsilon, D) = 0$ . Choose a decreasing sequence of continuous functions  $g_j$  on  $D^c$  with  $0 \leq g_j \leq 1$ , support  $g_j \downarrow F_\epsilon$  and  $g_j \downarrow \chi_{F_\epsilon}$ . Then  $v \in \mathcal{P}_{\epsilon+g_j}$  for each  $j$ . Therefore  $v(x) \leq L(\epsilon + g_j)(x) = \epsilon + Lg_j(x) = \epsilon + Ag_j(x)$  for each  $j$ . Since  $Ag_j(x) \downarrow \omega^x(F_\epsilon, D) = 0$  in  $D$ , we have  $v(x) \leq \epsilon$  in  $D$  for any  $\epsilon > 0$ . This proves the claim and thus the lemma.

## 2. Positive Harmonic Measure.

**Theorem 3.** *Let  $0 < \alpha < 2$ ,  $D$  be an open set and  $S$  be a compact subset of  $\partial D$  having positive  $\alpha$ -capacity. Suppose that  $C_\alpha(\{x \in D^c : 0 < d(x, S) \leq a\}) \rightarrow 0$  as  $a \rightarrow 0$ . Then  $\omega(S, D) > 0$ .*

*Proof.* We assume as we may that  $S \subseteq B(0, 1/2)$ . Let  $T = D^c \setminus S$ , and let  $T_a = \{x \in D^c : 0 < d(x, S) \leq a\}$  and  $T_{a,b} = \{x \in D^c : b \leq d(x, S) \leq a\}$  for  $0 < b < a < \frac{1}{10}$ . Then

$$C_\alpha(T_a) = \sup_{b>0} C_\alpha(T_{a,b}).$$

Choose and fix  $a > 0$  so that

$$C_\alpha(T_a) < 2^{-n+\alpha-1}(1 - 2^{-n+\alpha})C_\alpha(S),$$

and let  $b \in (0, a)$ .

Denote by  $\mu$  and  $\nu$  the  $\alpha$ -capacitory measures of  $S$  and  $T_{a,b}$  respectively. And let

$$U(x) = \int_S |x - y|^{-n+\alpha} d\mu(y),$$

and

$$V(x) = \int_{T_{a,b}} |x - y|^{-n+\alpha} d\nu(y).$$

Recall that  $U$  and  $V$  are positive  $\alpha$ -harmonic in  $\mathbb{R}^n \setminus S$  and  $\mathbb{R}^n \setminus T_{a,b}$  respectively, with  $U \equiv 1$  on  $S$  and  $V \equiv 1$  on  $T_{a,b}$  except possibly on some sets of zero  $\alpha$ -capacity, and  $\mu(S) = C_\alpha(S)$  and  $\nu(T_{a,b}) = C_\alpha(T_{a,b})$ . Hence

$$\nu(T_{a,b}) < 2^{-n+\alpha-1}(1 - 2^{-n+\alpha})\mu(S).$$

Note that for  $|x| \geq 3$ ,

$$\begin{aligned}
(2.1) \quad V(x) &\leq (|x| - 1)^{-n+\alpha} \nu(T_{a,b}) \\
&< (|x| - 1)^{-n+\alpha} 2^{-n+\alpha-1} (1 - 2^{-n+\alpha}) \mu(S) \\
&< 2^{-n+\alpha-1} (1 - 2)^{-n+\alpha} \left( \frac{|x| - 1}{|x| + 1} \right)^{-n+\alpha} U(x) \\
&\leq 2^{-1} (1 - 2^{-n+\alpha}) U(x).
\end{aligned}$$

Denote by  $u(x) = \omega^x(S, B(0, 10))$ ,  $v(x) = \omega^x(T_{a,b}, B(0, 10))$  and by  $\tilde{u}$  the function in  $\mathbb{R}^n$   $\alpha$ -harmonic in  $B(0, 10)$  with  $\tilde{u} \equiv U$  on  $B(0, 10)^c$ , i.e.

$$\tilde{u}(x) = \int_{B(0,10)^c} U(y) d\omega^x(y, B(0, 10)) \quad \text{in } B(0, 10).$$

Note that

$$u(x) > U(x) - \tilde{u}(x) \quad \text{on } B(0, 10).$$

Since  $U(y) \leq 9^{-n+\alpha} \mu(S)$  for  $|y| \geq 10$ , we have  $\tilde{u} \leq 9^{-n+\alpha} \mu(S)$ . Therefore for  $|x| < 10$ ,

$$\begin{aligned}
u(x) &> U(x) - 9^{-n+\alpha} \mu(S) \\
&= (1 - 2^{-n+\alpha}) U(x) + 2^{-n+\alpha} U(x) - 9^{-n+\alpha} \mu(S).
\end{aligned}$$

Note that for  $|x| = 3$ ,  $U(x) > 4^{-n+\alpha} \mu(S)$ , and

$$u(x) > (1 - 2^{-n+\alpha}) U(x) > 2V(x) > 2v(x)$$

from (2.1). Therefore for  $|x| = 3$ ,

$$\begin{aligned}
&\omega^x(S \cup T_b, B(0, 10) \setminus (S \cup T_a)) \\
&\geq \omega^x(S, B(0, 10) \setminus (S \cup T_{a,b})) \\
&\geq u(x) - v(x) > \frac{1}{2} u(x) > \frac{1}{2} (1 - 2^{-n+\alpha}) U(x) \\
&> c(n, \alpha) \mu(S)
\end{aligned}$$

for some  $c(n, \alpha) > 0$ . Let  $b \rightarrow 0^+$ , we obtain

$$\omega^x(S, B(0, 10) \setminus (S \cup T_a)) \geq c(n, \alpha) \mu(S) \quad \text{for } |x| = 3.$$

Hence by Lemma 1,

$$(2.2) \quad \sup\{\omega^x(S, B(0, 10) \setminus (S \cup T_a)) : x \in B(0, 10) \setminus (S \cup T_a)\} = 1.$$

If  $T_a = T \cap B(0, 10)$ , then  $B(0, 10) \setminus (S \cup T_a) = D \cap B(0, 10)$ , thus

$$\omega^x(S, D) \geq \omega^x(S, B(0, 10) \cap D) > 0 \quad \text{in } D \cap B(0, 10).$$

If  $T \cap B(0, 10) \setminus T_a \neq \phi$ , let

$$A = \sup\{\omega^x(S, B(0, 10) \setminus (S \cup T_a)) : x \in B(0, 10) \cap T \setminus T_a\}.$$

Because  $\text{dist}(S, B(0, 10) \cap T \setminus T_a) \geq a > 0$ , we have  $0 < A < 1$ . It follows from (2.2) that there exists  $x_0 \in B(0, 10) \setminus (S \cup T_a)$  so that  $\omega^{x_0}(S, B(0, 10) \setminus (S \cup T_a)) > A$ . Since  $D^c$  is the union of two disjoint sets  $S \cup T_a$  and  $T \setminus T_a$ , it follows from the Markov property and the choice of  $A$  and  $x_0$  that

$$\begin{aligned} & \omega^{x_0}(S, B(0, 10) \cap D) \\ &= \omega^{x_0}(S, B(0, 10) \setminus (S \cup T_a)) \\ & \quad - \int_{B(0, 10) \cap T \setminus T_a} \omega^x(S, B(0, 10) \setminus (S \cup T_a)) d\omega^{x_0}(x, B(0, 10) \setminus (S \cup T_a)) > 0. \end{aligned}$$

This implies that  $\omega^{x_0}(S, D) > 0$  and proves the theorem.

Proof of Theorem 3 shows in fact the following.

**Theorem 3'.** *There exists  $c(n, \alpha) > 0$  so that for any open set  $D$  and compact set  $S \subseteq \partial D$ , if  $C_\alpha(S) > 0$  and*

$$C_\alpha(\{x \in D^c : 0 < d(x, S) \leq a\}) < c(n, \alpha)C_\alpha(S)$$

*for some  $a > 0$ , then  $\omega(S, D) > 0$ .*

### 3. Examples.

Our theorems suggest that it is the relative sizes of  $S$  and  $D^c \setminus S$  that determine the vanishing of  $\omega(S, D)$ . Examples 1 and 2 reinforce this observation, and use all three theorems in the reasoning.

First we give the details for Example 1.

Since  $\lambda > \frac{n+\alpha}{n-\alpha}$ ,  $(\lambda+1)\frac{n-\alpha}{n} > 2$ . Choose  $\epsilon, \delta, a_k, x_k$  so that  $0 < \delta < \epsilon < (\lambda+1)\frac{n-\alpha}{n} - 2$ ,

$$a_k^{n-\alpha} = 2^{-nk} k^{1+\delta}$$

and

$$x_k^{n-\alpha} = k^{-2-\epsilon}.$$

Calculations show that

$$kx_k^n = k^{1-(2+\epsilon)\frac{n}{n-\alpha}} > k^{-\lambda}$$

and that for large  $k$ , say  $k \geq k_0$ ,

$$2a_{k+1} + 3x_k a_k < a_k.$$

Let  $S$  be the Cantor set in the unit cube constructed as follows. Let  $Q_0^1 = [0, 1]^n$ . Assume that the cubes  $Q_k^j$  ( $1 \leq j \leq 2^{nk}$ ) of side length  $a_k$  have been constructed. Inside

each  $Q_k^j$ , choose  $2^n$  closed subcubes of side length  $a_{k+1}$  at vertices of  $Q_k^j$ . Call this collection of subcubes  $Q_{k+1}^i (1 \leq i \leq 2^{n(k+1)})$ . Let

$$S = \bigcap_k \left( \bigcup_j Q_k^j \right).$$

Let  $B_k^j$  be the closed ball concentric to  $Q_k^j$  of radius  $x_k a_k$  and let

$$D = \mathbb{R}^n \setminus (S \cup \cup' B_k^j)$$

where  $\cup' B_k^j$  denotes the union  $\cup_{k \geq k_0} (\cup_j B_k^j)$ . Note that  $\partial D = S \cup \cup' \partial B_k^j$ .

The volume condition (1.7) follows easily from the estimate:

$$\begin{aligned} |D^c \cap Q_k^j| &= \sum_{m=k}^{\infty} 2^{n(m-k)} |B_m^1| \cong \sum_{m=k}^{\infty} 2^{-nk} 2^{nm} x_m^n a_m^n \\ &\cong k x_k^n a_k^n \geq k^{-\lambda} a_k^n. \end{aligned}$$

It is clear that  $\dim S = n - \alpha$ .

To see  $C_\alpha(S) > 0$ , we let  $\mu$  be the measure on  $S$  so that  $\mu(S \cap Q_k^j) = 2^{-kn}$  and observe that  $\mu(S \cap B(x, t)) \leq C(n) t^{n-\alpha} (\log \frac{2\sqrt{n}}{t})^{-1-\delta}$  for any  $x \in S$  and  $0 < t \leq \sqrt{n}$ . Therefore

$$\int_S |x - y|^{-n+\alpha} d\mu(y) \lesssim \int_0^{\sqrt{n}} \frac{\mu(B(x, t))}{t^{n-\alpha+1}} dt \leq c(n, \alpha, \delta) < +\infty$$

for all  $x \in S$ ; this proves that  $C_\alpha(S) > 0$ .

We now prove that

$$(3.1) \quad C_\alpha(S \cap Q_k^j) \cong 2^{-kn} k^\delta = a_k^{n-\alpha} k^{-1},$$

a fact needed in later discussion. Consider the restriction of  $\mu$  (from the last paragraph) onto  $Q_k^j$ ; then for any  $x \in S \cap Q_k^j$ ,

$$\int_{S \cap Q_k^j} |x - y|^{-n+\alpha} d\mu(y) \lesssim \int_0^{\sqrt{na_k}} \frac{\mu(B(x, t))}{t^{n-\alpha+1}} dt \cong k^{-\delta}.$$

For each  $x \in S \cap Q_k^j$ , a mutually disjoint sequence of cubes  $\{Q_m^{i(m,x)} : m \geq k+1\}$  can be found in  $Q_k^j$  so that  $\text{dist}(x, Q_m^{i(m,x)}) \cong a_m$ ; therefore

$$\begin{aligned} \int_{S \cap Q_k^j} |x-y|^{-n+\alpha} d\mu(y) &\gtrsim \sum_{m=k+1}^{\infty} \text{dist}(x, Q_m^{i(m,x)})^{-n+\alpha} \mu(Q_m^{i(m,x)}) \\ &\cong \sum_{m=k+1}^{\infty} a_m^{-n+\alpha} 2^{-mn} \cong k^{-\delta}. \end{aligned}$$

Since  $\mu(S \cap Q_k^j) = 2^{-kn}$ , this proves (3.1).

Denote by  $W_k = \cup_{m \geq k} (\cup_j B_m^j)$  for  $k \geq k_0$ . Then by the subadditivity of capacities,

$$C_\alpha(W_k) \leq \sum_{m=k}^{\infty} \sum_j C_\alpha(B_m^j) \lesssim \sum_{m=k}^{\infty} 2^{nm} x_m^{n-\alpha} a_m^{n-\alpha} \leq \sum_{m=k}^{\infty} m^{-1-\epsilon+\delta} \cong k^{-\epsilon+\delta}.$$

In fact  $C_\alpha(W_k) \cong k^{-\epsilon+\delta}$ ; this is not needed in our argument and the detail is not given.

Since  $C_\alpha(S) > 0$  and  $C_\alpha(W_k) \rightarrow 0$  as  $k \rightarrow \infty$ , it follows from Theorem 3 that

$$\omega(S, D) > 0.$$

Since the exterior of the open set  $\bar{D}^c$  satisfies the *VDC* and  $\partial D$  has zero volume, it follows from Theorem 1 that  $\omega(\partial D, \bar{D}^c) = 0$ .

When  $n \geq 3$ , or when  $n = 2$  and  $0 < \alpha \leq 1$ , a line segment has zero  $\alpha$ -capacity. Fix a closed ball  $B_0$  centered at  $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$  disjoint from  $S \cup \cup' B_k^j$ . First connect  $B_0$  to each  $B_{k_0}^i$  by a thin wire  $L_{k_0}^i$  in the unit cube and then connect each  $B_k^j (k \geq k_0)$  to every  $B_{k+1}^i$  contained in  $Q_k^j$  by a thin wire  $L_{k+1}^i$  in  $Q_k^j$  so that every wire meets exactly two balls, the entire collection of wires are mutually disjoint and  $S \cup B_0 \cup \cup' (B_k^j \cup L_k^j)$  is the image of a closed ball under a global homeomorphism. Let

$$D = \mathbb{R}^n \setminus (S \cup B_0 \cup \cup' (B_k^j \cup L_k^j)).$$

When wires are chosen thin enough, all assertions in Example 1 are still valid.

## Example 2.

If we retain  $S$  from Example 1 and replace each  $B_k^j$  by a closed set in  $Q_k^j \setminus \cup_i Q_{k+1}^i$  having  $\alpha$ -capacity  $\cong a_k^{n-\alpha}$ , then, in contrast with Example 1,  $\omega(S, D) = 0$ . This assertion follows from Theorem 2, the statement (3.1) and the fact that for the new set  $D$ ,  $D^c$  satisfies  $\alpha - CDC$ .

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