

Boundary Harnack Principle for Symmetric Stable Processes

Renming Song* and Jang-Mei Wu†

Department of Mathematics

University of Illinois

Urbana, IL 61801

Email: rsong@math.uiuc.edu

wu@math.uiuc.edu

Abstract

In this paper we study potential-theoretic properties of the symmetric α -stable processes ($0 < \alpha < 2$): establishing the boundary Harnack principle for ratios of α -harmonic functions on any open sets, identifying the Martin boundary with the Euclidean boundary for open sets with a certain interior fatness property, and extending earlier results on intrinsic ultracontractivity and conditional gauge theorem to certain open sets.

Key words: symmetric stable processes, boundary Harnack principle, Martin boundary, conditional gauge theorem, intrinsic ultracontractivity.

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*The research of this author is supported in part by NSF Grant DMS-9803240

†The research of this author is supported in part by NSF Grant DMS-9705227

1 Introduction

A symmetric α -stable process X on \mathbf{R}^n is a Lévy process whose transition density $p(t, x - y)$ relative to Lebesgue measure is uniquely determined by its Fourier transform

$$\int_{\mathbf{R}^n} e^{ix \cdot \xi} p(t, x) dx = e^{-t|\xi|^\alpha}.$$

Here α must be in the interval $(0, 2]$. When $\alpha = 2$, we get a Brownian motion running with a time clock twice as fast as the standard one. From now on, symmetric stable processes are restricted to the case $0 < \alpha < 2$. Unlike the generator Δ of Brownian motion whose time clock is twice as fast as the standard one, the generator of a symmetric α -stable process with $0 < \alpha < 2$ is the fractional Laplacian $-(-\Delta)^{\alpha/2}$ which is non-local. A symmetric stable process has discontinuous sample paths and heavy tails, while Brownian motion has continuous sample paths and exponentially decaying tails. X_t has infinite variance and when $0 < \alpha \leq 1$, $|X_t|$ has infinite mean.

Considerable progress has been made recently in extending potential-theoretic properties of Brownian motion to symmetric α -stable processes ($0 < \alpha < 2$): on Lipschitz domains, the boundary Harnack principle holds, the Martin boundary coincides with the Euclidean boundary and the conditional gauge theorem is true; and the semigroup of the Feynman-Kac semigroup for killed symmetric stable processes in a Hölder domain of order 0 is intrinsically ultracontractive. See [?], [?], [?], [?], [?], [?], [?], [?] for recent results on symmetric stable processes. For related results on Brownian motions see [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?] and [?].

Because the symmetric α -stable processes ($0 < \alpha < 2$) are discontinuous, it is more natural to study these potential-theoretic properties on general open sets, with no assumptions on the connectedness of the sets or the smoothness of the boundaries.

We prove in this paper the boundary Harnack principle on any open set D : for nonnegative functions u and v in \mathbf{R}^n , positive regular α -harmonic on $D \cap B(z, 2r)$, vanishing on $D^c \cap B(z, 2r)$, $z \in \partial D$ and $r > 0$, we have

$$C^{-1} \frac{u(x)}{u(y)} \leq \frac{v(x)}{v(y)} \leq C \frac{u(x)}{u(y)}, \quad x, y \in D \cap B(z, r),$$

for some constant C depending on n , α , and the ratio between r and the radius of the largest ball in $D \cap B(z, r)$. Our proof takes advantage of the discontinuity of the process and does not rely explicitly on standard inequalities on the ratio of harmonic measure and the Green function.

We also prove, with respect to symmetric stable processes, that the Martin boundary and the minimal Martin boundary coincide with the Euclidean boundary for bounded open sets satisfying a certain fatness condition near the boundary; these include, for example, the set $\{|x| < 1\} \setminus \{x_n = 0\}$ and a disconnected analogue of John domains.

Finally we extend the intrinsic ultracontractivity of the Feynman-Kac semigroup of the killed symmetric stable processes to open sets on which a certain distance function is inte-

grable; and extends the 3G theorem and the conditional gauge theorem to bounded open sets which are fat near the boundary.

The discontinuity of the process is crucial in obtaining results on disconnected sets, but it imposes many technical complications, e.g., the usual maximum principle does not hold for α -harmonic functions. Bogdan overcame some of the difficulties in his papers [?] and [?] on Lipschitz domains. The influence of [?] and [?] in our work will be apparent.

2 Preliminaries

In this paper we always assume that X is a symmetric α -stable process on \mathbf{R}^n with $n \geq 2$ and $0 < \alpha < 2$. The process X is transient and we use G to denote the Green function of X :

$$G(x, y) = A_1(n, \alpha)|x - y|^{\alpha-n},$$

where

$$A_1(n, \alpha) = 2^{-\alpha} \pi^{-\frac{n}{2}} \Gamma\left(\frac{n - \alpha}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)^{-1}.$$

Let Cap denote the capacity associated with the process X (or equivalently, with the Riesz kernel $A_1(n, \alpha)|x - y|^{\alpha-n}$). A function u is said to be quasi-continuous if there exists for any $\epsilon > 0$ an open set U such that $\text{Cap}(U) < \epsilon$ and $u|_{U^c}$ is continuous. It is well known that the Dirichlet form $(\mathcal{E}, \mathcal{F}^{\mathbf{R}^n})$ associated with X is given by

$$\begin{aligned} \mathcal{E}(u, v) &= A_2(n, \alpha) \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\alpha}} dx dy \\ \mathcal{F}^{\mathbf{R}^n} &= \left\{ u \in L^2(\mathbf{R}^n) : \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} dx dy < \infty \right\}, \end{aligned}$$

where

$$A_2(n, \alpha) = \frac{\alpha 2^{\alpha-1} \Gamma\left(\frac{\alpha+n}{2}\right)}{\pi^{n/2} \Gamma\left(1 - \frac{\alpha}{2}\right)}.$$

We also know that for any function $u \in \mathcal{F}^{\mathbf{R}^n}$, there is a quasi-continuous function \tilde{u} which is equal to u outside a set of Lebesgue measure 0. For any $u \in \mathcal{F}^{\mathbf{R}^n}$, we always use \tilde{u} to denote the quasi-continuous function above. As usual, E^x stands for the expectation with respect to the process starting from $x \in \mathbf{R}^n$, τ_A stands for the first exit time from A by the process X : $\tau_A = \inf\{t > 0 : X_t \notin A\}$. T_A stands for the first hitting time of A by the process X : $T_A = \inf\{t > 0 : X_t \in A\}$.

For any open set D , we use X^D to denote the symmetric α -stable process X killed upon leaving D . The Dirichlet form associated with X^D is $(\mathcal{E}, \mathcal{F})$, where

$$\mathcal{F} = \{u \in \mathcal{F}^{\mathbf{R}^n} : \tilde{u} \text{ is equal to } 0 \text{ outside a set of zero capacity on } D^c\}.$$

It is well known that there is a continuous symmetric function $G_D(\cdot, \cdot)$ defined on $D \times D$ except along the diagonal such that for any Borel measurable function $f \geq 0$,

$$E^x \left[\int_0^{\tau_D} f(X_s) ds \right] = \int_D G_D(x, y) f(y) dy.$$

G_D is called the Green function of X in D . Recall that a point $z \in \partial D$ is said to be a regular boundary point if $P^z(\tau_D = 0) = 1$ and an irregular boundary point if $P^z(\tau_D = 0) = 0$. It is well known that if $z \in \partial D$ is regular, then for any $x \in D$, $G_D(x, y) \rightarrow 0$ as $y \rightarrow z$. For any $x \in D$, we use ω_D^x to denote the α -harmonic measure on D^c associated with D :

$$\omega_D^x(A) = P^x(X(\tau_D) \in A), \quad A \subset D^c.$$

The α -harmonic measure ω_D^x is absolutely continuous on $(\overline{D})^c$ with respect to the Lebesgue measure and has a density

$$k_D(x, z) = A_2(n, \alpha) \int_D \frac{G_D(x, y)}{|y - z|^{n+\alpha}} dy \quad (2.1)$$

(see for instance [?]). It is proved in [?] that the boundary of a Lipschitz domain has zero harmonic measure. In the case of a ball $B = B(0, r)$,

$$k_B(x, y) = \begin{cases} A_3(n, \alpha) \left[\frac{r^2 - |x|^2}{|y|^2 - r^2} \right]^{\alpha/2} |x - y|^{-n}, & |y| > r, \\ 0, & |y| \leq r, \end{cases} \quad (2.2)$$

where

$$A_3(n, \alpha) = \Gamma(n/2) \pi^{-\frac{n}{2}-1} \sin(\pi\alpha/2).$$

Definition 2.1 Let D be an open set in \mathbf{R}^n . A locally integrable function f defined on \mathbf{R}^n taking values in $(-\infty, \infty]$ and satisfying the condition $\int_{\{|x|>1\}} |f(x)| |x|^{-(n+\alpha)} dx < \infty$ is said to be α -harmonic in D if f is continuous in D and for each $x \in D$ and each ball $B(x, r)$ with $\overline{B(x, r)} \subset D$,

$$f(x) = E^x f(X(\tau_{B(x, r)})).$$

An α -harmonic function f in D is said to be regular α -harmonic in D if

$$f(x) = E^x [f(X_{\tau_D})], \quad \forall x \in D;$$

and is said to be singular α -harmonic in D if it vanishes outside D .

If we set $\omega_D^x(A)$ to be $1_A(x)$ for $x \in D^c$, then $\omega_D^x(A)$ is regular α -harmonic in D . If we set G_D equal to zero on the diagonal of $D \times D$ and outside $D \times D$, then for any $x \in D$, $G_D(x, \cdot) = G_D(\cdot, x)$ is singular α -harmonic in $D \setminus \{x\}$.

If f is α -harmonic in a bounded open set D , then f is regular α -harmonic in D_1 for any $D_1 \subset \overline{D_1} \subset D$. If f is α -harmonic in a bounded open set D and continuous on \overline{D} , then h is regular α -harmonic in D . When D is a Lipschitz domain, every nonnegative α -harmonic function f in D can be written uniquely as $r + s$ where r and s are nonnegative, respectively regular and singular α -harmonic in D . These and other properties of α -harmonic functions can be found in [?] and [?].

The following Harnack inequality is due to Bogdan [?].

Lemma 2.1 *Let $r > 0$ and x_1, x_2 be points in \mathbf{R}^n satisfying $|x_1 - x_2| < Mr$. Let $u \geq 0$ be a function in \mathbf{R}^n which is α -harmonic in $B(x_1, r) \cup B(x_2, r)$. Then*

$$C^{-1}M^{-n-\alpha}u(x_2) \leq u(x_1) \leq CM^{n+\alpha}u(x_2)$$

for some constant $C = C(n, \alpha) > 1$.

In this paper, we use $B(x, r)$ to denote the open ball of radius r centered at x , and use $R(x, r_1, r_2)$ to denote the spherical shell $\{y : r_1 < |y - x| < r_2\}$. It is understood that all constants depend at most on n and α unless otherwise mentioned; for simplicity, especially in the proofs, we use C and C' to denote positive constants depending at most on n and α , and use $C(\cdot)$ to denote constants depending also on the variables inside the parentheses in addition to n and α ; the values of the constants may change from one line to another.

3 Boundary Harnack Principle

We state the main result of this section.

Theorem 3.1 (*Boundary Harnack Principle.*) *Suppose that D is an open set, $z \in \partial D$, $r > 0$ and that $B(A, \kappa r)$ is a ball in $D \cap B(z, r)$. Then for any functions $u, v \geq 0$ in \mathbf{R}^n , positive regular α -harmonic in $D \cap B(z, 2r)$ and with value 0 in $D^c \cap B(z, 2r)$, we have*

$$C^{-1}\kappa^{n+\alpha}\frac{u(A)}{v(A)} \leq \frac{u(x)}{v(x)} \leq C\kappa^{-n-\alpha}\frac{u(A)}{v(A)}, \quad x \in D \cap B(z, \frac{r}{2}),$$

for some constant $C = C(n, \alpha) > 1$.

Definition 3.1 *Let $\kappa \in (0, 1)$. We say that an open set D in \mathbf{R}^n is κ -fat if there exists $R > 0$ such that for each $z \in \partial D$ and $r \in (0, R)$, $D \cap B(z, r)$ contains a ball $B(A_r(z), \kappa r)$.*

Bounded κ -fat open sets may be regarded as the disconnected analogue of John domains.

Theorem 3.2 *Suppose that D is a bounded open set which is κ -fat for some $\kappa \in (0, 1)$. Then there exist positive constants $R, C = C(n, \alpha, \kappa)$ and $\nu = \nu(n, \alpha, \kappa)$ such that for all $z \in \partial D$ and $r \in (0, R)$, functions $u, v \geq 0$ in \mathbf{R}^n , regular α -harmonic in $D \cap B(z, 2r)$, vanishing on $D^c \cap B(z, 2r)$ and satisfying $u(A_r(z)) = v(A_r(z)) > 0$, we have*

$$\left| \frac{u(x)}{v(x)} - \frac{u(y)}{v(y)} \right| \leq C \left(\frac{|x - y|}{r} \right)^\nu, \quad x, y \in D \cap B(z, r).$$

In particular, the limit $\lim_{D \ni x \rightarrow w} u(x)/v(x)$ exists for every $w \in \partial D \cap B(z, r)$.

Essential to the proof of Theorem ?? are a comparison of harmonic measures (Lemma ??) and a Carleson type estimate (Lemma ??).

Lemma 3.3 *Let D be an open set. Suppose that $B(A, \kappa r) \subset D \subset B(0, r)$ for some $r > 0$ and $\kappa \in (0, 1)$. Then*

$$\omega_D^x(B(0, r)^c) \leq C \kappa^{-n} \omega_{D \setminus B(A, \kappa r)}^x(B(A, \kappa r)), \quad x \in D \cap B(0, \frac{r}{2}),$$

for some constant $C = C(n, \alpha) > 0$.

Proof. For any $\phi \in C_0^\infty(\mathbf{R}^n)$, we have (see [?])

$$-(-\Delta)^{\frac{\alpha}{2}} \phi(x) = A_2(n, \alpha) \int_{\mathbf{R}^n} \frac{\phi(x+y) - \phi(x) - (\nabla \phi(x) \cdot y) 1_{B(0,1)}(y)}{|y|^{n+\alpha}} dy.$$

Take a sequence of functions ϕ_m in $C_0^\infty(\mathbf{R}^n)$ such that $0 \leq \phi_m \leq 1$,

$$\phi_m(x) = \begin{cases} 0, & |x| < 1/2 \\ 1, & 1 \leq |x| \leq m+1 \\ 0, & |x| > m+2, \end{cases}$$

and that $\sum_{i,j} |\frac{\partial^2}{\partial x_i \partial x_j} \phi_m|$ is uniformly bounded. Then there exists a constant $C > 0$ such that

$$|(-\Delta)^{\frac{\alpha}{2}} \phi_m| \leq C, \quad m \geq 1. \quad (3.1)$$

In fact,

$$\begin{aligned} & \left| \int_{\{|y| \geq 1\}} \frac{\phi_m(x+y) - \phi_m(x) - (\nabla \phi_m(x) \cdot y) 1_{B(0,1)}(y)}{|y|^{n+\alpha}} dy \right| \\ &= \left| \int_{\{|y| \geq 1\}} \frac{\phi_m(x+y) - \phi_m(x)}{|y|^{n+\alpha}} dy \right| \\ &\leq \int_{\{|y| \geq 1\}} \frac{dy}{|y|^{n+\alpha}} \leq C; \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\{|y| \leq 1\}} \frac{\phi_m(x+y) - \phi_m(x) - (\nabla \phi_m(x) \cdot y) 1_{B(0,1)}(y)}{|y|^{n+\alpha}} dy \right| \\ &\leq C \int_{\{|y| \leq 1\}} \frac{|y|^2}{|y|^{n+\alpha}} dy \leq C. \end{aligned}$$

Define $\phi_{m,r}(y) = \phi_m(\frac{y}{r})$, then by a change of variable we obtain

$$(-\Delta)^{\frac{\alpha}{2}} \phi_{m,r}(y) = r^{-\alpha} (-\Delta)^{\frac{\alpha}{2}} \phi_m(\frac{y}{r}), \quad y \in \mathbf{R}^n,$$

therefore by (??) we get for $m \geq 1$,

$$|(-\Delta)^{\frac{\alpha}{2}} \phi_{m,r}| \leq C r^{-\alpha}. \quad (3.2)$$

Fix $x \in D \cap B(0, \frac{r}{2})$, let $g(y) = G_D(x, y)$. Using

$$(-\Delta)^{\frac{\alpha}{2}}(A_1(n, \alpha)|x - y|^{\alpha-n}) = \delta_x,$$

and the fact that $\phi_{m,r}(x) = 0$, one checks easily (see page 67 of [?]) that

$$\int_{D^c} \phi_{m,r}(y) \omega_D^x(dy) = - \int_D g(y) (-\Delta)^{\frac{\alpha}{2}} \phi_{m,r}(y) dy, \quad m \geq 1.$$

Combining this with (??) we get that

$$\omega_D^x(B(0, r)^c) = \lim_{m \rightarrow \infty} \omega_D^x(R(0, r, (m+1)r)) \leq Cr^{-\alpha} \int_D g(y) dy.$$

We assume as we may that $x \notin B(A, \kappa r)$, otherwise the conclusion of the lemma holds trivially. Since g is α -harmonic in $D \setminus \{x\}$,

$$g(A) = \int_{D \cap B^c} k_B(A, y) g(y) dy \geq \int_{D \cap B(A, \frac{3\kappa r}{4})^c} k_B(A, y) g(y) dy,$$

where $B = B(A, \frac{\kappa r}{2})$. Since $\frac{3\kappa r}{4} \leq |y - A| \leq 2r$ for $y \in B(A, \frac{3\kappa r}{4})^c \cap D$, it follows from (??) that

$$\begin{aligned} g(A) &\geq C \int_{D \cap B(A, \frac{3\kappa r}{4})^c} \frac{(\kappa r)^\alpha}{|y - A|^{n+\alpha}} g(y) dy \\ &\geq C \kappa^\alpha r^{-n} \int_{B(A, \frac{3\kappa r}{4})^c \cap D} g(y) dy. \end{aligned}$$

The Harnack inequality yields

$$\int_{B(A, \frac{3\kappa r}{4})} g(y) dy \leq C \kappa^n r^n g(A).$$

Combining these two estimates we get that

$$\omega_D^x(B(0, r)^c) \leq C \kappa^{-\alpha} r^{n-\alpha} g(A).$$

Let $\Omega = D \setminus B(A, \frac{\kappa r}{2})$ and note from (??) that for $y \in B(A, \frac{\kappa r}{4})$,

$$2^{-n-\alpha} \leq \frac{k_\Omega(x, y)}{k_\Omega(x, A)} \leq 2^{n+\alpha}.$$

Since $G_D(\cdot, A)$ is α -harmonic in $D \setminus \{A\}$,

$$\begin{aligned} g(A) &= G_D(x, A) = \int_{B(A, \frac{\kappa r}{2})} G_D(y, A) \omega_\Omega^x(dy) \\ &\leq C \int_{B(A, \frac{\kappa r}{4})} G_D(y, A) k_\Omega(x, A) dy + \int_{\{\frac{\kappa r}{4} \leq |y-A| \leq \frac{\kappa r}{2}\}} G_D(y, A) \omega_\Omega^x(dy) \\ &\leq C \kappa^\alpha r^\alpha k_\Omega(x, A) + C' \kappa^{\alpha-n} r^{\alpha-n} \omega_\Omega^x(\{\frac{\kappa r}{4} \leq |y-A| \leq \frac{\kappa r}{2}\}) \\ &\leq C \kappa^{\alpha-n} r^{\alpha-n} \int_{B(A, \frac{\kappa r}{4})} k_\Omega(x, y) dy + C' \kappa^{\alpha-n} r^{\alpha-n} \omega_\Omega^x(\{\frac{\kappa r}{4} \leq |y-A| \leq \frac{\kappa r}{2}\}) \\ &\leq C \kappa^{\alpha-n} r^{\alpha-n} \omega_\Omega^x(B(A, \frac{\kappa r}{2})). \end{aligned}$$

This implies that

$$\omega_D^x(B(0, r)^c) \leq C\kappa^{-n}\omega_{D \setminus B(A, \kappa r)}^x(B(A, \kappa r)).$$

The following result on a Carleson type estimate is adapted from Lemma 4 of [?]. The proof there does not use the Lipschitz condition, works for open sets and gives the constant $\kappa^{-\alpha}$. ■

Lemma 3.4 *Let D be an open set. Suppose that $B(A, \kappa r) \subset D \cap B(0, r)$ for some $r > 0$ and $\kappa \in (0, 1)$. Then there exists a constant $C = C(n, \alpha) > 0$ such that for any nonnegative function u which is regular α -harmonic in $D \cap B(0, 2r)$ with $u = 0$ in $D^c \cap B(0, 2r)$, we have*

$$u(x) \leq C\kappa^{-\alpha}u(A), \quad x \in D \cap B(0, \frac{3r}{2}).$$

Lemma 3.5 *Let D be an open set. Assume that $B(A, \kappa r) \subset D \cap B(0, r)$ for some $r > 0$ and $\kappa \in (0, 1)$. Suppose that $u \geq 0$ is regular α -harmonic in $D \cap B(0, 2r)$ and $u = 0$ in $D^c \cap B(0, 2r)$. If w is a regular α -harmonic function in $D \cap B(0, r)$ such that*

$$w(x) = \begin{cases} u(x), & x \in B(0, \frac{3r}{2})^c \cup (D^c \cap B(0, r)) \\ 0, & r \leq |x| < \frac{3r}{2}, \end{cases}$$

then

$$w(A) \geq C\kappa^\alpha u(A),$$

for some constant $C = C(n, \alpha) > 0$.

Proof. It was noted in [?] that

$$\int_{\frac{3r}{2}}^{2r} \int_{R(0, \sigma, 2r)} (|y|^2 - \sigma^2)^{-\frac{\alpha}{2}} u(y) dy d\sigma \leq Cr^{1-\alpha} \int_{R(0, \frac{3r}{2}, 2r)} u(y) dy,$$

thus there is a $\sigma \in (\frac{5r}{3}, \frac{7r}{4})$ such that

$$\int_{R(0, \sigma, 2r)} (|y|^2 - \sigma^2)^{-\frac{\alpha}{2}} u(y) dy \leq Cr^{-\alpha} \int_{R(0, \frac{3r}{2}, 2r)} u(y) dy.$$

Since u is regular α -harmonic in $D \cap B(0, 2r)$, we have

$$\begin{aligned} u(A) &= \int_{B(0, \sigma)^c} k_{D \cap B(0, \sigma)}(A, y) u(y) dy \\ &\leq \int_{B(0, \sigma)^c} k_{B(0, \sigma)}(A, y) u(y) dy \\ &\leq C \left(\int_{R(0, \sigma, 2r)} + \int_{B(0, 2r)^c} \right) \frac{(\sigma^2 - |A|^2)^{\frac{\alpha}{2}}}{(|y|^2 - \sigma^2)^{\frac{\alpha}{2}}} \frac{1}{|y - A|^n} u(y) dy. \end{aligned}$$

Note that we have $|y - A| \geq \frac{|y|}{3}$ on $B(0, \sigma)^c$ and $|y| - \sigma \geq \frac{|y|}{8}$ on $B(0, 2r)^c$. Hence

$$u(A) \leq Cr^\alpha \int_{B(0, \frac{3r}{2})^c} \frac{u(y)}{|y|^{n+\alpha}} dy.$$

From the definition of w we know that

$$\begin{aligned} w(A) &= \int_{B(0, \frac{3r}{2})^c} k_{D \cap B(0, r)}(A, y) u(y) dy \\ &\geq \int_{B(0, \frac{3r}{2})^c} k_{B(A, \kappa r)}(A, y) u(y) dy \\ &= C \int_{B(0, \frac{3r}{2})^c} \frac{(\kappa r)^\alpha}{(|y - A|^2 - (\kappa r)^2)^{\frac{\alpha}{2}}} \frac{1}{|y - A|^n} u(y) dy. \end{aligned}$$

Note that $|y - A| \leq 2|y|$ on $B(0, 2r)^c$. Hence

$$w(A) \geq C\kappa^\alpha r^\alpha \int_{B(0, \frac{3r}{2})^c} \frac{u(y)}{|y|^{n+\alpha}} dy.$$

Therefore

$$w(A) \geq C\kappa^\alpha u(A).$$

Proof of Theorem ??. Without loss of generality we assume that $z = 0$ and $u(A) = v(A)$. Define u_1 and u_2 to be regular α -harmonic functions in $D \cap B(0, r)$ such that

$$u_1(x) = \begin{cases} u(x), & r \leq |x| < \frac{3r}{2} \\ 0, & x \in B(0, \frac{3r}{2})^c \cup (D^c \cap B(0, r)) \end{cases}$$

and

$$u_2(x) = \begin{cases} u(x), & x \in B(0, \frac{3r}{2})^c \cup (D^c \cap B(0, r)) \\ 0, & r \leq |x| < \frac{3r}{2}, \end{cases}$$

and note that $u = u_1 + u_2$. If $D \cap \{r \leq |y| < \frac{3r}{2}\}$ is empty, then $u_1 = 0$ and the inequality (??) below holds trivially. So we assume $D \cap \{r \leq |y| < \frac{3r}{2}\}$ is not empty. Then by Lemma ??,

$$u(y) \leq C\kappa^{-\alpha} u(A), \quad y \in D \cap B(0, \frac{3r}{2}).$$

For $x \in D \cap B(0, \frac{r}{2})$, we have

$$\begin{aligned} u_1(x) &= E^x(u(X(T_{(D \cap B(0, r))^c})) : X(T_{(D \cap B(0, r))^c}) \in D \cap \{r \leq |y| < \frac{3r}{2}\}) \\ &\leq \sup_{D \cap \{r \leq |y| < \frac{3r}{2}\}} u(y) \cdot \omega_{D \cap B(0, r)}^x(B(0, r)^c) \\ &\leq C\kappa^{-\alpha} u(A) \cdot \omega_{D \cap B(0, r)}^x(B(0, r)^c). \end{aligned}$$

Now using Lemma ?? we get

$$u_1(x) \leq C\kappa^{-n-\alpha}u(A)\omega_{D \setminus B(A, \frac{\kappa r}{2})}^x(B(A, \frac{\kappa r}{2})), \quad x \in D \cap B(0, \frac{r}{2}). \quad (3.3)$$

The Harnack inequality implies that

$$u(y) \geq Cu(A), \quad y \in B(A, \frac{\kappa r}{2}),$$

therefore

$$u(x) \geq Cu(A)\omega_{D \setminus B(A, \frac{\kappa r}{2})}^x(B(A, \frac{\kappa r}{2})), \quad x \in D \cap B(0, \frac{r}{2}). \quad (3.4)$$

From (??), the analogue of (??) for v and the assumption that $u(A) = v(A)$, it follows that for $x \in D \cap B(0, \frac{r}{2})$,

$$u_1(x) \leq C\kappa^{-n-\alpha}v(A)\omega_{D \setminus B(A, \frac{\kappa r}{2})}^x(B(A, \frac{\kappa r}{2})) \leq C\kappa^{-n-\alpha}v(x). \quad (3.5)$$

For $x \in D \cap B(0, r)$,

$$\begin{aligned} u_2(x) &= \int_{B(0, \frac{3r}{2})^c} k_{D \cap B(0, r)}(x, y)u(y)dy \\ &= A_2(n, \alpha) \int_{B(0, \frac{3r}{2})^c} \left(\int_{D \cap B(0, r)} \frac{G_{D \cap B(0, r)}(x, z)}{|z - y|^{n+\alpha}} dz \right) u(y)dy. \end{aligned}$$

Let

$$s(x) = A_2(n, \alpha) \int_{D \cap B(0, r)} G_{D \cap B(0, r)}(x, z)dz,$$

then we have

$$C^{-1} \leq \frac{u_2(x)}{u_2(A)} / \frac{s(x)}{s(A)} \leq C. \quad (3.6)$$

Applying (??) to u and v and Lemma ?? to v and v_2 , we obtain for $x \in D \cap B(0, \frac{r}{2})$,

$$u_2(x) \leq Cu_2(A) \frac{s(x)}{s(A)} \leq C \frac{u_2(A)}{v_2(A)} v_2(x) \leq C \frac{u(A)}{\kappa^\alpha v(A)} v_2(x) = C\kappa^{-\alpha}v_2(x), \quad (3.7)$$

Combining (??) and (??), we have

$$u(x) \leq C\kappa^{-n-\alpha}v(x), \quad x \in D \cap B(0, \frac{r}{2}).$$

■

The following result is adapted from Lemma 5 of [?]. The proof there works for bounded open sets satisfying the κ -fatness condition.

Lemma 3.6 *Suppose that D is a bounded open set which is κ -fat for some $\kappa \in (0, 1)$. Then there exist positive constants R , $C = C(n, \alpha, \kappa)$ and $\gamma = \gamma(n, \alpha, \kappa) < \alpha$ such that for all $z \in \partial D$ and $r \in (0, R)$, and function $u \geq 0$, α -harmonic in $D \cap B(z, r)$, we have*

$$u(A_s(z)) \geq C(s/r)^\gamma u(A_r(z)), \quad s \in (0, r).$$

Now Theorem ?? follows from the boundary Harnack principle and Lemma ?? in very much the same way as in the case of Lipschitz domains proved in Lemma 16 of [?]. We omit the details.

4 Martin Boundary

Suppose that D is a bounded open set in \mathbf{R}^n . In this section we apply the main results of last section to the study of the Martin boundary of D with respect to the symmetric stable process X . The argument in this section is standard, the complication arises from the presence of irregular boundary points.

Let $x_0 \in D$ be fixed and set

$$M_D(x, y) = \frac{G_D(x, y)}{G_D(x_0, y)}, \quad x, y \in D, y \neq x_0.$$

There is a compactification D^M of D , unique up to a homeomorphism, such that $M_D(x, y)$ has a continuous extension to $D \times (D^M \setminus \{x_0\})$ and $M_D(\cdot, z_1) = M_D(\cdot, z_2)$ if and only if $z_1 = z_2$. (See, for instance, 1.XII.3 of [?] or [?].) The set $\partial^M D = D^M \setminus D$ is called the Martin boundary of D . For $z \in \partial^M D$, set $M_D(\cdot, z)$ to be zero in D^c .

A positive singular α -harmonic function u on D is minimal if, whenever v is a positive α -harmonic function on D with $v \leq u$ on \mathbf{R}^n , then $u = cv$ for some constant c . The set of points $z \in \partial^M D$ such that $M_D(\cdot, z)$ is minimal α -harmonic in D is called the minimal Martin boundary of D .

The main result of this section is as follows:

Theorem 4.1 *Suppose that D is a bounded open set which is κ -fat for some $\kappa \in (0, 1)$. Then both the Martin boundary and the minimal Martin boundary of D can be identified with the Euclidean boundary of D .*

As a consequence of this theorem, we conclude that for every nonnegative singular α -harmonic function h in D , there exists a unique finite measure μ on ∂D such that

$$h(x) = \int_{\partial D} M_D(x, z) \mu(dz), \quad x \in D. \quad (4.1)$$

(See [?].)

Before we give the proof of Theorem ??, we establish the following simple lemma.

Lemma 4.2 *Suppose that h is a bounded singular α -harmonic function in a bounded open set D . If there is a set N of zero capacity such that for any $z \in \partial D \setminus N$,*

$$\lim_{D \ni x \rightarrow z} h(x) = 0,$$

then h is identically zero.

Proof. Take an increasing sequence of open sets $\{D_m\}_{m \geq 1}$ satisfying $\overline{D_m} \subset D_{m+1}$ and $\bigcup_{m=1}^{\infty} D_m = D$. Set $\tau_m = \tau_{D_m}$. Then $\tau_m \uparrow \tau_D$ and $\lim_{m \rightarrow \infty} X_{\tau_m} = X_{\tau_D}$ by quasi-left continuity of X . Since N has zero capacity, we have

$$P^x(X(\tau_D) \in N) = 0, \quad x \in D.$$

Therefore by the bounded convergence theorem we have for any $x \in D$,

$$\begin{aligned} h(x) &= \lim_{m \rightarrow \infty} E^x(h(X_{\tau_m}), \tau_m < \tau_D) \\ &= \lim_{m \rightarrow \infty} E^x(h(X_{\tau_m})1_{\partial D \setminus N}(X(\tau_D)); \tau_m < \tau_D) = 0. \end{aligned}$$

Proof of Theorem ??. For each fixed $z \in \partial D$ and $x \in D$, the existence of the limit

$$M_D(x, z) = \lim_{D \ni \xi \rightarrow z} \frac{G_D(x, \xi)}{G_D(x_0, \xi)}$$

is an immediate consequence of Theorem ??. For each $z \in \partial D$, set $M_D(x, z)$ to be zero for $x \in D^c$. Arguing along the lines of Lemma 6 of [?] and replacing Lemma 3 and Lemma 4 there by our Theorem ?? and Lemma ?? respectively, we conclude that (a) $M_D(\cdot, z)$ is α -harmonic in D for any $z \in \partial D$; (b) the mapping $(x, z) \mapsto M_D(x, z)$ is continuous on $D \times \partial D$; and (c) if $z, w \in \partial D$, $z \neq w$ and w is a regular boundary point, then $M_D(x, z) \rightarrow 0$ as $x \rightarrow w$. Thus the Martin boundary of D can be identified with a subset of the Euclidean boundary ∂D .

Now we show that each Euclidean boundary point corresponds to a different nonnegative α -harmonic function, hence the Martin boundary of D coincides with the Euclidean boundary ∂D .

Let I be the set of irregular boundary points of D . It is well known that $\text{Cap}(I) = 0$. Choose a decreasing sequence of open set Δ_m containing I such that

$$\lim_{m \rightarrow \infty} \text{Cap}(\Delta_m) = 0.$$

Hence

$$\lim_{m \rightarrow \infty} P^x(T_{\Delta_m} < \infty) = 0, \quad x \notin \bigcap_{m=1}^{\infty} \Delta_m.$$

Define a sequence D_k of subsets of D by

$$D_k = \left\{ x \in D : \text{dist}(x, D^c) > \frac{1}{k} \right\}, \quad k = 1, 2, \dots$$

Without loss of generality we may assume that $x_0 \in D_1 \setminus \overline{\Delta_1}$, thus

$$\omega_{D_k}^{x_0}(\Delta_m \cap D_k^c) = P^{x_0}(X(T_{D_k^c}) \in \Delta_m),$$

and hence

$$\lim_{m \rightarrow \infty} \sup_k \omega_{D_k}^{x_0}(\Delta_m \cap D_k^c) = 0.$$

For any $z \in \partial D$, we define

$$\nu_k^z(dy) = M_D(y, z) \omega_{D_k}^{x_0}(dy), \quad k = 1, 2, \dots$$

As $k \rightarrow \infty$, the measure ν_k^z converges weakly to the unit point mass δ_z . Indeed, for every $k \geq 1$, $\nu_k^z(\mathbf{R}^n) = \int M_D(y, z) \omega_{D_k}^{x_0}(dy) = M_D(x_0, z) = 1$. Also, for each ball $B = B(z, r)$ and each $m \geq 1$

$$\begin{aligned} \nu_k^z(B^c) &= \int_{B^c} M_D(y, z) \omega_{D_k}^{x_0}(dy) \\ &= \int_{(B^c \cap \Delta_m) \cap D_k^c} M_D(y, z) \omega_{D_k}^{x_0}(dy) + \int_{(B^c \setminus \Delta_m) \cap D_k^c} M_D(y, z) \omega_{D_k}^{x_0}(dy) \\ &\leq \sup_{y \in B^c} M_D(y, z) \cdot \omega_{D_k}^{x_0}(\Delta_m \cap D_k^c) + \sup_{y \in (B^c \setminus \Delta_m) \cap D_k^c} M_D(y, z). \end{aligned}$$

From the boundary Harnack principle we see that $M_D(\cdot, z)$ is bounded on B^c . For any $\epsilon > 0$, we can choose an $m > 0$ so that

$$\sup_k \omega_{D_k}^{x_0}(\Delta_m \cap D_k^c) \leq \epsilon / \sup_{y \in B^c} M_D(y, z).$$

Note from (b) and (c) in the first paragraph of this proof that

$$\lim_{k \rightarrow \infty} \sup_{y \in (B^c \setminus \Delta_m) \cap D_k^c} M_D(y, z) = 0.$$

Hence

$$\lim_{k \rightarrow \infty} \nu_k^z(B^c) = 0.$$

Consequently if $M_D(\cdot, z_1) = M_D(\cdot, z_2)$ then $\delta_{z_1} = \delta_{z_2}$ which implies that $z_1 = z_2$.

So far we have shown that the Martin boundary coincides with the Euclidean boundary and the Martin kernels $M_D(\cdot, z)$, $z \in \partial D$, are all α -harmonic. Thus it follows from [?] that for every nonnegative singular α -harmonic function h in D , there exists a finite measure μ on ∂D such that (??) holds.

Finally we show that, for every $z \in \partial D$, $M_D(\cdot, z)$ is a minimal α -harmonic function, hence the minimal Martin boundary of D can be identified with the Euclidean boundary. Fix $z \in \partial D$ and suppose that $h \leq M_D(\cdot, z)$, where h is nonnegative and α -harmonic in D . Then there is a finite μ on ∂D such that

$$h(\cdot) = \int_{\partial D} M_D(\cdot, w) \mu(dw).$$

If μ is not a multiple of δ_z , then there is a positive measure $\nu \leq \mu$ such that $\text{dist}(z, \text{supp}(\nu)) > 0$. Let

$$u(\cdot) = \int_{\partial D} M_D(\cdot, w) \nu(dw).$$

Then u is a positive α -harmonic function in D and is bounded above by $M_D(\cdot, z)$. Take $\epsilon = \frac{1}{2}\text{dist}(z, \text{supp}(\nu))$. Then by the boundary Harnack principle, $M_D(\cdot, z)$ is bounded on $B(z, \epsilon)^c$ and so is u . Again from the boundary Harnack principle we see that $M_D(\cdot, \cdot)$ is bounded on $B(z, \epsilon) \times \text{supp}(\nu)$, so u is also bounded on $B(z, \epsilon)$. Since $M_D(x, z) \rightarrow 0$ as x approaches any regular boundary point different from z ; and consequently $u(x) \rightarrow 0$ as x approaches any regular boundary point different from z . From Lemma ?? we see that u is identically zero. Therefore $\mu = c\delta_z$ for some c and $M_D(\cdot, z)$ is minimal α -harmonic. ■

5 Intrinsic Ultracontractivity

In this section we shall study intrinsic ultracontractivity on open sets satisfying a condition on a distance function. We follow the approach using log-Sobolev inequality by Bañuelos [?] for Brownian motion and by Chen and Song [?] for symmetric stable processes. The additional technical detail needed is a Harnack inequality for positive solutions of a Schrödinger equation on disjoint balls (Lemma ??).

Let us first recall the definition of intrinsic ultracontractivity, due to Davies and Simon [?]. Suppose that H is a semibounded self-adjoint operator on $L^2(D)$ with D being an open set in \mathbf{R}^n and that $\{e^{Ht}, t > 0\}$ is an irreducible positivity-preserving semigroup with integral kernel $a(t, x, y)$. Assume that the top of the spectrum μ_0 of H is an eigenvalue. In this case, μ_0 has multiplicity one and the corresponding eigenfunction φ_0 , normalized by $\|\varphi_0\|_2 = 1$, can be chosen to be positive almost everywhere on D . φ_0 is called the ground state of H .

Let U be the unitary operator U from $L^2(D, \varphi_0^2(x)dx)$ to $L^2(D)$ given by $Uf = \varphi_0 f$ and define \widetilde{H} on $L^2(D, \varphi_0^2(x)dx)$ by

$$\widetilde{H} = U^{-1}(H - \mu_0)U.$$

Then $e^{\widetilde{H}t}$ is an irreducible symmetric Markov semigroup on $L^2(D, \varphi_0^2(x)dx)$ whose integral kernel with respect to the measure $\varphi_0^2(x)dx$ is given by

$$\frac{e^{-\mu_0 t} a(t, x, y)}{\varphi_0(x)\varphi_0(y)}.$$

H is said to be ultracontractive if e^{Ht} is a bounded operator from $L^2(D)$ to $L^\infty(D)$ for all $t > 0$. H is said to be intrinsically ultracontractive if \widetilde{H} is ultracontractive; that is, $e^{\widetilde{H}t}$ is a bounded operator from $L^2(D, \varphi_0^2(x)dx)$ to $L^\infty(D, \varphi_0^2(x)dx)$ for all $t > 0$.

In this section we assume that q is in the Kato class $\mathcal{K}_{n,\alpha}$, i.e.,

$$\limsup_{r \downarrow 0} \int_{x \in \mathbf{R}^n} \int_{|x-y| \leq r} \frac{|q(y)|}{|x-y|^{n-\alpha}} dy = 0. \quad (5.1)$$

We fix a bounded open set D in this section and we use φ_0 to denote the eigenfunction corresponding to the first eigenvalue λ_0 , i.e., the first eigenfunction of the Feynman-Kac semigroup

$$T_t f(x) = E^x \left(\exp \left(\int_0^{t \wedge \tau_D} q(X_s) ds \right) f(X_t) \right).$$

Our main results of this section are as follows.

Theorem 5.1 *Suppose that D is a bounded open set and that there is a $\gamma > n/\alpha$ such that*

$$\int_D \left(\log \left(\frac{\text{diam}(D)}{\text{dist}(x, \partial D)} \right) \right)^\gamma dx < \infty. \quad (5.2)$$

Then for any $\varepsilon > 0$, we have

$$\int_D f^2 \log \frac{|f|}{\varphi_0} dx \leq \eta \mathcal{E}(f, f) + \beta(\eta) \|f\|_2^2 + \|f\|_2^2 \log \|f\|_2, \quad f \in \mathcal{F} \cap L^\infty(D, dx)$$

with

$$\beta(\varepsilon) = -\frac{n}{2\alpha} \log \varepsilon + C(n, \alpha, D, \gamma, q) \varepsilon^{-n/(\gamma\alpha-n)} + C'(n, \alpha, D, \gamma, q).$$

Theorem 5.2 *Suppose D is a bounded open set satisfying (??) for some $\gamma > n/\alpha$. Then $L^D + q$ is intrinsically ultracontractive. More precisely,*

$$\frac{e^{-\lambda_0 t} u_q(t, x, y)}{\varphi_0(x) \varphi_0(y)} \leq e^{2M(t/2)} \quad \text{for all } x, y \in D \text{ and } t > 0,$$

where

$$M(t) = \frac{1}{t} \int_0^t A(\varepsilon) d\varepsilon$$

with

$$A(\varepsilon) = \begin{cases} -\frac{n}{2\alpha} \log \varepsilon + C(n, \alpha, D, \gamma, q) \varepsilon^{-n/(\gamma\alpha-n)} + C'(D, \gamma, q) & \text{for } \varepsilon \leq 1, \\ C(D, \gamma, q) + C'(n, \alpha, D, \gamma, q) & \text{for } \varepsilon > 1. \end{cases}$$

Theorem ?? follows from Theorem ?? in much the same way as in the proof of Theorem 3.5 of [?]. We omit the details.

To prove Theorem ?? we need a Harnack inequality for φ_0 .

Definition 5.1 *A bounded Borel function f defined on \mathbf{R}^n is said to a solution to the equation*

$$(L^D + q)f(x) = 0, \quad \text{on } D \quad (5.3)$$

if it is continuous on D and in any open domain D_0 , $D_0 \subset \overline{D_0} \subset D$,

$$f(x) = E^x \left[\exp \left(\int_0^{\tau_{D_0}} q(X_s) ds \right) f(X_{\tau_{D_0}}) \right].$$

Clearly φ_0 is a positive solution of

$$(L^D + q + \lambda_0)f(x) = 0, \quad x \in D.$$

From (??) and the estimates of the Green function(see (2.5) of [?]), it follows that there exists an $R_0 > 0$ such that for any ball B of radius less than R_0 ,

$$\sup_{x \in B, z \in \overline{B}^c} \int_B \frac{G_B(x, y) |q(y)| k_B(y, z)}{k_B(x, z)} dy \leq \frac{1}{2}. \quad (5.4)$$

We fix such a R_0 in the rest of this section. Then we have the following Harnack inequality for positive solutions to $(L^D + q)f(x) = 0$.

Lemma 5.3 *Let $r \in (0, R_0)$, $x_1, x_2 \in \mathbf{R}^n$ satisfy $|x_1 - x_2| < Mr$ for some positive integer k . Let $f \geq 0$ be a function which is a solution of*

$$(L^D + q)f(x) = 0, \quad x \in B(x_1, r) \cup B(x_2, r).$$

Then

$$C^{-1}M^{-n-\alpha}f(x_2) \leq f(x_1) \leq CM^{n+\alpha}f(x_2),$$

for some constant $C = C(n, \alpha) > 1$.

Proof. We may assume $|x_1 - x_2| > \frac{3}{2}r$, otherwise the conclusion follows from the Harnack inequality (Theorem 2.5 of [?]). The Harnack inequality also yields

$$f(x) \geq Cf(x_2), \quad x \in B(x_2, r/2).$$

By (??) we get, for any $x \in B(x_2, r/2)$,

$$K_{B(x_1, r/2)}(0, x) \geq CM^{-n-\alpha}r^{-n}.$$

Using (??) and Jensen's inequality we get that

$$e^{-1/2} \leq \inf_{x \in B(x_1, r/2), z \in B(x_1, r/2)^c} E_z^x [\exp(\int_0^{\tau_{B(x_1, r/2)}} q(X(t))dt)].$$

Here E_z^x stands for the expectation with respect to the $k_{B(0, \frac{r}{2})}(\cdot, z)$ -conditioned stable process starting from x . Therefore

$$\begin{aligned} f(x_1) &= E^{x_1} [\exp(\int_0^{\tau_{B(x_1, r/2)}} q(X(t))dt) f(X(\tau_{B(x_1, r/2)}))] \\ &= \int_{B(x_1, r/2)^c} E_x^{x_1} [\exp(\int_0^{\tau_{B(x_1, r/2)}} q(X(t))dt)] f(x) K_{B(x_1, r/2)}(0, x) dx \\ &\geq e^{-1/2} \int_{B(x_1, r/2)^c} K_{B(x_1, r/2)}(0, x) f(x) dx \\ &\geq e^{-1/2} \int_{B(x_2, r/2)} K_{B(x_1, r/2)}(0, x) f(x) dx \\ &\geq CM^{-n-\alpha} f(x_2), \end{aligned}$$

and, by symmetry, the assertion of the lemma follows. ■

Proof of Theorem ??. In this proof C and C' denote positive constants which depend on n, α, D, γ and q . Let x_0 be a fixed point in one of the largest Whitney cubes in the Whitney decomposition W of D . For any $x \in D$, we have

$$|x - x_0| \leq \frac{\text{diam}(D)}{\text{dist}(x, \partial D)} \text{dist}(x, \partial D)$$

Therefore by Lemma ?? above,

$$\varphi_0(x) \geq C \left(\frac{\text{dist}(x, \partial D)}{\text{diam}(D)} \right)^{n+\alpha} \varphi_0(x_0), \quad x \in D. \quad (5.5)$$

Let

$$\rho(x) = \log \frac{\text{diam}(D)}{\text{dist}(x, \partial D)}, \quad x \in D.$$

Then using (??) and arguing along the same lines as in Lemma 3.2 of [?], we can show that for $p = \gamma\alpha/n$,

$$\int_D (\rho(x))^p f^2(x) dx \leq C \mathcal{E}(f, f), \quad f \in \mathcal{F}. \quad (5.6)$$

Let $p = \gamma\alpha/n$, and let $p' = \gamma\alpha/(\gamma\alpha - n)$ be its conjugate. It follows from (??), (??) and the elementary inequality $ab \leq a^p + b^{p'}$ that for any $\varepsilon > 0$ and $f \in \mathcal{F}$,

$$\begin{aligned} & \int_D f^2 \log \frac{1}{\varphi_0} dx \\ & \leq C \int_D \rho(x) f(x)^2 dx + \log\left(\frac{C'}{\varphi_0(x_0)}\right) \int_D f^2(x) dx \\ & = C \int_D \frac{\varepsilon^{1/p}}{\varepsilon^{1/p}} \rho(x) f(x)^2 dx + \log\left(\frac{C'}{\varphi_0(x_0)}\right) \int_D f^2(x) dx \\ & \leq C\varepsilon \int_D (\rho(x))^p f(x)^2 dx + C\varepsilon^{-p'/p} \int_D f^2(x) dx + \log\left(\frac{C'}{\varphi_0(x_0)}\right) \int_D f^2(x) dx \\ & \leq C\varepsilon \mathcal{E}(f, f) + C\varepsilon^{-p'/p} \int_D f^2(x) dx + \log\left(\frac{C'}{\varphi_0(x_0)}\right) \int_D f^2(x) dx. \end{aligned}$$

This shows that for any $\varepsilon > 0$ we have

$$\int_D f^2 \log \frac{1}{\varphi_0} dx \leq \varepsilon \mathcal{E}(f, f) + \beta(\varepsilon) \|f\|_2^2, \quad f \in \mathcal{F} \quad (5.7)$$

with

$$\beta(\varepsilon) = C\varepsilon^{-n/(\gamma\alpha-n)} + C'. \quad (5.8)$$

It has been proved in Theorem Theorem 4.1 of [?] that for any $\varepsilon > 0$,

$$\int_D f^2 \log |f| dx \leq \varepsilon \mathcal{E}(f, f) + \beta(\varepsilon) \|f\|_2^2, \quad f \in \mathcal{F} \quad (5.9)$$

with

$$\beta(\varepsilon) = -\frac{n}{2\alpha} \log \eta + C. \quad (5.10)$$

Combining (??)–(??) we arrive at the conclusion of the theorem. ■

We now give some examples of open sets that satisfy condition (??).

(A) If D is a bounded domain in \mathbf{R}^n whose boundary can be represented locally as a graph of a function having modulus of continuity no worse than $(\log \frac{1}{t})^{-\gamma}$, for some $\gamma > \frac{n}{\alpha}$, then condition (??) is satisfied for any $\gamma' \in (\frac{n}{\alpha}, \gamma)$. As a consequence of this, if D is a bounded domain in \mathbf{R}^n whose boundary can be represented locally as a graph of a Hölder continuous function then condition (??) is satisfied.

(B) Let D be a bounded open set and N_k be the number of closed dyadic cubes of side-length 2^{-k} that intersects ∂D . If for some $\gamma > \frac{n}{\alpha}$,

$$\sum N_k 2^{-kn} k^\gamma < \infty, \quad (5.11)$$

then condition (??) is satisfied. In particular, condition (??) is satisfied when the Minkowski dimension of ∂D is less than n . To see this, let $W = \{Q_j\}$ be a dyadic Whitney decomposition of D and M_k be the number of cubes of side-length 2^{-k} in W , then $M_k \leq CN_k$ and condition (??) is equivalent to

$$\sum M_k 2^{-kn} k^\gamma < \infty, \quad (5.12)$$

(see [?].)

(C) Suppose that D is a bounded open set which is κ -fat for some $\kappa \in (0, 1)$. Then

$$\int_D \text{dist}(x, \partial D)^{-\nu} dx < \infty, \quad (5.13)$$

for some $\nu = \nu(n, \kappa) > 0$. This clearly implies (??).

To see (??), we adapt an argument by Smith and Stegenga [?] for John domains. Fix a dyadic Whitney decomposition $W = \{Q_j\}$ of D satisfying

$$1 \leq \frac{\text{dist}(x, \partial D)}{\text{diam}(Q)} \leq 5, \quad x \in Q \in W. \quad (5.14)$$

Define for $m = 1, 2, \dots$,

$$\Omega_m = \{Q \in W : \frac{\kappa}{5} 2^{-m} \leq \frac{\text{diam}(Q)}{\text{diam}(D)} \leq 2^{-m}\},$$

and

$$D_m = \bigcup_{Q \in \Omega_m} Q.$$

Then D is the union of the D_m 's. Let m_0 be the smallest integer m such that $2^{-m} \text{diam}(D) \leq R$. If $m \geq m_0$ and $x \in \cup_{k=m}^{\infty} D_k$, then

$$\text{dist}(x, \partial D) \leq 5 \cdot 2^{-m} \text{diam}(D). \quad (5.15)$$

Choose $x^* \in \partial D$ such that $\text{dist}(x, \partial D) = |x - x^*|$. Since D is κ -fat, $D \cap B(x^*, 2^{-m} \text{diam}(D))$ contains a ball $B(A, \kappa 2^{-m} \text{diam}(D))$. Let Q be one of the Whitney cubes containing A . Note from (??) and (??) that $x \in \frac{50}{\kappa} Q$ and $Q \in \Omega_m$. Therefore

$$\bigcup_{k=m}^{\infty} D_k \subset \bigcup_{Q \in \Omega_m} \frac{50}{\kappa} Q, \quad m \geq m_0. \quad (5.16)$$

Hence

$$\sum_{k=m}^{\infty} |D_k| \leq C(n, \kappa) |D_m|, \quad m \geq m_0,$$

We conclude from Lemma 1 of [?] that

$$|D_m| \leq C(n, \kappa) |D_{m_0}| e^{-(m-m_0)/C(n, \kappa)}, \quad m \geq m_0. \quad (5.17)$$

This implies that $\int_D \text{dist}(x, \partial D)^{-\nu} dx < \infty$ for some $\nu > 0$.

6 3G Theorem and Conditional Gauge Theorem

The proof of the 3G theorem in [?] extends to any bounded open sets that has the boundary Harnack principle and the κ -fatness condition. We omit the details.

Theorem 6.1 (*3G Theorem*) *Suppose that D is a bounded open set which is κ -fat for some $\kappa \in (0, 1)$. Then there exists a constant $C = C(n, \alpha, D) > 0$ such that*

$$\frac{G_D(x, y)G_D(y, z)}{G_D(x, z)} \leq C \frac{|x - z|^{n-\alpha}}{|x - y|^{n-\alpha}|y - z|^{n-\alpha}}, \quad x, y, z \in D, \quad (6.1)$$

and

$$\frac{G_D(x, y)M_D(y, z)}{M_D(x, z)} \leq C(|x - y|^{\alpha-n} + |y - z|^{\alpha-n}), \quad x, y \in D, z \in \partial D. \quad (6.2)$$

The conditional gauge theorem follows the 3G Theorem and Theorem ??, see Section 5 of [?].

Theorem 6.2 (*Conditional Gauge Theorem*). *Suppose that q belongs to the Kato class $\mathcal{K}_{n, \alpha}$ and that D is a bounded open set which is κ -fat for some $\kappa \in (0, 1)$. If the function*

$$x \mapsto E^x[\exp(\int_0^{\tau_D} q(X_s) ds)]$$

is not identically infinite on D , then there exists a constant $C = C(n, \alpha, D, q) > 1$ such that

$$C^{-1} \leq \inf_{(x, z) \in D \times \mathbf{R}^n} E_z^x \left[\exp\left(\int_0^{\tau_D} q(X(t)) dt\right) \right] \leq \sup_{(x, z) \in D \times \mathbf{R}^n} E_z^x \left[\exp\left(\int_0^{\tau_D} q(X(t)) dt\right) \right] \leq C,$$

where for $z \in \partial D$, E_z^x stands for the expectation with respect to the $M_D(\cdot, z)$ -conditioned stable process starting from $x \in D$, while for $z \in (\overline{D})^c$, E_z^x stands for the expectation with respect to the $k_D(\cdot, z)$ -conditioned stable process starting from $x \in D$.

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