QUASICONFORMAL DIMENSIONS OF SELF-SIMILAR SETS

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Abstract. The Sierpinski gasket and other self-similar subsets of $\mathbb{R}^d$, $d \geq 2$, can be mapped by quasiconformal self-maps of $\mathbb{R}^d$ onto sets of Hausdorff dimension arbitrarily close to one. In $\mathbb{R}^2$ we construct explicit mappings. In $\mathbb{R}^d$, $d \geq 3$, the results follow from general theorems on the equivalence of invariant sets for iterated function systems under quasisymmetric maps and global quasiconformal maps. More specifically, we present geometric conditions ensuring that (i) isomorphic systems have quasisymmetrically equivalent invariant sets, and (ii) one-parameter isotopies of systems have invariant sets which are equivalent under global quasiconformal maps.

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Contents

1. Introduction

The distortion of Hausdorff dimension by quasiconformal mappings has been a subject of interest within geometric function theory for some time. Gehring and Väisälä [9] established K-dependent bounds for the distortion of the Hausdorff dimension of a fixed subset of $\mathbb{R}^d$ by a K-quasiconformal self-map of $\mathbb{R}^d$. The spectacular results of Astala [2] provide sharp bounds in the planar case. In recent years dilatation-independent bounds for Hausdorff dimension distortion have been considered. Bishop [4] showed that for sets of positive dimension there is never an obstruction to raising dimension by quasiconformal maps. On the other hand, examples of Tyson [25] and Bishop–Tyson [6] (also [10, §15]) show that the corresponding statement for lowering dimension can fail. For each $\alpha \in [1, d]$, there exists a compact set $E \subset \mathbb{R}^d$ for which $\dim F(E) \geq \dim E = \alpha$ for every quasiconformal map $F : \mathbb{R}^d \to \mathbb{R}^d$. Here and henceforth we denote by "dim" the Hausdorff dimension. In fact, these examples have the stronger property that the dimension cannot be reduced by any quasisymmetric map to another metric space. Recall that a homeomorphism F between metric spaces (X, d) and (Y, d') is said to be *quasisymmetric* if there exists a homeomorphism η of $[0, \infty)$ onto itself so that

$$d(x,y) \leq t d(x,z) \quad \Rightarrow \quad d'(F(x),F(y)) \leq \eta(t) d'(F(x),F(z)) \tag{1.1}$$

for all $x, y, z \in X$. See Tukia–Väisälä [24]. Every quasiconformal map $F : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 2$, is quasisymmetric and the restriction of a quasisymmetric map to a subset is again quasisymmetric.

For a fixed metric space X, the *conformal dimension* $\mathcal{C}\dim X$ of X is the infimum of the Hausdorff dimensions of all metric spaces quasisymmetrically equivalent to X:

$$\mathcal{C}\dim X = \inf\{\dim Y : \exists F : X \to Y \text{ quasisymmetric}\}.$$

This concept was introduced by Pansu [22]. For further information regarding conformal dimension, see Tyson [25], [26], Bishop–Tyson [5], [6], Balogh [3], Keith–Laakso [13], and the recent work of Bonk and Kleiner [8], [7].

For a fixed set $E \subset \mathbb{R}^d$ we define the *quasiconformal dimension* $\mathcal{QC}\dim E$ to be the infimum of the Hausdorff dimensions of all images of E under quasiconformal self-maps of $\mathbb{R}^d$:

$$\mathcal{QC}\dim E = \inf\{\dim F(E) : \ F : \mathbb{R}^d \to \mathbb{R}^d \text{ quasiconformal}\}.$$

By the above remarks,

$$\mathcal{C}\dim E \le \mathcal{QC}\dim E \le \dim E$$

for every E; and for each $\alpha \in [1, d]$ there exists $E \subset \mathbb{R}^d$ with $\mathcal{C}\dim E = \mathcal{QC}\dim E = \dim E = \alpha$.

In this paper we show that the quasiconformal dimension is one for a class of self-similar subsets of Euclidean spaces. Our starting example is the *Sierpinski gasket* SG (Figure 1), which can be characterized as the unique nonempty compact set K in $\mathbb{R}^2$ satisfying $K = f_0(K) \cup f_1(K) \cup f_2(K)$, where

$$f_0(z) = \tfrac{1}{2}z + \tfrac{1}{2}, \quad f_1(z) = \tfrac{1}{2}z + \tfrac{1}{2}e^{2\pi i/3}, \quad f_2(z) = \tfrac{1}{2}z + \tfrac{1}{2}e^{-2\pi i/3}. \tag{1.2}$$

The Hausdorff dimension of SG is $\frac{\log 3}{\log 2} = 1.5849\ldots$ We prove the following

Theorem 1.3. $\mathcal{QC}\dim SG = 1$.

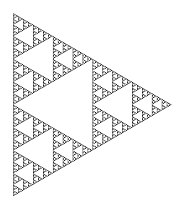

FIGURE 1. The Sierpinski gasket SG

Laakso [16] has earlier shown that $\mathcal{C}\dim SG = 1$. The strict inequality $\mathcal{QC}\dim SG < \dim SG$ was previously established by Meyer [20], Laakso [16] and Tyson, independently.[1]

In the plane, we also prove the corresponding theorem for the polygaskets $PG(N) \subset \mathbb{R}^2$ (see Figure 2) which are generated by regular N-sided polygons.

Theorem 1.4. *For each* $N \ge 3$, $N \not\equiv 0 \pmod 4$, $\mathcal{QC}\dim PG(N) = 1$.

In higher dimensions, an analogous result holds for the d-dimensional Sierpinski gaskets $SG^d \subset \mathbb{R}^d$.

Theorem 1.5. *For each* $d \ge 3$, $\mathcal{QC}\dim SG^d = 1$.

In each of these three theorems, we (i) construct self-similar sets of Hausdorff dimension arbitrarily close to one, (ii) build quasisymmetric maps from the given set onto these new sets, and (iii) extend the quasisymmetric map to a global quasiconformal map. Steps two and three rely on general theorems

[1] In June 2003, Laakso indicated to us an alternate proof for Theorem 1.3.

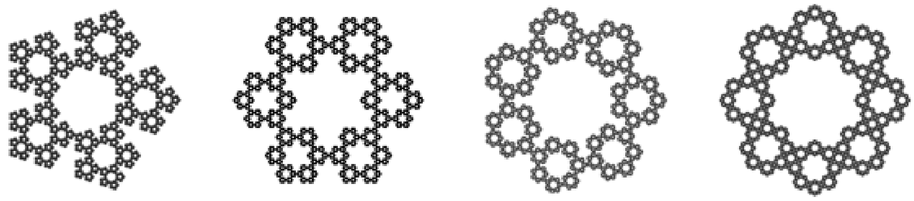

FIGURE 2. Polygaskets $PG(N)$ for $N = 5, 6, 7, 8$

on the quasisymmetric equivalence (Theorem 1.7) and global quasiconformal equivalence (Theorem 1.9) of invariant sets for iterated function systems. These results will be stated momentarily. The application of the general quasiconformal extension theorem to the case of the higher-dimensional gaskets is very complicated, and the construction of the self-similar sets in step (i) in the case of the polygaskets is also very involved. These results are given in sections 6 and 7, respectively.

Before stating our general results, we recall that an *iterated function system* (IFS) is a family of strictly contractive similarities $\mathcal{F} = \{f_1, \dots, f_M\}$ of a Euclidean space $\mathbb{R}^d$. The *invariant set* $K = K(\mathcal{F})$ is the unique nonempty compact set which is invariant under the action of $\mathcal{F}$. Certain geometric conditions serve to limit the geometric complexity of invariant sets. First is the *open set condition* (OSC), introduced by Moran [21] and rediscovered by Hutchinson [11], which postulates the existence of a bounded nonempty open set O in $\mathbb{R}^d$ such that the images of O under the maps in $\mathcal{F}$ are disjoint proper subsets of O. By limiting the size of the overlap sets $f_i(K) \cap f_j(K)$, $i \neq j$, the open set condition allows for the explicit computation of $\dim K$ as the unique positive solution s to the equation

$$\sum_{i=1}^{M} \lambda_i^s = 1, \tag{1.6}$$

where λ_i denotes the contraction ratio associated with $f_i \in \mathcal{F}$. Next is the notion of *post-critical finiteness* (PCF), introduced by Kigami [14]. Post-critically finite systems are essentially characterized by the requirement that the *critical set* $\bigcup_{i\neq j} f_i(K) \cap f_j(K)$ and its full backward image under $\mathcal{F}$ is finite. For the precise definition, see section 3.

Our results hold for *gasket type* iterated function systems, a class which we introduce in Definition 3.19. The term 'gasket type' is motivated by the strong geometric similarities between the invariant sets of such systems and that of the classical Sierpinski gasket. Gasket type systems are post-critically finite and satisfy the open set condition.

The precise statements of Theorems 1.7 and 1.9 use the language of symbolic dynamics, which we now review. The dynamical attributes of an iterated function system $\mathcal{F} = \{f_1, \ldots, f_M\}$ are encoded via its symbolic representation as a quotient of the sequence space $\Sigma = A^\infty$, $A = \{1, \ldots, M\}$. Indeed, there exists a continuous mapping $\pi_\mathcal{F}$ from Σ onto the invariant set $K = K(\mathcal{F})$. Moreover, $\pi_\mathcal{F}$ descends to a homeomorphism between $\Sigma / \stackrel{\mathcal{F}}{\sim}$ and $K(\mathcal{F})$, where the quotient space $\Sigma / \stackrel{\mathcal{F}}{\sim}$ is defined by the equivalence relation $w \stackrel{\mathcal{F}}{\sim} w'$ if and only if $\pi_\mathcal{F}(w) = \pi_\mathcal{F}(w')$.

Each iterated function system $\mathcal{F}$ contains certain distinguished elements which comprise the *boundary* of $\mathcal{F}$. See section 3 for the precise definition. An iterated function system is called *boundary congruent* if all of the boundary share a common contraction ratio and have no rotation.

Following Kigami [15], we say that two iterated function systems $\mathcal{F} = \{f_1, \ldots, f_M\}$ and $\mathcal{G} = \{g_1, \ldots, g_M\}$ are *isomorphic* if, up to a reordering of the indices, the equivalence relations $\Sigma \stackrel{\mathcal{F}}{\sim}$ and $\Sigma \stackrel{\mathcal{G}}{\sim}$ coincide. In this case the canonically defined map $\pi_\mathcal{G} \circ \pi_\mathcal{F}^{-1}$ is a well-defined homeomorphism between the invariant sets. We now state our theorem on quasisymmetric equivalence of invariant sets.

Theorem 1.7. *Let $\mathcal{F}$ and $\mathcal{G}$ be iterated function systems, each of which is boundary congruent and of gasket type. If $\mathcal{F}$ and $\mathcal{G}$ are isomorphic, then $F = \pi_\mathcal{G} \circ \pi_\mathcal{F}^{-1} : K(\mathcal{F}) \to K(\mathcal{G})$ is quasisymmetric.*

Corollary 1.8. *Let $\mathcal{F}$ be a boundary congruent, gasket type system with invariant set K. Then*

$$\mathcal{C} \dim(K) \le \inf \dim K(\mathcal{G}),$$

where the infimum is taken over all boundary congruent, gasket type systems $\mathcal{G}$ isomorphic to $\mathcal{F}$.

Our next result concerns the global quasiconformal equivalence of invariant sets. Here we require a stronger assumption about the manner in which the two systems are related.

Theorem 1.9. *Let $\mathcal{F}$ and $\mathcal{G}$ be iterated function systems, each of which is boundary congruent and of gasket type. If $\mathcal{F}$ and $\mathcal{G}$ are equivalent through a nondegenerate isotopy of boundary congruent, gasket type systems, then $F = \pi_\mathcal{G} \circ \pi_\mathcal{F}^{-1} : K(\mathcal{F}) \to K(\mathcal{G})$ admits a quasiconformal extension to $\mathbb{R}^d$.*[2]

Corollary 1.10. *Let $\mathcal{F}$ be a boundary congruent, gasket type iterated function system with invariant set K. Then*

$$\mathcal{QC} \dim(K) \le \inf \dim K(\mathcal{G}),$$

[2]Note that in the case of a **complex** isotopy $\{\mathcal{F}^t\}_{t \in \mathbb{D}}$ of holomorphically varying IFS's in the plane, the conclusion of Theorem 1.9 is guaranteed by Slodkowski's extension [23] of the celebrated λ-lemma of Mañé, Sad and Sullivan [17]. Theorem 1.9 can be viewed as a version of the λ-lemma for real isotopies in any dimension, in the setting of self-similar invariant sets.

where the infimum is taken over all boundary congruent, gasket type iterated function systems $\mathcal{G}$ equivalent to $\mathcal{F}$ through some nondegenerate isotopy of such systems.

Two systems $\mathcal{F}$ and $\mathcal{G}$ are said to be *equivalent through an isotopy* of iterated function systems if there exists a continuously varying one-parameter family of systems $\mathcal{F}^t = \{f_1^t, \dots, f_M^t\}_{0\le t\le 1}$, so that $\mathcal{F} = \mathcal{F}^0$, $\mathcal{G} = \mathcal{F}^1$, and $\mathcal{F}^{t'}$ and $\mathcal{F}^{t''}$ are isomorphic for any $0 \le t' < t'' \le 1$. *Nondegeneracy* of the isotopy means that the dimension of the affine subspace spanned by the invariant set $K(\mathcal{F}^t)$ is constant in t. For the topology on the space of iterated function systems which we use, see Remark 3.6.

It would be interesting to know when two isomorphic systems can be joined by such an isotopy. For further discussion, see Remark 3.40.

The boundary congruence condition is essential; both Theorems 1.7 and 1.9 fail in its absence. See Example 4.5.

The definition of gasket-type systems (Definition 3.19) includes IFS's with totally disconnected invariant sets. A theorem of MacManus [18] states that every uniformly disconnected subset of $\mathbb{R}^2$ is equivalent with the Cantor ternary set via a quasiconformal map of $\mathbb{R}^2$. The notion of *uniform disconnectedness* is a quantitative and scale-invariant strengthening of total disconnectedness. In the setting of self-similar invariant sets, total disconnectedness automatically improves to uniform disconnectedness. It is well-known that the Cantor ternary set has quasiconformal dimension equal to zero; consequently, every totally disconnected self-similar invariant set in $\mathbb{R}^2$ has quasiconformal dimension equal to zero. Corollary 1.10 includes this earlier result as a special case.

The restriction $N \not\equiv 0 \pmod 4$ in Theorem 1.4 is imposed because the IFS for $PG(N)$ is not post-critically finite when $N \equiv 0 \pmod 4$ (cf. the case $N = 8$ in Figure 2). In the absence of post-critical finiteness, it is difficult to construct quasisymmetric deformations of invariant sets. The Sierpinski carpet SC (Figure 3) is another example of an invariant set for a non-PCF iterated function system. Note that $\mathcal{C}\dim SC \ge 1+\log 2/\log 3 > 1$ since SC contains the product of $[0,1]$ with the Cantor ternary set, and such product spaces are known to be minimal for conformal dimension [25]. Keith and Laakso [13] have shown that $\mathcal{C}\dim SC < \dim SC = \log 8/\log 3$. The exact value of $\mathcal{C}\dim SC$ is unknown, and it is not known whether $\mathcal{QC}\dim SG$ and $\dim SC$ are equal.

This paper is organized as follows. In section 2 we prove that the global quasiconformal dimension of the Sierpinski gasket is one. Section 3 contains preliminary definitions and basic notation. The intrinsic quasisymmetry of the canonical homeomorphism on invariant sets (Theorem 1.7) and the quasiconformal extensions to the ambient spaces (Theorem 1.9) are proved in sections 4 and 5, respectively. Two theorems of Väisälä from [27], which provide sufficient conditions for extending a quasisymmetric map to a global quasiconformal map, play a crucial role in the proof of Theorem 1.9.

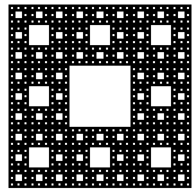

FIGURE 3. The Sierpinski carpet SC

Finally, in sections 6 and 7 we prove Theorems 1.5 and 1.4. The constructions of the deformed iterated function systems for the polygasket and the d-dimensional gasket are quite intricate, involving many technical combinatorial/geometric estimates.

2. THE SIERPINSKI GASKET

Fix a natural number $n \in \mathbb{N}$. Instead of the contractions $\mathcal{F}_0 = \{f_0, f_1, f_2\}$ in (1.2), we view the Sierpinski gasket as the invariant set for the following iterated function system consisting of $6n+3$ planar contractive similarities:
(2.1)
$$\mathcal{F} = \{f_j^{(n+1)} : j = 0,1,2\} \cup \{f_j^{(m)} \circ f_k : j,k = 0,1,2, j \neq k, m = 1,\dots,n\}.$$

Here $f^{(p)}$ denotes the p-fold composition of f. Figure 4(c) shows the images of the triangle T_0 with vertices 1, $\omega = e^{2\pi i/3}$ and $\omega^2 = e^{4\pi i/3}$ under the mappings in $\mathcal{F}$ in the case $n = 2$.

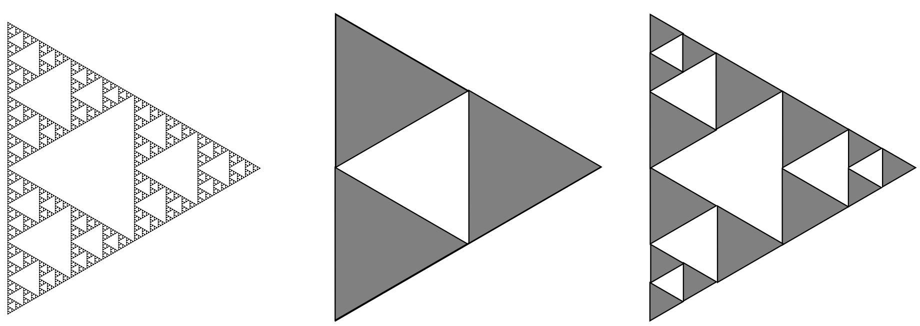

FIGURE 4. (a) SG; (b) the defining triangles from $\mathcal{F}_0$ for SG; (c) the defining triangles from $\mathcal{F}$ ($n = 2$) for SG.

We define a deformation $\mathcal{G}$ of $\mathcal{F}$ which generates an invariant set SG_n. $\mathcal{F}$ and $\mathcal{G}$ are isomorphic and the homeomorphism from SG to SG_n is quasisymmetric and can easily be extended to a quasiconformal map of $\mathbb{R}^2$. Finally, $\lim_{n\to\infty} \dim SG_n = 1$. Taken together, these results prove Theorem 1.3.

To define $\mathcal{G}$ we replace the geometrically decreasing sequence of triangles $f_j^{(m)} \circ f_k(T_0)$, $m = 1,\dots,n$, with a row of equally sized triangles. The deformation can be viewed as a discrete analog of the conformal map $z \mapsto \log z$. Explicitly, consider the family of planar contraction mappings

$$\mathcal{G} = \{g_j : j = 0,1,2\} \cup \{h_{jkm} : j,k = 0,1,2, j \neq k, m = 1,\dots,n\},$$

where

$$g_j(z) = \frac{1}{n+2} z + \frac{n+1}{n+2}\omega^j, \tag{2.2}$$

$$h_{jkm}(z) = \frac{1}{(n+2)\sqrt{3}} e^{\epsilon_{jk}\pi i/6}(z-\omega^j) + \frac{1}{n+2}\omega^k + \frac{m+1}{n+2}\omega^j, \tag{2.3}$$

and $\epsilon_{01} = \epsilon_{12} = \epsilon_{20} = 1$, $\epsilon_{10} = \epsilon_{21} = \epsilon_{02} = -1$. See Figure 5 for the case $n = 2$.

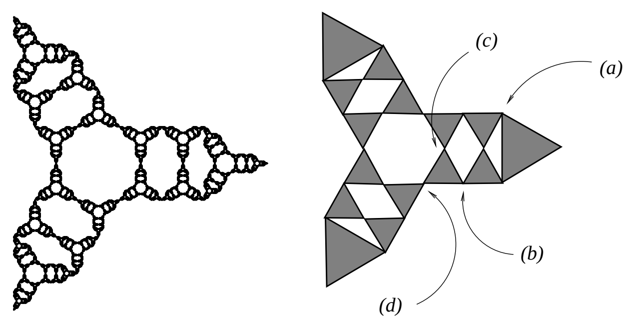

FIGURE 5. (a) SG_2; (b) the defining triangles from $\mathcal{G}$ for SG_2.

According to a result from the following section (Proposition 3.37), in order to show that $\mathcal{F}$ and $\mathcal{G}$ are isomorphic it suffices to verify that the vertices of the triangles $g_j(T_0), h_{jkm}(T_0)$ are identified according to the same rule whereby the vertices of the triangles $f_j^{(n+1)}(T_0), f_j^{(m)} \circ f_k(T_0)$ are identified. That is, we must verify that

(a) $h_{jkn}(\omega^j) = g_j(\omega^k)$,
(b) $h_{jk,m+1}(\omega^k) = h_{jkm}(\omega^j)$, $1 \le m \le n-1$,
(c) $h_{jkm}(\omega^l) = h_{jlm}(\omega^k)$, $1 \le m \le n$,
(d) $h_{jk1}(\omega^k) = h_{kj1}(\omega^j)$,

for all $j,k,l = 0,1,2$, $j \neq k$, $j \neq l$. (An example of an intersection of each of these types is indicated in Figure 5(b).) We leave to the reader the straightforward verification of (a)–(d). Observe that the scaling factors $\lambda = \frac{1}{n+2}$ and $\mu = \frac{1}{(n+2)\sqrt{3}}$ in (2.2) and (2.3) were chosen precisely to make (d) an identity; the relevant fact is that

$$e^{\epsilon_{jk}\pi i/6} + e^{\epsilon_{kj}\pi i/6} = 2\cos\frac{\pi}{6} = \sqrt{3} = \frac{\lambda}{\mu}.$$

By Theorem 1.7 there is a canonically defined quasisymmetric homeomorphism $F_n : SG \to SG_n$. It is easy to see what this homeomorphism must be. First, there is a natural identification between the vertices of $f_j^{(n+1)}(T_0)$ or $f_j^{(m)} \circ f_k(T_0)$ and $g_j(T_0)$ or $h_{jkm}(T_0)$, respectively. Iterating gives an identification between corresponding vertices of triangles at each level of the construction. This correspondence extends by density to a homeomorphism between SG and SG_n. This is the canonical homeomorphism F_n.

Since the components of $\hat{\mathbb{C}} \setminus SG$ are quasidiscs, the quasisymmetric map F_n extend to a quasiconformal map in each component by Ahlfors' Extension Theorem [1]. By [28, Theorem 3.3], the extended map is quasiconformal on all of $\mathbb{R}^2$.

To complete the proof of Theorem 1.3, we will estimate the Hausdorff dimensions of the deformed gaskets SG_n. Since $\mathcal{G}$ satisfies the open set condition, the dimension of SG_n is the unique positive solution to the equation

$$3(\frac{1}{n+2})^{s_n} + 6n(\frac{1}{(n+2)\sqrt{3}})^{s_n} = 1.$$

Then $\frac{9n}{(n+2)^{s_n}} \geq 1$ whence $s_n \leq \frac{\log(9n)}{\log(n+2)} \to 1$ as $n \to \infty$.

Remark 2.4. The linear interpolation $\mathcal{F}^t = \{g_j^t, h_{jkm}^t\}_{0 \leq t \leq 1}$ defined by

$$\begin{aligned} g_j^t &= tg_j + (1-t)f_j^{(n+1)}, \\ h_{jkm}^t &= th_{jkm} + (1-t)f_k \circ f_j^{(m)}, \end{aligned} \tag{2.5}$$

is a nondegenerate isotopy between $\mathcal{F}$ and $\mathcal{G}$. Thus the existence of the quasiconformal extension would also follow from Theorem 1.9. However, the above argument via Ahlfors' Extension Theorem [1] is simpler.

3. Definitions and notation

3.1 Basic notation. In sections 3–5 we always work in a fixed Euclidean space $\mathbb{R}^d$. For $1 \leq p \leq d$ we view $\mathbb{R}^p$ as a subset of $\mathbb{R}^d$ via the embedding $(x_1, \dots, x_p) \mapsto (x_1, \dots, x_p, 0, \dots, 0)$. For a subset A of $\mathbb{R}^p$, or more generally of a p-dimensional affine subspace of $\mathbb{R}^d$, we use $|A|_p$ to denote the p-dimensional Lebesgue measure of A. We abbreviate $|A| = |A|_d$ and we denote the measure of the unit ball in $\mathbb{R}^d$ by Ω_d.

We denote the $d \times d$ identity matrix by I_d. For a $d \times d$ matrix A we denote by $||A|| := \sup\{|Av| : v \in \mathbb{R}^d, |v| \leq 1\}$ the operator norm.

We say that a map $f : \mathbb{R}^d \to \mathbb{R}^d$ is a *similarity* if there exists $\lambda > 0$ so that the identity $|f(x) - f(y)| = \lambda|x - y|$ holds for all $x, y \in \mathbb{R}^d$. It is well-known that any such map can be written in the form

$$f(x) = f(0) + \lambda\, A \cdot x \tag{3.2}$$

for some matrix A in the orthogonal group $O(d)$.

A similarity is called *strict* if $\lambda < 1$. Each strict similarity $f : \mathbb{R}^d \to \mathbb{R}^d$ has a unique fixed point $a = (I_d - \lambda A)^{-1} \cdot f(0)$, and can be written in the form

$$f(x) = a + \lambda A(x - a). \tag{3.3}$$

Conversely, each triple (a, λ, A) with $a \in \mathbb{R}^d$, $0 < \lambda < 1$, and $A \in O(d)$ gives rise to a strict similarity f of $\mathbb{R}^d$ via (3.3). The space $\mathcal{SS}(\mathbb{R}^d)$ of all strict similarities of $\mathbb{R}^d$ is thus finite-dimensional. The topology on $\mathcal{SS}(\mathbb{R}^d)$ is the canonical topology arising from the representations of elements in the form (3.3). Observe that this topology is metrizable, for example, it is generated by the metric

$$D(f,g) = |a - a'| + |\lambda - \lambda'| + ||A - A'||, \qquad f, g \in \mathcal{SS}(\mathbb{R}^d), \tag{3.4}$$

where $f = (I_d - \lambda A)a + \lambda A$ and $g = (I_d - \lambda' A')a' + \lambda' A'$. The resulting topology on $\mathcal{SS}(\mathbb{R}^d)$ coincides with the topology of local uniform convergence. Observe, however, that $\mathcal{SS}(\mathbb{R}^d)$ is not a complete metric space.

3.5 Iterated function systems. We review the basic theory of iterated function systems as developed in Hutchinson [11]. For a modern treatment, see Kigami [15, Chapter 1]. Our notation is consistent with [15].

An *iterated function system (IFS)* consists of a finite collection $\mathcal{F} = \{f_1, \ldots, f_M\}$ of strict similarities of the Euclidean space $\mathbb{R}^d$. Associated to each iterated function system $\mathcal{F}$ there is a unique nonempty compact set $K = K(\mathcal{F}) \subset \mathbb{R}^d$ which is invariant under the transformations $f_1, \ldots, f_M$, i.e.,

$$K = \bigcup_{i=1}^{M} K_i, \qquad K_i = f_i(K).$$

This follows from the completeness of the space of all compact subsets of $\mathbb{R}^d$ equipped with the Hausdorff metric, see [19, 4.13] or [15, Theorem 1.1.4].

Remark 3.6. For fixed M, the space $\mathcal{IFS}(\mathbb{R}^d, M)$ of all iterated function systems $\mathcal{F} = \{f_1, \ldots, f_M\}$ in $\mathbb{R}^d$ is naturally identified with the quotient $\mathcal{SS}(\mathbb{R}^d)^M / S_M$, where S_M denotes the symmetric group of permutations of M letters. Via this identification, the topology on $\mathcal{SS}(\mathbb{R}^d)$ described above induces a topology on $\mathcal{IFS}(\mathbb{R}^d, M)$, which is generated by the metric

$$D(\mathcal{F}, \mathcal{G}) = \min_{\iota \in S_M} \max_{k=1}^{M} D(f_k, g_{\iota_k}), \qquad \mathcal{F}, \mathcal{G} \in \mathcal{IFS}(\mathbb{R}^d, M). \tag{3.7}$$

In the remainder of this section we fix an iterated function system $\mathcal{F}$ in $\mathcal{IFS}(\mathbb{R}^d, M)$ with invariant set K and denote by (a_j, λ_j, A_j) the data from (3.3) associated with an element $f_j \in \mathcal{F}$, $j = 1, \ldots, M$. To avoid some trivial cases we assume that $M \geq 2$ and $a_1 \neq a_2$ so that $\operatorname{diam} K > 0$.

3.8 Symbolic dynamics. Let A be an alphabet consisting of the letters $1, \ldots, M$ and let $W_m = A^m$, $m \geq 1$ (resp. $\Sigma = A^{\mathbb{N}}$), denote the space of words of length m (resp. words of infinite length) with letters drawn from A. We denote elements of these spaces by concatenation of letters, i.e., we write $w = w_1 w_2 \cdots w_m \in W_m$ or $w = w_1 w_2 \cdots \in \Sigma$, where $w_j \in A$. We set $W = \cup_{m \geq 1} W_m$ to be the set of all words of finite length. For a fixed letter $i \in A$ we denote by $\overline{i}$ the infinite word $iii \cdots$.

We denote by σ the shift map on Σ:

$$\sigma(w_1w_2w_3\cdots) = w_2w_3\cdots .$$

Fix an IFS $\mathcal{F}$ as above. For each finite word $w = w_1\cdots w_m$ let $f_w = f_{w_1}\circ\cdots\circ f_{w_m}$ and $\lambda_w = \lambda_{w_1}\cdots\lambda_{w_m}$. For an arbitrary set $S\subset\mathbb{R}^d$, let $S_w = f_w(S)$. If S is a nonempty compact set with $f_i(S)\subset S$ for all $i\in A$, then

$$\lim_{m\to\infty}\max_{w\in W_m}\operatorname{diam} S_w = 0 \tag{3.9}$$

and

$$K = \bigcap_{m\geq 0}\bigcup_{w\in W_m} S_w.$$

If $w = w_1w_2\cdots$ is an infinite word, then $S_w := \cap_m S_{w_1\cdots w_m}$ is nonempty; by (3.9) S_w contains precisely one point.

We consider on the space Σ the product topology induced by the discrete topology on A and we define a map $\pi = \pi_{\mathcal{F}} : \Sigma\to K$ by setting $\pi(w)$ equal to the unique point in K_w. Then π is a continuous surjection between compact sets [15, Theorem 1.2.3]. Observe that

$$\pi(w) = \lim_{m\to\infty} f_{w_1\cdots w_m}(x_0), \qquad w = w_1w_2\cdots\in\Sigma, \tag{3.10}$$

where x_0 is an arbitrarily chosen point in $\mathbb{R}^d$.

Proposition 3.11. *$\pi(w) = \pi(w')$ for $w\neq w'\in\Sigma$ if and only if $\pi(\sigma^s w) = \pi(\sigma^s w')$, where $s = s(w,w') := \min\{m : w_m\neq w'_m\} - 1$.*

This is Proposition 1.2.5 in [15]. Observe that $s = s(w,w')$ if and only if $w_i = w'_i$ for $1\leq i\leq s$ and $w_{s+1}\neq w'_{s+1}$. Moreover, $\pi(\sigma^s w) = \pi(\sigma^s w')\in\bigcup_{i\neq j} K_i\cap K_j$.

3.12 Uniform perfectness. For future reference we record the following connectedness-type property of invariant sets. A metric space X is *c-uniformly perfect*, $c>0$, if for every $x\in X$ and every $0<r<\operatorname{diam} X$ the annulus $B(x,r)\setminus B(x,cr)$ is nonempty. For example, every connected space is 1/2-uniformly perfect.

In connection with the following proposition, recall our standing assumptions that $M\geq 2$ and $\operatorname{diam} K>0$.

Proposition 3.13. *The invariant set K is $\frac{1}{2}\lambda_{min}$-uniformly perfect, where $\lambda_{min} := \min\{\lambda_1,\ldots,\lambda_M\}$.*

Proof. Let $x\in K$ and $r>0$ and choose $\tau\in\Sigma$, $\pi(\tau) = x$. Let $w = w_1\cdots w_m$ be the shortest subword of τ for which $K_w\subset B(x,r)$. If w is the empty word, then $K\subset B(x,r)$. In this case we may choose a point $y\in K$ with $|x-y|\geq\frac{1}{2}r$ and the result follows. If w is not the empty word, then

$$\operatorname{diam} K_w \geq \lambda_{\min}\operatorname{diam} K_{w_1\cdots w_{m-1}}\geq\lambda_{\min} r.$$

Again, there exists $y\in K_w$ with $|x-y|\geq\frac{1}{2}\lambda_{\min} r$. The proof is complete. $\square$

3.14 Post-critical finiteness and the open set condition. Following Hutchinson [11], we say that an IFS $\mathcal{F} = \{f_1, \dots, f_M\}$ satisfies the *open set condition* (OSC) if there exists a (nonempty) bounded open set O such that $O_j \subset O$ for all j and $O_j \cap O_k = \emptyset$ for all $j \neq k$, where $O_j = f_j(O)$. Recall that for an IFS which satisfies the open set condition, the dimension of the invariant set can be computed using (1.6). See, e.g., Theorem 9.3 in [19] or Theorem 1.5.7 in [15].

For an IFS $\mathcal{F}$, let

$$C := \cup_{i \neq j} K_i \cap K_j$$

denote the *critical set* for the images $K_i = f_i(K)$, and let

$$P := \bigcup_{w \in W} f_w^{-1}(C) \cap K \tag{3.15}$$

denote the *post-critical set*. The symbolic preimages of these sets are the *critical symbols* $\pi^{-1}(C)$ and the *post-critical symbols*

$$\pi^{-1}(P) = \bigcup_{m \geq 1} \sigma^m(\pi^{-1}(C)) \tag{3.16}$$

which are defined as subsets of the sequence space Σ.[3] (For the equality of the expressions in (3.16), see [15, Proposition 1.3.5].) An IFS is said to be *post-critically finite* (PCF) if it has finitely many post-critical symbols.

On some occasions we will refer to the post-critical set as the *boundary* of $\mathcal{F}$. We denote the set of maps $f \in \mathcal{F}$ whose fixed point lies in P by $\partial\mathcal{F}$.

The following result is a special case of Lemma 1.3.14 in [15].

Proposition 3.17. *Let $\mathcal{F}$ be post-critically finite and let $f_i \in \mathcal{F}$ with fixed point a_i. Then $\pi^{-1}(a_i) = \{\bar{i}\}$.*

The prototypical PCF system satisfying the open set condition is the standard IFS $\mathcal{F} = \{f_0, f_1, f_2\}$ in (1.2) defining the Sierpinski gasket SG. See Figure 4(a). Here the critical and post-critical sets are $C = \{d, e, f\}$ and $P = \{a, b, c\}$. Observe that the interior of the initial triangle T_0 with vertices at a, b and c verifies the open set condition.

Note that in this example the post-critical set P coincides with the set of fixed points of the maps in $\mathcal{F}$ and also with the vertices of T_0. These identifications need not hold in general. The concept of a gasket type IFS, which we introduce next, axiomatizes certain relations among these three sets.

3.18 Gasket-type systems. Let $\mathcal{F}$ be an IFS with invariant set K. Let $\mathrm{Fix}(\mathcal{F}) = \{a_1, \dots, a_M\}$ denote the set of fixed points of the maps in $\mathcal{F}$. Denote by Π the convex hull of K. As usual, let $\Pi_w = f_w(\Pi)$ for $w \in W$.

[3]Our definition of post-critical finiteness agrees with that of [15], however, some of our terminology differs. In particular, in [15] the terms critical and post-critical set refer to the sets $\pi^{-1}(C)$ and $\pi^{-1}(P)$.

Definition 3.19. We say that $\mathcal{F}$ is a system of *gasket type* if the following conditions hold:

(i) Π is a polyhedron,
(ii) each vertex of Π is in $\mathrm{Fix}(\mathcal{F})$.
(iii) for all $i,j \in A$, $i \neq j$, $K_i \cap K_j = \Pi_i \cap \Pi_j$, and when this intersection is nonempty, it contains exactly one point which is a common vertex of Π_i and Π_j,
(iv) for $v \in C$ the set $S_v := \bigcup_{m \geq 1} \sigma^m \pi^{-1}(v)$ contains at most two words.

Condition (iii) implies that the post-critical set P is contained in the set of vertices $V(\Pi)$ of the polyhedron Π. Thus

$$P \subset V(\Pi) \subset \mathrm{Fix}(\mathcal{F})$$

for gasket type systems. Both inclusions may be strict. Observe that we do not require that the post-critical set be nonempty; in particular, we allow the possibility that a gasket type IFS is totally disconnected.

Gasket type systems are post-critically finite. Indeed, $\pi^{-1}(P) = \cup_{v \in C} S_v$ has cardinality at most $M(M-1)$. In fact, condition (iv) is equivalent with the following:

(iv') $\mathcal{F}$ is post-critically finite and $\Pi_i \cap \Pi_j \cap \Pi_k = \emptyset$ whenever $i,j,k \in A$, $i \neq j$, $i \neq k$, $j \neq k$.

Lemma 3.20. *Let $\mathcal{F}$ be a gasket-type IFS. If $v \in K_i \cap K_j$ with $i \neq j$, then $S_v = \{\overline{k}, \overline{l}\}$ and $v = f_i(a_k) = f_j(a_l)$, where a_k, a_l denote the fixed points of f_k, f_l, respectively.*

Proof. Suppose that $v \in K_i \cap K_j$. Then $f_i^{-1}(v), f_j^{-1}(v) \in P \subset \mathrm{Fix}(\mathcal{F})$ and $v = f_i(a_k) = f_j(a_l)$ for some $f_k, f_l \in \partial\mathcal{F}$. Then $v = \pi(i\overline{k}) = \pi(j\overline{l})$ and $\overline{k}, \overline{l} \in S_v$. □

Examples 3.21. Set $\omega = e^{2\pi i/3}$. For $j = 1, \ldots, 6$ let $f_j(z) = \frac{2}{3}a_j + \frac{1}{3}z$, where $a_1 = 1$, $a_2 = (1+\omega)/2$, $a_3 = (1+\omega^2)/2$, $a_4 = \omega$, $a_5 = -1/2$ and $a_6 = \omega^2$. Also let $\tilde{f}_2(z) = \frac{1}{3}\omega^2(z-1)$ and $\tilde{f}_3(z) = \frac{1}{3}\omega(z-1)$. Finally, let $\hat{f}_1(z) = \frac{1}{3} + \frac{2}{3}z$.

(1) The IFS $\mathcal{F} = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ is a post-critically finite system which fails to satisfy the second assertion in condition (iv').
(2) The gasket type IFS $\mathcal{F} = \{f_1, \tilde{f}_2, \tilde{f}_3, f_4, f_6\}$ shows that the case "S_v is a singleton" in condition (iv) of Definition 3.19 can occur.
(3) The IFS $\mathcal{F} = \{\hat{f}_1, f_4, f_5, f_6\}$ shows that in general elements of the critical set need not be vertices of every polyhedron in which they lie. Thus the last phrase in condition (iii) is not a consequence of the remaining parts of the definition.

Proposition 3.22. *Gasket type systems satisfy the open set condition.*

Proof. If the topological dimension p of Π is equal to d, we choose O to be the interior of Π. If $p < d$ we write $\mathbb{R}^d = \Lambda \oplus \Lambda^\perp$ as the direct sum of the p-dimensional affine subspace Λ which spans Π and its orthogonal complement $\Lambda^\perp$, with origin chosen so that $\Lambda \times \{0\}$ coincides with Λ. Then we may choose $O = O' \times B^{d-p}(0,1) \subset \mathbb{R}^d$, where O' denotes the interior of Π in the subspace topology on Λ. □

Remark 3.23. Note also that $\Pi_v \subset \Pi_w$ if and only if v is formed by adding additional letters after w; in this case we call Π_v a *descendant* or a *subpolyhedron* of Π_w. By the *generation* of a descendant Π_v of Π_w we mean the number of additional letters in v, i.e., the length of v minus the length of w. The *children* of Π_w are its first generation descendants. If Π_v is a child of Π_w, we denote by $\widehat{\Pi_v} = \Pi_w$ the *parent* of Π_v. When Π_v and $\Pi_{v'}$ are two distinct descendants of Π_w, then either (i) $\Pi_v \cap \Pi_{v'} = \emptyset$ or (ii) $\Pi_v \cap \Pi_{v'}$ contains exactly one point, or (iii) $\Pi_v \subset \Pi_{v'}$ or $\Pi_{v'} \subset \Pi_v$. When Π_v and $\Pi_{v'}$ are two distinct children of Π_w, then either (i) or (ii) must be true. We call two subpolyhedra *adjacent* if they intersect in exactly one point. Observe that condition (ii) of Definition 3.19 implies that each vertex of Π_w can be a vertex of at most one child Π_{wi}.

3.24 Geometric data associated with a gasket type system. Let $\mathcal{F} = \{f_1, \ldots, f_M\}$ be a gasket type system. Let

$$\lambda_{\min} := \min\{\lambda_1, \ldots, \lambda_M\}, \quad \lambda_{\max} := \max\{\lambda_1, \ldots, \lambda_M\}, \tag{3.25}$$

be the *minimal* and *maximal scaling ratios* associated with the elements of $\mathcal{F}$. Next, let

$$\delta := \frac{\min\{\operatorname{dist}(\Pi_i, \Pi_j) : \Pi_i \cap \Pi_j = \emptyset, i, j \in A\}}{\operatorname{diam} \Pi} \tag{3.26}$$

be the *minimal relative distance between non-adjacent children* and let

$$\theta := \min\{\angle xvy : x \in \Pi_i, y \in \Pi_j, \Pi_i \cap \Pi_j = \{v\} \neq \emptyset, i, j \in A\} \tag{3.27}$$

be the *minimal angle between adjacent children*. If all of the children Π_i are disjoint, the minimum in (3.27) is over the empty set; in this case we set $\theta = \pi$ so that $\sin(\theta/2) = 1$. (See, e.g., Lemmas 3.29–3.32 and (5.2).)

Finally, let

$$\kappa := \lambda_{\min} \cdot \frac{s_{\min}(\Pi)}{\operatorname{diam} \Pi}, \tag{3.28}$$

where

$$s_{\min}(\Pi) := \min\{|v - w| : v, w \text{ distinct vertices of } \Pi\}.$$

Observe that all five of these quantities are positive and $\lambda_{\max} < 1$; $\theta > 0$ follows from the fact that the sets Π_i are convex polyhedra which meet only at vertices. Moreover, as functions of the IFS $\mathcal{F}$, the quantities in (3.25)–(3.28) are functions of either the contraction ratios λ_i or the vertices of Π, and thus are continuous with respect to the topology on $\mathcal{IFS}(\mathbb{R}^d, M)$ described in Remark 3.6.

In the following three lemmas we fix a subpolyhedron $\Pi_w = f_w(\Pi)$.

Lemma 3.29. *Let Π_{wi} and Π_{wj} be two adjacent children of Π_w which intersect at v. If $x \in \Pi_{wi}$ and $y \in \Pi_{wj}$, then*

$$|x-y| \le |x-v| + |y-v| \le \csc(\theta/2)|x-y|.$$

Proof. It follows from the definition of θ and the self similarity that $\angle xvy \ge \theta > 0$ for all $x \in \Pi_{wi}$ and $y \in \Pi_{wj}$. The stated inequalities then follow from the Law of Cosines and the triangle inequality. □

Lemma 3.30. *Let Π_{wi}, Π_{wj} and Π_{wk} be three distinct children of Π_w. Then*

$$\min\{\delta, \frac{1}{2}\kappa \sin(\theta/2)\} \operatorname{diam} \Pi_w \le \max\{|x-y|, |x-z|\} \le \operatorname{diam} \Pi_w \tag{3.31}$$

for $x \in \Pi_{wi}$, $y \in \Pi_{wj}$ and $z \in \Pi_{wk}$.

Proof. Suppose two of these children are not adjacent. Without loss of generality let them be Π_{wi} and Π_{wj}. Then

$$|x-y| \ge \operatorname{dist}(\Pi_{wi}, \Pi_{wj}) \ge \delta \operatorname{diam} \Pi_w.$$

Suppose next that all three children are pairwise adjacent. We may assume that $|x-y| < \frac{1}{2}\sin(\theta/2)s_{\min}(\Pi_{wi})$. Let $\{v\} = \Pi_{wi} \cap \Pi_{wj}$ and $\{v'\} = \Pi_{wi} \cap \Pi_{wk}$, and note from condition (iv') in Definition 3.19 that $v \ne v'$. By the previous lemma,

$$\begin{aligned}\csc(\theta/2)|x-z| &\ge |x-v'| \ge |v-v'| - |x-v| \\ &\ge s_{\min}(\Pi_{wi}) - \csc(\theta/2)|x-y| \\ &> \frac{1}{2}s_{\min}(\Pi_{wi}) \ge \frac{1}{2}\kappa \operatorname{diam} \Pi_w.\end{aligned}$$

□

Lemma 3.32. *Let v be a vertex of Π_w and let Π_{wi} and Π_{wj} be distinct children of Π_w so that v is a vertex of Π_{wj}. Then*

$$\min\{\delta, \kappa \sin(\theta/2)\} \operatorname{diam} \Pi_w \le \operatorname{dist}(v, \Pi_{wi}) \le \operatorname{diam} \Pi_w. \tag{3.33}$$

Observe that under the hypotheses of Lemma 3.32, v is not a vertex of Π_{wi}. See Remark 3.23.

Proof. Let $x \in \Pi_{wi}$. If Π_{wi} and Π_{wj} are not adjacent, then $|x-v| \ge \delta \operatorname{diam} \Pi_w$. Suppose that Π_{wi} and Π_{wj} are adjacent, with $\Pi_{wi} \cap \Pi_{wj} = \{v_1\}$. As discussed above, $v_1 \ne v$. By Lemma 3.29,

$$|x-v| \ge \sin(\theta/2)|v-v_1| \ge \sin(\theta/2)s_{\min}(\Pi_{wj}) \ge \kappa \sin(\theta/2) \operatorname{diam} \Pi_w.$$

□

3.34 Boundary congruence. Let $\mathcal{F}$ be a gasket type system. We say that $\mathcal{F}$ is *boundary congruent* if all of the similarities in $\partial\mathcal{F}$ share a common contraction ratio and have trivial rotation matrix. In other words, there exists $0 < \lambda_\partial < 1$ so that each $f \in \partial\mathcal{F}$ satisfies $f(x) = a + \lambda_\partial(x-a)$ for some $a \in P$.

Remark 3.35. One could generalize the definition of boundary congruence by requiring that all of the maps in $\partial\mathcal{F}$ share a common contraction ratio λ_∂ and a common rotation matrix $A_\partial \in O(d)$ (not necessarily the identity matrix), i.e., that each $f \in \partial\mathcal{F}$ satisfy $f(x) = a + \lambda_\partial A_\partial(x-a)$ for some $a \in P$. However, it can be shown that this *a priori* weaker requirement is in fact equivalent with the stated condition. (Here the standing assumptions that $M \geq 2$ and $\mathrm{Fix}(\mathcal{F})$ contains at least two elements are needed.)

The boundary congruence property is essential for proving quasisymmetry of the canonical homeomorphism F between isomorphic invariant sets. Without it, the relative distances $|x-v|/|y-v|$ and $|F(x)-F(v)|/|F(y)-F(v)|$ will be incommensurable at a point v in the critical set of $\mathcal{F}$. See Example 4.5.

3.36 Isomorphic systems. Fix $M \geq 1$ and denote by S_M the symmetric group on M letters. Each permutation $\iota \in S_M$ induces a bijection of $\Sigma = \{1,\dots,M\}^\infty$ by the formula $w_1w_2\cdots \mapsto \iota_{w_1}\iota_{w_2}\cdots$. We use the same notation ι for this bijection of Σ.

Let $\mathcal{F},\mathcal{G} \in \mathcal{IFS}(\mathbb{R}^d, M)$. Following Kigami [15, Definition 1.3.2], we say that $\mathcal{F}$ and $\mathcal{G}$ are *isomorphic* if the map $F := \pi_\mathcal{G} \circ \iota \circ \pi_\mathcal{F}^{-1}$ is a well-defined homeomorphism between $K(\mathcal{F})$ and $K(\mathcal{G})$. In other words, there exists a permutation $\iota \in S_M$ so that $\pi_\mathcal{G}\circ\iota(w) = \pi_\mathcal{G}\circ\iota(w')$ whenever $w,w' \in \Sigma$ satisfy $\pi_\mathcal{F}(w) = \pi_\mathcal{F}(w')$.

To simplify matters, we assume henceforth that for isomorphic systems $\mathcal{F}$ and $\mathcal{G}$, the elements of $\mathcal{G}$ are ordered so that ι is the identity. Then the canonical homeomorphism from $K(\mathcal{F})$ to $K(\mathcal{G})$ is

$$F = \pi_\mathcal{G} \circ \pi_\mathcal{F}^{-1}.$$

Proposition 3.37 gives an effective way to verify that two gasket type systems are isomorphic.

Proposition 3.37. *Let $\mathcal{F}$ and $\mathcal{G}$ be two gasket type iterated function systems. For each $i = 1,\dots,M$, denote the fixed point of f_i by a_i and the fixed point of g_i by a'_i. Then the following are equivalent:*

(i) *$\mathcal{F}$ and $\mathcal{G}$ are isomorphic,*
(ii) *for each $v \in C$, if $v = f_i(a_k) = f_j(a_l)$ for some $i \neq j$ (as in Lemma 3.20), then $g_i(a'_k) = g_j(a'_l)$.*

Proof. First assume that $\mathcal{F}$ and $\mathcal{G}$ are isomorphic, and let $v = f_i(a_k) = f_j(a_l) \in C$. Then $\pi_\mathcal{F}(i\overline{k}) = v = \pi_\mathcal{F}(j\overline{l})$ whence $\pi_\mathcal{G}(i\overline{k}) = \pi_\mathcal{G}(j\overline{l})$ and $g_i(a'_k) = g_j(a'_l)$. Thus (i) implies (ii).

To see why the converse holds, suppose that $w,w' \in \Sigma$, $w \neq w'$ and $\pi_\mathcal{F}(w) = \pi_\mathcal{F}(w')$. By Proposition 3.11, $v := \pi_\mathcal{F}(\sigma^s w) = \pi_\mathcal{F}(\sigma^s w') \in C$, where $s = s(w,w')$. By Lemma 3.20, $S_v = \{\overline{k},\overline{l}\}$ and $v = f_i(a_k) = f_j(a_l)$ for some $i \neq j$ and k and l. Thus $\pi^{-1}(v) = \{i\overline{k}, j\overline{l}\} \supset \{\sigma^s w, \sigma^s w'\}$. Since $\sigma^s w \neq \sigma^s w'$ we may assume without loss of generality that $\sigma^s w = i\overline{k}$ and

$\sigma^s w' = j\bar{l}$. By the hypothesis $\pi_{\mathcal{G}}(\sigma^s w) = \pi_{\mathcal{G}}(\sigma^s w')$ and so $\pi_{\mathcal{G}}(w) = \pi_{\mathcal{G}}(w')$ as required. □

3.38 Isotopies of iterated function systems. Now assume that $\mathcal{F}^t = \{f_1^t, \dots, f_M^t\}$, $0 \le t \le 1$, is a one-parameter family of iterated function systems in $\mathbb{R}^d$ which vary continuously in t, that is, the map $t \mapsto \mathcal{F}^t$ is continuous with respect to the topology on the space $\mathcal{IFS}(\mathbb{R}^d, M)$ discussed in Remark 3.6. We say that $\{\mathcal{F}^t\}$ is an *isotopy* if $\mathcal{F}^{t'}$ and $\mathcal{F}^{t''}$ are isomorphic whenever $0 \le t' < t'' \le 1$. We call such an isotopy *nondegenerate* if the dimension of the affine subspace Λ_t of $\mathbb{R}^d$ spanned by the invariant set $K(\mathcal{F}^t)$ is constant in t. We denote this common dimension by p.

Remark 3.39. Nondegeneracy of the isotopy ensures that the invariant sets $K(\mathcal{F}^t)$ are "uniformly thick" in dimension p. For a precise statement along these lines, see Remark 5.5.

A typical example of a degenerate isotopy is the planar family $\mathcal{F}^t = \{f_1^t, f_2^t, f_3^t, f_4^t\}$, where $f_1^t(z) = \frac{3+t}{12}z$, $f_2^t(z) = \frac{3+t}{12} + \frac{(\sqrt{3}+\sqrt{t}i)^2}{12}z$, $f_3^t(z) = \frac{9-t}{12} + \frac{(\sqrt{3}-\sqrt{t}i)^2}{12}(z-1)$, and $f_4^t(z) = 1 + \frac{3+t}{12}(z-1)$. This isotopy interpolates between $\mathcal{F}^0$, whose invariant set is the unit line segment $[0,1]$, and $\mathcal{F}^1$, whose invariant set is the classical von Koch snowflake.

On the other hand, every isotopy of gasket type systems for which the polyhedron Π^t is identical for all t, is nondegenerate. This covers, for example, the isotopies in Remark 2.4 and section 6.

Remark 3.40. It is not immediately clear when two isomorphic systems can be joined by a nondegenerate isotopy. In the planar case, linear isotopies (as in (2.5)) always generate similarities provided all of the contraction maps involved are orientation-preserving, i.e. analytic. However, it is not easy to determine whether the intermediate IFS's are all isomorphic to the original system and whether the isotopy is nondegenerate.

In higher dimensions linear isotopies do not give similarity maps for $0 < t < 1$. This stems from the fact that the Cauchy-Riemann system defining conformality in higher dimensions is nonlinear (see [12, (1.22)]). As before, if all of the maps in question have coherent orientations, nonlinear isotopies of similarity maps can be found. But again the questions of determining when the intermediate systems are all isomorphic, and when the isotopy is nondegenerate are formidable. In the case of the d-dimensional gaskets (section 6), we will construct explicit nondegenerate isotopies. See Proposition 6.16.

4. Quasisymmetric maps between invariant sets

In this section we prove Theorem 1.7, which asserts that the canonical homeomorphism $F = \pi_{\mathcal{G}} \circ \pi_{\mathcal{F}}^{-1}$ between the invariant sets of two isomorphic, boundary congruent, gasket type systems $\mathcal{F}$ and $\mathcal{G}$, is quasisymmetric.

For brevity, we write $K = K(\mathcal{F})$ and $K' = K(\mathcal{G})$. In general, we will put a prime over a quantity that has been defined for $\mathcal{F}$ to denote the corresponding quantity for $\mathcal{G}$. For example, $\{\lambda'_j\}$ will denote the contraction ratios for the maps in $\mathcal{G}$ and λ'_∂ will denote the contraction ratio associated with the similarities whose fixed points lie in $\partial\mathcal{G}$. Let Π and Π' be the convex hulls of K and K', respectively. Without loss of generality we may assume that $\operatorname{diam}\Pi = \operatorname{diam}\Pi' = 1$.

The proof of Theorem 1.7 relies on a number of geometric lemmas (Lemmas 4.1–4.3) which describe the metric distortion induced by the canonical homeomorphism. The setting for these lemmas is the following. We fix a word $w \in W$ and let $Q = \Pi_w = f_w(\Pi)$ and $Q' = g_w(\Pi')$. We denote by $d = \operatorname{diam} Q = \lambda_w$ and $d' = \operatorname{diam} Q' = \lambda'_w$, and by $Q_\tau = f_{w\tau}(\Pi)$ and $Q'_\tau = g_{w\tau}(\Pi')$ for a word $\tau \in W$.

In the statements of the lemmas, we write $A \preceq B$ (resp. $A \succeq B$, resp. $A \simeq B$) if $A \le CB$ (resp. $A \ge B/C$, resp. $A/C \le B \le CA$) for some constant $C < \infty$ which may depend on $\mathcal{F}$ and $\mathcal{G}$, but does not depend on the length of the word w.

Lemma 4.1. *Let $x, y \in Q$. If $|x - y| \succeq d$, then $|F(x) - F(y)| \simeq d'$.*

Lemma 4.2. *Let $x \in Q$ and let v be the common vertex of two children of Q. If $|x - v| \simeq \lambda_\partial^k d$ for some $k \in \mathbb{N}$, then $|F(x) - F(v)| \simeq (\lambda'_\partial)^k d'$.*

Lemma 4.3. *Let Q_τ and Q_η be adjacent subpolyhedra in Q, not necessarily in the same generation. If $x \in Q_\tau$ and $y \in Q_\eta$ with $|x - y| \simeq \lambda_\partial^k d$ for some $k \in \mathbb{N}$, then $|F(x) - F(y)| \simeq (\lambda'_\partial)^k d'$.*

Proof of Lemma 4.1. Let $x, y \in Q$ with $|x - y| \ge cd$, for some positive constant c depending only on $\mathcal{F}$. Let n be the smallest index such that x and y are contained in nonadjacent nth generation subpolyhedra Q_τ and Q_η respectively. Then the parents $\widehat{Q_\tau}$ and $\widehat{Q_\eta}$ are either adjacent or identical. It follows that $|x - y| \le \operatorname{diam}\widehat{Q_\tau} + \operatorname{diam}\widehat{Q_\eta} \le 2\lambda_{\max}^{n-1} d$, and so $\lambda_{\max}^{n-1} \ge \frac{1}{2}c$.

Since $\mathcal{F}$ and $\mathcal{G}$ are isomorphic, Q'_τ and Q'_η are nonadjacent. This implies that $|F(x) - F(y)| \ge \delta\lambda_{\min}^{n-1} d' \succeq d'$. The reverse inequality $|F(x) - F(y)| \le d$ is trivial. □

Proof of Lemma 4.2. Let $x \in Q$, let v be the common vertex of two children of Q, and assume that $\lambda_\partial^k / cd \le |x - v| \le c\lambda_\partial^k d$ for some $k \in \mathbb{N}$. Let Q_τ, $\tau \in W_n$, be the smallest descendant of Q which contains x and v. Then

$$\min\{\delta, \kappa \sin(\theta/2)\} \operatorname{diam} Q_\tau \le |x - v| \le \operatorname{diam} Q_\tau$$

by Lemma 3.32. The boundary congruence assumption implies that $\operatorname{diam} Q_\tau = \lambda_\partial^n d$, whence $|k - n| \simeq 1$.

Since $\mathcal{F}$ and $\mathcal{G}$ are isomorphic, Q'_τ is the smallest descendant of Q' which contains $F(x)$ and $F(v)$. Applying Lemma 3.32 again yields

$$\min\{\delta', e(\Pi')\lambda'_{\min} \sin(\frac{\theta'}{2})\} \operatorname{diam} Q'_\tau \le |F(x) - F(v)| \le \operatorname{diam} Q'_\tau.$$

As before, $\operatorname{diam} Q'_\tau = (\lambda'_\partial)^n d$ and $|k-n| \simeq 1$. The proof is complete. $\square$

Proof of Lemma 4.3. Let Q_τ and Q_η be two adjacent subpolyhedra of Q with common vertex v, and let $x \in Q_\tau$ and $y \in Q_\eta$ satisfy $|x-y| \simeq \lambda_\partial^k d$. By Lemma 3.29, $|x-v| + |y-v| \simeq |x-y| \simeq \lambda_\partial^k d$. Assume as we may that $|y-v| \le |x-v| \simeq \lambda_\partial^k d$. By Lemma 4.2, $|F(y)-F(v)| \preceq (\lambda'_\partial)^k d' \simeq |F(x)-F(v)|$. Applying Lemma 3.29 again yields $|F(x)-F(y)| \simeq |F(x)-F(v)| + |F(y)-F(v)| \simeq (\lambda'_\partial)^k d'$ as desired. $\square$

Proof of Theorem 1.7. To prove that F is quasisymmetric, it suffices to show that there exists a constant C so that

$$x,y,z \in K, \quad \frac{1}{2} \le \frac{|x-y|}{|x-z|} \le 2 \Rightarrow \frac{1}{C} \le \frac{|F(x)-F(y)|}{|F(x)-F(z)|} \le C. \tag{4.4}$$

Indeed, assume that F satisfies (4.4) and assume that $x,y,z \in K$ satisfy $|x-y| \le |x-z|$. If in addition $\frac{1}{2}|x-z| \le |x-y|$ then (4.4) immediately yields

$$\frac{|F(x)-F(y)|}{|F(x)-F(z)|} \le C,$$

so assume that $|x-y| \le \frac{1}{2}|x-z|$. Then $|y-z| \ge \frac{1}{2}|x-z|$ and $|y-z| \le |y-x| + |x-z| \le 2|x-z|$ so

$$\frac{|F(x)-F(y)|}{|F(x)-F(z)|} \le 1 + \frac{|F(y)-F(z)|}{|F(x)-F(z)|} \le C+1$$

by (4.4). Thus we have shown that F is weakly quasisymmetric. Since K and K' are doubling and uniformly perfect (see Proposition 3.13), F is necessarily quasisymmetric. See [10, Theorem 10.19].

Let $x,y,z \in K$ with $\frac{1}{2}|x-z| \le |x-y| \le 2|x-z|$ and assume that Q is the smallest subpolyhedron of Π which contains x, y and z. Let Q_i be a child of Q containing x. (If x is in the critical set C then choose either of the children containing x.) Then all possible locations of x, y and z are covered by the following four cases (and the variation of (4) in which y and z are switched):

1. at least one of y and z is contained in a child of Q which is disjoint from Q_i;
2. $y \in Q_j$ and $z \in Q_k$ for two different children Q_j, Q_k of Q each of which is adjacent to Q_i;
3. y and z are elements of the same child Q_j of Q which is adjacent to Q_i;
4. y is an element of a child Q_j of Q which is adjacent to Q_i and $z \in Q_i$.

In each of these cases, we will show that the ratio $\frac{|F(x)-F(y)|}{|F(x)-F(z)|}$ is bounded away from zero and infinity.

Case (1) is the simplest. If, say, y is contained in a child disjoint from Q_i, then $|x-z| \ge \frac{1}{2}|x-y| \ge \frac{1}{2}\delta d$. Consequently $|F(x)-F(y)| \simeq d'$ and $|F(x)-F(z)| \simeq d'$ by Lemma 4.1.

For case (2), Lemma 3.30 yields $\max\{|x-y|,|x-z|\} \succeq d$. Then this case can be completed as in case (1).

In cases (3) and (4) x and y are in different subpolyhedra and all three points are in $Q_i \cup Q_j$. Let $Q_i \cap Q_j = \{v\}$. Choose integers k, l and m so that $|x-v| \simeq \lambda_\partial^k d$, $|y-v| \simeq \lambda_\partial^l d$ and $|z-v| \simeq \lambda_\partial^m d$. Then Lemma 4.2 (applied to $Q = Q_i$ and $Q = Q_j$) implies that $|F(x)-F(v)| \simeq (\lambda'_\partial)^k d'$, $|F(y)-F(v)| \simeq (\lambda'_\partial)^l d'$ and $|F(z)-F(v)| \simeq (\lambda'_\partial)^m d'$. We divide these two cases into further subcases (i), (ii) and (iii) according to which of the quantities $|x-v|$, $|y-v|$ or $|z-v|$ is the largest.

Case (i): Assume first that $|x-v|$ is the largest. Since $|x-v|+|y-v| \simeq |x-y|$ by Lemma 3.29, we have $|x-y| \simeq |x-v| \simeq \lambda_\partial^k a$. By Lemma 4.3, $|F(x)-F(y)| \simeq (\lambda'_\partial)^k d'$.

By assumption, $|x-z| \simeq |x-y| \simeq \lambda_\partial^k d$ as well. In case (3) (x and z are in different subpolyhedra Q_i and Q_j) we apply Lemma 4.3, while in case (4) (x and z are in the same subpolyhedron Q_i) we apply Lemma 4.1 on the smallest subpolyhedron of Q_i containing v, x and z. In either case we conclude that $|F(x)-F(z)| \simeq (\lambda'_\partial)^k d'$ as desired.

Case (ii): The case when $|y-v|$ is largest is handled by interchanging x and y in case (i).

Case (iii): Finally, we consider the case when $|z-v|$ is the largest. In case (4) we use Lemma 3.29 as before to conclude that $|z-y| \succeq \lambda_\partial^m d$. Thus $|z-x|+|x-y| \succeq \lambda_\partial^m d$ and since both terms on the left hand side are comparable, each in turn is $\succeq \lambda_\partial^m d$. Since all of the distances $|x-y|$, $|x-z|$ and $|y-z|$ are bounded above by this same quantity we conclude that

$$|x-y| \simeq |x-z| \simeq |y-z| \simeq |z-v| \simeq \lambda_\partial^m d.$$

We now apply Lemma 4.3 and Lemma 4.1 to the smallest subpolyhedron containing v, x and z as above to conclude that

$$|F(x)-F(z)| \simeq |F(y)-F(z)| \simeq (\lambda'_\partial)^m d'.$$

Finally, in case (3) we use Lemma 3.29 again to conclude that $|z-x| \succeq \lambda_\partial^m d$, whence $|y-x| \succeq \lambda_\partial^m d$ by the hypothesis. Applying Lemma 4.3 twice yields the same conclusion

$$|F(x)-F(z)| \simeq |F(y)-F(z)| \simeq (\lambda'_\partial)^m d'.$$

All cases have now been covered and the proof of Theorem 1.7 is complete. □

Example 4.5. The assumption of boundary congruence in Theorem 1.7 is necessary. To see this, consider the IFS's $\mathcal{F} = \{f_1, f_2\}$ and $\mathcal{G} = \{g_1, g_2\}$ on the real line, where $f_1(x) = x/2$, $f_2(x) = x/2 + 1/2$ and $g_1(x) = x/3$, $g_2(x) = x/3 + 2/3$. The invariant set for both of these IFS's is the unit line segment $[0,1]$, and the induced canonical homeomorphism $F = \pi_{\mathcal{G}} \circ \pi_{\mathcal{F}}^{-1}$ is not quasisymmetric.

5. Global quasiconformal extensions

In this section we study the question: when does the canonical homeomorphism between the invariant sets of two isomorphic IFS's $\mathcal{F}$ and $\mathcal{G}$ admit a global quasiconformal extension? The main result, Theorem 1.9, provides the existence of such an extension when $\mathcal{F}$ and $\mathcal{G}$ are boundary congruent gasket type systems which are joined by a nondegenerate isotopy of such systems.

Väisälä's theory of s-quasisymmetric maps and extension theorems [27] is crucial in our proof of Theorem 1.9. A map $f: X \to Y$ between metric spaces is said to be *s-quasisymmetric*, $s > 0$, if it is η-quasisymmetric for some increasing homeomorphism η satisfying $\eta(t) \le t + s$ for $0 \le t \le 1/s$. The notion of s-quasisymmetry provides a quantitative measure for the degree of "closeness" of a quasisymmetric map to the space of similarities.

The following theorem asserts that the canonical homeomorphism is s-quasisymmetric for some small s, provided the original systems are sufficiently close in the natural topology described in Remark 3.6.

Theorem 5.1. *Let $d \ge 1$ and $M \ge 2$. There exists a continuous function $\epsilon : \mathcal{IFS}(\mathbb{R}^d, M) \to (0,1)$ so that the following statement holds:*

Let $\mathcal{F}$ and $\mathcal{G}$ be iterated function systems, each of which is boundary congruent and of gasket type. Assume that $\mathcal{F}$ and $\mathcal{G}$ are isomorphic and $D(\mathcal{F},\mathcal{G}) < \epsilon(\mathcal{F})$. Then the canonical homeomorphism $F = \pi_{\mathcal{G}} \circ \pi_{\mathcal{F}}^{-1} : K(\mathcal{F}) \to K(\mathcal{G})$ is s-quasisymmetric with $s \to 0$ as $D(\mathcal{F},\mathcal{G}) \to 0$.

Here D denotes the metric on $\mathcal{IFS}(\mathbb{R}^d, M)$ defined in (3.7).

We may choose

$$\epsilon(\mathcal{F}) = \frac{1}{5000000}\delta\kappa \sin^2(\theta/2)\lambda_{\min}^8(1-\lambda_{\max})^3, \tag{5.2}$$

where $\lambda_{\min}$, $\lambda_{\max}$, δ, θ, and κ are the values defined in (3.25)–(3.28) for $\mathcal{F}$. Observe that $\epsilon(\mathcal{F})$ is a continuous function of $\mathcal{F}$.

Theorem 5.3 (Väisälä). *Let Λ be a p-dimensional affine subspace of $\mathbb{R}^d$ and let A be a compact, thick subset of Λ. There exists $s_0 > 0$ such that if $0 \le s \le s_0$, then each s-quasisymmetric map $F : A \to F(A) \subset \mathbb{R}^d$ admits an s_1-quasisymmetric extension $\tilde{F} : \mathbb{R}^d \to \mathbb{R}^d$.*

This is Theorem 6.2 of [27]. Here, for a p-dimensional affine subspace $\Lambda \subset \mathbb{R}^d$, we say that a set $A \subset \Lambda$ is *thick in* Λ if there exist constants $r_0 > 0$ and $\beta > 0$ such that for any $a \in A$ and any $0 < r \le r_0$, there is a simplex $\Delta \subset \Lambda$ whose vertices lie in $A \cap \overline{B(a,r)}$ and which has p-volume $|\Delta|_p \ge \beta r^p$. See [27, §6.1].

Lemma 5.4. *Let K be the invariant set for an iterated function system $\mathcal{F}$ in $\mathbb{R}^d$, and assume that* $\operatorname{diam} K > 0$*. Then K is thick in Λ, where Λ denotes the affine subspace of $\mathbb{R}^d$ spanned by K.*

Proof. Without loss of generality we may assume that $\operatorname{diam} K = 1$. Fix a specific simplex Δ with vertices in K and p-volume $|\Delta|_p > 0$. We will show that K is thick in Λ with parameters $r_0 = 1$ and $\beta = \lambda_{\min}^p |\Delta|_p$.

Fix $x \in K$ and $0 < r \leq 1$. By the proof of Proposition 3.13, there exists $w \in W$ with $K_w \subset \overline{B(x,r)}$ and $\operatorname{diam} K_w \geq \lambda_{\min} r$. The simplex $\Delta_w = f_w(\Delta)$ has vertices in K_w and p-volume $|\Delta|_p (\operatorname{diam} K_w)^p$. Then $|\Delta_w|_p \geq \beta r^p$ as desired. □

Remark 5.5. When $\{\mathcal{F}^t\}$ is a nondegenerate isotopy of isomorphic systems, there exist simplices $\Delta_t \subset \Lambda_t$ with vertices in $K(\mathcal{F}^t)$ and $\inf_{0 \leq t \leq 1} |\Delta_t|_p > 0$. To see this, observe that $p + 1 \leq M$ (since Λ_t is contained in the span of the M fixed points of the maps in $\mathcal{F}$) and for each t,

$$\max_{\{i_1,\ldots,i_{p+1}\} \subset A} |\operatorname{co}(a^t_{i_1}, \ldots, a^t_{i_{p+1}})|_p > 0, \tag{5.6}$$

where co denotes convex hull. For each fixed set $\{i_1, \ldots, i_{p+1}\} \subset A$, the p-volume of $\operatorname{co}(a^t_{i_1}, \ldots, a^t_{i_{p+1}})$ is a continuous function of t. Since the maximum of finitely many continuous functions is again continuous, the expression in (5.6) is continuous in $t \in [0,1]$, and thus has a positive lower bound independent of t.

In this case, the proof of Lemma 5.4 shows the following stronger statement: when $\{\mathcal{F}^t\}$ is a nondegenerate isotopy of isomorphic systems, the invariant sets $K(\mathcal{F}^t)$ are all thick, with parameters r_0 and β independent of t. Therefore the value of s_0 in Theorem 5.3 can be chosen independent of t, when the theorem is applied to $A = \mathcal{F}^t$, $0 \leq t \leq 1$.

Assuming temporarily the validity of Theorem 5.1, we now give the proof of Theorem 1.9.

Proof of Theorem 1.9. Let $\{\mathcal{F}^t\}_{0 \leq t \leq 1}$ be a nondegenerate isotopy of boundary congruent, gasket type IFS's joining $\mathcal{F} = \mathcal{F}^0$ to $\mathcal{G} = \mathcal{F}^1$.

Since the maps $t \mapsto \mathcal{F}^t$ from $[0,1]$ to $\mathcal{IFS}(\mathbb{R}^d, M)$ and $\mathcal{F} \mapsto \epsilon(\mathcal{F})$ from $\mathcal{IFS}(\mathbb{R}^d, M)$ to $(0,\infty)$ are continuous, $\epsilon_0 := \inf_t \epsilon(\mathcal{F}^t) > 0$. For each $\epsilon < \epsilon_0$, choose $N = N(\epsilon)$ so that $D(\mathcal{F}^a, \mathcal{F}^b) < \epsilon$ for all $0 \leq a < b \leq 1$ with

$$b - a \leq \frac{1}{N}. \tag{5.7}$$

(Later in the proof we will fix a particular choice of ϵ, but until then the proof is valid for all $\epsilon < \epsilon_0$.) We abbreviate $K(t) = K(\mathcal{F}^t)$. By Theorem 5.1, for any a and b satisfying (5.7) the map $F_{a,b} : K(a) \to K(b)$ is $s(\epsilon)$-quasisymmetric with $s(\epsilon) \to 0$ as $\epsilon \to 0$.

Combining Theorem 5.3 with Lemma 5.4 and Remark 5.5, we conclude that each of the maps $F_{a,b} : K(a) \to K(b) \subset \mathbb{R}^d$ extends to an $s_1(\epsilon)$-quasisymmetric map $\widetilde{F_{a,b}} : \mathbb{R}^d \to \mathbb{R}^d$. Moreover, $s_1(\epsilon) \to 0$ as $\epsilon \to 0$.

We are now in a position to fix the parameter ϵ. Choose $0 < \epsilon < \epsilon_0$ so that $s(\epsilon) \leq s_0$ (the value from Theorem 5.3) and $s_1(\epsilon) \leq 1$. Now the index $N = N(\epsilon)$ is also fixed.

To complete the proof of Theorem 1.9, apply the previous discussion to each of the maps $F_{(j-1)/N,j/N} : K((j-1)/N) \to K(j/N)$, $j = 1, \dots, N$, to deduce the existence of $s_1(\epsilon)$-quasisymmetric extensions of these maps to $\mathbb{R}^d$. As before, denote these extensions by $\widetilde{F_{(j-1)/N,j/N}}$. Every s-quasisymmetric self-map of $\mathbb{R}^d$, $s \le 1$, is $(1+s)$-quasiconformal. Hence the composition $\tilde{F} := \widetilde{F_{0,1/N}} \circ \cdots \circ \widetilde{F_{(N-1)/N,1}}$ is $(1+s_1(\epsilon))^N$-quasiconformal. This completes the proof. □

Remark 5.8. The proof shows that the extension $\tilde{F}$ is $1+o(1)$-quasiconformal when $D(\mathcal{F},\mathcal{G}) \to 0$. Indeed, note that when $D(\mathcal{F},\mathcal{G}) < \epsilon$, then $N = 1$ and the extension is $1 + s_1(\epsilon)$-quasiconformal.

To prove Theorem 5.1 we will make use of the following result which gives a sufficient condition for a function to be s-quasisymmmetric.

Theorem 5.9 (Väisälä). *Let $X \subset \mathbb{R}^p$ be c-uniformly perfect and let $0 < \varkappa \le \frac{1}{144}c^4$. Let $f : X \to f(X) \subset \mathbb{R}^d$ be a map such that for every bounded $A \subset X$ there is a similarity embedding $h : \mathbb{R}^p \hookrightarrow \mathbb{R}^d$ with scale factor λ_h for which $||h - f||_{L^\infty(A)} \le \varkappa\lambda_h \operatorname{diam} A$. Then f is s-quasisymmetric, where $s = s(\varkappa) \to 0$ as $\varkappa \to 0$.*

This theorem is due to Väisälä. In [27, Theorem 3.9] it is proved for connected sets X. The extension to uniformly perfect sets requires only minor modifications; we include this extension here to show the dependence of the constants.

Proof of Theorem 5.9. Following the proof from [27], we deduce that f is injective and satisfies $t' \le t + 4\varkappa(1+t)^2$ for any $a, b, x \in X$, where $t' = |f(a) - f(x)|/|f(b) - f(x)|$ and $t = |a - x|/|b - x|$. If $t \le \varkappa^{-1/4}$ then $t' \le t + 9\varkappa^{1/2}$. Hence if f is quasisymmetric, then it is s-quasisymmetric with $s = s(\varkappa) = \max\{\varkappa^{1/4}, 9\varkappa^{1/2}\}$.

To show that f is quasisymmetric, it suffices to verify the conditions in Theorem 3.10 of [24]. Set $h = 2/c$ and $H = 4/c$. Since X is c-uniformly perfect, it is $(\frac{1}{2}c, \frac{1}{2})$-homogeneously dense. If $t \le 2/c$ then $t \le \varkappa^{-1/4}$ and $t' \le t + 9\varkappa^{1/2} < 4/c$. If $t \le c/4$ then $t' \le c/4 + (c^4/36)(1 + c/4)^2 \le c/2$. The quasisymmetry of f follows. □

The following technical proposition plays a key role in the proof of Theorem 5.1. For ease of exposition, we have deferred the proof of this proposition to the end of this section.

Proposition 5.10. *Let $\mathcal{F}$ and $\mathcal{G}$ be iterated function systems, each of which is boundary congruent and of gasket type. Assume that $\mathcal{F}$ and $\mathcal{G}$ are isomorphic and $D(\mathcal{F},\mathcal{G}) \le \frac{1}{2}\min\{\lambda_{min}, 1 - \lambda_{max}\}$, where $\lambda_{min} = \lambda_{min}(\mathcal{F})$ and $\lambda_{max} = \lambda_{max}(\mathcal{F})$. Assume also that $\operatorname{diam} K = 1$ and $0 \in K$, where $K = K(\mathcal{F})$. Let $i, j, k, l \in A$ satisfy $\pi_{\mathcal{F}}(i\overline{k}) = \pi_{\mathcal{F}}(j\overline{l})$. Let $n \in \mathbb{N}$ be arbitrary and set $\tau = i\underbrace{k\cdots k}_{n}$ and $\eta = j\underbrace{l\cdots l}_{n}$.*

Then

$$|| \operatorname{id} - \Phi ||_{L^\infty(\Pi(\mathcal{F}))} \leq C D(\mathcal{F}, \mathcal{G}),$$

where $\Phi = g_\eta^{-1} \circ g_\tau \circ f_\tau^{-1} \circ f_\eta$ *and*

$$C = \frac{500}{\lambda_{min}(1 - \lambda_{max})^3}. \tag{5.11}$$

Corollary 5.12. *With hypotheses as in Proposition 5.10, if* $D(\mathcal{F}, \mathcal{G}) < \epsilon(\mathcal{F})$*, then*

$$|| \operatorname{id} - \Phi ||_{L^\infty(\Pi(\mathcal{F}))} \leq \frac{1}{10000} \delta \kappa \sin^2(\theta/2) \lambda_{min}^7. \tag{5.13}$$

Let $\mathcal{F}$ and $\mathcal{G}$ be isomorphic iterated function systems with $D(\mathcal{F}, \mathcal{G}) \leq \epsilon(\mathcal{F})$. For future reference we record the estimates

$$\lambda_{\max}(\mathcal{G}) \leq \epsilon(\mathcal{F}) + \lambda_{\max} \leq \frac{1}{2}(1 + \lambda_{\max}) \tag{5.14}$$

and

$$\lambda_{\min}(\mathcal{G}) \geq \lambda_{\min} - \epsilon(\mathcal{F}) \geq \frac{1}{2} \lambda_{\min} \tag{5.15}$$

which follow from the inequalities $\epsilon(\mathcal{F}) \leq \frac{1}{2}\lambda_{\min}$ and $\epsilon(\mathcal{F}) \leq \frac{1}{2}(1 - \lambda_{\max})$.

Proof of Theorem 5.1. Let $\epsilon : \mathcal{IFS}(\mathbb{R}^d, M) \to (0, \infty)$ be the function given in (5.2). Let $\mathcal{F} = \{f_1, \dots, f_M\}$ and $\mathcal{G} = \{g_1, \dots, g_M\}$ be isomorphic IFS's as in the statement of the theorem. By Proposition 3.13, $K = K(\mathcal{F})$ is c-uniformly perfect with $c = \frac{1}{2}\lambda_{\min}$. Since the theorem is invariant under similarity mappings of $\mathbb{R}^d$, we may assume that $\operatorname{diam} K = 1$ and $0 \in K$. The conclusion will follow after we verify the assumptions of Theorem 5.9 for the canonical homeomorphism $F : K \to K(\mathcal{G}) \subset \mathbb{R}^d$ with

$$\varkappa = \frac{1}{144} c^4 = \frac{1}{2304} \lambda_{\min}^4. \tag{5.16}$$

Let A be a subset of K.

Case 1 ($A = K$): Since $K \subset \overline{B(0,1)}$ and $D(\mathcal{F}, \mathcal{G}) < \epsilon(\mathcal{F})$,

$$\max_i ||f_i - g_i||_{L^\infty(K)} \leq 3\epsilon(\mathcal{F}). \tag{5.17}$$

We will verify the hypothesis of Theorem 5.9 with h equal to the identity. Let $x \in K$ and choose $w = w_1 w_2 \cdots \in \Sigma$ with $\pi_{\mathcal{F}}(w) = x$. Then $\pi_{\mathcal{G}}(w) = F(x)$ and

$$\begin{aligned} |x - F(x)| &= |\pi_{\mathcal{F}}(w) - \pi_{\mathcal{G}}(w)| \\ &= \lim_{m \to \infty} |f_{w_1} \circ f_{w_2} \circ \cdots \circ f_{w_m}(x_0) - g_{w_1} \circ g_{w_2} \circ \cdots \circ g_{w_m}(x_0)| \end{aligned}$$

by (3.10). By the triangle inequality and (5.17),

$$|x - F(x)| \leq 3\epsilon(\mathcal{F}) + \lambda_{g_{w_1}} \lim_{m \to \infty} |f_{w_2} \circ \cdots \circ f_{w_m}(x_0) - g_{w_2} \circ \cdots \circ g_{w_m}(x_0)|.$$

Iterating yields

$$\begin{aligned}|x - F(x)| &\le 3\epsilon(\mathcal{F})(1 + \lambda_{g_{w_1}} + \lambda_{g_{w_1}}\lambda_{g_{w_2}} + \cdots) \\ &\le \frac{3}{1-\lambda_{\max}(\mathcal{G})}\epsilon(\mathcal{F}) \le \frac{6}{1-\lambda_{\max}}\epsilon(\mathcal{F}) \le \varkappa\end{aligned}$$

by the choice of $\epsilon(\mathcal{F})$ in (5.2) and (5.14).

Thus the hypothesis in Theorem 5.9 is verified in this case.

Case 2 ($A = K_w$ for some $w \in W$): The canonical homeomorphism $F : K(\mathcal{F}) \to K(\mathcal{G})$ verifies the identity

$$F|\, K_w = g_w \circ F \circ f_w^{-1}.$$

By the previous case, $||\operatorname{id} - F||_{L^\infty(K)} \le \varkappa \operatorname{diam} K$. Choose $h = g_w \circ f_w^{-1}$. The proof of this case is then finished by the calculation

$$||h - F||_{L^\infty(K_w)} = \lambda_{g_w} ||\operatorname{id} - F||_{L^\infty(K)} \le \lambda_{g_w}\varkappa \operatorname{diam} K = \varkappa\lambda_h \operatorname{diam} K_w.$$

Case 3 (general A): Let Π_w be the smallest subpolyhedron of Π containing A. If A meets three distinct children of Π_w then

$$\operatorname{diam} A \ge \min\{\delta, \frac{1}{2}\kappa\sin(\theta/2)\} \operatorname{diam}\Pi_w \tag{5.18}$$

by Lemma 3.30. Also if A meets two nonadjacent children of Π_w, then $\operatorname{diam} A \ge \delta \operatorname{diam}\Pi_w$ and so again (5.18) holds. Choosing $h = g_w \circ f_w^{-1}$ as in the previous case and applying that case, we conclude

$$\begin{aligned}||h - F||_{L^\infty(A)} \le ||h - F||_{L^\infty(K_w)} &\le \frac{6}{1-\lambda_{\max}}\epsilon(\mathcal{F})\lambda_h \operatorname{diam} K_w \\ &\le \frac{6}{1-\lambda_{\max}} \cdot \frac{1}{\frac{1}{2}\kappa\delta\sin(\theta/2)}\epsilon(\mathcal{F})\lambda_h \operatorname{diam} A \\ &\le \varkappa\lambda_h \operatorname{diam} A\end{aligned}$$

by the choice of $\epsilon(\mathcal{F})$.

The final (and most difficult) case occurs when A is contained entirely within two adjacent children Π_{wi} and Π_{wj} of Π_w. As before, let $\Pi_{wi} \cap \Pi_{wj} = \{v\}$. We consider the smallest pair of subpolyhedra $\Pi_{w\tau}$ and $\Pi_{w\eta}$ of Π_w satisfying the following conditions:

(i) $\Pi_{w\tau}$ is contained in Π_{wi} and $\Pi_{w\eta}$ is contained in Π_{wj},
(ii) $\Pi_{w\tau} \cap \Pi_{w\eta} = \{v\}$,
(iii) $A \subset \Pi_{w\tau} \cup \Pi_{w\eta}$, and
(iv) τ and η have equal length.

The boundary congruence of $\mathcal{F}$ implies that

$$\lambda_{\min} \operatorname{diam}\Pi_{w\eta} \le \operatorname{diam}\Pi_{w\tau} \le \frac{1}{\lambda_{\min}} \operatorname{diam}\Pi_{w\eta}. \tag{5.19}$$

Also, a combination of Lemmas 3.29 and 3.32 and the choice of $\Pi_{w\tau}$ and $\Pi_{w\eta}$ reveal that

$$\begin{aligned}\operatorname{diam} A &\geq \sin(\theta/2)\min\{\delta, \kappa\sin(\theta/2)\}\min\{\operatorname{diam}\Pi_{w\tau}, \operatorname{diam}\Pi_{w\eta}\} \\ &\geq \delta\kappa\sin^2(\theta/2)\lambda_{\min}\operatorname{diam}\Pi_{w\tau}\end{aligned} \tag{5.20}$$

by (5.19).

We claim that the similarity $h = g_{w\tau} \circ f_{w\tau}^{-1}$ verifies the condition in Theorem 5.9. For the part of A which lies within $\Pi_{w\tau}$ this is immediate:

$$\begin{aligned}||h - F||_{L^\infty(A\cap\Pi_{w\tau})} &\leq ||h - F||_{L^\infty(K_{w\tau})} \\ &\leq \frac{6}{1-\lambda_{\max}}\epsilon(\mathcal{F})\lambda_h\operatorname{diam}\Pi_{w\tau} \\ &\leq \frac{6}{1-\lambda_{\max}}\cdot\frac{1}{\delta\kappa\sin^2(\theta/2)\lambda_{\min}}\epsilon(\mathcal{F})\lambda_h\operatorname{diam}A \\ &\leq \varkappa\lambda_h\operatorname{diam}A\end{aligned} \tag{5.21}$$

by (5.20). For the part of A which lies within $\Pi_{w\eta}$ we proceed in two steps. Let $h_1 = g_{w\eta} \circ f_{w\eta}^{-1}$. We will show that

$$||h_1 - F||_{L^\infty(A\cap\Pi_{w\eta})} \leq \frac{1}{2}\varkappa\lambda_h\operatorname{diam}A \tag{5.22}$$

and

$$||h_1 - h||_{L^\infty(A\cap\Pi_{w\eta})} \leq \frac{1}{2}\varkappa\lambda_h\operatorname{diam}A. \tag{5.23}$$

The proof of (5.22) is similar to that of (5.21); we use

$$\operatorname{diam}A \geq \delta\kappa\sin^2(\theta/2)\lambda_{\min}\operatorname{diam}\Pi_{w\eta}$$

in place of (5.20) as well as the estimate

$$\lambda_h^{-1}\lambda_{h_1} = \frac{\lambda'_w\lambda'_\eta\lambda_w\lambda_\tau}{\lambda_w\lambda_\eta\lambda'_w\lambda'_\tau} = \frac{\lambda'_j\lambda_i}{\lambda_j\lambda'_i} \leq \frac{\lambda_{\max}\,\lambda_{\max}(\mathcal{G})}{\lambda_{\min}\,\lambda_{\min}(\mathcal{G})} \leq \frac{2}{\lambda_{\min}^2}$$

which follows from (5.15).

To complete the proof it suffices to verify (5.23). In fact we will prove that

$$||h_1 - h||_{L^\infty(\Pi_{w\eta})} \leq \frac{1}{2}\varkappa\cdot\delta\kappa\sin^2(\theta/2)\lambda_{\min}\cdot\lambda_h\operatorname{diam}\Pi_{w\eta}; \tag{5.24}$$

observe that (5.24) and (5.20) together imply (5.23). Using the definitions of h and h_1 we reduce (5.24) to the statement

$$||g_\eta \circ f_\eta^{-1} - g_\tau \circ f_\tau^{-1}||_{L^\infty(\Pi_\eta)} \leq \frac{1}{2}\varkappa\cdot\delta\kappa\sin^2(\theta/2)\lambda_{\min}\cdot\frac{\lambda_h\operatorname{diam}\Pi_{w\eta}}{\lambda'_w}. \tag{5.25}$$

By the choice of $\Pi_{w\tau}$ and $\Pi_{w\eta}$ we have

$$\tau = i\underbrace{k\cdots k}_{n} \qquad \text{and} \qquad \eta = j\underbrace{l\cdots l}_{n}$$

for some $n \in \mathbb{N}$. Setting $\Phi = g_\eta^{-1} \circ g_\tau \circ f_\tau^{-1} \circ f_\eta$ as in Proposition 5.10, we see that (5.25) in turn is equivalent to

$$|| \operatorname{id} - \Phi||_{L^\infty(\Pi)} \le \frac{1}{2}\varkappa \cdot \delta\kappa \sin^2(\theta/2)\lambda_{\min} \cdot \frac{\lambda_h \operatorname{diam} \Pi_{w\eta}}{\lambda'_w \lambda'_\eta}. \tag{5.26}$$

Observe that

$$\frac{\lambda_h \operatorname{diam} \Pi_{w\eta}}{\lambda'_w \lambda'_\eta} = \frac{\lambda'_\tau \lambda_\eta}{\lambda_\tau \lambda'_\eta} \ge \lambda_{\min}\lambda_{\min}(\mathcal{G}) \ge \frac{1}{2}\lambda^2_{\min}.$$

Thus it suffices to verify the inequality

$$\begin{aligned} || \operatorname{id} - \Phi||_{L^\infty(\Pi)} &\le \frac{1}{4}\varkappa \cdot \delta\kappa \sin^2(\theta/2)\lambda^3_{\min} \\ &= \frac{1}{9216}\delta\kappa \sin^2(\theta/2)\lambda^7_{\min}. \end{aligned} \tag{5.27}$$

By the choice of $\varkappa$ in (5.16), (5.27) follows from (5.13) which was the conclusion of Corollary 5.12. The proof of Theorem 5.1 is complete. □

To conclude this section we give the proof of Proposition 5.10.

Assume that the maps in $\mathcal{G}$ are ordered so that $D(\mathcal{F},\mathcal{G}) = \max_{i \in A} D(f_i, g_i)$. Recall that $D(f,g) = |\lambda - \lambda'| + |a - a'| + ||A - A'||$, where $f = (I - \lambda A)a + \lambda A$ and $g = (I - \lambda' A')a' + \lambda' A'$. The proof of the proposition consists in estimating $|z - \Phi(z)|$, for $z \in K$, by a series of terms involving the data λ_i, a_i, A_i and λ'_i, a'_i, A'_i. To this end we first collect several lemmas providing estimates for terms of this form. In all of these lemmas, we assume the hypotheses of Proposition 5.10.

To ease notation we write $D = D(\mathcal{F},\mathcal{G})$.

In our first lemma, we estimate the operator norm of $B_\tau - B'_\tau$, where $B_\tau := I_d - \lambda_\tau A_\tau$ and $B'_\tau := I_d - \lambda'_\tau A'_\tau$ and τ is as in the hypotheses of Proposition 5.10.

Lemma 5.28. $||B_\tau - B'_\tau|| \le \frac{2}{1-\lambda_{max}} D$.

Proof. Since $\lambda_\tau = \lambda_i \lambda_\partial^n$, $A_\tau = A_i$, etc. (recall that boundary mappings have trivial rotation matrix A_∂), we find

$$\begin{aligned} ||B_\tau - B'_\tau|| &= ||\lambda_\tau A_\tau - \lambda'_\tau A'_\tau|| \\ &\le (n+2) \max\{\lambda_{\max}, \lambda_{\max}(\mathcal{G})\}^n D \\ &\le 2(n+1)(\frac{1+\lambda_{\max}}{2})^n D \le \frac{2}{1-\lambda_{\max}} D \end{aligned}$$

by repeated applications of the triangle inequality. Here we used the elementary estimate $2(n+1)x^n \le 1/(1-x)$, $n \ge 1$, $0 < x < 1$. □

Remark 5.29. In a similar manner, we can prove $||B_w - B'_w|| \le \frac{2}{1-\lambda_{\max}} D$ for any $w \in W$, where $B_w = I_d - \lambda_w A_w$, etc.

Since Φ is a similarity, $\Phi = \Phi(0) + \lambda_\Phi A_\Phi$ for some $\lambda_\Phi > 0$ and $A_\Phi \in O(d)$. Our next lemma estimates the operator norm of $B_\Phi := I_d - \lambda_\Phi A_\Phi$.

Lemma 5.30. $||B_\Phi|| \le \frac{10}{\lambda_{min}} D$.

Proof. Since the contraction ratio and rotation matrix for a composition of similarities are the product of the corresponding quantities for the composands,

$$\lambda_\Phi = (\lambda'_\eta)^{-1}\lambda'_\tau\lambda_\tau^{-1}\lambda_\eta = \frac{\lambda_j\lambda'_i}{\lambda'_j\lambda_i} \tag{5.31}$$

and

$$A_\Phi = (A'_\eta)^{-1}\cdot A'_\tau \cdot A_\tau^{-1}\cdot A_\eta = (A'_j)^{-1}\cdot A'_i\cdot A_i^{-1}\cdot A_j. \tag{5.32}$$

Here we have used the boundary congruence to observe that the contributions to λ_Φ from the boundary similarities f_k, f_l, g_k, g_l vanish. Observe that $\lambda_k = \lambda_l = \lambda_\partial$ and $\lambda'_k = \lambda'_l = \lambda'_\partial$.

From (5.31) and (5.32) we estimate

$$\begin{aligned}\lambda_\Phi &\le \left(1 + \frac{|\lambda_i - \lambda'_i|}{\lambda_i}\right)\left(1 - \frac{|\lambda_j - \lambda'_j|}{\lambda_j}\right)^{-1}\\ &\le \left(1+\frac{D}{\lambda_{\min}}\right)\left(1 - \frac{D}{\lambda_{\min}}\right)^{-1} \le 1 + \frac{4}{\lambda_{\min}}D \le 3.\end{aligned}$$

and

$$||I_d - A_\Phi|| = ||A'_j\cdot A_j^{-1} - A'_i\cdot A_i^{-1}|| \le ||A_j - A'_j|| + ||A_i - A'_i|| \le 2D.$$

Combining these inequalities gives

$$||B_\Phi|| \le |\lambda_\Phi - 1| + \lambda_\Phi||I_d - A_\Phi|| \le \frac{10}{\lambda_{\min}}D.$$

□

Next, we estimate $|v - v'|$, where $v = f_i(a_k) = f_j(a_l)$ and $v' = g_i(a'_k) = g_j(a'_l)$.

Lemma 5.33. $|v - v'| \le 7D$.

Proof. This is an easy consequence of the representations $v = B_i a_i + \lambda_i A_i a_k$ and $v' = B'_i a'_i + \lambda'_i A'_i a'_k$, where $B_i := I_d - \lambda_i A_i$ and $B'_i := I_d - \lambda'_i A'_i$. □

For our final lemma, we estimate the distance between the fixed points a_τ and a'_τ for f_τ and g_τ.

Lemma 5.34. $|a_\tau - a'_\tau| \le \frac{36}{(1-\lambda_{max})^3} D$.

Proof. The fixed point for $f_\tau = f_i \circ f_k^{(n)}$ is

$$a_\tau = B_\tau^{-1}\cdot(B_i a_i + (B_\tau - B_i)a_k).$$

Thus

$$|a_\tau - a'_\tau| \le |B_\tau^{-1}B_i a_i - (B'_\tau)^{-1}B'_i a'_i| + |B_\tau^{-1}(B_\tau - B_i)a_k - (B'_\tau)^{-1}(B'_\tau - B'_i)a'_k|.$$

In the first term, we use the triangle inequality together with the estimates $||B_i|| \le 2$, $||B_i'|| \le 2$, $||B_i - B_i'|| \le 2D$ to deduce

$$|B_\tau^{-1}B_ia_i - (B_\tau')^{-1}B_i'a_i'| \le 2 \cdot ||B_\tau^{-1} - (B_\tau')^{-1}|| + ||(B_\tau')^{-1}|| \cdot 4D.$$

In a similar manner, we use the above estimates together with $||B_\tau - B_i|| \le 2$, $||B_\tau' - B_i'|| \le 2$ and Lemma 5.28 to control the second term:

$$\begin{aligned}
&|B_\tau^{-1}(B_\tau - B_i)a_k - (B_\tau')^{-1}(B_\tau' - B_i')a_k'| \\
&\quad \le ||B_\tau^{-1} - (B_\tau')^{-1}|| \cdot ||B_\tau - B_i|| + ||(B_\tau')^{-1}|| \cdot \big(||B_\tau - B_\tau'|| + ||B_i - B_i'|| + ||B_\tau' - B_i'||D\big) \\
&\quad \le 2 \cdot ||B_\tau^{-1} - (B_\tau')^{-1}|| + ||(B_\tau')^{-1}|| \cdot \frac{6}{1-\lambda_{\max}}D.
\end{aligned}$$

Finally, we use Lemma 5.28 and (5.14) to get

$$\begin{aligned}
||B_\tau^{-1} - (B_\tau')^{-1}|| &\le ||B_\tau^{-1}|| \cdot ||B_\tau' - B_\tau|| \cdot ||(B_\tau')^{-1}|| \\
&\le \frac{2}{(1-\lambda_{\max})^2(1-\lambda_{\max}(\mathcal{G}))}D \le \frac{4}{(1-\lambda_{\max})^3}D
\end{aligned}$$

and

$$||(B_\tau')^{-1}|| \le \frac{1}{1-\lambda_{\max}(\mathcal{G})} \le \frac{2}{1-\lambda_{\max}}.$$

Combining these estimates gives the desired conclusion. □

Remark 5.35. A similar proof shows that $|a_w - a_w'| \le CD$ for any $w \in W$, where a_w and a_w' denote the fixed points for f_w and g_w, respectively and C denotes a constant depending only on $\lambda_{\min}$ and $\lambda_{\max}$. Since the collection of fixed points for the maps f_w, $w \in W$, is dense in K, it follows that

$$||\pi_{\mathcal{F}} - \pi_{\mathcal{G}}||_{L^\infty(\Sigma)} \le CD(\mathcal{F}, \mathcal{G})$$

for any isomorphic, boundary congruent, gasket type IFS's $\mathcal{F}$ and $\mathcal{G}$.

Proof of Proposition 5.10. For any $z \in \Pi = \Pi(\mathcal{F})$, we have

$$|z - \Phi(z)| \le |\Phi(0)| + |B_\Phi(z)| \le |\Phi(0)| + ||B_\Phi||. \tag{5.36}$$

Since the second term was bounded in Lemma 5.30, it suffices to provide an estimate for $|\Phi(0)|$. To this end, we calculate

$$\begin{aligned}
\Phi(0) &= (I_d - \frac{1}{\lambda_\eta'}A_\eta'^{-1})a_\eta' + \frac{1}{\lambda_\eta'}A_\eta'^{-1}(I_d - \lambda_\tau' A_\tau')a_\tau' \\
&\quad + \frac{\lambda_\tau'}{\lambda_\eta'}A_\eta'^{-1}A_\tau'(I_d - \frac{1}{\lambda_\tau}A_\tau^{-1})a_\tau + \frac{\lambda_\tau'}{\lambda_\eta'\lambda_\tau}A_\eta'^{-1}A_\tau'A_\tau^{-1}(I_d - \lambda_\eta A_\eta)a_\eta \\
&= -\frac{1}{\lambda_\eta'}A_\eta'^{-1}B_\eta'(a_\eta' - v') + \frac{1}{\lambda_\eta'}A_\eta'^{-1}B_\tau'(a_\tau' - v') \\
&\quad - \frac{\lambda_\tau'}{\lambda_\eta'\lambda_\tau}A_\eta'^{-1}A_\tau'A_\tau^{-1}B_\tau(a_\tau - v) + \frac{\lambda_\tau'}{\lambda_\eta'\lambda_\tau}A_\eta'^{-1}A_\tau'A_\tau^{-1}B_\eta(a_\eta - v) \\
&\quad + (I_d - \frac{\lambda_\tau'}{\lambda_\eta'}(A_\eta')^{-1}A_\tau')(v' - v) + B_\Phi(v)
\end{aligned}$$

Since $v = f_\tau(a_k) = f_\eta(a_l)$ we see that $a_\tau - v = \lambda_\tau A_\tau(a_\tau - a_k)$, etc. Hence

$$\begin{aligned}\Phi(0) = &-B'_\eta(a'_\eta - a'_l) + \frac{\lambda'_\tau}{\lambda'_\eta} A'^{-1}_\eta A'_\tau B'_\tau(a'_\tau - a'_k) - \frac{\lambda'_\tau}{\lambda'_\eta} A'^{-1}_\eta A'_\tau B_\tau(a_\tau - a_k) \\ &+ \lambda_\Phi A_\Phi B_\eta(a_\eta - a_l) + (I_d - \frac{\lambda'_\tau}{\lambda'_\eta} A'^{-1}_\eta A'_\tau)(v' - v) + B_\Phi(v) \\ = &-B'_\eta(a'_\eta - a_\eta + a_l - a'_l) + \frac{\lambda'_\tau}{\lambda'_\eta} A'^{-1}_\eta A'_\tau B'_\tau(a'_\tau - a_\tau + a_k - a'_k) \\ &+ \frac{\lambda'_\tau}{\lambda'_\eta} A'^{-1}_\eta A'_\tau (B'_\tau - B_\tau)(a_\tau - a_k) + \left(B_\eta - B'_\eta - B_\eta B_\Phi\right)(a_\eta - a_l) \\ &+ (I_d - \frac{\lambda'_\tau}{\lambda'_\eta} A'^{-1}_\eta A'_\tau)(v' - v) + B_\Phi(v) \\ = &\, I + II + III + IV + V + VI.\end{aligned}$$

Using the lemmas, we estimate

$$|I| \le 2(|a_\eta - a'_\eta| + |a_l - a'_l|) \le \frac{74}{(1-\lambda_{\max})^3} D,$$

$$|II| \le \frac{2}{\lambda'_{\min}}(|a_\tau - a'_\tau| + |a_k - a'_k|) \le \frac{296}{\lambda_{\min}(1-\lambda_{\max})^3} D,$$

$$|III| \le \frac{1}{\lambda'_{\min}} ||B_\tau - B'_\tau|| \le \frac{4}{\lambda_{\min}(1-\lambda_{\max})} D,$$

$$|IV| \le ||B_\eta - B'_\eta|| + 2||B_\Phi|| \le \frac{22}{\lambda_{\min}(1-\lambda_{\max})} D,$$

$$|V| \le \left(1 + \frac{1}{\lambda'_{\min}}\right)|v' - v| \le \frac{21}{\lambda_{\min}} D,$$

and

$$|VI| \le ||B_\Phi|| \le \frac{10}{\lambda_{\min}} D.$$

Thus

$$|\Phi(0)| \le \frac{427}{\lambda_{\min}(1-\lambda_{\max})^3} D. \tag{5.37}$$

Using (5.37) and Lemma 5.30 in (5.36) completes the proof of Proposition 5.10. □

6. Higher-dimensional gaskets

Fix an integer $d \ge 2$ and consider the IFS $\mathcal{F}_0$ in $\mathbb{R}^d$ defined by the $d+1$ conformal contractions

$$f_j(x) = p_j + \frac{1}{2}(x - p_j), \qquad j = 0, \dots, d,$$

where $\{p_0, \dots, p_d\}$ is a collection of points in $\mathbb{R}^d$ satisfying $|p_i - p_j| = 1$ for all $i \ne j$. Using a convenient abuse of notation, we will occasionally write p_j to denote the vector based at the origin with terminus p_j.

We denote the invariant set for $\mathcal{F}_0$ by SG^d and call this the *d-dimensional Sierpinski gasket.* Observe that

$$\dim SG^d = \frac{\log(d+1)}{\log 2}.$$

Furthermore, for each $n \in \mathbb{N}$, SG^d is the invariant set for the IFS

(6.1)
$$\mathcal{F} = \{f_j^{(n+1)} : j = 0, \dots, d\} \cup \{f_j^{(m)} \circ f_k : j, k = 0, \dots, d, j \neq k, m = 1, \dots, n\}$$

in $\mathcal{IFS}(\mathbb{R}^d, M)$, $M = (d+1)(nd+1)$.

In this section we prove Theorem 1.5 which states that the global quasiconformal dimension of SG^d is equal to one for each $d \geq 3$. To prove Theorem 1.5, we will construct, for each n, a system isomorphic to $\mathcal{F}$ (Proposition 6.14) so that the invariant sets of the deformed IFS's have Hausdorff dimensions tending to one (Proposition 6.15). We then show that each deformed system can be joined to the corresponding undeformed system $\mathcal{F}$ by a nondegenerate isotopy (Proposition 6.16). Theorem 1.5 then follows from Corollary 1.10.

6.2 Geometry of the d-dimensional gasket. Denote by Δ_0 the initial simplex obtained as the closed convex hull of the points $p_0, \dots, p_d$. Without loss of generality, we may assume that the centroid of Δ_0 is the origin in $\mathbb{R}^d$. Denote the distance from any vertex p_j to the centroid by

$$\sigma_d := |p_j|, \tag{6.3}$$

denote the length of the altitude at p_j (that is, the distance from p_j to the hyperplane $H_j \subset \mathbb{R}^d$ spanned by the points $p_0, \dots, p_{j-1}, p_{j+1}, \dots, p_d$) by

$$a_d := \operatorname{dist}(p_j, H_j), \tag{6.4}$$

and denote the angle between the altitude at p_j and any of the edges $\overline{p_j p_k}$ by θ_d.

Lemma 6.5. $\sigma_d = \sqrt{\frac{d}{2d+2}}$, $a_d = \sqrt{\frac{d+1}{2d}}$, *and*

$$\cos\theta_d = a_d = \frac{1}{2\sigma_d}. \tag{6.6}$$

In what follows we never use the exact formulas for σ_d and a_d but we do use (6.6) repeatedly.

Proof. In the planar case we have $\sigma_2 = 1/\sqrt{3}$, $a_2 = \sqrt{3}/2$ and $\theta_2 = \pi/6$. Now let $d \geq 3$ be arbitrary. By considering the triangle with vertices at p_j, p_k and the origin, we see that $\sigma_d \cos\theta_d = 1$.

On the other hand, considering the triangle with vertices at p_j, p_k and the centroid of $\Delta_0 \cap H_j$, we see that $\cos\theta_d = a_d$ and $\sigma_{d-1}^2 + a_d^2 = 1$. The conclusion follows by induction on d. $\square$

In the remainder of this section, we assume that the dimension $d \geq 3$ is fixed and write $\sigma = \sigma_d$, $\theta = \theta_d$, etc.

For each $j, k = 0, \dots, d$, $j \neq k$, let Λ_{jk} be the plane spanned by the vectors p_j and p_k and let $A_{jk} \in O(d)$ be the rotation matrix that maps the vector $p_k - p_j$ onto the vector $-\frac{p_j}{|p_j|} = -\frac{p_j}{\sigma}$ and acts as the identity on $\Lambda_{jk}^{\perp}$. Thus A_{jk} rotates vectors in Λ_{jk} by angle θ.

In the deformed IFS (defined in the following subsection), the matrices A_{jk} will occur as the rotation matrices associated with similarities h_{jkm} (compare with (2.3)). To guarantee that the simplices $h_{jkm}(\Delta_0)$ and $h_{jlm}(\Delta_0)$ intersect at precisely one vertex, we must verify the identities $h_{jkm}(p_l) = h_{jlm}(p_k)$. These identities will follow from the next lemma.

Lemma 6.7. *For each $j, k, l = 0, \dots, d$, $j \neq k$, $j \neq l$, $k \neq l$,*

$$(A_{jk} + A_{jl})(p_l - p_k) = \frac{2 + 3\sigma}{1 + 2\sigma}(p_l - p_k).$$

Proof. The matrices A_{jk}, A_{jl} leave invariant the orthogonal complement of the subspace Λ of $\mathbb{R}^d$ spanned by the vectors p_j, p_k and p_l. It thus suffices to restrict our attention to this three-dimensional subspace, and it is convenient to identify Λ with $\mathbb{R}^3$ and choose coordinates in Λ so that $p_j = (0, 0, \sigma)$,

$$p_k = \left(\frac{\sqrt{3\sigma^2 - 1}}{2\sigma}, \frac{1}{2}, \sigma - \frac{1}{2\sigma} \right)$$

and

$$p_l = \left(\frac{\sqrt{3\sigma^2 - 1}}{2\sigma}, -\frac{1}{2}, \sigma - \frac{1}{2\sigma} \right),$$

or possibly with p_k and p_l reversed. (Note that $|p_j| = |p_k| = |p_l| = \sigma$ and $|p_j - p_k| = |p_j - p_l| = |p_k - p_l| = 1$.) From the definition of A_{jk} we have

$$\begin{aligned} &A_{jk}(p_k - p_j) = -\frac{p_j}{\sigma}, \\ (6.8) \qquad &A_{jk}(-\frac{p_j}{\sigma}) = 2\cos\theta(-\frac{p_j}{\sigma}) - (p_k - p_j) = (1 - \frac{1}{\sigma^2})p_j - p_k, \\ &A_{jk}(p_j \times p_k) = p_j \times p_k, \end{aligned}$$

where $p_j \times p_k$ denotes the cross product of p_j and p_k within the subspace Λ. The vectors $p_k - p_j$, $-p_j/\sigma$ and $p_j \times p_k$ span Λ (note that they are not an orthonormal basis) and hence from the values in (6.8) we compute

$$(6.9) \quad A_{jk}(p_l - p_k) = \frac{1 + \sigma - \sigma^2}{\sigma(1 + 2\sigma)} p_j + \frac{2 + 3\sigma}{2(1 + 2\sigma)}(p_l - p_k) + \frac{\sigma}{2(1 + 2\sigma)}(p_k + p_l).$$

The result follows by symmetry in k and l. □

6.10 Deformations of d-dimensional gaskets. We now describe the deformed versions of the d-dimensional gasket SG^d which will figure in the proof of Theorem 1.5. Fix a natural number $n \in \mathbb{N}$. The deformed gasket

SG_n^d is the invariant set for a new IFS $\mathcal{G} \in \mathcal{IFS}(\mathbb{R}^d, M)$ with $M = (d+1)(nd+1)$. Choose λ and μ so that

$$\frac{\lambda}{\mu} = \frac{2+3\sigma}{1+2\sigma}$$

is the eigenvalue from Lemma 6.7 and

$$2\lambda + \frac{n\mu}{\sigma} = 1; \tag{6.11}$$

and define the IFS

$$\mathcal{G} = \{g_j : j = 0, \dots, d\} \cup \{h_{jkm} : j,k = 0,\dots,d, j \neq k, m = 1,\dots,n\},$$

where

$$g_j(x) = \lambda x + (1-\lambda)p_j \tag{6.12}$$

and

$$h_{jkm}(x) = \mu A_{jk}(x - p_j) + \lambda p_k + (\lambda + m\mu/\sigma)p_j. \tag{6.13}$$

(Compare with (2.2) and (2.3) and recall that $|p_j| = \sigma$.)

Observe that the contraction ratios μ associated with the maps h_{jkm} are constant. If we define $\mathcal{R}'_{j,m} = \{h_{jkm}(\Delta_0) : k = 0,\dots,d, k \neq j\}$ and $\mathcal{A}'_j = \cup_{m=1}^n \mathcal{R}'_{j,m} \cup \{g_j(\Delta_0)\}$, then the deformed gasket SG_n^d has the form of a "starfish" with $d+1$ arms $\mathcal{A}'_0, \dots, \mathcal{A}'_d$. Each arm $\mathcal{A}'_j$ consists of n blocks $\mathcal{R}'_{j,m}$ of equal diameter and a cap $g_j(\Delta_0)$ of diameter λ. Each block $\mathcal{R}'_{j,m}$ is a simplicial complex comprised of d simplices $h_{jkm}(\Delta_0)$ of diameter μ. The dual graph for $\mathcal{R}'_{j,m}$ is the complete graph on d vertices.[4]

Proposition 6.14. *$\mathcal{F}$ and $\mathcal{G}$ are isomorphic.*

Proof. In view of Proposition 3.37, it suffices to verify the identities

(a) $h_{jkn}(p_j) = g_j(p_k)$,
(b) $h_{jk,m+1}(p_k) = h_{jkm}(p_j)$, $1 \le m \le n-1$,
(c) $h_{jkm}(p_l) = h_{jlm}(p_k)$, $1 \le m \le n$,
(d) $h_{jk1}(p_k) = h_{kj1}(p_j)$,

for all $j,k,l = 0,\dots,d$, $j \neq k$, $j \neq l$. These identities follow directly from the definitions of g_j and h_{jkm}, the first identity in (6.8), and the eigenvalue condition in Lemma 6.7. For example, to verify (c) we use the definition of h_{jkm} to compute

$$\begin{aligned} h_{jkm}(p_l) - h_{jlm}(p_k) &= \mu A_{jk}(p_l - p_j) + \lambda p_k - \mu A_{jl}(p_k - p_j) - \lambda p_l \\ &= \mu A_{jk}(p_l - p_k) + \mu A_{jl}(p_l - p_k) - \lambda(p_l - p_k) \\ &= 0. \end{aligned}$$

The proofs of (a), (b) and (d) are similar. ☐

[4]By the *dual graph* for a simplicial complex K we mean the abstract graph Γ whose vertices are in one-to-one correspondence with the simplices in K with the following adjacency matrix: two vertices in Γ are connected by an edge if and only if the corresponding simplices in K meet at a vertex.

Proposition 6.15. $\dim SG_n^d \to 1$ *as* $n \to \infty$.

Proof. Since each of the sets SG_n^d satisfies the open set condition, the dimension s_n^d of SG_n^d is the unique positive number satisfying

$$(d+1)\lambda^{s_n^d} + d(d+1)n\mu^{s_n^d} = 1.$$

Since $\mu < \lambda < 2/n$ it follows that

$$1 \leq (d+1)^2 n(\frac{2}{n})^{s_n^d}$$

whence $s_n^d \leq \frac{\log((d+1)^2 n)}{\log(n/2)} \to 1$ as $n \to \infty$. □

Proposition 6.16. *There exists a nondegenerate isotopy of isomorphic systems* $\{\mathcal{F}^t\}_{0\leq t\leq 1}$ *joining* $\mathcal{F} = \mathcal{F}^0$ *to* $\mathcal{G} = \mathcal{F}^1$.

Proof. Let A_{jk}^t be the orthogonal matrix which is the identity on $\Lambda_{jk}^\perp$ and acts as a rotation on Λ_{jk} by angle $t\theta$. Thus $\{A_{jk}^t\}$ interpolates between the identity matrix and A_{jk}.

For $0 \leq t < 1$, set

$$\alpha^t = \frac{\sin(1-t)\theta}{\sin\theta} \qquad \text{and} \qquad \beta^t = \frac{\sin t\theta}{\sin\theta}.$$

Lemma 6.17. $A_{jk}^t(p_k - p_j) = \alpha^t(p_k - p_j) + \beta^t(-\frac{p_j}{\sigma})$.

Proof. Fix j and k and let $A = A_{jk}$ and $\Lambda = \Lambda_{jk}$. Let $v = p_k - p_j$ and $w = -p_j/\sigma$. Then $A^t v$ lies in Λ, so $A^t v = \alpha v + \beta w$ for some $\alpha, \beta \in \mathbb{R}$. By definition, the angle between $A^t v$ and v is $t\theta$ and the angle between $A^t v$ and w is $(1-t)\theta$. Thus

$$\cos t\theta = A^t v \cdot v = \alpha + \beta\cos\theta$$

and

$$\cos(1-t)\theta = A^t v \cdot w = \alpha\cos\theta + \beta$$

from which it follows that $\alpha = \alpha^t$ and $\beta = \beta^t$. □

Lemma 6.18. $(A_{jk}^t + A_{jl}^t)(p_l - p_k) = \frac{6\sigma^2 + 2\sigma^2\cos t\theta - 2}{4\sigma^2 - 1}(p_l - p_k)$.

The proof is similar to that of Lemma 6.7, using the identities

$$A_{jk}^t(p_k - p_j) = \alpha^t(p_k - p_j) + \beta^t(-\frac{p_j}{\sigma})$$

and

$$A_{jk}^t(-\frac{p_j}{\sigma}) = (\alpha^t + 2\beta^t\cos\theta)(-\frac{p_j}{\sigma}) - \beta^t(p_k - p_j)$$

in place of the first and second equations of (6.8). Observe that the eigenvalue in Lemma 6.18 is equal to 2 when $t = 0$ and is equal to the value $\frac{2+3\sigma}{1+2\sigma}$ when $t = 1$ (since $2\cos\theta = 1/\sigma$).

We now define the deformed iterated function systems. Fix $0 \leq t < 1$ and define

(6.19)

$$\mathcal{F}^t = \{g_j^t : j = 0, \ldots, d\} \cup \{h_{jkm}^t : j, k = 0, \ldots, d, j \neq k, m = 1, \ldots, n\},$$

where

$$g_j^t(x) := \lambda^t x + (1-\lambda^t)p_j \tag{6.20}$$

and

$$\begin{aligned} h_{jkm}^t(x) &:= \mu^t(\rho^t)^{n-m}A_{jk}^t(x-p_j) + \lambda^t(\rho^t)^{n-m}p_k \\ &\quad + (1+\nu^t/\sigma - (\lambda^t+\nu^t/\sigma)(\rho^t)^{n-m})p_j, \end{aligned} \tag{6.21}$$

and the data λ^t, μ^t, ν^t, and $\rho^t > 1$ are determined by the equations

$$\alpha^t + \frac{\lambda^t}{\mu^t} = \frac{6\sigma^2 + 2\sigma^2\cos t\theta - 2}{4\sigma^2 - 1}, \tag{6.22}$$

$$\mu^t\alpha^t = \lambda^t(\rho^t - 1), \tag{6.23}$$

$$\mu^t\beta^t = \nu^t(\rho^t - 1), \tag{6.24}$$

and

$$(2\lambda^t + \nu^t/\sigma)(\rho^t)^n = 1 + \nu^t/\sigma. \tag{6.25}$$

The geometric intuition behind (6.20) and (6.21) is very simple. For $0 \le t < 1$ let $\mathcal{R}_{j,m}^t := \{h_{jkm}^t(\Delta_0) : k = 0,\dots,d, k \ne j\}$. For fixed j and m, the simplices in $\mathcal{R}_{j,m}^t$ form a simplicial complex whose dual graph is the complete graph on d vertices. These simplices are tilted from $h_{jkm}^0(\Delta_0)$ according to the rotation matrix A_{jk}^t and by an angle $t\theta$ relative to the axis $\overline{0p_j}$. Scaled in order for $\mathcal{R}_{j,m}^t$ and $\mathcal{R}_{j,m+1}^t$ to intersect properly, the diameters $\mu^t(\rho^t)^{n-m}$ of the simplices in $\mathcal{R}_{j,m}^t$ must decrease geometrically in m. The overall scaling ratios λ^t and μ^t are chosen to make the entire picture fit precisely within the original simplex Δ_0 and to guarantee the required intersections among the child simplices. This ensures that the IFS's $\mathcal{F}^t$ and $\mathcal{F}^{t'}$ are isomorphic for $t \ne t'$.

As before, to establish the proposition it suffices to verify the identities

(a) $h_{jkn}^t(p_j) = g_j^t(p_k)$,
(b) $h_{jk,m+1}^t(p_k) = h_{jkm}^t(p_j)$, $1 \le m \le n-1$,
(c) $h_{jkm}^t(p_l) = h_{jlm}^t(p_k)$, $1 \le m \le n$,
(d) $h_{jk1}^t(p_k) = h_{kj1}^t(p_j)$,

for all $j,k,l = 0,\dots,d$, $j \ne k$, $j \ne l$.

Condition (a) follows directly from the definitions, while condition (b) follows from Lemma 6.17, (6.23) and (6.24). For condition (c) we use Lemmas 6.17 and 6.18 as well as (6.22). Finally, condition (d) follows from Lemma 6.17 and equations (6.23), (6.24) and (6.25). We leave the details to the reader. □

Remark 6.26. For fixed $t < 1$ the quantities λ^t/μ^t, ρ^t and ν^t/μ^t are independent of n. From (6.25) it follows that $\lambda^t, \mu^t, \nu^t = O(\rho^{-n})$.

Now assume that $n \in \mathbb{N}$ is fixed. It is straightforward to verify that the IFS $\mathcal{F}^0$ from (6.19) coincides with the IFS $\mathcal{F}$ from (6.1). To verify that the

IFS $\mathcal{F}^t$ converges to the IFS $\mathcal{G}$ as $t \to 1$, we analyze the limiting behavior of the equations (6.22)–(6.25). Set $\epsilon = 1 - t$. Beginning with (6.22) and using (6.6) we find

$$\frac{\lambda^t}{\mu^t} = \frac{\lambda}{\mu} - \frac{\theta}{2\sin\theta}\epsilon + O(\epsilon^2).$$

From (6.23) we find

$$\rho^t = 1 + \frac{\mu\theta}{\lambda\sin\theta}\epsilon + \frac{\mu^2\theta^2}{2\lambda^2\sin^2\theta}\epsilon^2 + O(\epsilon^3)$$

and from (6.24) we find

$$\frac{\nu^t}{\mu^t} = \frac{\lambda\sin\theta}{\mu\theta}\cdot\frac{1}{\epsilon} - \left(\frac{1}{2} + \frac{\lambda\cos\theta}{\mu}\right) + O(\epsilon).$$

Dividing (6.25) through by μ^t and substituting the values for λ^t/μ^t, ρ^t and ν^t/μ^t from above, we find after some work that

$$\frac{1}{\mu^t} = \left(\frac{n}{\sigma} + \frac{2\lambda}{\mu}\right) + O(\epsilon) = \frac{1}{\mu} + O(\epsilon) \tag{6.27}$$

by (6.11). Hence $\mu^t = \mu + O(\epsilon)$ and $\lambda^t = \lambda + O(\epsilon)$.

7. The polygaskets

Fix an integer $N \geq 3$ and consider the IFS $\mathcal{F}_0$ generated by the N contractions $f_j(z) = \omega^j + \lambda(N)(z - \omega^j)$, $j = 0, \dots, N-1$. Here $\omega = e^{2\pi i/N}$ denotes the principal Nth root of unity and

$$\lambda(N) := \frac{1-\omega}{(1-\omega^q)(1+\omega^{1-q})} = \frac{\sin(\pi/N)}{2\sin(\pi q/N)\cos(\pi(q-1)/N)}, \tag{7.1}$$

where

$$q = q(N) := 1 + \left[\frac{N}{4}\right]. \tag{7.2}$$

The *polygasket* $PG(N)$ is the invariant set for $\mathcal{F}_0$. By (1.6),

$$\dim PG(N) = \frac{\log N}{\log 1/\lambda}.$$

Denote by Π the initial N-gon which is the closed convex hull of the Nth roots of unity. The contraction ratio $\lambda(N)$ in (7.1) is chosen so that neighboring polygons $f_j(\Pi)$ and $f_{j+1}(\Pi)$ meet on the boundary but not in their interiors.[5] Note that two polygons $f_j(\Pi)$, $f_{j+1}(\Pi)$ meet at either a single point or along a line segment, according to whether $N \not\equiv 0 \pmod 4$ or $N \equiv 0 \pmod 4$. (See also Lemma 7.11.) It follows that $\mathcal{F}_0$ satisfies the

[5] By convention, all arithmetic computations involving indices are taken mod N.

open set condition for any N (choose O to be the interior of Π), and is of gasket type provided $N \not\equiv 0 \pmod 4$.[6]

For the remainder of this section, we fix an integer $N \geq 5$ with $N \not\equiv 0 \pmod 4$. We will prove Theorem 1.4 which asserts that the global quasiconformal dimension of $PG(N)$ is equal to one. To simplify notation we suppress the dependence on N, writing $\lambda = \lambda(N)$, $q = q(N)$, etc.

7.3 Sketch of the proof. Fix an integer $n \geq 1$ and consider $PG(N)$ as the invariant set for the following IFS in $\mathcal{IFS}(\mathbb{R}^2, M)$, $M = N(N-1)n + N$:

$$\begin{aligned} \mathcal{F} := \{ f_j^{(n+1)} &: j = 0, \ldots, N-1\} \\ \cup \{ f_j^{(m)} \circ f_k &: j,k = 0, \ldots, N-1, j \neq k, m = 1, \ldots, n\}. \end{aligned} \tag{7.4}$$

See Figure 6 for the case $N = 9$, $n = 2$.

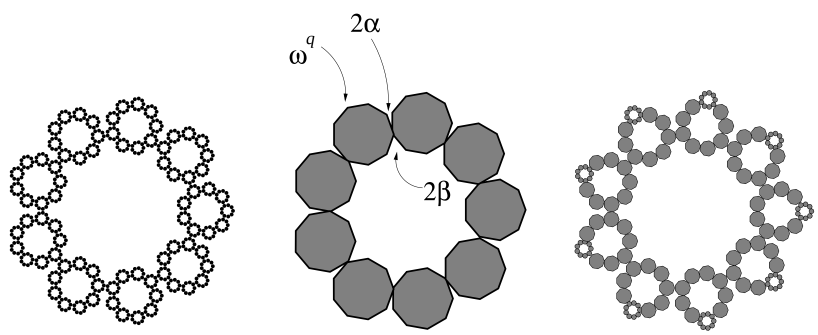

FIGURE 6. (a) $PG(9)$; (b) the defining nonagons from $\mathcal{F}_0$ for the nonagasket; (c) the defining nonagons from $\mathcal{F}$ ($n = 2$) for the nonagasket

Set $\mathcal{P} := \cup_{f \in \mathcal{F}} f(\Pi)$. The critical set C_0 for $\mathcal{F}$ separates $\mathcal{P}$ into N "arms"

$$\mathcal{A}_j = f_j^{(n+1)}(\Pi) \cup \bigcup_{m=1}^{n} \bigcup_{k=1, k\neq j}^{N} f_j^{(m)} \circ f_k(\Pi), \qquad j = 0, \ldots, N-1.$$

Equivalently, $\mathcal{A}_j$ can be characterized as the union of those polygons $f(\Pi)$, $f \in \mathcal{F}$, which are contained in the sector $\{re^{i\theta} : |\theta - 2\pi j/N| \leq \pi/N\}$. Within a fixed arm $\mathcal{A}_j$, the two point sets $\{f_j^{(m)}(\omega^{j+q}), f_j^{(m)}(\omega^{j-q})\}$, $m = 2, \ldots, n+1$, separate $\mathcal{A}_j$ into n *intermediate blocks*

$$\mathcal{B}_{j,m} = \bigcup_{k=1, k\neq j}^{N} f_j^{(m)} \circ f_k(\Pi),$$

[6]When $N \equiv 0 \pmod 4$, two adjacent similarity pieces $f_j(PG(N))$ and $f_{j+1}(PG(N))$ intersect either along a line segment (if $N = 4$) or along a Cantor set (if $N = 8, 12, 16, \ldots$). $PG(4)$ is a closed square. For $PG(4k)$, $k \geq 2$, we do not know whether the dimension can be reduced by a quasiconformal map of the plane. See Figure 2 for the case $N = 8$.

$m = 1, \ldots, n$, and one *terminal block*

$$\mathcal{B}_{j,n+1} = f_j^{(n+1)}(\Pi).$$

Observe that all of the $N-1$ polygons in a block $B_{j,m}$, $m = 1, \ldots, n$, have the same size $\lambda^{m+1} \operatorname{diam} \Pi$ and that these sizes form a geometrically decreasing sequence. The polygon in the terminal block $\mathcal{B}_{j,n+1}$ has the same size as those in the final intermediate block $\mathcal{B}_{j,n}$.

Our goal is to construct a new gasket type IFS

$$\begin{aligned} \mathcal{G} := \{g_j : j = 0, \ldots, N-1\} \\ \cup \{h_{jkm} : j, k = 0, \ldots, N-1, j \neq k, m = 1, \ldots, n\} \end{aligned}$$

which is isomorphic to $\mathcal{F}$. The precise details of this construction are quite complicated and will occupy the bulk of this section. In brief, the idea is to replace the geometrically decreasing blocks $\mathcal{B}_{j,m}$ in each arm with blocks of constant size. The overall shape of the resulting invariant set $PG_n(N)$ will be essentially that of an N-armed starfish. See Figure 7 for a picture of one of the deformed arms in the case $N = 9$, $n = 3$.

In order to show that $\mathcal{F}$ and $\mathcal{G}$ are isomorphic, we will verify the condition in Proposition 3.37. See Proposition 7.32. Theorem 1.7 then ensures that the canonical homeomorphism $F : PG(N) \to PG_n(N)$ is quasisymmetric. A computation (see Proposition 7.33) yields $\lim_{n\to\infty} \dim PG_n(N) = 1$.

It is likely that the deformed IFS's may be joined to the original IFS by isotopies, however, establishing this fact rigorously is technically difficult. For this reason, we use the more well-developed theory of quasiconformal maps in the plane to directly construct quasiconformal extensions of the canonical homeomorphisms from $PG(N)$ to $PG_n(N)$. See Proposition 7.34.

The remainder of this section is organized as follows. We first collect some additional basic geometric observations regarding the structure of the N-gasket which will be important in the proof. We then prove Theorem 1.4 by filling in the details in the above sketch.

7.5 Geometry of the polygasket. The critical set for $PG(N)$ as generated by $\mathcal{F}_0$ is given explicitly as

$$C_0 = \{f_j(\omega^{j+q}) = f_{j+1}(\omega^{j+1-q}) : j = 1, \ldots, N\}. \tag{7.6}$$

For $m \geq 0$ let $C_m = \cup_{w \in W_m} f_w(C_0)$.

Lemma 7.7. *Each subpolygon Π_w, $w \in W_m$, meets $\bigcup_{w' \in W_m, w' \neq w} \Pi_{w'}$ in either two or three points. This intersection contains three points if and only if $m \geq 2$ and Π_w has a vertex in $C_0 \cup \cdots \cup C_{m-2}$. In this case Π_w has precisely one vertex v in $C_0 \cup \cdots \cup C_{m-2}$.*

We leave the proof to the reader.

Within the polygon Π we distinguish a particular (directed) chord $\overline{\omega^q \omega^{-q}}$. Similarly, for each Π_w, $w \in W$, we distinguish the chord $\overline{f_w(\omega^q) f_w(\omega^{-q})}$. In general, we say that a polygon Π' *has chord* $\overline{xy}$ if there is a similarity which maps Π' onto Π taking x to ω^q and y to ω^{-q}.

For three points a, b, c in the plane, denote by $\angle abc \in [0, 2\pi)$ the oriented angle from $\overline{ba}$ to $\overline{bc}$. If $0 < \angle abc < \pi$ let $\angle_e abc = \pi - \angle abc$ be the exterior angle at the vertex b for the triangle with vertices a, b, c.

Set

$$\gamma = \frac{\pi}{2} - \frac{\pi q}{N}, \tag{7.8}$$

$$\alpha = \frac{\pi}{2} - \frac{2(q-1)\pi}{N} > 0, \tag{7.9}$$

and

$$\beta = \frac{2q\pi}{N} - \frac{\pi}{2} > 0. \tag{7.10}$$

Here $\gamma = \angle \omega^q 10$, while the quantities 2α and 2β give the angles between adjacent N-gons (see Figure 6(b)). Note that $\alpha + \beta = 2\pi/N = \angle_e xyz$ is the exterior angle at the vertex of a regular N-gon, or equivalently, the exterior angle at y for the triangle with vertices x, y, z, where $\overline{xy}$ and $\overline{yz}$ are the directed chords associated with a pair of adjacent children of Π (with respect to the original IFS $\mathcal{F}_0$).

In connection with the following lemma, recall our standing assumption that $N \geq 5$, $N \not\equiv 0 \pmod 4$.

Lemma 7.11. (i) $N \equiv 1$, 2 *or* $3 \pmod 4$ *if and only if* $2\alpha < 2\pi/N$, $2\alpha = 2\pi/N$ *or* $2\alpha > 2\pi/N$, *respectively;*

(ii) $\frac{\gamma}{q-1} < \min\{2\alpha, 2\pi/N\}$ *and* $\frac{\gamma}{q} < 2\beta$*;*

(iii) $\frac{2\gamma}{N-2q} = 2\alpha$ *if* $N \equiv 1 \pmod 4$ *and* $\frac{2\gamma}{N-2q} < 2\pi/N$ *if* $N \equiv 2, 3 \pmod 4$.

Proof. These facts are easy computations using the definitions (7.8), (7.9), (7.10) together with the definition of q in (7.2). $\square$

The equality in Lemma 7.11(iii) when $N \equiv 1 \pmod 4$ introduces additional complications into the construction of the deformed gaskets in this case. See the remarks following Claim 7.24.

Lemma 7.12. *Let* Π' *and* Π'' *be polygons with chords* $\overline{xy}$ *and* $\overline{yz}$ *respectively, where* $0 < \angle xyz < \pi$. *If*

$$\max\{0, \frac{2\pi}{N} - 2\alpha\} < \angle_e xyz < \frac{2\pi}{N} + 2\beta,$$

then $\Pi' \cap \Pi'' = \{y\}$.

In the context of this lemma it is important to note that we do **not** require $|x - y| = |y - z|$.

Proof. Let a and b be vertices of Π' which are neighbors of y so that a, y, b occur in clockwise order around the boundary of Π'. Let c and d be the vertices of Π'' which are neighbors of y so that c, y, d occur in clockwise order. It suffices to prove that $\angle dya > 0$ and $\angle byc > 0$. A computation gives $\angle ayx = \angle zyd = (2q-1)\pi/N$ and $\angle xyb = \angle cyz = (N-2q-1)\pi/N$. Thus

$$\begin{aligned}\angle dya &= 2\pi - \angle ayx - \angle xyz - \angle zyd \\ &> \pi - \frac{2(2q-1)\pi}{N} + \max\{0, \frac{2\pi}{N} - 2\alpha\} \geq 0.\end{aligned}$$

Similarly

$$\begin{aligned}\angle byc &= \angle xyz - \angle xyb - \angle cyz \\ &> \pi - \frac{2(N-2q-1)\pi}{N} - \frac{2\pi}{N} - 2\beta \geq 0.\end{aligned}$$

□

7.13 Notation and definitions. We restrict our attention to a single arm of the polygasket; without loss of generality assume that it is $\mathcal{A}_0$. For $m \geq 0$ and $k = 0, \ldots, N-1$ we set

$$\Pi_k^m := f_0^{(m)}(\Pi_k)$$

and

$$X_k^m := f_0^{(m)} \circ f_k(\omega^{k+q}) = f_0^{(m)} \circ f_{k+1}(\omega^{k+1-q}).$$

Thus the critical set $C_0 = \{X_k^0 : 0 \leq k \leq N-1\}$. Furthermore, the blocks within $\mathcal{A}_0$ are $\mathcal{B}_m := \mathcal{B}_{0,m} = \{\Pi_k^m : 1 \leq k \leq N-1\}$ and $\mathcal{B}_{n+1} := \mathcal{B}_{0,n+1} = \{\Pi_0^n\}$, and $X_k^m = \Pi_k^m \cap \Pi_{k+1}^m$. Figure 7 shows a single arm in the case $N = 9$, $n = 3$.

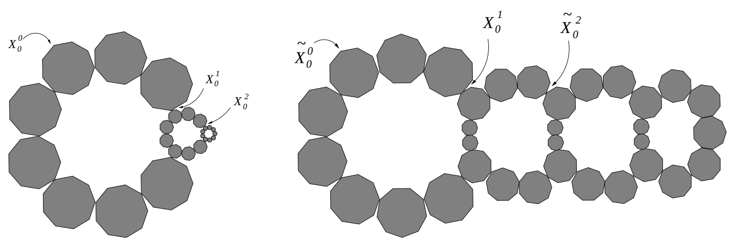

FIGURE 7. (a) One arm of the undeformed gasket $PG(9)$, showing the junction points X_0^0, X_0^1, X_0^2, …; (b) one arm of the deformed hexagasket $PG_3(9)$, showing the junction points $\tilde{X}_0^0$, $\tilde{X}_0^1$, $\tilde{X}_0^2$, …

Within $\mathcal{B}_m$ we define certain distinguished polygons. The *right joint* (resp. *left joint*) is Π_q^m (resp. Π_{N-q}^m). The *right side* (resp. *left side*), consists of the polygons Π_k^m with $1 \leq k \leq q-1$ (resp. $N-q+1 \leq k \leq N-1$).[7] Finally, the *bridge* consists of the polygons Π_k^m with $q+1 \leq k \leq N-q-1$. Observe that all of the polygons Π_k^m, $1 \leq k \leq N-1$, fall into one of these five categories.

[7]The language here is from the viewpoint of an observer standing at the point $z = 1$.

Note that the points $\omega^q, f_0(\omega^q) = X_0^0, f_0^{(2)}(\omega^q) = X_0^1, \dots, 1$ are collinear, as are the points $\omega^{-q}, f_0(\omega^{-q}) = X_{N-1}^0, f_0^{(2)}(\omega^{-q}) = X_{N-1}^1, \dots, 1$. Note also that the set $\mathcal{S}_R$ (resp. $\mathcal{S}_L$) which is the union of all of the right joints and right arms (resp. the left joints and left arms) is connected.

7.14 Initial attempt to construct $PG_n(N)$**.** A natural approach to the construction of the deformed arm $\mathcal{A}_1'$ is the following:

1. Apply a similarity h_1 to S_R of the form
$$h_1(z) = X_0^0 + e^{i\gamma}(z - X_0^0),$$
where γ is defined in (7.8). Note that h_1 maps the line containing $X_0^0, X_0^1, \dots, 1$ to the line containing $X_0^0 = h_1(X_0^0), h_1(X_0^1), \dots, h_1(1)$ which is parallel to the x-axis.
2. Apply a continuous map h_2 to $h_1(\mathcal{S}_R)$, where h_2 is defined inductively as follows: h_2 is the identity on $h_1(\mathcal{B}_1 \cap \mathcal{S}_R)$ and $h_2(z) = h_2(X_0^{m-1}) + \lambda^{-m}(z - h_2(X_0^{m-1}))$ for $z \in h_1(\mathcal{B}_m \cap \mathcal{S}_R)$, $2 \le m \le n$. Observe that each set $h_2(\mathcal{B}_m \cap \mathcal{S}_R)$ is a horizontal translate of $h_2(\mathcal{B}_1 \cap \mathcal{S}_R)$.
3. Repeat parts 1 and 2 for S_L.
4. Adjust the bridges within each block $\mathcal{B}_m$ so that the vertices which anchor the bridge to the left and right joints coincide with the new positions of those vertices within $h_1(\mathcal{S}_R)$.
5. Adjust the size of the last block $\mathcal{B}_{n+1}$ so that the vertices which anchor it to the left and right sides of $\mathcal{B}_m$ are correct.

However, there is a small problem with this plan. The image of the first block $\mathcal{B}_1$ is no longer contained within the sector $\{re^{i\theta} : |\theta| \le \pi/N\}$ due to the large rotation angle γ; thus the images of neighboring arms will overlap. This complicates the construction. To correct this problem we use smaller rotations (by angle $\gamma/(q-1)$) of each of the $q-1$ polygons in the right side, and similarly in the left side of the first block.

We now give the proof of Theorem 1.4 in detail. We will describe the construction of the deformed blocks $\mathcal{B}_m'$ within the arm $\mathcal{A}_0'$ block by block. The other deformed arms $\mathcal{A}_1', \dots, \mathcal{A}_{N-1}'$ are obtained from $\mathcal{A}_0'$ by rotation.

7.15 Block 1**.** In this block, we leave the right and left joints as well as the bridge unchanged. Thus X_k^1, $q-1 \le k \le N-q$, are unchanged. Replace the remaining points X_k^1 by points $\tilde{X}_k^1$ (and for convenience, denote by $\tilde{X}_k^1 = X_k^1$ the unchanged points also) so that the exterior angles

$$\angle_e \tilde{X}_{k+1}^1 \tilde{X}_k^1 \tilde{X}_{k-1}^1 = \frac{2\pi}{N} - \frac{\gamma}{q-1} \tag{7.16}$$

for $1 \le k \le q-1$ and $N-q \le k \le N-2$, and that the distances $|X_k^1 - X_{k+1}^1|$ are equal for all $0 \le k \le N-2$. Replace each Π_k^1 (for $1 \le k \le q-1$ and

$N-q+1 \le k \le N-1$) by the regular N-gon $\tilde{\Pi}_k^1$ with directed chord $\overline{\tilde{X}_k^1\tilde{X}_{k-1}^1}$; and again denote by $\tilde{\Pi}_k^1 = \Pi_k^1$ the unchanged polygons as well.

Let $\mathcal{B}_1' = \{\tilde{\Pi}_k^1 : 1 \le k \le N-1\}$.

Claim 7.17. *The only points of intersection of the polygons in $\mathcal{B}_1'$ are $\{\tilde{X}_k^1\} = \tilde{\Pi}_k^1 \cap \tilde{\Pi}_{k+1}^1$, $k = 1, \dots, N-2$.*

From (7.16) and Lemma 7.11(ii) it follows that

$$\frac{2\pi}{N} - 2\alpha < \angle_e \tilde{X}_{k+1}^1\tilde{X}_k^1\tilde{X}_{k-1}^1 < \frac{2\pi}{N}$$

and the claim follows from Lemma 7.12.

Claim 7.18. *Each polygon $\tilde{\Pi}_k^1$, $1 \le k \le N-1$, is contained in the sector $\{re^{i\theta} : |\theta| \le \pi/N\}$ and*

$$\bigcup_{k=1}^{N-1} \tilde{\Pi}_k^1 \cap \{re^{\pm i\pi/N}\} = \{X_0^0, X_{N-1}^0\}.$$

Let L be the ray $\{re^{i\pi/N} : r \ge 0\}$. Calculations show that the angle φ_k from the chord $\overline{\tilde{X}_k^1\tilde{X}_{k-1}^1}$ to L increases as k decreases from q to 1. When $N \equiv 2, 3 \pmod 4$ the φ_k's are all positive; when $N \equiv 1 \pmod 4$ all of these values except φ_q are positive and $|\varphi_q| < \varphi_{q-1}$. These facts show that

$$\begin{aligned} \operatorname{dist}(X_k, L) &\ge \max_{k=q-1,q} \{\operatorname{dist}(X_q, L), \operatorname{dist}(X_{q-1}, L)\} \\ &= \operatorname{dist}(X_q, L) \end{aligned}$$

for $k = 0, \dots, q-2$. From this, Claim 7.18 follows easily.

Claim 7.19. *The chord $\overline{X_0^1X_1^1}$ has been rotated by an angle γ, i.e., the angle between $\overline{X_0^1X_1^1}$ and $\overline{\tilde{X}_0^1\tilde{X}_1^1}$ is γ.*

This angle of rotation is built up through the cumulative effect of $q-1$ rotations by angle $\gamma/(q-1)$.

7.20 Blocks 2 **to** $n-1$**.** Let $\tilde{h}_1$ be the similarity

$$\tilde{h}_1(z) = X_0^1 + \Lambda e^{i\gamma}(z - X_0^1), \tag{7.21}$$

where $\Lambda \ge 1$ is a constant to be determined later. Then $X_0^1 = \tilde{h}_1(X_0^1)$, $\tilde{h}_1(X_0^2), \dots, \tilde{h}_1(1)$ lie on a horizontal line. Apply $\tilde{h}_1$ to $\mathcal{S}_R \setminus \mathcal{B}_1$. (Recall that $\mathcal{S}_R$ denotes the union of all of the right joints and right arms.) Then the sets $\tilde{h}_1(\mathcal{S}_R \cap \mathcal{B}_m)$, $m = 2, \dots, n$, are all similar to one another.

Let $\tilde{h}_2$ be a continuous map of $\tilde{h}_1(\mathcal{S}_R \setminus \mathcal{B}_1)$ which translates $\tilde{h}_1(\mathcal{S}_R \cap \mathcal{B}_m)$ horizontally and expands $\tilde{h}_1(\mathcal{S}_R \cap \mathcal{B}_m)$ by a factor λ^{2-m}. More precisely, $\tilde{h}_2$ is defined inductively as follows: $\tilde{h}_2$ is the identity on $\tilde{h}_1(\mathcal{B}_2)$, and after $\tilde{h}_2$ has been defined on $\tilde{h}_1(\mathcal{B}_{m-1})$ (and hence defined at $\tilde{h}_1(X_0^{m-1})$),

$$\tilde{h}_2(z) = \tilde{h}_2\tilde{h}_1(X_0^{m-1}) + \lambda^{2-m}(z - \tilde{h}_2\tilde{h}_1(X_0^{m-1}))$$

for $z \in \tilde{h}_1(\mathcal{B}_m)$, $3 \le m \le n$.

Let $\tilde{\Pi}_k^m = \tilde{h}_2 \tilde{h}_1(\Pi_k^m)$, $1 \le k \le q$, be the new right side and right joint for $\mathcal{B}'_m$, $2 \le m \le n$. Observe that $\tilde{\Pi}_k^m$ has chord $\overline{\tilde{X}_k^m \tilde{X}_{k-1}^m}$, where

$$\tilde{X}_k^m = \tilde{h}_2 \tilde{h}_1(X_k^m), \qquad 1 \le k \le q. \tag{7.22}$$

Define the left joint and left side for $\mathcal{B}'_m$ analogously. Note that $\tilde{X}_0^1, \tilde{X}_0^2, \dots, \tilde{X}_0^n$ are collinear.

To complete the discussion for these blocks, we must describe what happens to the bridges.

Consider first the bridge in $\mathcal{B}_2$. Using the formula in (7.22), we observe that the angle between the vectors $\overline{\tilde{X}_{N-q-1}^2 \tilde{X}_{N-q}^2}$ and $\overline{\tilde{X}_q^2 \tilde{X}_{q-1}^2}$ is equal to 2γ plus the angle between $\overline{X_{N-q-1}^2 X_{N-q}^2}$ and $\overline{X_q^2 X_{q-1}^2}$ due to the rotations resulting from $\tilde{h}_2$. Therefore points $\tilde{X}_k^2$, $q+1 \le k \le N-q-2$, can be chosen so that

(i) the distances $|\tilde{X}_k^2 - \tilde{X}_{k-1}^2|$ are equal for all $q+1 \le k \le N-q-1$; and

(ii) $\angle_e \tilde{X}_{k+1}^2 \tilde{X}_k^2 \tilde{X}_{k-1}^2 = \frac{2\pi}{N} - \frac{2\gamma}{N-2q}$ for $q \le k \le N-q-1$.

Now let $\tilde{\Pi}_k^2$, $q+1 \le k \le N-q-1$, be the regular N-gon with directed chord $\overline{\tilde{X}_k^2 \tilde{X}_{k-1}^2}$. This defines the new bridge in $\mathcal{B}'_2$.

The new bridges in $\mathcal{B}'_m$ for $3 \le m \le n$ are defined to be translates of the bridge in $\mathcal{B}'_2$ in the obvious manner.

We have now defined the deformed blocks $\mathcal{B}'_m$ for all $2 \le m \le n$. However, the sides of $\mathcal{B}'_n$ will be modified later in order to fit in the terminal block $\mathcal{B}'_{n+1}$.

Claim 7.23. *For each $m = 2\dots, n$ any two polygons in $\mathcal{B}'_m$, not in the bridge, meet at most at one point.*

This is clear from the construction.

Claim 7.24. *For each $m = 2\dots, n$ any two polygons in $\mathcal{B}'_m$, not in the sides, meet at most at one point.*

If $N \equiv 2, 3 \pmod 4$ this follows from the choice of the points $\tilde{X}_k^m$, Lemma 7.11(ii) and Lemma 7.12, since

$$\frac{2\pi}{N} - 2\alpha \le 0 < \angle_e \tilde{X}_{k+1}^m \tilde{X}_k^m \tilde{X}_{k-1}^m < \frac{2\pi}{N}.$$

If $N \equiv 1 \pmod 4$ we have

$$\frac{2\pi}{N} - 2\alpha = \angle_e \tilde{X}_{k+1}^m \tilde{X}_k^m \tilde{X}_{k-1}^m$$

and the hypothesis in Lemma 7.12 fails; neighboring polygons in the bridge share a side. Therefore we must make the exterior angles between neighboring chords in the bridge slightly larger than $2\pi - 2\gamma/(N-2q)$. This can be accomplished by a small clockwise rotation of $\tilde{\Pi}_q^2$ about the point $\tilde{X}_0^1$, and a small counterclockwise rotation of $\tilde{\Pi}^2_{N-q}$ about the point $\tilde{X}^1_{N-1}$. Subsequently, we must make small adjustments to the points $\tilde{X}^2_k$ for $1 \le k \le q$ and $N-q \le k \le N-1$. We require, and indeed it is possible, that (i) $\overline{\tilde{X}^1_0\tilde{X}^2_0}$ and $\overline{\tilde{X}^1_{N-1}\tilde{X}^2_{N-1}}$ remain horizontal, (ii) the newly positioned chords $\overline{\tilde{X}^2_k\tilde{X}^2_{k-1}}$, $2 \le k \le q-1$ or $N-q+2 \le k \le N-1$, for the sides have equal length, and (iii) the modification of all chords involved can be as small as future needs require. In view of (i)–(iii), it is again possible to choose the subsequent blocks $\mathcal{B}'_m$, $3 \le m \le n$, as horizontal translates of $\mathcal{B}'_2$.

Claim 7.25. *The right side of $\mathcal{B}'_1$ and the union of the right joints and right sides of $\mathcal{B}'_m$, $2 \le m \le n$, intersect precisely at the point $\tilde{X}^1_0$.*

This is due to the fact that the two segments $\overline{X^1_0X^1_1}$ and $\overline{X^1_0 1}$ are each rotated by the angle γ in the process of forming $\mathcal{B}'_1$ and $\cup^n_{m=2}\mathcal{B}'_m$.

Claim 7.26. *For each $2 \le m \le n-1$, $\mathcal{B}'_m$ and the union of the right joints and right sides of $\mathcal{B}'_{m+1}$, intersect precisely at $\tilde{X}^m_0$.*

This is again obvious from the construction. The statements for the left side are analogous.

The scaling factor Λ in (7.21) can be chosen so that the next three claims are satisfied.

Claim 7.27. *$\mathcal{B}'_1$ is disjoint from the bridge of $\mathcal{B}'_2$.*

Claim 7.28. *For each $2 \le m \le n-1$, $\mathcal{B}'_m$ is disjoint from the bridge of $\mathcal{B}'_{m+1}$.*

Claim 7.29. *The deformed arm is contained in the sector $\{re^{i\theta} : |\theta| \le \pi/N\}$ and intersects the boundary of this sector only at the points $\tilde{X}^0_0$ and $\tilde{X}^0_{N-1}$.*

7.30 Blocks n and $n+1$. Finally, we describe the terminal block $\mathcal{B}'_{n+1}$. In order to properly fit this block into the deformed arm $\mathcal{A}'_0$ it is necessary to make some small modifications to the penultimate block $\mathcal{B}'_n$.

The joints and the bridge $\tilde{\Pi}^n_k$, $q \le k \le N-q$, in Block n will remain unchanged. In particular, the points $\tilde{X}^n_k$, $q-1 \le k \le N-q$, are unchanged. For $0 \le k \le q-2$ and $N-q+1 \le k \le N-1$ replace $\tilde{X}^n_k$ by new points, again called $\tilde{X}^n_k$, so that (i) the distances $|\tilde{X}^n_k - \tilde{X}^n_{k-1}|$ are equal for all $1 \le k \le q-1$ and $N-q+1 \le k \le N-1$, and (ii) $\angle_e \tilde{X}^n_{k+1}\tilde{X}^n_k\tilde{X}^n_{k-1} = \frac{2\pi}{N} + \frac{\gamma}{q}$ for $0 \le k \le q-1$ and $N-q \le k \le N-1$. (In the case $N \equiv 1 \pmod 4$) the angle in (ii) is slightly less than $2\pi/N + \gamma/q$ due to the adjustments in connection with Claim 7.24.) For $0 \le k \le q-1$ and $N-q+1 \le k \le N-1$, redefine $\tilde{\Pi}^n_k$ to be the regular N-gon with chord $\overline{\tilde{X}^n_k\tilde{X}^n_{k-1}}$.

Now replace the block $\mathcal{B}'_n$ defined earlier by the new collection

$$\mathcal{B}'_n := \{\tilde{\Pi}^n_k : 1 \le k \le N-1\}$$

and set

$$\mathcal{B}'_{n+1} := \{\tilde{\Pi}^n_0\}.$$

Claim 7.31. $\tilde{\Pi}^n_{k+1} \cap \tilde{\Pi}^n_k = \{\tilde{X}^n_k\}$ *for* $0 \le k \le q-1$ *and* $N-q+1 \le k \le N-1$.

Recall that indices are taken modulo N so $\tilde{\Pi}^n_N = \tilde{\Pi}^n_0$.

Claim 7.31 follows from the definition of the points $\tilde{X}^n_k$ together with Lemma 7.11(iii) and Lemma 7.12.

Finally, let

$$\mathcal{A}'_0 := \cup_{m=1}^{n+1} \mathcal{B}'_m$$

and

$$\mathcal{A}'_j := \{e^{2\pi i j/N} z : z \in \mathcal{A}'_0\}.$$

The deformed IFS $\mathcal{G} = \{g_j\} \cup \{h_{jkm}\}$ is defined as follows. Rescale the polygons $\tilde{\Pi}^m_k$ so that the closed convex hull of $\cup_j \mathcal{A}'_j$ is again equal to the original polygon Π. Continue to denote the rescaled polygons by $\tilde{\Pi}^m_k$. Let g_0 and h_{0km} map Π onto $\tilde{\Pi}^n_0$ and $\tilde{\Pi}^m_k$ respectively preserving the directed chords. Then define g_j and h_{jkm} as

$$g_j(z) = e^{2\pi i j/N} g_0(e^{-2\pi i j/N} z)$$

and

$$h_{jkm}(z) = e^{2\pi i j/N} h_{0km}(e^{-2\pi i j/N} z).$$

Proposition 7.32. *$\mathcal{F}$ and $\mathcal{G}$ are isomorphic.*

This follows from Claims 7.17–7.31 and Proposition 3.37.

The invariant set for the deformed IFS $\mathcal{G}$ is denoted $PG_n(N)$. To complete the proof of Theorem 1.4 it suffices to show the following two results.

Proposition 7.33. $\dim PG_n(N) \le 1 + C_0/\log n$, *where* $C_0 < \infty$ *depends only on* N.

Proposition 7.34. *For each* $n \ge 2$ *and* $N \ge 5$, $N \not\equiv 0 \pmod 4$, *the canonical homeomorphism* $F : PG(N) \to PG_n(N)$ *may be extended to a quasiconformal map of* $\mathbb{R}^2$.

Proof of Proposition 7.33. Let $\tilde{\lambda}^m_k = \operatorname{diam} \tilde{\Pi}^m_k / \operatorname{diam} \Pi$ for $1 \le k \le N-1$, $1 \le m \le n$ or $k = 0$, $m = n$.

Claim 7.35. *The contraction ratios $\tilde{\lambda}^m_k$ take on only five values when* $N \equiv 2, 3 \pmod 4$ *and six values when* $N \equiv 1 \pmod 4$.

In Block 1 there is one value associated with the joints and the bridge and a second value associated with the sides. In Blocks 2 through $n-1$ there is one value associated with the bridge and a second value associated with the joints, which is the same for the sides when $N \equiv 2, 3 \pmod 4$ and a third value associated with the sides when $N \equiv 1 \pmod 4$. Finally, in Blocks n and $n+1$ the value associated with the joints and the bridge is the same as that for Block $n-1$, and there is an additional value associated with the sides and with Block $n+1$.

Returning to the proof of Proposition 7.33, there exist constants $0 < C_1 < 1 < C_2 < \infty$, depending on N but not on n, so that

$$\frac{C_1}{n} \le \tilde{\lambda}_k^m \le \frac{C_2}{n} \tag{7.36}$$

for all relevant k and m and all $n \ge 0$.

Since the deformed IFS $\mathcal{G}$ is of gasket type, it satisfies the open set condition and hence $\dim PG_n(N) =: s_n$ is the unique positive solution to the equation

$$N \sum_{k=1}^{N-1} \sum_{m=1}^{n} (\tilde{\lambda}_k^m)^{s_n} + (\tilde{\lambda}_0^n)^{s_n} = 1. \tag{7.37}$$

Combining (7.36) and (7.37), we find

$$1 \le N^2 n C_2^2 (\frac{1}{n})^{s_n}$$

which implies $s_n \le 1 + 2\log(C_2 N)/\log n$ as desired. □

Next, we turn to the proof of Proposition 7.34. In what follows we work in the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $U_\emptyset$ be the component of $\hat{\mathbb{C}} \setminus PG(N)$ which contains the origin, and for each $w \in W$, let U_w be the component of $\hat{\mathbb{C}} \setminus PG(N)$ which contains the center of Π_w. Furthermore, let U_∞ be the unbounded component of $\hat{\mathbb{C}} \setminus PG(N)$. Observe that all of the domains $U_\emptyset$ and U_w, $w \in W$, are pairwise similar.

A topological circle $\Gamma \subset \hat{\mathbb{C}}$ is said to satisfy the *Ahlfors three-point condition* with constant $a < \infty$ if for each pair of points $z, z' \in \Gamma$, one of the two components of $\Gamma \setminus \{z, z'\}$ has diameter at most $a|z - z'|$.

Lemma 7.38. *There exists a constant $a = a(N) < \infty$ so that the boundary of each of the complementary components of the N-gasket satisfies the Ahlfors condition with constant a.*

Proof. It suffices to verify that $\partial U_\emptyset$ and ∂U_∞ satisfy the Ahlfors condition with some constant $a < \infty$. For simplicity we only consider $\partial U_\emptyset$; the other case is virtually identical.

Let δ and θ be the geometric data associated with the polygasket as in (3.26) and (3.27).

For each $w \in W$, $\Pi_w \cap \partial U_\emptyset$ is either empty or is a topological arc. Let $z, z' \in \partial U_\emptyset$ and let w be the longest word in W for which $\Pi_w \ni z, z'$. Then $z \in \Pi_{wi}$ and $z' \in \Pi_{wj}$ for some $i \ne j$. There is a component γ of $\partial U_\emptyset \setminus \{z, z'\}$ contained in Π_w. We break the proof into two cases:

Case (i): Π_{wi} and Π_{wj} are disjoint. In this case $|z - z'| \ge \operatorname{dist}(\Pi_{wi}, \Pi_{wj}) \ge \delta \operatorname{diam} \Pi_w \ge \delta \operatorname{diam} \gamma$, and the Ahlfors condition holds with $a = 1/\delta$.

Case (ii): Π_{wi} and Π_{wj} are adjacent. Let $\Pi_{wi} \cap \Pi_{wj} = \{\zeta\}$. Choose σ, τ such that $z, \zeta \in \Pi_{w\sigma}$, $z', \zeta \in \Pi_{w\tau}$, $|z - \zeta| \succeq \operatorname{diam} \Pi_{w\sigma}$ and $|z' - \zeta| \succeq \operatorname{diam} \Pi_{w\tau}$.

Since $\gamma \subset \Pi_{w\sigma} \cup \Pi_{w\tau}$ we conclude that

$$\operatorname{diam} \gamma \leq \operatorname{diam} \Pi_{w\sigma} + \operatorname{diam} \Pi_{w\tau} \preceq |z-\zeta| + |z'-\zeta| \preceq |z-z'|$$

by Lemma 3.29. This completes the proof of Lemma 7.38. □

Proof of Theorem 7.34. Let $F : PG(N) \to PG_n(N)$ be the canonical map. By Proposition 7.32 and Theorem 1.7, F is quasisymmetric. By Lemma 7.38 and a theorem of Ahlfors [1], each component of $\hat{\mathbb{C}} \setminus PG(N)$ is a K-quasidisc with $K < \infty$ depending on $a(N)$ only. It follows from another theorem of Ahlfors [1] that F can be extended to a function $\tilde{F}$, continuous on $\hat{\mathbb{C}}$ and K_1-quasiconformal on each component of $\hat{\mathbb{C}} \setminus PG(N)$, with K_1 depending only K and the quasisymmetry data of F on $PG(N)$. A theorem of Väisälä [28, Theorem 3.3] guarantees that $\tilde{F}$ is quasiconformal in $\hat{\mathbb{C}}$. □

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