

GEOMETRY AND QUASISYMMETRIC PARAMETRIZATION OF SEMMES SPACES

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ABSTRACT. We consider decomposition spaces \mathbb{R}^3/G that are manifold factors and admit defining sequences consisting of cubes-with-handles. Metrics on \mathbb{R}^3/G constructed via modular embeddings into Euclidean spaces promote the controlled topology to a controlled geometry.

The quasisymmetric parametrizability of the metric space $\mathbb{R}^3/G \times \mathbb{R}^m$ imposes quantitative topological constraints, in terms of the circulation and growth, to the defining sequences for \mathbb{R}^3/G . We give a necessary condition and a sufficient condition for the existence of parametrization.

The necessary condition answers a question of Heinonen and Semmes on quasisymmetric parametrizability of spaces associated to the Bing double. The sufficient condition gives new examples of quasispheres in \mathbb{S}^4 .

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1. INTRODUCTION

1.1. A homeomorphism $f: X \rightarrow Y$ between metric spaces (X, d_X) and (Y, d_Y) is called *quasisymmetric* if there exists a homeomorphism $\eta: [0, \infty) \rightarrow$

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$[0, \infty)$ so that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta \left(\frac{d_X(x, y)}{d_X(x, z)} \right)$$

for all triples $\{x, y, z\}$ in X . Quasisymmetry generalizes quasiconformality, from Euclidean spaces to general metric spaces. A metric space (X, d) is called a *metric n -sphere* if it is homeomorphic to \mathbb{S}^n .

When is a metric n -sphere (X, d) quasisymmetrically equivalent to the standard \mathbb{S}^n ? The goal is to find qualitative metric properties of the space (X, d) that recognize such geometric equivalence. A complete characterization of quasispheres is known only for dimensions 1 and 2.

In dimension 1, a result of Tukia and Väisälä [21] states *a metric 1-sphere (X, d) is quasisymmetrically equivalent to \mathbb{S}^1 if and only if X is doubling and is of bounded turning*. Bonk and Kleiner [5, Theorem 1.1] give a characterization in dimension 2. A consequence of their theorems states that *a metric 2-sphere (X, d) is quasisymmetrically equivalent to \mathbb{S}^2 if X is linearly locally contractible and Ahlfors 2-regular*. Semmes proved this result earlier for metric spaces with some added smoothness properties [17, Section 5]. Wildrick proved recently an analogue of Bonk and Kleiner's result for \mathbb{R}^2 [22].

A metric space (X, d) is said to be *linearly locally contractible* if every ball of radius r is contractible in a concentric ball of radius Cr , for a fixed $C > 1$; and X is said to be *Ahlfors 2-regular* if there exists a measure μ on the space so that the μ -measure of every ball of radius r is uniformly comparable to r^2 .

Could a metric space which is homeomorphic to \mathbb{S}^n , or \mathbb{R}^n , and resembles \mathbb{S}^n , or \mathbb{R}^n , geometrically (linearly locally contractible), measure-theoretically (Ahlfors n -regular), and analytically (supports Poincaré and Sobolev inequalities) in dimensions $n \geq 3$, fail to be quasisymmetrically equivalent to \mathbb{S}^n , or \mathbb{R}^n ?

Semmes' counterexample [19] to this natural question in dimension 3 is a geometrically self-similar space modeled on the decomposition space \mathbb{R}^3/Bd associated to the Bing double Bd. The classical construction of R.H. Bing in geometric topology gives an example of a wild involution in \mathbb{R}^3 . As a topological space \mathbb{R}^3/Bd is homeomorphic to \mathbb{R}^3 .

Semmes shows that this space admits a metric that is smooth Riemannian outside a totally disconnected closed set and, in many ways, indistinguishable from the standard metric on \mathbb{R}^3 , and yet the space is not quasisymmetrically equivalent to \mathbb{R}^3 . In Semmes' metric the 2^k tori at k -th stage of the construction of \mathbb{R}^3/Bd are uniformly round and thick, whereas under any homeomorphism from \mathbb{R}^3/Bd to \mathbb{R}^3 , there exists a sequence of tori that are distorted into a shape longer and thinner than allowed by any fixed quasisymmetry. Semmes' construction is robust and essentially available in all decomposition spaces of \mathbb{R}^3 arising from topologically self-similar constructions.

The natural conditions for metric n -spheres listed earlier are also insufficient in higher dimensions. The decomposition space \mathbb{R}^3/Wh associated to the Whitehead continuum Wh is not homeomorphic to \mathbb{R}^3 , but $\mathbb{R}^3/\text{Wh} \times \mathbb{R}$ is homeomorphic to \mathbb{R}^4 . In [12] Heinonen and the second author showed

that the decomposition space \mathbb{R}^3/Wh associated to the Whitehead continuum Wh admits a linearly locally contractible and Ahlfors 3-regular metric, but $(\mathbb{R}^3/\text{Wh}) \times \mathbb{R}^m$ is not quasisymmetrically equivalent to \mathbb{R}^{3+m} for any $m \geq 1$. The metric on \mathbb{R}^3/Wh is due to Semmes, as in the case of \mathbb{R}^3/Bd it makes the tori in the construction of the Whitehead continuum uniformly round and thick. The Whitehead link, formed by a meridian of the first torus and the core of the second torus, however prevents the *conformal modulus* of a sequence of surface families over longitudes of the nested tori from being quasi-preserved under any homeomorphism $\mathbb{R}^3/\text{Wh} \times \mathbb{R}^m \rightarrow \mathbb{R}^{3+m}$.

The decomposition space \mathbb{R}^3/Wh is only one example of an exotic manifold factor of \mathbb{R}^4 . By a theorem of Edwards and Miller [7], decomposition spaces that are exotic factors of \mathbb{R}^4 exist in abundance. In fact, cell-like closed 0-dimensional upper semicontinuous decomposition spaces \mathbb{R}^3/G are manifold factors of \mathbb{R}^4 , that is, $\mathbb{R}^3/G \times \mathbb{R}$ is homeomorphic to \mathbb{R}^4 . Furthermore, under mild assumptions on the decomposition, these spaces are definable by nested sequences $\mathcal{X} = (X_k)_{k \geq 0}$ of unions of cubes-with-handles, i.e., the degenerate part of the decomposition G is $\bigcap_{k \geq 0} X_k$; see Lambert and Sher [14] and Sher and Alford [20]. This class of decomposition spaces provides a natural environment for testing the quasisymmetric parametrization.

1.2. In this article, we consider a subclass of decomposition spaces \mathbb{R}^3/G that are manifold factors and admit defining sequences of finite type. A defining sequence $\mathcal{X} = (X_k)_{k \geq 0}$ is of *finite type* if the components of all difference sets $X_k \setminus \text{int} X_{k+1}$ for $k \geq 0$ have finitely many PL-homeomorphism classes.

We take up a systematic study of geometrical realizations of these spaces that promote the controlled topology to a controlled geometry. This upgrading is obtained by introducing the concept of a *welding structure*. A welding structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ consists of *condensers* \mathcal{C} , an *atlas* \mathcal{A} consisting of *charts*, and *weldings* \mathcal{W} determined by the atlas \mathcal{A} . Whereas the condensers can be seen as fixed geometric realizations of PL-homeomorphism equivalence classes of components of differences $X_k \setminus \text{int} X_{k+1}$ in the defining sequence \mathcal{X} , the charts in the atlas \mathcal{A} determine the parametrization of these components. The weldings, in turn, are transition maps between the charts; see Section 4.1.

Although every defining sequence of finite type \mathcal{X} admits a natural welding structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$, the collection of weldings \mathcal{W} need not even be homotopically finite. Using results from classical geometric topology, we construct for every defining sequence of finite type a geometrically simple welding structure, called *Semmes structure*, with only translations as weldings; see Theorem 5.2.

Semmes structures allow natural geometrization of the decomposition space \mathbb{R}^3/G . Given a Semmes structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ and a *scaling factor* $\lambda \in (0, 1)$, we show that there exists a *modular embedding* $\mathbb{R}^3/G \rightarrow \mathbb{R}^n$ that respects the atlas \mathcal{A} and the chosen scaling factor λ ; see Theorem 5.3. We call the metrics induced on \mathbb{R}^3/G by modular embeddings *Semmes metrics* and the corresponding metric spaces *Semmes spaces*.

For a fixed Semmes structure, the Semmes spaces $(\mathbb{R}^3/G, d_\lambda)$ for all scalings are mutually quasisymmetric; see Proposition 6.10. We find it appealing that, although \mathbb{R}^3/G does not admit a canonical metric, *there exists a natural class of metrics on \mathbb{R}^3/G respecting \mathcal{X} whose quasisymmetry equivalence classes are essentially parametrized by Semmes structures on \mathcal{X} .*

The metrics in this paper naturally extend the class of metrics constructed by Semmes in [19].

As in the self-similar case of Semmes, the metric spaces $(\mathbb{R}^3/G, d_\lambda)$ for the given scaling factors λ , are quasiconvex, Ahlfors 3-regular for a range of values of λ depending on the growth of the defining sequence, and linearly locally contractible when a local contractibility is assumed on the defining sequence; see Section 6. A defining sequence $\mathcal{X} = (X_k)_{k \geq 0}$ is *locally contractible* if every component of X_{k+1} is contractible in X_k for all $k \geq 0$. Using the terminology of Semmes, spaces $(\mathbb{R}^3/G, d_\lambda)$ carry thick pencils of curves (Remark 5.9). Thus, by a general result of Semmes, spaces $(\mathbb{R}^3/G, d_\lambda)$ support $(1, 1)$ -Poincaré inequality. In particular, spaces $(\mathbb{R}^3/G, d_\lambda)$ are *Loewner spaces* in the sense of Heinonen and Koskela [9]. We discuss these properties in more detail in Section 6.4.

1.3. Having this general theory at our disposal, we now discuss the problem of quasisymmetric parametrization.

Due to the quasi-invariance of the conformal modulus, the existence of a quasisymmetric homeomorphism between $\mathbb{R}^3/G \times \mathbb{R}^m$ and \mathbb{R}^{3+m} imposes a constraint between geometry (growth of the handlebodies and fixed scaling factor) and topology (circulation of the handlebodies).

The *order of growth* can be viewed as the growth of the number of components of X_k in the sequence as k tends to infinity; see Definition 4.1. The *order of circulation* of \mathcal{X} describes the growth of the unsigned linking numbers of longitudes of handlebodies of $X_{k'}$ with respect to the meridians of X_k , for $k' > k$; see Definition 8.1. For the Whitehead continuum and the Bing double, these quantities are $(\omega_{\text{Wh}}, \gamma_{\text{Wh}}) = (2, 1)$ and $(\omega_{\text{Bd}}, \gamma_{\text{Bd}}) = (2, 2)$ respectively.

Given a Semmes metric d_λ on \mathbb{R}^3/G , the space $\mathbb{R}^3/G \times \mathbb{R}^m$ is equipped with the product metric $d_{\lambda, m}((x, v), (y, w)) = d_\lambda(x, y) + |v - w|$.

Theorem 1.1. *Let \mathbb{R}^3/G be a decomposition space of finite type associated to a locally contractible defining sequence \mathcal{X} . Suppose that the order of growth of the defining sequence \mathcal{X} is at most $\gamma_\mathcal{X}$, the order of circulation is at least $\omega_\mathcal{X}$, and*

$$(1.1) \quad \omega_\mathcal{X}^3 > \gamma_\mathcal{X}^2.$$

Then there exists a Semmes metric on \mathbb{R}^3/G so that $\mathbb{R}^3/G \times \mathbb{R}^m$ is a linearly locally contractible, Ahlfors $(3 + m)$ -regular, Loewner space but not quasisymmetrically equivalent to \mathbb{R}^{3+m} for any $m \geq 0$.

In particular, Theorem 1.1 answers a question of Heinonen and Semmes in [11, Question 11].

Theorem 1.2. *The decomposition space \mathbb{R}^3/Bd associated to the Bing double admits a metric that is Ahlfors 3-regular and linearly locally contractible*

but none of the spaces $\mathbb{R}^3/\text{Bd} \times \mathbb{R}^m$ for $m \geq 1$ is quasisymmetrically equivalent to \mathbb{R}^{3+m} .

When applying Theorem 1.1, estimating the order of circulation from below for a particular decomposition space can be a challenging topological problem on its own. In the decomposition space associated to the Bing double in [19], or the Whitehead decomposition space in [12], or Bing's dogbone space to be discussed in Section 15, the circulation is estimated by adapting a theorem of Freedman and Skora [8] on the relative homologies and essential intersections.

Theorem 1.1 gives a necessary topological condition for the quasisymmetric parametrizability of a Semmes space $\mathbb{R}^3/G \times \mathbb{R}^m$ associated to a defining sequence. In the opposite direction, additional Euclidean restrictions on the welding structure yield positive parametrization results for \mathbb{R}^3/G without stabilization. These geometric assumptions on the defining sequence are encapsulated in the notion of the *strong welding structure*; see Section 7.

Theorem 1.3. *Let \mathbb{R}^3/G be a decomposition space of finite type that admits a defining sequence with a strong welding structure in \mathbb{R}^3 . Then there exists an Ahlfors 3-regular linearly locally contractible metric on \mathbb{R}^3/G so that \mathbb{R}^3/G is quasisymmetric to \mathbb{R}^3 . Moreover, there exist an isometric embedding $\theta: \mathbb{R}^3/G \rightarrow \mathbb{R}^4$ and a quasisymmetric homeomorphism $f: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ so that $f(\mathbb{R}^3) = \theta(\mathbb{R}^3/G)$.*

These decomposition spaces give new examples of *quasispheres* in \mathbb{R}^4 as formulated in the second part of the theorem.

In light of Theorem 1.3, we ask about the sharpness of the condition (1.1) in Theorem 1.1, especially for a fixed m . In case of \mathbb{R}^3 (i.e. $m = 0$) the construction of Antoine's necklaces G using I linked tori yields a class of decomposition spaces where the circulation and the growth can be calculated; $(\omega_G, \gamma_G) = (2, I)$. Semmes' result on the Bing double implies that the decomposition space $(\mathbb{R}^3/G, d)$ associated with a necklace of two tori, when equipped with a Semmes metric d , is not quasisymmetric to \mathbb{R}^3 .

The existence of a quasisymmetric parametrization of \mathbb{R}^3/G when I is large has been observed by Heinonen and Rickman [10] using round similar tori. Using rectangular tori in place of round tori, we prove in Section 13 that *for every $I \geq 10$, the decomposition space \mathbb{R}^3/G associated to Antoine's I -necklace may be equipped with a Semmes metric so that it is quasisymmetrically equivalent to \mathbb{R}^3* ; see Theorem 13.1.

Having these examples at hand, the real test for the sharpness of Theorem 1.1 seems to be the quasisymmetric parametrizability of the decomposition space associated to the Antoine's 3-necklace.

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2. PRELIMINARIES

Unless otherwise stated, we assume that \mathbb{R}^n , $n \geq 1$, is equipped with the Euclidean metric with and the standard basis (e_1, \dots, e_n) . We denote

by $B^n(x, r)$ the closed Euclidean ball in \mathbb{R}^n of radius r and center x . For brevity, the closed balls about origin are denoted $B^n(r) = B^n(0, r)$ for $r > 0$ and $\mathbb{B}^n = B^n(1)$. Similarly, $S^{n-1}(x, r) = \partial B^n(x, r)$ is the Euclidean sphere of radius r and center x in \mathbb{R}^n , and, $S^{n-1}(r) = S^{n-1}(0, r)$ for $r > 0$ and $\mathbb{S}^{n-1} = S^{n-1}(1)$.

For all $1 \leq m < n$, we identify \mathbb{R}^m with the subspace $\mathbb{R}^m \times \{0\}$ in \mathbb{R}^n where $\{0\}$ is the origin in \mathbb{R}^{n-m} , and identify a set $A \subset \mathbb{R}^m$ with the set $A \times \{0\}$ in $\mathbb{R}^m \times \mathbb{R}^{n-m}$. When \mathbb{R}^n is expressed as $\mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^q$ with $m, p, q > 0$, $m + p + q = n$, a subset of \mathbb{R}^n in the form $A \times B \times C$ is understood to have the property that $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^p$, and $C \subset \mathbb{R}^q$.

By a map, we always mean a continuous map. Given a map $F: X \times [0, 1] \rightarrow Y$, we denote by $F_t: X \rightarrow Y$ the map $F_t(x) = F(x, t)$. We say that a homotopy $F: X \times [0, 1] \rightarrow Y$ is an *isotopy* if F_t is a homeomorphism for all $t \in [0, 1]$.

We call a map $\alpha: I \rightarrow X$ from an interval in \mathbb{R} into a metric space X as a *path* and maps $\mathbb{S}^1 \rightarrow X$ as *loops*. If there is no confusion we do not make distinction between a map and its image. Images of paths and loops are also called as *curves*. A loop $\mathbb{S}^1 \rightarrow X$ is *simple* if it is an embedding.

Given a set E in a metric space (X, d) and a number $a > 0$, we call

$$N_d(E, a) = \{x \in X : \text{dist}_d(x, E) < a\}$$

the a -neighborhood of E in X . When $X = \mathbb{R}^n$ and d is the Euclidean metric, we write $N^n(E, a)$ for $N_d(E, a)$. We denote by $\mathcal{C}(E)$ the set of all connected components of E .

Given a metric space (X, d) so that points in the space can be connected by rectifiable paths, we denote by \hat{d} the *path metric* of (X, d) defined by

$$\hat{d}(x, y) = \inf_{\gamma} \ell_d(\gamma)$$

for $x, y \in X$, where $\ell_d(\gamma)$ is the length of path γ in metric d and the infimum is taken over all paths γ connecting x and y in X . A metric space (X, d) is called *quasiconvex* if $\text{id}: (X, \hat{d}) \rightarrow (X, d)$ is bilipschitz.

A metric space (X, d) is *Ahlfors Q -regular* if there exist a Borel measure μ in X and a constant $C \geq 1$ so that

$$\frac{1}{C}r^Q \leq \mu(B(x, r)) \leq Cr^Q$$

for every ball $B(x, r)$ of radius $r \leq \text{diam } X$ about x in X . Furthermore, the space (X, d) is *locally linearly contractible* if there exists $C \geq 1$ so that the ball $B(x, r)$ in X is contractible in $B(x, Cr)$ for all $r < 1/C$.

We say that a mapping $f: (X, d_X) \rightarrow (Y, d_Y)$ between metric spaces is a (λ, L) -*quasisimilarity* if

$$\frac{\lambda}{L}d_X(x, y) \leq d_Y(f(x), f(y)) \leq \lambda L d_X(x, y)$$

for all $x, y \in X$. Clearly, quasisimilarities are a subclass of quasisymmetries. As usual, we call $(\lambda, 1)$ -quasisimilarities as *similarities* and 1-similarities as *isometries*. The $(1, L)$ -quasisimilarities are *L -bilipschitz mappings*. In what follows, we abuse the notation and denote $|x - y| = d(x, y)$ when there is no ambiguity on the metric in question.

In a metric measure space (X, d, μ) we define the *p-modulus of an m-chain family* as follows. In what follows, we consider only Lipschitz chains of multiplicity one, that is, we consider only m -chains σ so that $\sigma = \sum_{i=1}^k \sigma_i$, where $\sigma_i: [0, 1]^m \rightarrow X$ is a Lipschitz map for $i = 1, \dots, k$.

Given a family Σ of m -chains in a X , the p -modulus of Σ is

$$(2.1) \quad \text{Mod}_p(\Sigma) = \inf_{\rho} \int_X \rho^p \, d\mu,$$

where ρ is a non-negative Borel function satisfying

$$(2.2) \quad \sum_{i=1}^k \int_{\sigma_i([0,1]^m)} \rho \, d\mathcal{H}^m \geq 1$$

for all $\sigma = \sum_{i=1}^k \sigma_i \in \Sigma$.

The handlebodies we consider are three dimensional piecewise linear cubes-with-handles embedded into \mathbb{R}^n . For this we assume in what follows that \mathbb{R}^n is given a fixed PL-structure for every $n \geq 3$.

We use the following topological facts on cubes-with-handles; see [13, Chapter 2] for more details. We say that H is a cube-with-handles if it is a regular neighborhood of an embedded *rose* $\iota(\bigvee^g \mathbb{S}^1)$, where $\iota: \bigvee^g \mathbb{S}^1 \rightarrow \mathbb{R}^3$ is a PL-embedding. Here $\bigvee^g \mathbb{S}^1$ is the *wedge of g circles*, that is, identification of g circles at a point; $\bigvee^0 \mathbb{S}^1$ is a point. The number g of circles in the rose is called the *number of handles of H* or the *genus of H* . The image $\iota(\bigvee^g \mathbb{S}^1)$ is called a *core of H* .

The genus of H is also the maximal number of essentially embedded 2-disks that do not separate H . We say that a disk D in H is *essentially embedded* if there exists an embedding $\varphi: (\mathbb{B}^2, \partial\mathbb{B}^2) \rightarrow (H, \partial H)$ so that $\varphi|_{\partial\mathbb{B}^2}: \partial\mathbb{B}^2 \rightarrow \partial H$ is not null-homotopic in ∂H .

The genus of H is a topological invariant. Two cube-with-handles H and H' in \mathbb{R}^3 are PL homeomorphic if and only if they have the same number of handles and both are either orientable or non-orientable ([13, Theorem 2.2]). We denote by $g(H)$ the *genus of H* .

Three dimensional cube-with-handles in \mathbb{R}^n need not be orientable for $n > 3$, but three dimensional cubes-with-handles in \mathbb{R}^3 inherit orientation from \mathbb{R}^3 and are therefore orientable.

3. DECOMPOSITION SPACES

We begin this section by reviewing some classical results on decomposition spaces relevant to our study. We do not aim at the full generality and refer to Daverman [6] for details.

A *decomposition* G of a topological space X is a partition of X . Associated with G is the decomposition space X/G equipped with the topology induced by the quotient map $\pi_G: X \rightarrow X/G$, the richest topology for which π_G is continuous, see [6, p. 8].

A decomposition G is *upper semi-continuous* (usc) if each $g \in G$ is closed and if for every $g \in G$ and every neighborhood U of g in X there exists a neighborhood V of g contained in U so that every $g' \in G$ intersecting V is contained in U . If G is *upper semi-continuous* then X/G is metrizable [6,

Definition I.2 and Proposition I.2.2]; however there is not a canonical metric on X/G .

Suppose that G is a usc decomposition of an n -manifold M and d is a metric on M/G . The decomposition map $\pi_G: M \rightarrow M/G$ can be approximated by homeomorphisms if and only if G satisfies Bing's shrinkability criterion [6, Theorem II.5.2]. In particular, M/G is homeomorphic to M .

Bing's *shrinkability criterion* states that for every $\varepsilon > 0$ there is a homeomorphism $h: M \rightarrow M$ such that

- (1) $\text{diam } h(g) < \varepsilon$ for each $g \in G$, and
- (2) $d(\pi_G h(x), \pi_G(x)) < \varepsilon$ for every $x \in M$.

Suppose M is an n -manifold. If G is a shrinkable usc decomposition then each $g \in G$ is cellular, therefore cell-like [6, Proposition II.6.1 and Corollary III.15.2B]. A subset Z of M is *cellular* if for each open $U \supset Z$ there is an n -cell E such that $Z \subset \text{int} E \subset E \subset U$; recall that an n -cell is a subset homeomorphic to \mathbb{B}^n . A compact set Z in a space X is *cell-like in X* if Z can be contracted to a point in every neighborhood of Z .

Certain decomposition spaces can be constructed from *defining sequences*. A *defining sequence for a decomposition* of an n -manifold M is a sequence $\mathcal{X} = (X_k)_{k \geq 0}$ of compact sets satisfying $\text{int } X_k \supset X_{k+1}$. The *decomposition* G associated to the defining sequence \mathcal{X} consists of the components of $X_\infty = \bigcap_{k \geq 0} X_k$ and the singletons from $M \setminus X_\infty$, see [6, p.61]. Then the decomposition space M/G associated to \mathcal{X} is upper semi-continuous and $\overline{\pi_G(X_\infty)}$ is compact and 0-dimensional, see [6, Proposition II.9.1].

In the context of defining sequences, a sufficient condition for \mathbb{R}^3/G to be homeomorphic with \mathbb{R}^3 is the following shrinking criterion: *For each $k \geq 1$ and each $\epsilon > 0$, there exist $\ell \geq 1$ and a homeomorphism h of \mathbb{R}^3 onto itself satisfying $h|(\mathbb{R}^3 \setminus X_k) = \text{id}$, and $\text{diam } h(H) < \epsilon$ for all components H of $X_{k+\ell}$.*

We fix some notations for following sections. Let $\mathcal{X} = (X_k)_{k \geq 0}$ be a defining sequence. We denote by $\mathcal{C}(\mathcal{X}) = \bigcup_k \mathcal{C}(X_k)$ all components of the defining sequence $\mathcal{X} = (X_k)_{k \geq 0}$; recall that $\mathcal{C}(E)$ the set of all components of E .

Given $H \in \mathcal{C}(\mathcal{X})$ there is a unique index $k \geq 0$ so that $H \in \mathcal{C}(X_k)$. We call the index k the *level* of H and denote $\text{level}(H) = k$. For every $H \in \mathcal{C}(\mathcal{X})$, we denote

$$H^{\text{diff}} = H \setminus \text{int} X_{\text{level}(H)+1}.$$

Then $\mathcal{C}(H \setminus \text{int} H^{\text{diff}})$ consists of all components of $X_{\text{level}(H)+1}$ contained in H .

Given handlebodies H and H' in $\mathcal{C}(\mathcal{X})$, we have

$$H = H', \quad H' \subset \text{int} H, \quad H \subset \text{int} H', \quad \text{or} \quad H \cap H' = \emptyset.$$

Thus $\partial H \cap X_\infty = \emptyset$ for every $H \in \mathcal{C}(\mathcal{X})$. Since X_∞ is closed in \mathbb{R}^3 , there exists, for every $H \in \mathcal{C}(\mathcal{X})$, a neighborhood $\Omega_{\partial H}$ of ∂H in \mathbb{R}^3 so that $\pi_G|_{\Omega_{\partial H}}$ is an embedding.

At times we shall write \mathbb{R}^3/X_∞ for \mathbb{R}^3/G for simplicity, in particular when X_∞ is a Whitehead continuum, a necklace, a the Bing double or a Bing's dogbone space.

3.1. Decomposition spaces as manifold factors. Our main interest lies in decomposition spaces \mathbb{R}^3/G that are homeomorphic to \mathbb{R}^3 or that $\mathbb{R}^3/G \times \mathbb{R}^m$ is homeomorphic to \mathbb{R}^{3+m} for some $m > 0$. Decomposition spaces of the latter type are called as *manifold factors of Euclidean spaces*.

By results of Sher and Alford and Lambert and Sher ([20, Theorem 1] and [14]), if G is a cell-like usc decomposition of \mathbb{R}^3 so that the closure of all non-degenerate elements of G is 0-dimensional, then G admits a defining sequence with (unions of) cubes-with-handles. Subsequently Edwards and Miller [7, p.192] proved that if G satisfies the conditions of Lambert and Sher, then \mathbb{R}^3/G is a *factor* of \mathbb{R}^4 , that is,

$$(3.1) \quad \mathbb{R}^3/G \times \mathbb{R} \approx \mathbb{R}^4,$$

and $G \times \mathbb{R}$ is a shrinkable decomposition of \mathbb{R}^4 , see also [6, Section V. 27]. In particular, the quotient map $\pi' : \mathbb{R}^{3+m} \rightarrow \mathbb{R}^{3+m}/(G \times \mathbb{R}^m)$ can be approximated by homeomorphisms. The composition $(\pi_G \times \text{id}) \circ (\pi')^{-1} : \mathbb{R}^{3+m}/(G \times \mathbb{R}^m) \rightarrow \mathbb{R}^3/G \times \mathbb{R}^m$ is a homeomorphism [6, Proposition I.2.4]. Therefore

$$\mathbb{R}^3/G \times \mathbb{R}^m \approx \mathbb{R}^{3+m},$$

and $\pi_G \times \text{id} : \mathbb{R}^{3+m} \rightarrow \mathbb{R}^3/G \times \mathbb{R}^m$ can be approximated by homeomorphisms.

Let \mathbb{R}^3/G be a decomposition space associated to a locally contractible defining sequence $\mathcal{X} = (X_k)_{k \geq 0}$ consisting of unions of cubes-with-handles. Then, by Edwards-Miller, $\mathbb{R}^3/G \times \mathbb{R}$ is homeomorphic to \mathbb{R}^4 . Indeed, under this assumption on \mathcal{X} , the components of X_∞ are cell-like and $\pi_G(X_\infty)$ is compact and 0-dimensional.

3.2. Local contractibility. In this section, we apply the Edwards-Miller theorem to obtain local contractibility of \mathbb{R}^3/G from the local contractibility of its defining sequence. We prove a slightly stronger result than needed in our further applications.

Lemma 3.1. *Let $m \geq 1$ and let \mathbb{R}^3/G be a decomposition space associated to a locally contractible defining sequence $\mathcal{X} = (X_k)_{k \geq 0}$. Then, for every $k \geq 0$, $H' \in \mathcal{C}(X_k)$, $H \in \mathcal{C}(X_{k+1} \cap H')$ and $r > 0$, there exists a $(3+m)$ -cell E so that*

$$\pi_G(H) \times [-r, r]^m \subset E \subset \pi_G(H') \times (-2r, 2r)^m.$$

In particular, components of $\pi_G(X_{k+1})$ are contractible in $\pi_G(X_k)$ for $k \geq 0$.

The proof of the lemma is based on an approximation of the quotient map $\pi_G \times \text{id} : \mathbb{R}^{3+m} \rightarrow \mathbb{R}^3/G \times \mathbb{R}^m$ by homeomorphisms, and a classical Penrose-Whitehead-Zeeman lemma ([15, Lemma 2.7]): *Let M be an n -manifold and let $P \subset \text{int}M$ be an $(q-1)$ -dimensional polyhedron ($0 < 2q \leq n$) such that the inclusion map $i : P \rightarrow M$ is homotopic in M to a constant. Then there exists an n -cell $E \subset \text{int}M$ such that $P \subset \text{int}E$.*

Proof of Lemma 3.1. Let $r > 0$, $k \geq 0$, and let $H' \in \mathcal{C}(X_k)$ and $H \in \mathcal{C}(X_{k+1} \cap H')$ be handlebodies in \mathcal{X} . Let δ be any metric on the decomposition space \mathbb{R}^3/G , and δ_m be the product of δ with the Euclidean metric on $\mathbb{R}^3/G \times \mathbb{R}^m$. Let $a_0 = \min\{r, \text{dist}_\delta(\partial\pi_G(H), \partial\pi_G(H'))\}$.

We fix cores \mathcal{R} and \mathcal{R}' of H and H' , respectively, and choose a regular neighborhood H'' of \mathcal{R} and a regular neighborhood H''' of \mathcal{R}' so that

$$H \subset \text{int}H'' \subset H'' \subset \text{int}H''' \subset H''' \subset \text{int}H' \subset H'.$$

We may assume that $a_0/10 < \text{dist}_\delta(x, \partial\pi_G H) < a_0/9$ for all $x \in \partial\pi_G H''$ and $a_0/10 < \text{dist}_\delta(x, \partial\pi_G H') < a_0/9$ for all $x \in \partial\pi_G H'''$. By the uniqueness of regular neighborhoods ([16, Theorem 3.24]), H'' is a regular neighborhood of H and H' is a regular neighborhood of H''' . Since H is contractible in H' , we have that H'' is contractible in H''' .

By the Penrose-Whitehead-Zeeman lemma and the uniqueness of regular neighborhoods there exists a $(3+m)$ -cell E' so that

$$\mathcal{R} \times \{0\} \subset H'' \times \left(-\frac{5}{4}r, \frac{5}{4}r\right)^m \subset E' \subset H''' \times \left(-\frac{3}{2}r, \frac{3}{2}r\right)^m.$$

Since $\pi_G \times \text{id}$ can be approximated by homeomorphisms, by the Edwards-Miller theorem, we may fix a homeomorphism $h: \mathbb{R}^{3+m} \rightarrow \mathbb{R}^3/G \times \mathbb{R}^m$ so that

$$\max_{(x,v) \in X_0 \times [-3r, 3r]^m} \delta_m(h(x, v), (\pi_G(x), v)) < a_0/100.$$

Then $h^{-1}(\pi_G H \times [-r, r]^m) \subset H'' \times \left(-\frac{5}{4}r, \frac{5}{4}r\right)^m$ and $h(H''' \times \left(-\frac{3}{2}r, \frac{3}{2}r\right)^m) \subset \pi_G H' \times (-2r, 2r)^m$. Thus $E = h(E')$ is a $(3+m)$ -cell satisfying

$$\pi_G(H) \times [-r, r]^m \subset E \subset \pi_G(H') \times (-2r, 2r)^m.$$

Both claims now follow. \square

Convention. All decomposition spaces \mathbb{R}^3/G in this article are derived from defining sequences \mathcal{X} consisting of (unions of) cubes-with-handles. At times, we denote the space by $(\mathbb{R}^3/G, \mathcal{X})$ to emphasize the role of the sequence \mathcal{X} .

4. DECOMPOSITION SPACES OF FINITE TYPE

Recall, that a defining sequence $\mathcal{X} = (X_k)_{k \geq 0}$ has *finite type* if the components of all difference sets $X_k \setminus \text{int}X_{k+1}$ for $k \geq 0$ have finitely many PL-homeomorphism classes.

A defining sequence $\mathcal{X} = (X_k)_{k \geq 0}$ of finite type has a finite (*upper*) *growth*

$$(4.1) \quad \bar{\gamma}_{\mathcal{X}} = \max\{\#\mathcal{C}(X_{k+1} \cap H) : H \in \mathcal{C}(X_k), k \geq 0\}.$$

Definition 4.1. The *order of growth* $\gamma_{\mathcal{X}}$ of \mathcal{X} is defined to be

$$(4.2) \quad \gamma_{\mathcal{X}} = \limsup_{k \rightarrow \infty} (\#\mathcal{C}(X_k))^{1/k} \leq \bar{\gamma}_{\mathcal{X}}.$$

By the finiteness of the welding structure, the cubes-with-handles in $\mathcal{C}(\mathcal{X})$ have uniformly bounded genus; we denote

$$(4.3) \quad \bar{g}_{\mathcal{X}} = \max\{g(H) : H \in \mathcal{C}(\mathcal{X})\}.$$

4.1. Welding structures. Let $n \geq 3$. By abusing the standard terminology in potential theory, we say that a pair (A, B) is a *condenser in \mathbb{R}^n* if A is a 3-dimensional cube-with-handles in \mathbb{R}^n and B a disjoint union of 3-dimensional cubes-with-handles in \mathbb{R}^n so that $B \subset \text{int}A$. Given a condenser $\mathbf{c} = (A, B)$, we denote by

$$\mathbf{c}^{\text{diff}} = A \setminus \text{int}B.$$

Given two condensers $\mathbf{c} = (A, B)$ and $\mathbf{c}' = (A', B')$ in \mathbb{R}^n , a PL-embedding $\psi: \partial A' \rightarrow \partial B$ is said to be a *welding of \mathbf{c}' to \mathbf{c}* . Since $\partial A'$ is a closed surface

and ∂B is a disjoint union of closed surfaces in \mathbb{R}^n , $\psi(\partial A')$ is a component of ∂B . Here ∂M is the two dimensional manifold boundary of a 3-manifold M .

Let $(\mathbb{R}^3/G, \mathcal{X})$ be a decomposition space and \mathcal{C} a family of condensers in \mathbb{R}^n . Suppose that \mathcal{A} is a collection of PL-homeomorphisms $\{\varphi_H: H^{\text{diff}} \rightarrow \mathfrak{c}_H^{\text{diff}}: H \in \mathcal{C}(\mathcal{X}), \mathfrak{c}_H \in \mathcal{C}\}$ with the property that for every $H \in \mathcal{C}(\mathcal{X})$ there exists a unique φ_H in \mathcal{A} having H^{diff} as its domain and the difference set $\mathfrak{c}_H^{\text{diff}}$ of a particular condenser $\mathfrak{c} \in \mathcal{C}$ as its image. We call such a collection an *atlas for \mathcal{X}* , elements of \mathcal{A} *charts*.

The pair $(\mathcal{C}, \mathcal{A})$ induces a welding scheme \mathcal{W} , consisting of all transition maps $\varphi_H \circ \varphi_{H'}^{-1}|_{\partial A_{H'}}$ for $H \in \mathcal{C}(\mathcal{X})$ and $H' \in \mathcal{C}(H \cap X_{\text{level}(H)+1})$, where $\varphi_H: H^{\text{diff}} \rightarrow \mathfrak{c}_H^{\text{diff}}$ and $\varphi_{H'}: H'^{\text{diff}} \rightarrow \mathfrak{c}_{H'}^{\text{diff}}$ are charts in \mathcal{A} , and $\mathfrak{c}_H = (A_H, B_H)$ and $\mathfrak{c}_{H'} = (A_{H'}, B_{H'})$ are condensers in \mathcal{C} ; note that $\partial H' = H^{\text{diff}} \cap H'$, $\varphi_H \circ \varphi_{H'}^{-1}|_{\partial A_{H'}}$ is a welding of $(A_{H'}, B_{H'})$ to (A_H, B_H) , and

$$\begin{array}{ccc} & \partial H' & \\ \varphi_{H'}|_{\partial H'} \swarrow & & \searrow \varphi_H|_{\partial H'} \\ \partial A_{H'} & \xrightarrow{\varphi_H \circ \varphi_{H'}^{-1}|_{\partial A_{H'}}} & \partial B_H \end{array}$$

The triple $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ is called a *welding structure on \mathcal{X}* . We say that $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ is a *finite welding structure* if \mathcal{C} and \mathcal{W} are finite.

Remark 4.2. *A decomposition space $(\mathbb{R}^3/G, \mathcal{X})$ of finite type admits a welding structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ whose condensers are contained in \mathbb{R}^3 and are of different topological types.*

To see this, consider the collection \mathcal{D} of all condensers $(H, H \cap X_{\text{level}(H)+1})$ for $H \in \mathcal{C}(\mathcal{X})$ associated with the defining sequence \mathcal{X} . Let \mathcal{C} be a subcollection of \mathcal{D} that contains exactly one representative for each homeomorphism class in \mathcal{D} . For every $H \in \mathcal{C}(\mathcal{X})$, fix a PL-homeomorphism φ_H from $(H, H \cap X_{\text{level}(H)+1})$ to its representative $\mathfrak{c}_H = (A_H, B_H)$ in \mathcal{C} . The atlas \mathcal{A} consists of all charts $\varphi_H|_{H^{\text{diff}}}$, and weldings are induced by condensers and charts. Note that a priori this welding structure need not be finite

4.2. Self-Similar Spaces. *Self-similar decomposition spaces* are examples of decomposition spaces of finite type. Semmes' *initial packages* for defining self-similar decomposition spaces yield almost directly finite welding structures on the defining sequences if the initial packages are understood in the PL-category instead of smooth category; see [19, Section 3].

An *initial package* $(T, T_1, \dots, T_N, \phi_1, \dots, \phi_N)$ consists of cubes-with-handles T, T_1, \dots, T_N in \mathbb{R}^3 with $T_i \subset \text{int}T$ and $T_i \cap T_{i'} = \emptyset$ for $i \neq i'$, together with PL-embeddings $\phi_i: U \rightarrow T$ of a neighborhood U of T into T so that $\phi_i(T) = T_i$ and the images $\phi_i(U)$ are mutually disjoint neighborhoods of T_i 's. The defining sequence $\mathcal{X} = (X_k)_{k \geq 0}$ is given by $X_0 = T$ and

$$X_k = \bigcup_{\alpha} \phi_{\alpha}(T)$$

for $k \geq 1$, where $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, \dots, N\}^k$ and $\phi_{\alpha} = \phi_{\alpha_1} \circ \dots \circ \phi_{\alpha_k}$.

The *welding structure* $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ associated with the initial package consists of a single condenser $\mathfrak{c} = \left(T, \bigcup_{i=1}^k \phi_i(T)\right)$, the atlas

$$\mathcal{A} = \{(\phi_\alpha|T^{\text{diff}})^{-1} : \phi_\alpha(T)^{\text{diff}} \rightarrow T^{\text{diff}} : \alpha \in J\},$$

where $J = \bigcup_{k \geq 0} \{1, \dots, N\}^k$, and the weldings

$$\mathcal{W} = \{\phi_i|\partial T : 1 \leq i \leq N\}.$$

We refer to [19, Section 3] for more details on initial packages for self-similar decomposition spaces.

5. MODULAR EMBEDDINGS

In this section we discuss embeddings of decomposition spaces of finite type into Euclidean spaces. We show that a defining sequence of finite type admits a geometrically simple welding structure, called a *Semmes structure*. To a given Semmes structure we associate a *modular embedding* of \mathbb{R}^3/G to a Euclidean space respecting the quasisimilarity type of that structure. The embedding in turn defines a geometrically natural *modular metric* on the decomposition space \mathbb{R}^3/G .

Given a welding structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ on a decomposition space $(\mathbb{R}^3/G, \mathcal{X})$ of finite type and $0 < \lambda < 1$, we say that an embedding $\theta: \mathbb{R}^3/G \rightarrow \mathbb{R}^n$ is λ -*modular* (with respect to $(\mathcal{C}, \mathcal{A}, \mathcal{W})$) if $\theta \circ \pi_G|_{\mathbb{R}^3 \setminus X_0} = \text{id}$ and there exists $L \geq 1$ so that

$$(5.1) \quad \theta \circ \pi_G \circ \varphi_H^{-1} : \mathfrak{c}_H^{\text{diff}} \rightarrow \mathbb{R}^n$$

is a (λ^k, L) -quasisimilarity for every $H \in \mathcal{C}(X_k)$ and $k \geq 0$;

$$\begin{array}{ccc} H^{\text{diff}} & \xrightarrow{\varphi_H} & \mathfrak{c}_H^{\text{diff}} \\ \pi_G|_{H^{\text{diff}}} \downarrow & & \downarrow \theta \circ \pi_G \circ \varphi_H^{-1} \\ \mathbb{R}^3/G & \xrightarrow{\theta} & \mathbb{R}^n \end{array}$$

Given a λ -modular embedding $\theta: \mathbb{R}^3/G \rightarrow \mathbb{R}^n$ with respect to a welding structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$, we define the λ -*modular metric* d_θ on \mathbb{R}^3/G by

$$(5.2) \quad d_\theta(x, y) = |\theta(x) - \theta(y)|.$$

We need the notion of compatible atlases to compare modular metrics induced by modular embeddings with respect to two different welding structures. Welding structures $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ and $(\mathcal{C}', \mathcal{A}', \mathcal{W}')$ on \mathcal{X} are said to have *compatible atlases* if there exists $L \geq 1$ so that

$$\varphi'_H \circ \varphi_H^{-1}|_{\mathfrak{c}_H^{\text{diff}}} : \mathfrak{c}_H^{\text{diff}} \rightarrow (\mathfrak{c}'_H)^{\text{diff}}$$

is L -bilipschitz for every $H \in \mathcal{C}(\mathcal{X})$, where homeomorphisms $\varphi_H: H^{\text{diff}} \rightarrow \mathfrak{c}_H^{\text{diff}}$ and $\varphi'_H: H^{\text{diff}} \rightarrow (\mathfrak{c}'_H)^{\text{diff}}$ are charts in \mathcal{A} and \mathcal{A}' , respectively.

Lemma 5.1. *Let $(\mathbb{R}^3/G, \mathcal{X})$ be a decomposition space of finite type and $\lambda \in (0, 1)$. Suppose $(\mathcal{C}_i, \mathcal{A}_i, \mathcal{W}_i)$, $i = 1, 2$, are welding structures on \mathcal{X} having compatible atlases, and let $\theta_i: \mathbb{R}^3/G \rightarrow \mathbb{R}^{m_i}$ be λ -modular embeddings associated to $(\mathcal{C}, \mathcal{A}, \mathcal{W})$, respectively. Then path metrics \hat{d}_{θ_1} and \hat{d}_{θ_2} on \mathbb{R}^3/G are bilipschitz equivalent.*

Proof. Since welding structures $(\mathcal{C}_1, \mathcal{A}_1, \mathcal{W}_1)$ and $(\mathcal{C}_2, \mathcal{A}_2, \mathcal{W}_2)$ are compatible, there exists $L_0 \geq 1$ so that, for every $H \in \mathcal{C}(\mathcal{X})$, $\varphi_H^2 \circ (\varphi_H^1)^{-1}: (c_H^1)^{\text{diff}} \rightarrow (c_H^2)^{\text{diff}}$ is L_0 -bilipschitz, where $\varphi_H^i: H^{\text{diff}} \rightarrow (c_H^i)^{\text{diff}}$ is the chart for H in \mathcal{A}_i for $i = 1, 2$. Thus there exists $L'_0 \geq 1$ so that $\theta_1(H^{\text{diff}})$ and $\theta_2(H^{\text{diff}})$ are L'_0 -bilipschitz equivalent for every $H \in \mathcal{C}(\mathcal{X})$. Therefore path metrics on $\theta_1(\mathbb{R}^3/G)$ and $\theta_2(\mathbb{R}^3/G)$ are bilipschitz equivalent. \square

We call a welding structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ for a defining sequence \mathcal{X} of finite type in \mathbb{R}^n , $n \geq 4$, a *Semmes structure* if it is finite and

- (S1) all boundary components of differences $\{c^{\text{diff}}: c \in \mathcal{C}\}$ of the same genus are congruent,
- (S2) weldings in \mathcal{W} are translations,
- (S3) for every $c = (A, B) \in \mathcal{C}$ we have that $\partial A \subset \mathbb{B}^3 \subset \mathbb{R}^3 \subset \mathbb{R}^n$, $\partial B \subset \mathbb{B}^3 \times \{1\} \subset \mathbb{R}^4 \subset \mathbb{R}^n$, and $\text{int}(c^{\text{diff}}) \subset \mathbb{B}^3 \times (0, 1) \times \mathbb{R}^{n-4}$.

All defining sequences of finite type admit Semmes structures.

Theorem 5.2 (Existence of Semmes Structures). *Let $(\mathbb{R}^3/G, \mathcal{X})$ be a decomposition space of finite type. Then \mathcal{X} admits a Semmes structure in \mathbb{R}^{16} .*

We state the Modular Embedding Theorem in terms of a given Semmes structure as follows.

Theorem 5.3 (Modular Embedding Theorem). *Let $(\mathbb{R}^3/G, \mathcal{X})$ be a decomposition space of finite type and $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ a Semmes structure on \mathcal{X} . Then for every $0 < \lambda < 1$, there exists a λ -modular embedding $\theta: \mathbb{R}^3/G \rightarrow \mathbb{R}^n$ for some $n \geq 4$, whose image $\theta(\mathbb{R}^3/G)$ is quasiconvex in the Euclidean metric. Moreover, there exists $L = L(\theta) \geq 1$ so that, any two distinct points $x, y \in \theta(\mathbb{R}^3/G)$ are in an L -bilipschitz image of a closed Euclidean 3-ball of radius $|x - y|$.*

The proof of Modular Embedding Theorem is divided into two parts. First we consider a tree $\text{Tree}_{\mathcal{X}}$ derived from the combinatorial structure of the defining sequence \mathcal{X} and a bilipschitz embedding of $\text{Tree}_{\mathcal{X}}$ into some Euclidean space \mathbb{R}^n . In the second part, we obtain an embedding of \mathbb{R}^3/G into \mathbb{R}^{16+n} by gluing reshaped and rescaled Semmes condensers provided by Theorem 5.2 guided by the embedded tree.

As the dimension of the Euclidean spaces receiving the embeddings does not play a significant role in defining metrics, we do not attempt to obtain the optimal ambient dimensions for the Semmes structures. Under additional geometric assumptions on the welding structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$, there exists a modular embedding of \mathbb{R}^3/G into \mathbb{R}^4 ; see Section 7.

5.1. Semmes structures. In this section we prove the existence of Semmes structures for defining sequences of finite type. In the proof we use three classical results from geometric topology, we state them as lemmas.

To straighten the condensers we use a version of Zeeman's unknotting theorem [16, Corollary 5.9].

Lemma 5.4. *Suppose $q \geq 2m + 2$ and $f_0, f_1: M \rightarrow \text{int}Q$ are homotopic embeddings of a closed m -manifold M into a q -polyhedron Q . Then $f_0(M)$*

and $f_1(M)$ are ambient isotopic by an isotopy supported by a compact set in $\text{int}Q$.

To straighten the welding maps between condensers, we modify the charts using the *Klee trick*; see [6, Proposition II.4].

Lemma 5.5. *Let $n \geq 1$ and let X be a PL closed set in \mathbb{R}^n and $f: X \rightarrow X$ a bilipschitz PL-homeomorphism. Given a neighborhood U of X in \mathbb{R}^n , there exists a bilipschitz PL homeomorphism $h: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ so that $h|(\mathbb{R}^n \setminus U) \times \{0\} = \text{id}$ and $h|X \times \{0\} = f$.*

Finally, to obtain condensers satisfying (S3), we use the following lemma based on general position.

Lemma 5.6. *Suppose $\mathbf{c} = (A, B)$ is a condenser in \mathbb{R}^n , $n \geq 8$, so that $\partial A \subset \mathbb{B}^3 \times \{0\} \subset \mathbb{R}^4 \subset \mathbb{R}^n$ and $B \subset \mathbb{B}^3 \times \{1\} \subset \mathbb{R}^4 \subset \mathbb{R}^n$. Then there exists a PL-homeomorphism $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ so that $F|(\partial A \cup B) = \text{id}$ and $F(\mathbf{c}^{\text{diff}}) \subset \mathbb{B}^3 \times (0, 1) \times \mathbb{R}^{n-4}$.*

Proof. We fix $t > 1$ and $t' < 0$ so that $A \subset \mathbb{R}^3 \times (t', t) \times \mathbb{R}^{n-4}$.

Since $\partial B \times [1, t]$ is a 3-dimensional PL-manifold in $\mathbb{R}^4 \subset \mathbb{R}^n$ and \mathbf{c}^{diff} is 3-dimensional, we have, by general position, that there exists a PL-homeomorphism $h: \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^{n-4} \rightarrow \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^{n-4}$ so that $h|(\mathbb{R}^3 \times (\mathbb{R} \setminus (1, t)) \times \mathbb{R}^{n-4}) = \text{id}$ and $h(\partial B \times [1, t]) \cap \mathbf{c}^{\text{diff}} = \emptyset$.

Let $B' = B + te_4$ and $A' = (A \setminus B) \cup h(\partial B \times [1, t]) \cup B'$. Since $h(\partial B \times [1, t])$ is a one-sided collar of ∂B , there exists a PL-homeomorphism $H: A \rightarrow A'$ so that $H|\partial A = \text{id}$ and $H|B$ is the translation $(x, 1, y) \mapsto (x, t, y)$, where $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^{n-4}$.

Let $g: \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^{n-4} \rightarrow \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^{n-4}$ be the map $(x, s, y) \mapsto (x, s/t, y)$. Then $\mathbf{c}'' = g(\mathbf{c}') = (A'', B'')$ is a condenser so that $(\mathbf{c}'')^{\text{diff}} \subset \mathbb{R}^3 \times (-\infty, 1) \times \mathbb{R}^{n-4}$. Note that $g \circ H|(\partial A \cup B) = \text{id}$. Since the same argument can be applied to t' , we may assume that $(\mathbf{c}'')^{\text{diff}} \subset \mathbb{R}^3 \times (0, 1) \times \mathbb{R}^{n-4}$.

We fix a piecewise linear function $\nu: \mathbb{R} \rightarrow (0, 1)$ and a PL-homeomorphism $f: \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^{n-4} \rightarrow \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^{n-4}$, $f(x, s, y) = (\nu(s)x, s, y)$, so that $\nu(s) = 1$ for $s \notin (0, 1)$ and $f((\mathbf{c}'')^{\text{diff}} \cap \mathbb{R}^3 \times \{s\}) \subset \mathbb{B}^3 \times \{s\}$ for $s \in (0, 1)$. Since $f|\partial A \cup B = \text{id}$, the composition $F = f \circ g \circ H$ satisfies the requirements of the claim. \square

Proof of Theorem 5.2. Assume, as we may, that $(\mathcal{C}, A, \mathcal{W})$ is a welding structure for \mathcal{X} in \mathbb{R}^3 with finitely many condensers \mathcal{C} ; see Remark 4.2. As a preliminary step, we fix, for every $0 \leq g \leq \bar{g}_{\mathcal{X}}$, a cube-with-handles T_g of genus g in \mathbb{R}^3 .

Step 1: We straighten the boundary components of condensers by applying Lemma 5.4 on unlinking.

Let $\mathbf{c} = (A, B)$ be a condenser in \mathcal{C} . We fix a point $u_D \in \mathbb{R}^3 \times \{1\}$ for each $D \in \mathcal{C}(B)$ so that handlebodies $T_{g_D} + u_D$ are pair-wise disjoint, where g_D is the genus of D . Fix also a regular neighborhood of C of ∂A in \mathbb{R}^3 so that $C \cap B = \emptyset$ and an embedding $f: C \cup B \rightarrow \mathbb{R}^3 \times \{1\}$ such that $f(\partial A) = \partial T_{g_A}$ and $f(D) = T_{g_D} + u_D$.

Since f is homotopic to the natural embedding $\text{id}_{C \cup B}: C \cup B \hookrightarrow \mathbb{R}^4$, there exists an ambient isotopy in \mathbb{R}^8 from $\text{id}_{C \cup B}$ to f by Lemma 5.4. So there

exists a PL-homeomorphism $h_c: \mathbb{R}^8 \rightarrow \mathbb{R}^8$ such that $h_c(\partial A) = \partial T_{g_A}$ and $h_c(D) = T_{g_D} + u_D$ for every $D \in \mathcal{C}(B)$.

Homeomorphisms h_c induce a new welding structure with condensers $\tilde{\mathcal{C}} = \{(h_c(A), h_c(B)): c \in \mathcal{C}\}$, atlas $\tilde{\mathcal{A}} = \{h_{c_H} \circ \varphi_H: H \in \mathcal{C}(\mathcal{X})\}$, and the weldings $\tilde{\mathcal{W}}$ defined by $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{A}}$.

We denote the new finite structure $(\tilde{\mathcal{C}}, \tilde{\mathcal{A}}, \tilde{\mathcal{W}})$ in \mathbb{R}^4 again by $(\mathcal{C}, \mathcal{A}, \mathcal{W})$, and new condensers, charts and weldings again by c, φ_H , and $\varphi_H \circ \varphi_H^{-1}$ respectively.

Step 2: Using the Klee trick, we straighten the weldings from Step 1 to translations; during the process we have to add more condensers to the collection \mathcal{C} .

Fix a condenser $c = (A, B) \in \mathcal{C}$. For each $D \in \mathcal{C}(B)$, we fix a neighborhood U_D of D in \mathbb{R}^4 so that $U_D \cap \partial A = \emptyset$ and all neighborhoods U_D are pair-wise disjoint.

We first consider a single welding $\psi: \partial A' \rightarrow \partial B$ in \mathcal{W} from a condenser (A', B') to c , and let D be the component of B so that $\psi(\partial A') = \partial D$. Since $\partial A'$ and ∂D have the same genus, we have that $\partial A' = \partial T_{g_{A'}}$, $D = T_{g_{A'}} + v_D$ for a point $v_D \in \mathbb{R}^3 \times \{1\}$ and $f: \partial D \rightarrow \partial D$, $f(x) = \psi^{-1}(x) + v_D$, is a bilipschitz PL homeomorphism of ∂D . By Lemma 5.5, there exists a bilipschitz PL homeomorphism $h_\psi: \mathbb{R}^{16} \rightarrow \mathbb{R}^{16}$ so that $h_\psi|_{\mathbb{R}^4 \setminus U_D} = \text{id}$ and $h_\psi|_{\partial D} = f$. The homeomorphism h_ψ defines a condenser $(h_\psi(A), h_\psi(B))$ in \mathbb{R}^{16} and a welding $h_\psi \circ \psi: \partial A' \rightarrow \partial h_\psi(B)$. By construction, $h_\psi|(A^{\text{diff}} \setminus U_D) = \text{id}$, $h_\psi(\partial D) = \partial D$, and $h_\psi \circ \psi|_{\partial A'}$ is the translation $x \mapsto x + v_D$.

We now consider all welding combinations to the fixed condenser $c = (A, B)$, and add a collection of condensers, indexed by families of weldings in \mathcal{W} , to our welding structure as follows.

A family of weldings $W \subset \mathcal{W}$ is said to be *maximal* for (A, B) if

- (1) every $\psi \in W$ is a welding from a condenser $(A', B') \in \mathcal{C}$ to a component of ∂B , and
- (2) for every component D of B there is a unique $\psi \in W$ so that ∂D is the image of ψ .

Denote by $\mathcal{F}_{(A,B)}$ the collection of all maximal families of weldings for (A, B) . The number of families in $\mathcal{F}_{(A,B)}$ is countable, generally infinite.

Given a maximal family of weldings $W \in \mathcal{F}_{(A,B)}$, the composition (in any given order) of all mappings $\{h_\psi: \psi \in W\}$ gives rise to a bilipschitz PL homeomorphism $h_W: \mathbb{R}^{16} \rightarrow \mathbb{R}^{16}$ so that $(h_W(A), h_W(B))$ is a condenser, $h_W|_{\partial A} = \text{id}$ and, for every $\psi \in W$, welding $h_W \circ \psi$ is a translation. Here we have used the disjointness of the neighborhoods $U_D, D \in \mathcal{C}(B)$.

Having Lemma 5.6 at our disposal, we may assume that condensers $(h_W(A), h_W(B))$ satisfy (S1) and (S3).

Step 3: We denote by $(\hat{\mathcal{C}}, \hat{\mathcal{A}}, \hat{\mathcal{W}})$ the welding structure obtained from Step 2. That is,

$$\hat{\mathcal{C}} = \cup_{(A,B) \in \mathcal{C}} \bigcup \{(h_W(A), h_W(B)): W \in \mathcal{F}_{(A,B)}\},$$

and

$$\hat{\mathcal{W}} = \cup_{(A,B) \in \mathcal{C}} \bigcup \{h_W \circ \psi: \psi \in W \text{ and } W \in \mathcal{F}_{(A,B)}\}.$$

To define charts, for $H \in \mathcal{C}(\mathcal{X})$ let $\varphi_H: H^{\text{diff}} \rightarrow \mathfrak{c}_H^{\text{diff}}$ be the chart in \mathcal{A} . Corresponding to H^{diff} there exists a unique maximal family W_H of weldings from $\{\mathfrak{c}_{H'}: H' \in \mathcal{C}(H \cap X_{\text{level}(H)+1})\}$ to \mathfrak{c}_H . Note that $h_{W_H}(\partial A) \subset \mathbb{B}^3 \subset \mathbb{R}^4 \subset \mathbb{R}^{16}$, $h_{W_H}(\partial B) \subset \mathbb{B}^3 \times \{1\} \subset \mathbb{R}^4 \subset \mathbb{R}^{16}$, and $h_{W_H}(\text{int}A \setminus B) \subset \mathbb{B}^3 \times (0, 1) \times \mathbb{R}^{16}$. The charts are $\hat{\mathcal{A}} = \{h_{W_H} \circ \varphi_H: H \in \mathcal{C}(\mathcal{X})\}$.

Whereas weldings $h_W \circ \psi$ in $\hat{\mathcal{W}}$ are translations, the collection $\hat{\mathcal{C}}$ is generally infinite. However elements in $\hat{\mathcal{C}}$ have only finitely many topological types, because every condenser in $\hat{\mathcal{C}}$ is homeomorphic to a condenser in \mathcal{D} .

We choose and fix a subcollection \mathcal{C}' of $\hat{\mathcal{C}}$ that contains exactly one representative for each homeomorphism class in $\hat{\mathcal{C}}$. We fix also, for each $(h_W(A), h_W(B))$ in $\hat{\mathcal{C}}$, a PL-homeomorphism g_{W_H} from $(h_W(A), h_W(B))$ to its representative (E, F) in $\hat{\mathcal{C}}$, so that g_{W_H} is a translation on each boundary component of $h_W(A) \setminus h_W(B)$. The existence of g_{W_H} follows from Lemma 5.4.

The new charts are $\mathcal{A}' = \{g_{W_H} \circ h_{W_H} \circ \varphi_H: H \in \mathcal{C}(\mathcal{X})\}$; the new weldings \mathcal{W}' induced by \mathcal{C}' and \mathcal{A}' are clearly translations.

So $(\mathcal{C}', \mathcal{A}', \mathcal{W}')$ is a Semmes structure for \mathcal{X} . \square

5.2. Combinatorial trees. Let \mathbb{R}^3/G be a decomposition space with a defining sequence $\mathcal{X} = (X_k)$. We denote by $\text{Tree}_{\mathcal{X}}$ the tree with vertices $\mathcal{C}(\mathcal{X})$ and unoriented edges $\langle H, H' \rangle$, where $H \in \mathcal{C}(X_k)$ and $H' \in \mathcal{C}(H \cap X_{k+1})$.

Given $H, H' \in \mathcal{C}(\mathcal{X})$, we define

$$(5.3) \quad \rho_{\mathcal{X}}(H, H') = \max\{\text{level}(\hat{H}) \in \mathbb{Z}: H \cup H' \subset \hat{H} \in \mathcal{C}(\mathcal{X})\}.$$

Since $\text{Tree}_{\mathcal{X}}$ is a tree there exists a unique shortest chain $H = H_1, \dots, H_{\ell} = H'$ so that $\langle H_i, H_{i+1} \rangle$ is an edge in $\text{Tree}_{\mathcal{X}}$ for every $i = 1, \dots, \ell - 1$. In particular, there exists unique index $\hat{i} = \hat{i}(H, H')$ so that $\text{level}(H_{\hat{i}}) = \rho_{\mathcal{X}}(H, H')$.

Given $\lambda > 0$ we define the metric δ_{λ} on $\text{Tree}_{\mathcal{X}}$ by the formula

$$\delta_{\lambda}(H, H') = \sum_{i=1}^{\ell-1} \lambda^{\min\{\text{level}(H_i), \text{level}(H_{i+1})\}},$$

where $H, H' \in \mathcal{C}(\mathcal{X})$ and the sum is taken over the shortest chain $H = H_1, \dots, H_{\ell} = H'$. The metric δ_1 is the standard *graph distance* on $\text{Tree}_{\mathcal{X}}$. The definition of the metric δ_{λ} immediately yields a distance estimate

$$(5.4) \quad \lambda^{\rho_{\mathcal{X}}(H, H')} \leq \delta_{\lambda}(H, H') \leq C \lambda^{\rho_{\mathcal{X}}(H, H')}$$

for all $H, H' \in \mathcal{C}(\mathcal{X})$, $H \neq H'$, where $C = C(\lambda)$.

This distance estimate yields immediately that metric trees $(\text{Tree}_{\mathcal{X}}, \delta_{\lambda})$, $\lambda > 0$, are quasimetrically equivalent. We record this observation in the following lemma.

Lemma 5.7. *Let $\lambda_1, \lambda_2 > 0$. The identity map $(\text{Tree}_{\mathcal{X}}, \delta_{\lambda_1}) \rightarrow (\text{Tree}_{\mathcal{X}}, \delta_{\lambda_2})$ is η -quasisymmetric with $\eta(t) = Ct^p$, where $p = \log \lambda_2 / \log \lambda_1$ and $C = C(\lambda_1, \lambda_2)$.*

Metric trees $(\text{Tree}_{\mathcal{X}}, \delta_{\lambda})$ embed bilipschitzly into Euclidean spaces. Recall that (e_1, \dots, e_n) is the standard basis of \mathbb{R}^n for $n \geq 1$.

Lemma 5.8. *Let $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}))$ be a decomposition space of finite type and $0 < \lambda < 1$. Then there exist $n(\mathcal{X}, \lambda)$ and a map $e_{\mathcal{X}}: \mathcal{C}(\mathcal{X}) \rightarrow \{e_1, \dots, e_n\}$ so that the map $\vartheta: (\text{Tree}_{\mathcal{X}}, \delta_{\lambda}) \rightarrow \mathbb{R}^n$ defined inductively by $\vartheta(X_0) = 0$ and $\vartheta(H') = \vartheta(H) + \lambda^k e_{\mathcal{X}}(H')$ for $H \in \mathcal{C}(X_k)$ and $H' \in \mathcal{C}(H' \cap X_{k+1})$, is a bilipschitz embedding.*

Proof. Let $m_0 > 0$ be the smallest integer satisfying

$$(5.5) \quad \sum_{j=1}^{\infty} \lambda^{jm_0} < 1/4.$$

Since \mathcal{X} has finite type, there exists $n(\mathcal{C}, m_0)$ so that

$$\# \bigcup_{i=k}^{k+2m_0} \mathcal{C}(X_i \cap H) \leq n$$

for all $k \geq 0$ and $H \in \mathcal{C}(X_k)$. We fix a map $e_{\mathcal{X}}: \text{Tree}_{\mathcal{X}} \rightarrow \{e_1, \dots, e_n\}$ so that if $e(H) = e(H')$ then the graph distance $\delta_1(H, H') \geq m_0$.

We show now that the mapping $\vartheta: \text{Tree}_{\mathcal{X}} \rightarrow \mathbb{R}^n$, defined in the statement, is a bilipschitz embedding.

Let $H, H' \in \mathcal{C}(\mathcal{X})$ and let $H = H_0, \dots, H_{\ell} = H'$ be the unique shortest chain. Let $I_j = \{i: 0 \leq i \leq \ell, i \neq \hat{i}(H, H') \text{ and } e_{\mathcal{X}}(H_i) = e_j\}$ for $j = 1, \dots, n$. Then

$$\vartheta(H) - \vartheta(H') = \sum_{i=1}^{\ell-1} \vartheta(H_i) - \vartheta(H_{i+1}) = \sum_{j=1}^n \left(\sum_{i \in I_j} \vartheta(H_i) - \vartheta(H_{i'}) \right),$$

where i' is either $i-1$ or $i+1$ so that $\text{level} H_i = \text{level} H_{i'} + 1$.

By orthogonality,

$$|\vartheta(H) - \vartheta(H')| = \left(\sum_{j=1}^n \left| \sum_{i \in I_j} \vartheta(H_i) - \vartheta(H_{i'}) \right|^2 \right)^{1/2}.$$

Since $\vartheta(H_i) - \vartheta(H_{i'}) = \lambda^{\text{level}(H_i)} e_j$ for $i \in I_j$, we have

$$\frac{3}{4} \lambda^{k_j} \leq \left| \sum_{i \in I_j} \vartheta(H_i) - \vartheta(H_{i'}) \right| \leq \frac{5}{2} \lambda^{k_j},$$

where $k_j = \min\{\text{level}(H_i): i \in I_j\}$. Since

$$\rho_{\mathcal{X}}(H, H') = \min\{k_j: 1 \leq j \leq n\} - 1,$$

we have

$$\frac{3}{4} \lambda^{\rho_{\mathcal{X}}(H, H') + 1} \leq |\vartheta(H) - \vartheta(H')| \leq \frac{5\sqrt{n}}{2} \lambda^{\rho_{\mathcal{X}}(H, H') + 1}.$$

Thus, by (5.4), ϑ is bilipschitz. \square

5.3. Bending and reshaping of condensers. Suppose $\mathfrak{c} = (A, B)$ is a Semmes condenser in \mathbb{R}^m for some $m \geq 4$ and $e: \mathcal{C}(B) \rightarrow \{e_{m+1}, \dots, e_{m+n}\}$ is an injection, where (e_1, \dots, e_{m+n}) is an orthonormal basis of \mathbb{R}^{m+n} . We say that a bilipschitz PL-homeomorphism $b_{\mathfrak{c},e}: \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$ is a *bending of \mathfrak{c} by e* if

- (1) $b_{\mathfrak{c},e}|_{\partial A} = \text{id}$,
- (2) $b_{\mathfrak{c},e}|_D: x \mapsto x + e(D)$ for every $D \in \mathcal{C}(B)$, and
- (3) $b_{\mathfrak{c},e}(\text{int } \mathfrak{c}^{\text{diff}}) \subset \mathbb{B}^3 \times (0, 1) \times \mathbb{R}^{m+n-4}$.

Bendings of \mathfrak{c} by e can be easily found.

Let $k \geq 4$ and $\lambda \in (0, 1)$. We define the λ -reshaping $s_\lambda: \mathbb{R}^k \rightarrow \mathbb{R}^k$ by

$$s_\lambda(x, t, y) = (c(t)x, t, y)$$

for $(x, t, y) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^{k-4}$, where

$$c(t) = \begin{cases} \lambda, & t \geq 1 \\ 1 - (1 - \lambda)t, & 0 \leq t \leq 1 \\ 1, & t \leq 0. \end{cases}$$

5.4. Proof of the Modular Embedding Theorem. We complete the proof of Theorem 5.3 in this section. We construct first an auxiliary sequence of PL submanifolds (M_j) of a fixed Euclidean space which tends to a PL submanifold M_∞ . The image of the embedding $\theta|\pi_G(\mathbb{R}^3 \setminus X_\infty)$ will be the manifold M_∞ . This embedding is then extended to \mathbb{R}^3/G by continuity.

Assume, as we may by Theorem 5.2, that $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ is a Semmes structure for \mathcal{X} in \mathbb{R}^{16} .

Auxiliary sequence (M_j) . Let $e_\mathcal{X}: \mathcal{C}(\mathcal{X}) \rightarrow \{e_{16+1}, \dots, e_{16+n}\}$ be the map and $\vartheta: \text{Tree}_\mathcal{X} \rightarrow \{0\} \times \mathbb{R}^n$ be the embedding defined in Lemma 5.8, with a natural shift of coordinates; recall that $\vartheta(X_0) = 0$.

We enumerate the handlebodies in $\mathcal{C}(\mathcal{X})$ by H_0, H_1, \dots so that $H_0 = X_0$ and if $H_i \in \mathcal{C}(X_k)$ then $H_{i+1} \in \mathcal{C}(X_k) \cup \mathcal{C}(X_{k+1})$. We may assume that the condenser $\mathfrak{c}_0 = (A_0, B_0)$ in \mathcal{C} and the chart $\varphi_{X_0}: X_0^{\text{diff}} \rightarrow \mathfrak{c}_0^{\text{diff}}$ are chosen so that $\varphi_{X_0}|_{\partial X_0} = \text{id}$. We denote by $\mathfrak{c}_i = (A_i, B_i)$ the condensers $\mathfrak{c}_{H_i} \in \mathcal{C}$ and by φ_i the charts $\varphi_{H_i}: H_i^{\text{diff}} \rightarrow \mathfrak{c}_i^{\text{diff}}$ in \mathcal{A} for $i \geq 0$ in what follows.

Given $i \geq 0$, we denote by $\Phi_i: \mathcal{C}(H_i \cap X_{\text{level}(H_i)+1}) \rightarrow \mathcal{C}(B_i)$ the bijection between components induced by charts φ_{H_i} , that is, $\partial\Phi_i(H') = \varphi_{H_i}(\partial H')$ for $H' \in \mathcal{C}(H_i \cap X_{\text{level}(H_i)+1})$. Furthermore, we define $e^i: \mathcal{C}(B_i) \rightarrow \{e_{16+1}, \dots, e_{16+n}\}$ by $e^i = e_\mathcal{X} \circ \Phi_i^{-1}$;

$$\begin{array}{ccc} \mathcal{C}(H_i \cap X_{\text{level}(H_i)+1}) & \xrightarrow{\Phi_i} & \mathcal{C}(B_i) \\ \downarrow \text{incl.} & & \downarrow e^i \\ \mathcal{C}(\mathcal{X}) & \xrightarrow{e_\mathcal{X}} & \{e_{16+1}, \dots, e_{16+n}\} \end{array}$$

We then fix a family of bendings $\{b_i = b_{\mathfrak{c}_i, e^i}: \mathbb{R}^{16+m} \rightarrow \mathbb{R}^{16+m}: i \geq 0\}$ in such a way that if $(\mathfrak{c}_i, e^i) = (\mathfrak{c}_j, e^j)$ then $b_i = b_j$. Since $\{\mathfrak{c}_i: i \geq 0\}$ is finite, the family of bendings is also finite; thus bendings in $\{b_i, i \geq 0\}$ are uniformly bilipschitz.

Let $M_{-1} = (\mathbb{R}^3 \setminus X_0) \times \{0\} \subset \mathbb{R}^4 \subset \mathbb{R}^{16} \subset \mathbb{R}^{16+n}$ and let $\theta_{-1}: \mathbb{R}^3 \setminus X_0 \rightarrow \mathbb{R}^{16+n}$ to be the natural inclusion. We define M_0 by

$$M_0 = M_{-1} \cup \mathfrak{c}_0^{\text{diff}}$$

and embedding $\theta_0: \mathbb{R}^3 \setminus X_1 \rightarrow \mathbb{R}^{16+n}$ by $\theta_0|_{\mathbb{R}^3 \setminus X_0} = \theta_{-1}$ and $\theta_0|_{X_0^{\text{diff}}} = \varphi_{X_0}$.

Suppose now that we have defined manifolds M_{-1}, \dots, M_{j-1} and embeddings $\theta_{-1}, \dots, \theta_{j-1}$ so that, for $i = -1, \dots, j-1$,

(1) $M_i = M_{i-1} \cup f_i(\mathfrak{c}_i^{\text{diff}})$ where $H_i \in \mathcal{C}(X_{k_i})$, f_i is the quasimilarity

$$(5.6) \quad x \mapsto \lambda^{k_i}(s_\lambda \circ b_i)(x) + \vartheta(H_i) + w_i$$

and w_i a point in \mathbb{R}^{16} satisfying $|w_i| < C_0 \sum_{0 \leq r \leq k_i} \lambda^r$, where constant $C_0 > 0$ is chosen so that $B^{16+n}(C_0)$ contains all condensers, and

(2) the embedding $\theta_i: (\mathbb{R}^3 \setminus X_0) \cup (H_0^{\text{diff}} \cup \dots \cup H_i^{\text{diff}}) \rightarrow M_i$ is defined by $\theta_i|_{(\mathbb{R}^3 \setminus X_0) \cup (H_0^{\text{diff}} \cup \dots \cup H_{i-1}^{\text{diff}})} = \theta_{i-1}$ and $\theta_i|_{H_i^{\text{diff}}} = f_i \circ \varphi_i$.

We construct the set M_j and the embedding θ_j as follows. Suppose that $H_j \in \mathcal{C}(X_k)$ and that $H_i, i < j$, is the unique handlebody in $\mathcal{C}(X_{k-1})$ with the property $H_j \in \mathcal{C}(H_i \cap X_k)$; and let $\psi = \varphi_i \circ \varphi_j^{-1}$ be the welding map from \mathfrak{c}_j to $\mathfrak{c}_i^{\text{diff}}$. Since $(\mathcal{C}, \mathcal{W})$ is a Semmes scheme, ψ is a translation $x \mapsto x + v_\psi$ in \mathbb{R}^{16} , where $v_\psi \in \mathbb{R}^{16}$, $\langle v_\psi, e_4 \rangle = 1$, and $|v_\psi| < C_0$ as in the construction. By induction hypothesis, $f_i|_{\varphi_i(\partial H_j)}$ is a similarity

$$x \mapsto \lambda^k x + \lambda^k e_{\mathcal{X}}(H_j) + \vartheta(H_i) + w_i = \lambda^k x + \vartheta(H_j) + w_i;$$

here we use the fact that the bending $b_i|_{\varphi_i(\partial H_j)} = e^i(H_j) = e_{\mathcal{X}}(H_j)$ and the reshaping s_λ on $\partial \varphi_i(H_j)$ is a scaling by λ . We set f_j to be the quasimilarity

$$x \mapsto \lambda^k (s_\lambda \circ b_j)(x) + \vartheta(H_j) + \lambda^k v_\psi + w_i.$$

Since \mathfrak{c}_j is a Semmes condenser and $\langle v_\psi, e_4 \rangle = 1$, we have that $M_{j-1} \cap f_j(\mathfrak{c}_j)$ is a common boundary component of M_{j-1} and $f_j(\mathfrak{c}_j)$. Thus $M_j = M_{j-1} \cup f_j(\mathfrak{c}_j)$ is a connected manifold with boundary satisfying (1) in the induction hypothesis.

We set $w_j = \lambda^k v_\psi + w_i \in \mathbb{R}^{16}$, and note by induction hypothesis $|w_j| \leq C_0 \sum_{0 \leq r \leq k} \lambda^r$.

We define now the embedding $\theta_j: (\mathbb{R}^3 \setminus X_0) \cup (H_0^{\text{diff}} \cup \dots \cup H_j^{\text{diff}}) \rightarrow M_j$ by formula $\theta_j|_{\mathbb{R}^3 \cup (H_0^{\text{diff}} \cup \dots \cup H_{j-1}^{\text{diff}})} = \theta_{j-1}$ and $\theta_j|_{H_j^{\text{diff}}} = f_j \circ \varphi_j$. This completes the induction step.

Construction of M_∞ . Define now the limit manifold M_∞ by

$$M_\infty = \bigcup_{j \geq 0} M_j$$

and the limiting embedding $\theta_\infty: \mathbb{R}^3 \setminus X_\infty \rightarrow M_\infty$ by $\theta_\infty|_{M_j} = \theta_j$.

Since there exists $C > 0$ so that $\text{diam} \theta(H) \leq C \lambda^k$ for every $H \in \mathcal{C}(X_k)$, the components of $\overline{M_\infty} \setminus M_\infty$ are singletons. Thus $\theta_\infty \circ \pi_G^{-1}$ extends to a homeomorphism $\theta: \mathbb{R}^3/G \rightarrow \overline{M_\infty}$.

Since $\theta \circ \pi_G \circ \varphi_j^{-1} = f_j: \mathfrak{c}_j^{\text{diff}} \rightarrow \mathbb{R}^{16+n}$, it is $(\lambda^{\text{level}(H_j)}, L)$ -quasimilarity, for a constant $L \geq 1$ depending only on the family of bendings $\{b_i: i \geq 0\}$ and n . Thus θ is a λ -modular embedding.

Metric properties of $\theta(\mathbb{R}^3/G)$. We show now the last claim in the statement: given $x, y \in \theta(\mathbb{R}^3/G)$, there exists an L' -bilipschitz map $h: B^3(|x - y|) \rightarrow \theta(\mathbb{R}^3/G)$ so that $x, y \in h(B^3(|x - y|))$, where $L' = L'(\theta) \geq 1$. Then, in particular, $\theta(\mathbb{R}^3/G)$ is quasiconvex.

It suffices to consider the case $x, y \in \theta(\pi_G(\mathbb{R}^3 \setminus X_\infty))$; other cases are obtained by similar arguments.

We observe, by the $(\lambda^{\text{level}(H)}, L)$ -quasisimilarity of the mappings $\theta \circ \pi_G \circ \varphi_j^{-1}$ and the finiteness of condensers in \mathcal{C} , the following. If H and H' are condensers in $\mathcal{C}(\mathcal{X})$ satisfying $H^{\text{diff}} \cap H'^{\text{diff}} \neq \emptyset$, then any two points x and y in $\theta(\pi_G(H^{\text{diff}} \cup H'^{\text{diff}}))$ can be connected by a PL-curve contained in a 3-cell in $\theta(\pi_G(H^{\text{diff}} \cup H'^{\text{diff}}))$ that is L' -bilipschitz equivalent to a Euclidean ball of diameter $|x - y|$, where L' depends only on the data. In particular, the claim holds in this case.

We now assume $x \in \theta(\pi_G(H^{\text{diff}}))$, $y \in \theta(\pi_G(H'^{\text{diff}}))$ and $H^{\text{diff}} \cap H'^{\text{diff}} = \emptyset$. Let $H = H_0, \dots, H_\ell = H'$ be the unique shortest chain in $\text{Tree}_{\mathcal{X}}$ joining vertices H and H' , and $\rho_{\mathcal{X}}(H, H') = \min\{\text{level}(\hat{H}) \in \mathbb{Z}: H \cup H' \subset \hat{H} \in \mathcal{C}(\mathcal{X})\}$. Then by the construction of the embedding θ , there exists $C = C(\theta) \geq 1$ so that

$$(5.7) \quad C^{-1} \lambda^{\rho_{\mathcal{X}}(H, H')} \leq |x - y| \leq C \lambda^{\rho_{\mathcal{X}}(H, H')}.$$

There exist $C' = C'(\theta) \geq 1$ and points $x = x_0, \dots, x_\ell = y$ with $x_i \in \theta(\pi_G(H_i^{\text{diff}}))$ so that each x_i , $1 \leq i \leq \ell - 1$, is contained in a 3-cell $D_i \subset \theta(\pi_G(H_i^{\text{diff}}))$ that is L' -bilipschitz equivalent to $B^3(\lambda^{\text{level}(H_i)})$, and

$$C'^{-1} \lambda^{\text{level}(H_i)} \leq |x_i - x_{i+1}| \leq C \lambda^{\text{level}(H_i)}$$

for $0 \leq i \leq \ell - 1$. Consequently,

$$\sum_{i=0}^{\ell-1} |x_i - x_{i+1}| \leq C|x - y|.$$

By the argument in the last paragraph, we find PL 3-cells $E_i \subset \pi_G(H_i^{\text{diff}} \cup H_{i+1}^{\text{diff}})$ that are L' -bilipschitz equivalent to $B^3(|x_i - x_{i+1}|)$ and contain points x_i and x_{i+1} in their interiors in $\pi_G(H_i^{\text{diff}} \cup H_{i+1}^{\text{diff}})$, respectively. It is now easy to find a PL 3-cell $E \subset \bigcup_{i=1}^{\ell-1} D_i \cup \bigcup_{i=0}^{\ell-1} E_i$ that is L' -bilipschitz equivalent to $B^3(|x - y|)$ and contains points x and y . This concludes the proof of Theorem 5.3.

Remark 5.9. *The fact that any two points x, y in $\theta(\mathbb{R}^3/G)$ are contained in a 3-cell in $\theta(\mathbb{R}^3/G)$ that is L -bilipschitz equivalent to a Euclidean ball of diameter $|x - y|$, yields that $\theta(\mathbb{R}^3/G)$ has the Loewner property. We formulate this more precisely in Section 6.4.*

6. SEMMES SPACES

In this section we discuss Ahlfors regularity, linearly locally contractibility and the Loewner property of the modular metrics provided by the Modular Embedding Theorem.

Definition 6.1. Let $(\mathbb{R}^3/G, \mathcal{X})$ be a decomposition space of finite type, $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ a Semmes structure for \mathcal{X} , $\theta: \mathbb{R}^3/G \rightarrow \mathbb{R}^n$ a modular embedding

associated to $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ as in Theorem 5.3, and let d_θ be a λ -modular metric associated to θ . A metric space $(\mathbb{R}^3/G, d_\lambda)$ is called a *Semmes space* if d_λ is bilipschitz equivalent to d_θ . In this case we say d_λ is a *Semmes metric* with a *scaling factor* λ .

At times we say that $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\lambda)$ is a Semmes space in order to emphasize the relation to between the structure, the embedding, and the metric.

Product spaces $\mathbb{R}^3/G \times \mathbb{R}^m$ carry the natural product metric $d_{\lambda, m}$ defined by

$$(6.1) \quad d_{\lambda, m}((x, u), (y, v)) = d_\lambda(x, y) + |u - v|$$

for (x, u) and (y, v) in $\mathbb{R}^3/G \times \mathbb{R}^m$.

We observe that the metric space $(\mathbb{R}^3/G, d_\lambda)$ is quasiconvex; indeed, $\theta(\mathbb{R}^3/G)$ is a quasiconvex set in \mathbb{R}^n by Theorem 5.3. Together with Lemma 5.1 we have the bilipschitz equivalence of modular metric spaces associated to Semmes structures with compatible atlases. We record this observation as a lemma.

Lemma 6.2. *Let $\lambda \in (0, 1)$ and suppose that $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}_i, \mathcal{A}_i, \mathcal{W}_i), \theta_i, d_{\theta_i})$ are λ -modular spaces, $i = 1, 2$. The metrics d_{θ_1} and d_{θ_2} are bilipschitz equivalent if Semmes structures $(\mathcal{C}_1, \mathcal{A}_1, \mathcal{W}_1)$ and $(\mathcal{C}_2, \mathcal{A}_2, \mathcal{W}_2)$ have compatible atlases.*

6.1. Metric properties. We list some elementary metric and measure theoretic properties of Semmes spaces in the following remarks and the subsequent lemma. Let $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\lambda)$ be a Semmes space.

Remark 6.3. *By quasiconvexity of d_θ , the path metric space $(\mathbb{R}^3/G, \hat{d}_\theta)$ is a Semmes space. Similarly, the path metric space $(\mathbb{R}^3/G, \hat{d}_\lambda)$ of $(\mathbb{R}^3/G, d_\lambda)$ is a Semmes space.*

Remark 6.4. *By modularity of the embedding θ and quasiconvexity of the metric d_θ there exists a constant $C = C(d_\lambda)$ so that*

$$C^{-1}\lambda^{\rho_{\mathcal{X}}(H, H')} \leq d_\lambda(x, y) \leq C\lambda^{\rho_{\mathcal{X}}(H, H')}$$

for $x \in \pi_G(H^{\text{diff}})$ and $y \in \pi_G(H'^{\text{diff}})$ whenever $H, H' \in \mathcal{C}(\mathcal{X})$ satisfy $H^{\text{diff}} \cap H'^{\text{diff}} = \emptyset$.

Remark 6.5. *By the finiteness of the welding structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ and quasisimilarity property (5.1) of modular embeddings, there exists $C > 1$ so that for every $k \geq 0$ and $H \in \mathcal{C}(X_k)$,*

- (1) $C^{-1}\lambda^k \leq \text{dist}_{d_\lambda}(\partial\pi_G(H), \partial\pi_G(H')) \leq C\lambda^k$, if $H' \in \mathcal{C}(\mathcal{X})$, $H' \subset H$ and $H' \neq H$;
- (2) $C^{-1}\lambda^k \leq \text{diam}_{d_\lambda} \pi_G H^{\text{diff}} \leq C\lambda^k$;
- (3) $C^{-1}\lambda^{3k} \leq \mathcal{H}_{d_\lambda}^3(\pi_G(H^{\text{diff}})) \leq C\lambda^{3k}$; and
- (4) $C^{-1}r^3 \leq \mathcal{H}_{d_\lambda}^3(B_{d_\lambda}(x, r)) \leq Cr^3$, if $B_{d_\lambda}(x, r) \subset \pi_G(X_{k-1} \setminus X_{k+2})$

Observe also that components of $\pi_G(X_\infty)$ are singletons in $(\mathbb{R}^3/G, d_\lambda)$. Thus $\pi_G(X_\infty)$ is 0-dimensional.

Lemma 6.6. *Let $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\lambda)$ be a Semmes space. Then there exists $C > 1$ so that*

$$(6.2) \quad C^{-1}\lambda^k \leq \text{diam}_{d_\lambda} \pi_G H \leq C\lambda^k,$$

for every $k \geq 0$ and $H \in \mathcal{C}(X_k)$.

If, in addition, $\lambda^3\gamma_{\mathcal{X}} < 1$, then $\mathcal{H}_{d_\lambda}^3(\pi_G(X_\infty)) = 0$ and there exists $C > 1$ so that

$$(6.3) \quad C^{-1}\lambda^{3k} \leq \mathcal{H}_{d_\lambda}^3(\pi_G H) \leq C\lambda^{3k}$$

for every $k \geq 0$ and $H \in \mathcal{C}(X_k)$.

Proof. Since

$$\pi_G H = \overline{\bigcup_{i \geq k} \bigcup_{H' \in \mathcal{C}(X_i \cap H)} \pi_G(H'^{\text{diff}})}.$$

we have, by connectedness and Remark 6.5(2),

$$C^{-1}\lambda^k \leq \text{diam}_{d_\lambda} H^{\text{diff}} \leq \text{diam}_{d_\lambda} H \leq \sum_{i \geq k} C\lambda^i \leq C'\lambda^k,$$

Similarly, we have that

$$\mathcal{H}_{d_\lambda}^3(\pi_G(H)) = \mathcal{H}_{d_\lambda}^3(\pi_G(X_\infty \cap H)) + \sum_{i \geq k} \sum_{H' \in \mathcal{C}(X_i \cap H)} \mathcal{H}_{d_\lambda}^3(\pi_G(H'^{\text{diff}})).$$

Suppose now that $\lambda^3\gamma_{\mathcal{X}} < 1$. Then, by (6.2),

$$\mathcal{H}_{d_\lambda}^3(\pi_G(X_\infty)) \leq \limsup_{i \rightarrow 0} \sum_{H' \in \mathcal{C}(X_i)} (C\lambda^i)^3 \leq C^3 \limsup_{i \rightarrow 0} \lambda^{3i} \gamma_{\mathcal{X}}^i = 0.$$

By Remark 6.5(3), there exists $C > 1$, independent of H and k , so that

$$C^{-1}\lambda^{3k} \leq \sum_{i \geq k} \sum_{H' \in \mathcal{C}(X_i \cap H)} \mathcal{H}_{d_\lambda}^3(\pi_G(H'^{\text{diff}})) \leq \sum_{i \geq k} C\lambda^{3i} \gamma_{\mathcal{X}}^{i-k} \leq C\lambda^{3k}.$$

This concludes the proof. \square

Remark 6.7. *We observe, by Remark 6.5 and Lemma 6.6, that the number*

$$\hat{\epsilon}_\lambda = \min_{k \geq 0} \min_{H \in \mathcal{C}(X_k)} \left\{ \frac{\text{dist}_{d_\lambda}(\pi_G(\partial H), \pi_G(H \setminus H^{\text{diff}}))}{\lambda^k} \right\}.$$

is strictly positive. Furthermore, we may fix $\epsilon_\lambda = \epsilon_\lambda(d_\lambda) < \hat{\epsilon}/10$ so that $N_{d_\lambda}(\pi_G(\partial H), \epsilon_\lambda \lambda^{\text{level}(H)})$ is contained in a regular neighborhood of $\pi_G(\partial H)$ in $\pi_G(X_{\text{level}(H)-1}) \setminus \pi_G(H_{\text{level}(H)+1})$.

6.2. Ahlfors regularity. The Ahlfors regularity of Semmes spaces follows as in [19, Lemma 3.45]. We discuss the details for completeness of the exposition.

Proposition 6.8. *Let $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\lambda)$ be a Semmes space, and suppose that $0 < \lambda^3\gamma_{\mathcal{X}} < 1$. Then the space $(\mathbb{R}^3/G, d_\lambda)$ is Ahlfors 3-regular, and spaces $(\mathbb{R}^3/G \times \mathbb{R}^m, d_{\lambda, m})$ are Ahlfors $(3 + m)$ -regular for $m \geq 1$.*

Proof. It suffices to show that $(\mathbb{R}^3/G, d_\lambda)$ is Ahlfors 3-regular. Then the Ahlfors $(3 + m)$ -regularity of spaces $(\mathbb{R}^3/G \times \mathbb{R}^m, d_{\lambda,m})$, $m \geq 1$, follows by taking products.

By the bilipschitz invariance of Ahlfors regularity, we may assume that d_λ is the metric d_θ defined by a λ -modular embedding $\theta: \mathbb{R}^3/G \rightarrow \mathbb{R}^{8+n}$. To simplify the exposition, we assume that $X_0 = \mathbb{B}^3$, and denote by $X_{-j} = B^3(0, \lambda^{-j})$ for $j > 0$.

To show that

$$(6.4) \quad C^{-1}r^3 \leq \mathcal{H}_{d_\lambda}^3(B_{d_\lambda}(x, r)) \leq Cr^3$$

for all balls $B_{d_\lambda}(x, r)$ in $(\mathbb{R}^3/G, d_\lambda)$, we consider two cases: (a) $x \in \pi_G(X_\infty)$, and (b) $x \in \pi_G(\mathbb{R}^3 \setminus X_\infty)$.

Case (a) follows from (b). Indeed, since $\pi_G(\mathbb{R}^3 \setminus X_\infty)$ is dense in \mathbb{R}^3/G , given $x \in \pi_G(X_\infty)$ and $r > 0$ there exists $y \in \pi_G(\mathbb{R}^3 \setminus X_\infty)$ so that $d_\lambda(x, y) < r/2$. So $B_{d_\lambda}(y, r/2) \subset B_{d_\lambda}(x, r) \subset B_{d_\lambda}(y, 2r)$, and (6.4) follows by (b).

We next consider the case (b) $x \in \pi_G(X_0 \setminus X_\infty)$, and suppose $x \in \pi_G(H^{\text{diff}})$ and $H \in \mathcal{C}(X_k)$. By Remark 6.5(1), there exists a constant $C_1 = C_1(d_\lambda) \in (0, 1)$ so that if $r \leq C_1\lambda^k$ then $B_{d_\lambda}(x, r) \subset \pi_G(X_{k-1} \setminus X_{k+2})$. By (6.2) of Lemma 6.6, there exists a constant $C_2 = C_2(d_\lambda) > 1$ so that if $r \geq C_2\lambda^k$ then $\pi_G(H) \subset B_{d_\lambda}(x, r)$.

For $0 < r \leq C_1\lambda^k$, the claim follows from Remark 6.5(4). For $r \geq C_2\lambda^k$, we fix $m \in \mathbb{Z}$ so that $\lambda^{m+1} \leq r < \lambda^m$. Then, by Remark 6.5(1), there exist an integer $C_3 = C_3(d_\lambda) > 0$, and handlebodies $H' \in \mathcal{C}(X_{m+C_3})$ and $H'' \in \mathcal{C}(X_{m-C_3})$ so that

$$\pi_G(H') \subset B_{d_\lambda}(x, r) \subset \pi_G(H'').$$

Then, by Remark 6.5(3), there exists $C = C(d_\lambda) > 1$ so that

$$C^{-1}\lambda^{3(m+C_3)} \leq \mathcal{H}_{d_\lambda}^3(B_{d_\lambda}(x, r)) \leq C\lambda^{3(m-C_3)}.$$

In the remaining subcase $C_1\lambda^k < r < C_2\lambda^k$, $B_{d_\lambda}(x, r)$ contains the ball $B_{d_\lambda}(x, C_1\lambda^k)$ and is contained in a handlebody in $\mathcal{C}(X_{k-C_4})$ for some $C_4 = C_4(d_\lambda) > 0$. Then (6.4) follows by combining Remark 6.5 and (6.3) of Lemma 6.6. This concludes the proof. \square

6.3. Linear local contractibility. In this section we show that a Semmes space $(\mathbb{R}^3/G, (X_k)_{k \geq 0}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\lambda)$ is linearly locally contractible if \mathcal{X} is locally contractible. By Lemma 3.1, local contractibility of \mathcal{X} yields contractibility of components of $\pi_G(X_{k+1})$ in $\pi_G(X_k)$ for every $k \geq 0$.

Proposition 6.9. *Let $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\lambda)$ be a Semmes space and $m \geq 0$. Suppose that there exists $\ell \geq 1$ so that components of $\pi_G(X_{k+\ell})$ are contractible in $\pi_G(X_k)$ for every $k \geq k_0$. Then $(\mathbb{R}^3/G \times \mathbb{R}^m, d_{\lambda,m})$ is linearly locally contractible.*

As before, we assume as we may that $X_0 = \mathbb{B}^3$ and $X_{-j} = B^3(0, \lambda^{-j})$ for $j > 0$.

Proof. Since X_{-k} is a 3-cell for $k \leq 0$, we have by replacing ℓ with $\ell + k_0$ if necessary, that components of $\pi_G(X_{k+\ell})$ are contractible in $\pi_G(X_k)$ for every $k \in \mathbb{Z}$.

To show that there exists $C = C(d_{\lambda,m}) > 1$ so that every ball $B_{d_{\lambda,m}}(x, r)$ is contractible in $B_{d_{\lambda,m}}(x, Cr)$, we consider case (a) $x \in \pi_G(X_\infty) \times \mathbb{R}^m$ and case (b) $x \in \pi_G(\mathbb{R}^3 \setminus X_\infty) \times \mathbb{R}^m$.

Case (a) follows from (b). Indeed, let $x \in \pi_G(X_\infty) \times \mathbb{R}^m$ and $r > 0$. Then there exists $z \in \pi_G(\mathbb{R}^3 \setminus X_\infty) \times \mathbb{R}^m$ so that $d_{\lambda,m}(x, z) < r/2$. Hence $B_{d_{\lambda,m}}(x, r)$ is contained in a ball $B_{d_{\lambda,m}}(z, 2r)$ that is contractible in $B_{d_{\lambda,m}}(z, 2Cr) \subset B_{d_{\lambda,m}}(x, 4Cr)$, where $C = C(d_{\lambda,m})$ is as in case (b).

Let $x = (y, v) \in \pi_G(\mathbb{R}^3 \setminus X_\infty) \times \mathbb{R}^m$ with $y \in \pi_G(H^{\text{diff}})$ and $H \in C(X_k)$ as in case (b). We observe first that if $B_{d_\lambda}(x, r)$ contracts in $B_{d_\lambda}(x, Cr)$, where $C > 1$ depends only on the data, then $B_{d_{\lambda,m}}((x, v), r)$ contracts in $B_{d_\lambda}(x, Cr) \times (v + [-r, r]^m)$. Then the ball $B_{d_{\lambda,m}}((x, v), r)$ is contractible in $B_{d_{\lambda,m}}((x, v), (C + \sqrt{m})r)$ and the claim follows. Thus it suffices to find $C = C(d_\lambda, \ell) > 1$ so that $B_{d_\lambda}(x, r)$ contracts in $B_{d_\lambda}(x, Cr)$.

We note first that there exist constants $C_0 = C_0(d_\lambda) > 0$ and $C_1 = C_1(d_\lambda) > 1$ so that if $r < C_1\lambda^k$ then there exists a 3-cell $E \subset \pi_G(X_{k-1} \setminus X_{k+2})$ satisfying

$$B_{d_\lambda}(y, r) \subset E \subset B_{d_\lambda}(y, C_0r),$$

by the uniform quasimilarity of the modular embedding θ . Hence $B_{d_\lambda}(x, r)$ contracts in $B_{d_\lambda}(y, C_0r)$ if $r < C_1\lambda^k$.

Assume from now on that $r \geq C_1\lambda^k$. We fix handlebodies $H', H'' \in \mathcal{C}(\mathcal{X})$ satisfying $H \subset H' \subset H''$,

$$\text{level}(H') = \min\{\text{level}(K) : K \in \mathcal{C}(\mathcal{X}), B_{d_\lambda}(x, r) \subset \pi_G(K)\},$$

and $\text{level}(H'') = \text{level}(H') - \ell$. Then, in particular, $B_{d_\lambda}(x, r) \subset \pi_G(H')$ and $\pi_G(H')$ contracts in $\pi_G(H'')$.

Since $r \geq C_1\lambda^k$, by Remark 6.5 and Lemma 6.6, there exists $C_2 = C_2(d_\lambda) \geq 1$ so that $\text{diam}_{d_\lambda}(\pi_G(H'')) \leq C_2r$. Thus $\pi_G(H'') \subset B_{d_\lambda}(x, C_2r)$, and $B_{d_\lambda}(x, r)$ contracts in $B_{d_\lambda}(x, C_2r)$. This concludes the proof of case (b) and the proposition. \square

6.4. Loewner property. In this section, we briefly list some other analytical properties of Semmes spaces. We refer to [19], [18], and [9] for definitions and background. We assume in what follows that $(\mathbb{R}^3/G, d_\lambda)$ is Ahlfors 3-regular.

By the proof of the Modular Embedding Theorem (Theorem 5.3) that the space $(\mathbb{R}^3/G, d_\lambda)$ is not only quasiconvex but, in fact, possesses thick families of curves, since any pair of points $x, y \in \mathbb{R}^3/G$ is contained in a uniformly bilipschitz image of the Euclidean ball $B^3(|x - y|)$. This property is the same as [19, Lemma 3.70] for self-similar spaces. The argument of [19, Proposition 10.8] can now be applied almost verbatim to show that $(\mathbb{R}^3/G, d_\lambda)$ supports a (1, 1)-Poincaré inequality as formulated in [19, (10.9)]. Since the space \mathbb{R}^3/G is PL outside $\pi_G(X_\infty)$, the Poincaré inequality can be naturally formulated in terms of *generalized gradients* (*upper gradients*). We refer to [18, Appendix C] for a detailed treatment.

Ahlfors 3-regularity, quasiconvexity, and the (1, 1)-Poincaré inequality imply that $(\mathbb{R}^3/G, d_\lambda)$ is a Loewner space in the sense of Heinonen and Koskela ([9, Theorem 5.7]). A metric measure space (X, d, μ) of Hausdorff-dimension

Q is a *Loewner space* if there exists a function $\phi: (0, \infty) \rightarrow (0, \infty)$ so that

$$\text{Mod}_Q(E, F) \geq \phi(\Delta(E, F, X))$$

whenever E and F are disjoint continua in X , where

$$\Delta(E, F, X) = \frac{\text{dist}(E, F)}{\min\{\text{diam } E, \text{diam } F\}},$$

and $\text{Mod}_Q(E, F)$ is the Q -modulus of the family of paths connecting E and F in X .

Suppose now that the space $(\mathbb{R}^3/G \times \mathbb{R}^m, d_{\lambda, m})$ is Ahlfors $(3+m)$ -regular and homeomorphic to \mathbb{R}^{3+m} for some $m \geq 0$. Then $(\mathbb{R}^3/G \times \mathbb{R}^m, d_{\lambda, m})$ supports $(1, 1)$ -Poincaré inequality by a theorem of Semmes for manifolds [18, Theorem B.10(b)]. Thus $(\mathbb{R}^3/G \times \mathbb{R}^m, d_{\lambda, m})$ is a Loewner space by the aforementioned theorem of Heinonen and Koskela.

6.5. Quasisymmetric equivalence of Semmes metrics. In this section we prove the quasisymmetric equivalence of Semmes metrics on $(\mathbb{R}^3/G, \mathcal{X})$ associated to different welding structures and scaling factors.

Proposition 6.10. *Let $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}_i, \mathcal{A}_i, \mathcal{W}_i), \theta_i, d_{\lambda_i})$ be Semmes spaces for $i = 1, 2$, where $\lambda_1, \lambda_2 \in (0, 1)$. Suppose that $(\mathcal{C}_1, \mathcal{A}_1, \mathcal{W}_1)$ and $(\mathcal{C}_2, \mathcal{A}_2, \mathcal{W}_2)$ have compatible atlases. Then $\text{id}: (\mathbb{R}^3/G, d_{\lambda_1}) \rightarrow (\mathbb{R}^3/G, d_{\lambda_2})$ is quasisymmetric.*

Proof. Assume, as we may, that $X_0 = \mathbb{B}^3$ and define $X_{-j} = B^3(0, \lambda^{-j})$ for $j > 0$.

Since $\pi_G(\mathbb{R}^3 \setminus X_\infty)$ is dense in \mathbb{R}^3/G and metrics d_{λ_i} are bilipschitz equivalent to modular metrics d_{θ_i} for $i = 1, 2$, respectively, it suffices to show that there exists a homeomorphism $\eta: [0, \infty) \rightarrow [0, \infty)$ so that

$$(6.5) \quad \frac{|\theta_2(x) - \theta_2(y)|}{|\theta_2(x) - \theta_2(z)|} \leq \eta \left(\frac{|\theta_1(x) - \theta_1(y)|}{|\theta_1(x) - \theta_1(z)|} \right)$$

for all distinct points x, y , and z in $\pi_G(\mathbb{R}^3 \setminus X_\infty)$.

We divide the proof to different cases depending on relative distances between points x, y , and z . For brevity, say that points x and y in $\pi_G(\mathbb{R}^3 \setminus X_\infty)$ are *near* if there exists $H, H' \in \mathcal{C}(\mathcal{X})$ so that $\{x, y\} \subset \pi_G(H^{\text{diff}} \cup H'^{\text{diff}})$ and $H^{\text{diff}} \cap H'^{\text{diff}} \neq \emptyset$. Otherwise, we say that points x and y are *far*.

Let x, y , and z be distinct points in $\pi_G(\mathbb{R}^3 \setminus X_\infty)$.

Case I: Suppose that there are at least two pairs of points in the set $\{x, y, z\}$ are *near*.

Then there exists $H, H', H'' \in \mathcal{C}(\mathcal{X})$ so that $H^{\text{diff}} \cap H'^{\text{diff}} \neq \emptyset$, $H'^{\text{diff}} \cap H''^{\text{diff}} \neq \emptyset$ and $\{x, y, z\} \subset \pi_G(H^{\text{diff}} \cup H'^{\text{diff}} \cup H''^{\text{diff}})$. Then, by quasiconvexity of metrics d_{λ_i} , compatibility of atlases, and modularity of embeddings θ_i , there exists $C_1 = C_1(\theta_1, \theta_2) > 0$ so that (6.5) holds with $\eta = \eta_1$, where $\eta_1(t) = C_1 t$.

Case II: Suppose that all the points x, y , and z are *far* from each other. Then, by Remark 6.4, there exists $C_2 = C_2(\theta_1, \theta_2) > 0$ so that (6.5) holds with $\eta = \eta_2$, where $\eta_2(t) = C_2 t^p$ and $p = \log \lambda_2 / \log \lambda_1$.

Case III: Suppose now that there exists only one pair in $\{x, y, z\}$ where the points are *near* and that points in the other two pairs are *far*. We have three subcases.

Case III.1: Suppose that y and z are near. Then x and y are far and x and z are far. So there exists $C = C(\theta_1, \theta_2) > 0$ so that

$$\frac{1}{C} \leq \frac{|\theta_i(x) - \theta_i(y)|}{|\theta_i(x) - \theta_i(z)|} \leq C$$

for $i = 1, 2$. Thus (6.5) holds with $\eta = \eta_3$, where $\eta_3(t) = C_3 t$ with $C_3 = C_3(\theta_1, \theta_2) > 0$.

Case III.2: Suppose now that x and z are *near* and let $H, H' \in \mathcal{C}(\mathcal{X})$ be such that $\{x, z\} \subset \pi_G(H^{\text{diff}} \cup H'^{\text{diff}})$ and $H^{\text{diff}} \cap H'^{\text{diff}} \neq \emptyset$. Then, by modularity of embeddings θ_1 and θ_2 , there exist $C = C(\theta_1, \theta_2) > 1$ and $w \in \pi_G(H^{\text{diff}} \cup H'^{\text{diff}})$ so that

$$\min\{|\theta_i(x) - \theta_i(w)|, |\theta_i(z) - \theta_i(w)|\} \geq \frac{1}{C} \text{diam } \theta_i(\pi_G(H^{\text{diff}} \cup H'^{\text{diff}}))$$

for $i = 1, 2$.

Following the argument for cases I and II, there exists $C_4 = C_4(\theta_1, \theta_2) > 0$ so that

$$\frac{|\theta_2(x) - \theta_2(w)|}{|\theta_2(x) - \theta_2(z)|} \leq C_1 \eta_1 \left(\frac{|\theta_1(x) - \theta_1(w)|}{|\theta_1(x) - \theta_1(z)|} \right)$$

and

$$\frac{|\theta_2(x) - \theta_2(y)|}{|\theta_2(x) - \theta_2(w)|} \leq C_4 \eta_2 \left(\frac{|\theta_1(x) - \theta_1(y)|}{|\theta_1(x) - \theta_1(w)|} \right)$$

where homeomorphisms η_1 and η_2 are as in cases I and II.

Thus

$$\begin{aligned} \frac{|\theta_2(x) - \theta_2(y)|}{|\theta_2(x) - \theta_2(z)|} &= \frac{|\theta_2(x) - \theta_2(y)|}{|\theta_2(x) - \theta_2(w)|} \frac{|\theta_2(x) - \theta_2(w)|}{|\theta_2(x) - \theta_2(z)|} \\ &\leq C_1 C_4 \eta_2 \left(\frac{|\theta_1(x) - \theta_1(y)|}{|\theta_1(x) - \theta_1(w)|} \right) \eta_1 \left(\frac{|\theta_1(x) - \theta_1(w)|}{|\theta_1(x) - \theta_1(z)|} \right) \\ &\leq C_1 C_4 \eta_2 \left(C \frac{|\theta_1(x) - \theta_1(y)|}{|\theta_1(x) - \theta_1(z)|} \right) \eta_1 \left(C \frac{|\theta_1(x) - \theta_1(w)|}{|\theta_1(x) - \theta_1(z)|} \right), \end{aligned}$$

Thus (6.5) holds with $\eta = \eta_3$, where $\eta_3(t) = C_1 C_4 \eta_1(Ct) \eta_2(Ct)$.

Case III.3. The remaining case is that x and y are *near*. Let $H, H' \in \mathcal{C}(\mathcal{X})$ be such that $\{x, y\} \subset \pi_G(H^{\text{diff}} \cup H'^{\text{diff}})$ and $H^{\text{diff}} \cap H'^{\text{diff}} \neq \emptyset$. As in Case III.2, there exist $C = C(\theta_1, \theta_2) > 1$ and $w \in \pi_G(H^{\text{diff}} \cup H'^{\text{diff}})$ so that

$$(6.6) \quad \min\{|\theta_i(x) - \theta_i(w)|, |\theta_i(y) - \theta_i(w)|\} \geq \frac{1}{C} \text{diam } \theta_i(\pi_G(H^{\text{diff}} \cup H'^{\text{diff}})).$$

Furthermore, there exists $C_5 = C_5(\theta_1, \theta_2) > 0$ so that

$$\frac{|\theta_2(x) - \theta_2(y)|}{|\theta_2(x) - \theta_2(z)|} \leq C_5 \eta_1 \left(\frac{|\theta_1(x) - \theta_1(y)|}{|\theta_1(x) - \theta_1(w)|} \right) \eta_2 \left(\frac{|\theta_1(x) - \theta_1(w)|}{|\theta_1(x) - \theta_1(z)|} \right).$$

By (6.6) and assumptions on $\{x, y, z\}$, we have that

$$\max \left\{ \frac{|\theta_1(x) - \theta_1(y)|}{|\theta_1(x) - \theta_1(z)|}, \frac{|\theta_1(x) - \theta_1(y)|}{|\theta_1(x) - \theta_1(w)|}, \frac{|\theta_1(x) - \theta_1(w)|}{|\theta_1(x) - \theta_1(z)|} \right\} \leq C'$$

where $C' = C'(\theta_1, \theta_2)$. Assume first that

$$\frac{|\theta_1(x) - \theta_1(y)|}{|\theta_1(x) - \theta_1(w)|} \leq \left(\frac{|\theta_1(x) - \theta_1(y)|}{|\theta_1(x) - \theta_1(z)|} \right)^{1/2}.$$

Then

$$\frac{|\theta_2(x) - \theta_2(y)|}{|\theta_2(x) - \theta_2(z)|} \leq C_5 \eta_1 \left(\frac{|\theta_1(x) - \theta_1(y)|}{|\theta_1(x) - \theta_1(z)|} \right)^{1/2} \eta_2(C').$$

The case

$$\frac{|\theta_1(x) - \theta_1(w)|}{|\theta_1(x) - \theta_1(z)|} \leq \left(\frac{|\theta_1(x) - \theta_1(y)|}{|\theta_1(x) - \theta_1(z)|} \right)^{1/2}$$

is similar. So (6.5) holds with $\eta(t) = C_5 \max\{\eta_1(t^{1/2})\eta_2(C'), \eta_1(C')\eta_2(t^{1/2})\}$. This concludes case III.2 and the proof. \square

7. A SUFFICIENT CONDITION FOR QUASISYMMETRIC PARAMETRIZATION

In this section we show that a Semmes space $(\mathbb{R}^3/G, d_\lambda)$ admits a quasisymmetric parametrization by \mathbb{R}^3 if its defining sequence has a strong welding structure.

Definition 7.1. We say that a finite welding structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ is *strong* if condensers \mathcal{C} are in \mathbb{R}^3 and weldings \mathcal{W} are similarities.

Theorem 7.2. *Suppose that $(\mathbb{R}^3/G, \mathcal{X})$ is a decomposition space of finite type and the defining sequence \mathcal{X} has a strong welding structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$. Suppose also that components of X_{k+1} are contractible in X_k for $k \geq 0$. Then there exists $\lambda_0 \in (0, 1)$ depending on $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ for the following. For each $\lambda \in (0, \lambda_0)$, there is a λ -modular embedding $\theta: \mathbb{R}^3/G \rightarrow \mathbb{R}^4$ so that the metric space $(\mathbb{R}^3/G, d_\theta)$ is Ahlfors 3-regular, linearly locally contractible, and quasisymmetric to \mathbb{R}^3 . Furthermore, there exists a quasisymmetric map $f: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ so that $f(\mathbb{R}^3) = \theta(\mathbb{R}^3/G)$.*

Since $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ is not necessarily a Semmes structure, formally the quintuple $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\theta)$ is not a modular space in the sense of Section 6. It is easy to see, however, that there exists a Semmes structure with compatible atlas. Thus $(\mathbb{R}^3/G, d_\theta)$ in Theorem 7.2 is a Semmes space with scaling factor λ in the sense of Definition 6.1.

Remark 7.3. *The assumption on the existence of a strong welding structure is essential in Theorem 7.2. The decomposition space \mathbb{R}^3/Bd associated to the Bing double has a Semmes structure in \mathbb{R}^4 , however the associated Semmes spaces $(\mathbb{R}^3/\text{Bd}, d_\lambda)$ are not quasisymmetric to \mathbb{R}^3 for any $\lambda > 0$.*

The construction of the embedding from \mathbb{R}^3/G into \mathbb{R}^4 in Theorem 7.2 adapts Semmes' idea of building excellent packages for the self-similar decomposition spaces.

For the proof, we introduce the following notation related to condensers. Let $K \subset \mathbb{R}^3$ be a handlebody. We define $K^* = K \times [-\text{diam } K, \text{diam } K] \subset \mathbb{R}^4$, where $\text{diam } K$ is the Euclidean diameter of K . If B is a pair-wise disjoint union of handlebodies, we set $B^* = \bigcup_{K \in \mathcal{C}(B)} K^*$. Suppose (A, B) is a condenser in \mathbb{R}^3 , we will also call (A^*, B^*) a (4-dimensional) condenser.

7.1. Unlinking and repacking. As a preliminary step for the proof of Theorem 7.2, we discuss homeomorphisms of \mathbb{R}^4 that unlink and repack condensers. We assume *all condensers* $\mathfrak{c} = (A, B)$ have diameter $\text{diam } A = 1$. We call a similarity a c -similarity if it has scaling factor c .

Let $\mathfrak{c} = (A, B)$ be a condenser in \mathbb{R}^3 and $\lambda \in (0, 1)$. We say a PL-embedding $p_{\mathfrak{c}}: (\mathbb{R}^3 \setminus A) \cup B \rightarrow \mathbb{R}^3$ is a λ -*repacking* of \mathfrak{c} if there are pair-wise disjoint Euclidean balls $\{B_D \subset \text{int}A : D \in \mathcal{C}(B)\}$ such that

- (i) $p_{\mathfrak{c}}|_{\mathbb{R}^3 \setminus A} = \text{id}$,
- (ii) $p_{\mathfrak{c}}|_D$ is a similarity, and
- (iii) $p_{\mathfrak{c}}(D) \subset \text{int}B_D$ and $\text{diam} p_{\mathfrak{c}}(D) = \lambda$,

for each component D of B .

We say that a PL-homeomorphism $P_{\mathfrak{c}}: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is a **-stable λ -repacking of condenser \mathfrak{c} (or of condenser (A^*, B^*))* if

- (1) $P_{\mathfrak{c}}|_{(\mathbb{R}^3 \setminus A) \cup B}$ is a λ -repacking of \mathfrak{c} ,
- (2) $P_{\mathfrak{c}}|_{\mathbb{R}^4 \setminus A^*} = \text{id}$,
- (3) $P_{\mathfrak{c}}|_{D^*}$ is a similarity for each component D of B , in particular
- (4) $P_{\mathfrak{c}}(B^*) = P_{\mathfrak{c}}(B)^*$.

Lemma 7.4. *Let $\mathfrak{c} = (A, B)$ be a condenser in \mathbb{R}^3 so that components of B are contractible in A , and let $p_{\mathfrak{c}}: (\mathbb{R}^3 \setminus A) \cup B \rightarrow \mathbb{R}^3$ be a λ -repacking of a condenser $\mathfrak{c} = (A, B)$. Then there exists a *-stable λ -repacking $P_{\mathfrak{c}}: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ of \mathfrak{c} .*

Proof. We show first that, for every $D \in \mathcal{C}(B)$, there exists a PL-homeomorphism $f_D: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ so that $f_D|_{\mathbb{R}^4 \setminus A^*} = \text{id}$, $f_D|_D = p_{\mathfrak{c}}|_D$, and $f_D(D^*) = p_{\mathfrak{c}}(D)^*$. We combine these homeomorphisms together in the final step of the proof. We fix $c \in (0, 1)$ so that

$$B^* \cup (p_{\mathfrak{c}}(B))^* \subset \text{int}(A \times [-c, c]) \subset A \times [-c, c] \subset A \times [-1, 1] = A^*.$$

Let $D \in \mathcal{C}(B)$. Let $B_D = B^3(x_D, r_D) \subset A$ be the Euclidean ball containing $p_{\mathfrak{c}}(D)$ as in (iii). Since $p_{\mathfrak{c}}|_D$ is a similarity, it has a unique similarity extension $p_D: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with a scaling constant λ_D .

We denote again by $p_D: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ the extension $(x, t) \mapsto (p_D(x), \lambda_D t)$ of p_D . Let $\tilde{B}_D = p_D^{-1}(B^4(x_D, r_D))$.

Fix a core \mathcal{R}_D of D . Since D^* is a regular neighborhood of \mathcal{R}_D and \mathcal{R}_D is contractible in A , there exist, by the Penrose-Whitehead-Zeeman lemma (see Section 3.2), PL 4-cells E_D and E'_D in $A \times I$ so that

$$D^* \subset \text{int}E_D \subset E_D \subset \text{int}E'_D.$$

We fix $z_D \in \text{int}E_D$ and $0 < \varepsilon_D < r_D < r'_D$ so that $B^4(z_D, 2\varepsilon_D) \subset E_D$ and $B^4(z_D, r'_D) \subset \text{int}A^*$. Since E_D and E'_D are 4-cells there exists an orientation preserving PL homeomorphism $h_D: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ so that $h_D|_{\mathbb{R}^4 \setminus E'_D} = \text{id}$, $h_D(E_D) = B^4(z_D, \varepsilon_D)$ and $h_D(D^*) \subset B^4(z_D, \varepsilon_D/2)$. We fix also an orientation preserving PL homeomorphism $k_D: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ so that $k_D|_{D^*} = \text{id}$ and $k_D(E_D) = \tilde{B}_D$.

By standard isotopy results, there exists a PL-homeomorphism $\tau_D: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ so that $\tau_D|_{\mathbb{R}^4 \setminus (A \times I)} = \text{id}$ and $\tau_D|_{B^4(z_D, \varepsilon_D)}$ is the translation $x \mapsto x + (x_D - z_D)$ onto $B^4(x_D, \varepsilon_D)$.

Since

$$p_D \circ k_D \circ h_D^{-1} \circ \tau_D^{-1}(B^4(x_D, \varepsilon_D)) = p_D \circ k_D(E_D) = B_D$$

and p_D is orientation preserving, the map $\psi_D: B^4(x_D, \varepsilon_D) \rightarrow B_D$,

$$\psi_D = p_D \circ k_D \circ h_D^{-1} \circ \tau_D^{-1},$$

is an orientation preserving PL homeomorphism. Thus ψ_D extends to a homeomorphism $\psi_D: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ so that $\psi_D|_{\mathbb{R}^4 \setminus B^4(x_D, r'_D)} = \text{id}$.

Since $h_D(D^*) \subset B^4(z_D, \varepsilon_D/2)$ and $\tau_D|_{B^4(z_D, \varepsilon_D)}$ is an isometry, we have

$$\begin{aligned} \psi_D \circ \tau_D \circ h_D|_{D^*} &= p_D \circ k_D \circ h_D^{-1} \circ \tau_D^{-1} \circ \tau_D \circ h_D|_{D^*} \\ &= p_D \circ k_D|_{D^*} = p_D|_{D^*}. \end{aligned}$$

It is now easy to find a PL homeomorphism $f_D: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ so that $f_D|_{\mathbb{R}^4 \setminus (A \times I)} = \text{id}$ and $f_D|_{D^*} = p_D|_{D^*}$.

Final step: We define now homeomorphisms g_1, g_2 , and g_3 of \mathbb{R}^4 so that $P_c = g_3^{-1} \circ g_2 \circ g_1$ is a $*$ -stable λ -repacking of \mathfrak{c} . Recall that $\text{diam } A = 1$.

For every $D \in \mathcal{C}(B)$, we fix $c_D \in (c, 1)$ so that $c_D \neq c_{D'}$ for different components D and D' in $\mathcal{C}(B)$. We also fix $\delta > 0$ so that intervals $[c_D - \delta, c_D + \delta]$ are pair-wise disjoint and contained in $(c, 1)$. Let $\rho = \delta/(4c)$. We denote also $J_D = [c_D - \delta/4, c_D + \delta/4]$ for every $D \in \mathcal{C}(B)$.

We fix a PL-function $u: \mathbb{R}^3 \rightarrow [0, 1]$ so that $\text{spt } u \subset \text{int } A$ and $u|_D = c_D$ for every $D \in \mathcal{C}(B)$. It is now easy to find a PL-homeomorphism $g_1: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}$ so that $g_1|_{\mathbb{R}^4 \setminus A^*} = \text{id}$ and $g_1(x, t) = (x, u(x) + \rho t)$ for $(x, t) \in B \times I$. In particular,

$$g_1(D^*) \subset g_1(D \times I) = D \times J_D$$

for every $D \in \mathcal{C}(B)$.

The homeomorphism g_3 is defined similarly as g_1 with $(A, \bigcup_{D \in \mathcal{C}(B)} p_c(D))$ in place of (A, B) so that the PL-homeomorphism $g_3: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ satisfies $g_3|_{\mathbb{R}^4 \setminus A^*} = \text{id}$ and

$$g_3(p_c(D)^*) \subset g_3(p_c(D) \times I) \subset p_c(D) \times J_D$$

for $D \in \mathcal{C}(B)$.

Having g_1 and g_3 at our disposal, we construct a PL-homeomorphism g_2 as follows. For every $D \in \mathcal{C}(B)$, let $\zeta_D: \mathbb{R} \rightarrow \mathbb{R}$, be the piece-wise linear increasing function so that $\zeta_D(t) = \rho t + c_D$ for $t \in I$, and $\zeta_D(t) = t$ for $|t| > 1$. Let also $\xi_D: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the PL map $(x, t) \mapsto (x, \zeta_D(t))$. Then $\xi_D|_{D^*} = g_1|_{D^*}$ and $\xi_D|_{p_c(D)^*} = g_3|_{p_c(D)^*}$ for every $D \in \mathcal{C}(B)$.

Since $f_D|_{\mathbb{R}^4 \setminus \{-c < x_4 < c\}} = \text{id}$, we have

$$\xi_D \circ f_D \circ \xi_D^{-1}|_{\mathbb{R}^4 \setminus \{c_D - \delta/4 < x_4 < c_D + \delta/4\}} = \text{id}$$

for every $D \in \mathcal{C}(B)$. Thus the mapping $g_2: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ defined by taking the composition (in any order) of $\xi_D \circ f_D \circ \xi_D^{-1}$ for all $D \in \mathcal{C}(B)$, is a well-defined PL-homeomorphism. Moreover,

$$g_3^{-1} \circ g_2 \circ g_1|_{D^*} = (g_3^{-1} \circ \xi_D) \circ f_D \circ (\xi_D^{-1} \circ g_1)|_{D^*} = f_D|_{D^*} = p_D|_{D^*}$$

Thus $P_c = g_3^{-1} \circ g_2 \circ g_1$ is a $*$ -stable repacking of \mathfrak{c} . \square

Let $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ be a finite welding structure for a defining sequence \mathcal{X} . We say that $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ admits a λ -repacking if for every $\mathfrak{c} \in \mathcal{C}$ there exists a λ -repacking $p_c: (\mathbb{R}^3 \setminus A) \cup B \rightarrow \mathbb{R}^3$ of \mathfrak{c} . For finite welding structures $(\mathcal{C}, \mathcal{A}, \mathcal{W})$, λ -repacking of $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ exists for some $\lambda > 0$. The *repacking constant* $\lambda_{(\mathcal{C}, \mathcal{A}, \mathcal{W})}$ of $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ is defined to be the supremum of all $\lambda \in (0, 1)$ so that $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ admits a λ -repacking.

7.2. Proof of Theorem 7.2. We assume as we may that $X_0 = H_0$, that the condenser $\mathbf{c}_{H_0} = (A_{H_0}, B_{H_0})$ and the chart φ_{H_0} are chosen so that $A_{H_0} = H_0$ and $\varphi_{H_0}|_{\partial H_0} = \text{id}$, and that $\text{diam } A = 1$ for all $\mathbf{c} = (A, B)$ in the structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$.

We enumerate the handlebodies in $\mathcal{C}(\mathcal{X})$ by H_0, H_1, \dots so that if $H_j \in \mathcal{C}(X_k)$ then $H_{j+1} \in \mathcal{C}(X_k) \cup \mathcal{C}(X_{k+1})$. This enumeration provides a natural ordering for condensers, charts and the weldings as well. Denote by $\mathbf{c}_j = (A_j, B_j) = \mathbf{c}_{H_j}$ for condensers in \mathcal{C} and by $\varphi_j = \varphi_{H_j}: H_j^{\text{diff}} \rightarrow \mathbf{c}_j^{\text{diff}}$ the charts in \mathcal{A} for $j \geq 0$.

Let $k_j = \text{level}(H_j)$, and $q(j)$ be the index of the parent of H_j , that is, $\text{level}(H_{q(j)}) = k_j - 1$ and $H_j \in \mathcal{C}(H_{q(j)} \cap X_{k_j})$.

Let $w_j = \varphi_{q(j)} \circ \varphi_j^{-1}$ be the welding of (A_j, B_j) to its parent $(A_{q(j)}, B_{q(j)})$, for $j \geq 1$. Since w_j is a similarity and $B_{q(j)} \subset \mathbb{R}^3$, $w_j(A_j)$ is a component of $B_{q(j)}$. We extend w_j to a similarity $w_j: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ by $(x, t) \mapsto (w_j(x), \lambda_j t)$, where λ_j is the scaling factor of w_j . We call this extension w_j as a welding of (A_j^*, B_j^*) to $(A_{q(j)}^*, B_{q(j)}^*)$, and note that $w_j(A_j^*) = w_i(A_j^*)^*$ is a component of $B_{q(j)}^*$.

We construct now *cumulative welding maps* and *repacked cumulative welding maps*. The cumulative welding maps \hat{w}_j we define by $\hat{w}_0 = \text{id}$ and

$$\hat{w}_j = \hat{w}_{q(j)} \circ w_j$$

for $j \geq 1$.

Since $w_j|_{\partial A_j} = \varphi_{q(j)} \circ \varphi_j^{-1}|_{\partial A_j}$, we have

$$(7.1) \quad \begin{aligned} \hat{w}_j \circ \varphi_j|_{\partial H_j} &= \hat{w}_{q(j)} \circ w_j \circ \varphi_j|_{\partial H_j} \\ &= \hat{w}_{q(j)} \circ \varphi_{q(j)}|_{\partial H_j} \end{aligned}$$

for $j \geq 1$, and $\hat{w}_0 \circ \varphi_0|_{\partial H_0} = \text{id}$.

Since w_j is a similarity, $\hat{w}_j(A_j)$ is a component of $\hat{w}_{q(j)}(B_{q(j)})$. Since

$$\hat{w}_j(A_j^*) = \hat{w}_{q(j)}(w_j(A_j^*)) \subset \hat{w}_{q(j)}(B_{q(j)}^*),$$

it follows, by induction on the level of H , $H \in \mathcal{C}(\mathcal{X})$, that the images $\hat{w}_i(A_j^* \setminus B_j^*)$ are pair-wise disjoint for $j \geq 0$. Then

$$(7.2) \quad \mathbb{R}^4 \setminus \hat{F} = (\mathbb{R}^4 \setminus X_0^*) \cup \bigcup_{j=1}^{\infty} \hat{w}_j(A_j^* \setminus B_j^*)$$

is a disjoint union, where

$$\hat{F} = \bigcap_{k \geq 0} \left(\bigcup \{ \hat{w}_j(A_j) : (A_j, B_j) \in \mathcal{C}, \varphi_j^{-1}(A_j) = H_j \in \mathcal{C}(X_k) \} \right).$$

Since $\text{diam } \hat{w}_j(A_j) \rightarrow 0$ as $j \rightarrow \infty$, the components of \hat{F} are points.

We define now *repacked cumulative welding maps* \tilde{w}_j . Let $0 < \lambda < \lambda_{(\mathcal{C}, \mathcal{A}, \mathcal{W})}$. We show first that components of B_j are contractible in A_j . Let $D \in \mathcal{C}(B_j)$. Since $\varphi_j^{-1}(\partial D)$ is a boundary of a component of $H_j \cap X_{\text{level}(H_j)+1}$, $\varphi_j^{-1}(\partial D)$ is contractible in H_j . Thus ∂D is contractible in A_j . Let \mathcal{R}_D be a core of D that is contained in a collar Ω_D of ∂D in D . Since Ω_D retracts to ∂D , \mathcal{R}_D is contractible in A_j . Thus D is contractible in A_j .

Using Lemma 7.4, we fix a collection of $*$ -stable λ -repackings $\{P_c: \mathbb{R}^4 \rightarrow \mathbb{R}^4: c \in \mathcal{C}\}$. For simplicity, denote the $*$ -stable repacking for $c_j = (A_j, B_j)$ by $P_j = P_{c_j}$, for $j \geq 0$.

Given welding maps $w_j: \mathbb{R}^4 \rightarrow \mathbb{R}^4$, $j \geq 1$, and $*$ -stable repackings P_j ($j \geq 0$) we set $\tilde{w}_j: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ by

$$\tilde{w}_j = \tilde{w}_{q(j)} \circ w_j \circ P_j$$

for $j \geq 1$ and by $\tilde{w}_0 = P_0$.

Since the $*$ -stable repacking P_j is a similarity in each component of B_j^* , we have that $\tilde{w}_j|D^*$ is a similarity for every $D \in \mathcal{C}(B_j)$. Note that $\tilde{w}_{q(j)} \circ w_j|A_j^*$ is a similarity and that $\tilde{w}_j|A_j^* \setminus B_j^*$ is a composition of an L -bilipschitz map P_j with a similarity for every $j \geq 0$.

Since $P_j|\partial A_j = \text{id}$, we have, as in (7.1),

$$(7.3) \quad \tilde{w}_j \circ \varphi_j|\partial H_j = \tilde{w}_{q(j)} \circ \varphi_{q(j)}|\partial H_j$$

for $j \geq 1$; $\tilde{w}_0 \circ \varphi_0|\partial H_0 = P_0 \circ \varphi_0|\partial H_0 = \text{id}$.

We also have that $\tilde{w}_j(A_j)$ is a component of $\tilde{w}_{q(j)}(B_{q(j)})$ for $j \geq 1$ and images $\tilde{w}_j(A_j^* \setminus B_j^*)$ are pair-wise disjoint for $j \geq 0$. Thus we have a disjoint union

$$(7.4) \quad \mathbb{R}^4 \setminus \tilde{F} = (\mathbb{R}^4 \setminus X_0^*) \cup \bigcup_{j=1}^{\infty} \tilde{w}_j(A_j^* \setminus B_j^*)$$

where

$$\tilde{F} = \bigcap_{k \geq 0} \left(\bigcup \{ \tilde{w}_j(A_j) : (A_j, B_j) \in \mathcal{C}, \varphi_j^{-1}(A_j) = H_j \in \mathcal{C}(X_k) \} \right).$$

Similarly as for \hat{F} , we have that \tilde{F} is totally disconnected. Note also that the mapping $\tilde{w}_j|A_j$ is $L\lambda^j$ -Lipschitz for every $j \geq 1$, where L is the maximum of the Lipschitz constants of $*$ -stable repackings $\{P_c: c \in \mathcal{C}\}$.

Having (7.3) and (7.4) at our disposal, we define an embedding $\theta_\infty: \mathbb{R}^3 \setminus X_\infty \rightarrow \mathbb{R}^4$ by $\theta_\infty| \mathbb{R}^3 \setminus X_0 = \text{id}$ and $\theta_\infty|H_j^{\text{diff}} = \tilde{w}_j \circ \varphi_j$ for $j \geq 1$. Furthermore, θ_∞ descends (and then extends) to an embedding $\theta: \mathbb{R}^3/G \rightarrow \mathbb{R}^4$ so that $\theta(\pi_G(X_\infty)) = \tilde{F}$. The λ -modularity of θ follows directly from (uniform) quasimilarity of cumulative welding maps \tilde{w}_j . So $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\theta)$ is a Semmes space.

It remains to construct a quasisymmetric map $f: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ so that $f(\mathbb{R}^3) = \theta(\mathbb{R}^3/G)$. Since $P_j|\partial A_j^* = \text{id}$, we have

$$\tilde{w}_j \circ \hat{w}_j^{-1}|\hat{w}_j(\partial A_j^*) = (\tilde{w}_{q(j)} \circ w_j) \circ (w_j^{-1} \circ \hat{w}_{q(j)}^{-1})|\hat{w}_j(\partial A_j^*) = \tilde{w}_{q(j)} \circ \hat{w}_{q(j)}^{-1}|\hat{w}_j(\partial A_j^*)$$

for every $j \geq 1$. Thus the mapping $f_\infty: \mathbb{R}^4 \setminus \hat{F} \rightarrow \mathbb{R}^4 \setminus \tilde{F}$, defined by

$$f_\infty|\hat{w}_j(A_j^* \setminus B_j^*) = \tilde{w}_j \circ \hat{w}_j^{-1}|\hat{w}_j(A_j^* \setminus B_j^*)$$

and $f_\infty| \mathbb{R}^4 \setminus A_0^* = \text{id}$, is a well-defined homeomorphism. Since \hat{F} and \tilde{F} are totally disconnected, f_∞ extends to a homeomorphism $f: \mathbb{R}^4 \rightarrow \mathbb{R}^4$.

Since $f| \mathbb{R}^3 \setminus A_0 = \theta_\infty| \mathbb{R}^3 \setminus X_0$ and

$$f \circ \hat{w}_j \circ \varphi_j|X_j^{\text{diff}} = \tilde{w}_j \circ \hat{w}_j^{-1} \circ \hat{w}_j \circ \varphi_j|X_j^{\text{diff}} = \tilde{w}_j \circ \varphi_j|X_j^{\text{diff}} = \theta_\infty|X_j^{\text{diff}}$$

for every $j \geq 0$, we have

$$f(\mathbb{R}^3) = \theta(\mathbb{R}^3/G)$$

by continuity.

Since $\tilde{w}_j \circ \hat{w}_j^{-1}|_{\hat{w}_j(A_j^*)}$ is a (L, μ_j) -quasisimilarity for every $j \geq 0$, homeomorphism $f_\infty: \mathbb{R}^4 \setminus \hat{F} \rightarrow \mathbb{R}^4 \setminus \tilde{F}$ is quasiconformal. Moreover, homeomorphisms $f_j: \mathbb{R}^4 \rightarrow \mathbb{R}^4$, defined by

$$f_j|_{\mathbb{R}^4 \setminus \hat{w}_j(A_j^*)} = f_\infty|_{\mathbb{R}^4 \setminus \hat{w}_j(A_j^*)}$$

and

$$f_j|_{\hat{w}_j(A_j^*)} = \tilde{w}_j \circ \hat{w}_j^{-1}|_{\hat{w}_j(A_j^*)},$$

are uniformly quasiconformal. Thus there exists $\eta: [0, \infty) \rightarrow [0, \infty)$ so that homeomorphisms f_j are η -quasisymmetric. Since $f = \lim_{j \rightarrow \infty} f_j$, f is η -quasisymmetric. This completes the proof of Theorem 7.2.

Remark 7.5. *The quasisymmetric map $f: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ in Theorem 7.2 can be taken to be identity in a neighborhood of the infinity, that is, there exists $R > 0$ so that $f|_{\mathbb{R}^4 \setminus B^4(R)} = \text{id}$. Thus the quasisymmetric map $f: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ extends naturally to a quasiconformal map $f: \mathbb{S}^4 \rightarrow \mathbb{S}^4$ and $f(\mathbb{S}^3)$ is the one point compactification of $f(\mathbb{R}^3)$. Thus the embedding $\theta: \mathbb{R}^3/G \rightarrow \mathbb{R}^4$ extends to an embedding $\mathbb{S}^3/G \rightarrow \mathbb{S}^4$. So $\theta(\mathbb{S}^3/G)$ is a quasisphere, that is, $\theta(\mathbb{S}^3/G) = f(\mathbb{S}^3)$, where $f: \mathbb{S}^4 \rightarrow \mathbb{S}^4$ is a quasiconformal map.*

8. CIRCULATION OF HANDLEBODIES

In this section we introduce the notion of *circulation* of a union of handlebodies based on longitudes and meridians. This concept of circulation will be used in estimating conformal modulus of surface families.

8.1. Meridians and longitudes. Recall that a simple closed curve $\mathbb{S}^1 \rightarrow \partial\mathbb{B}^2 \times \mathbb{S}^1$ on the boundary of a torus $\mathbb{B}^2 \times \mathbb{S}^1$ is called a *meridian of $\mathbb{B}^2 \times \mathbb{S}^1$* if it is homotopic to the loop $e^{i\theta} \mapsto (e^{i\theta}, 1)$, on $\partial\mathbb{B}^2 \times \mathbb{S}^1$. In particular, a meridian is contractible in $\mathbb{B}^2 \times \mathbb{S}^1$ but not in $\partial\mathbb{B}^2 \times \mathbb{S}^1$.

A non-contractible loop in the solid torus $\mathbb{B}^2 \times \mathbb{S}^1$ is called a *longitude of $\mathbb{B}^2 \times \mathbb{S}^1$* . Longitudes are non-trivially linked with all meridians, that is, given a longitude σ and a meridian α of $\mathbb{B}^2 \times \mathbb{S}^1$ then $\sigma(\mathbb{S}^1) \cap \phi(\mathbb{B}^2) \neq \emptyset$ for every $\phi: \mathbb{B}^2 \rightarrow \mathbb{B}^2 \times \mathbb{S}^1$ satisfying $\phi|_{\partial\mathbb{B}^2} = \alpha$.

Let X be a union of cubes-with-handles. We call a simple closed PL curve $\alpha: \mathbb{S}^1 \rightarrow \partial X$ a *meridian of X* if $[\alpha] \neq 0$ in $\pi_1(\partial X)$ and $[\alpha] = 0$ in $\pi_1(X)$; that is, α is not contractible on ∂X but there exists a map $\phi: \mathbb{B}^2 \rightarrow X$ so that $\phi|_{\partial\mathbb{B}^2} = \alpha$.

Suppose $\alpha: \mathbb{S}^1 \rightarrow \partial X$ is a meridian of X . Departing slightly from the notion of mapping of pairs $(C, D) \rightarrow (E, F)$, we denote by $\phi: (\mathbb{B}^2, \partial\mathbb{B}^2) \rightarrow (X, \alpha)$ a mapping $\phi: \mathbb{B}^2 \rightarrow X$ that satisfies $\phi|_{\partial\mathbb{B}^2} = \alpha$. We denote by $\mathcal{E}(X, \alpha)$ the collection of all maps $\phi: (\mathbb{B}^2, \partial\mathbb{B}^2) \rightarrow (X, \alpha)$.

Let σ be an unweighted PL 1-cycle in a union of cubes-with-handles X , that is, $\sigma = \sigma_1 + \dots + \sigma_k$ where $\sigma_i: \mathbb{S}^1 \rightarrow X$ are PL loops for $i = 1, \dots, k$; we denote $|\sigma| = \bigcup_{i=1}^k \sigma_i(\mathbb{S}^1)$. We say that σ is a *longitude in X* if $|\sigma| \cap \phi(\mathbb{B}^2) \neq \emptyset$ for all $\phi: (\mathbb{B}^2, \partial\mathbb{B}^2) \rightarrow (X, \alpha)$ and all meridians α of X . Heuristically, longitudes are the 1-cycles in X that are linked with all meridians of X . We denote by

$$(8.1) \quad \Sigma(X) = \text{the family of all longitudes of } X.$$

Suppose that H_1, \dots, H_d are pair-wise disjoint handlebodies, then

$$(8.2) \quad \Sigma\left(\bigcup_{i=1}^d H_i\right) = \{\sigma_1 + \dots + \sigma_d : \sigma_i \in \Sigma(H_i), 1 \leq i \leq d\}.$$

8.2. Circulation with respect to meridians. Let H be a cube-with-handles, X a finite union of cubes-with-handles in H , and $\alpha: \mathbb{S}^1 \rightarrow \partial H$ a meridian of H . The *circulation of X in H with respect to meridian α* is defined to be

$$(8.3) \quad \text{circ}(X, \alpha, H) = \min_{\phi \in \mathcal{E}(H, \alpha)} \min_{\sigma \in \Sigma(X)} \#(|\sigma| \cap \phi(\mathbb{B}^2)).$$

Let $(\mathbb{R}^3/G, \mathcal{X})$ be a decomposition space. We say that \mathcal{M} is a *collection of meridians of \mathcal{X}* if every $\alpha \in \mathcal{M}$ is a meridian on H for some $H \in \mathcal{C}(\mathcal{X})$. Given a collection of meridians \mathcal{M} of \mathcal{X} and $H \in \mathcal{C}(\mathcal{X})$, we denote by $\mathcal{M}|H$ the meridians of H contained in \mathcal{M} .

Let $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}))$ be a decomposition space of finite type. We say that a collection of meridians \mathcal{M} of \mathcal{X} is *homotopically finite with respect to $(\mathcal{C}, \mathcal{A}, \mathcal{W})$* if for every $\mathbf{c} = (A, B) \in \mathcal{C}$ the set $\{[\varphi_H \circ \alpha] \in \pi_1(\partial A) : H \in \mathcal{C}(\mathcal{X}) \text{ satisfying } \varphi_H: H^{\text{diff}} \rightarrow \mathbf{c}^{\text{diff}} \text{ and } \alpha \in \mathcal{M}|H\}$ is finite.

Definition 8.1. Let $(\mathbb{R}^3/G, \mathcal{X})$ be a decomposition space of finite type. We say that the *order of circulation of \mathcal{X} is at least ω* , $\omega \geq 0$, if there exist a collection of meridians \mathcal{M} on \mathcal{X} and a welding structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ so that \mathcal{M} is homotopically finite with respect to $(\mathcal{C}, \mathcal{A}, \mathcal{W})$ and a constant $C > 0$ so that for every $\ell \geq 0$ there exist indices $k, k' \geq 0$ satisfying $k' - k \geq \ell$, a handlebody $H \in \mathcal{C}(X_k)$, and a meridian $\alpha \in \mathcal{M}|H$ so that

$$(8.4) \quad \text{circ}(X_{k'} \cap H, \alpha, H) \geq C\omega^{k'-k}.$$

The homotopical finiteness of a collection of meridians translates to geometric finiteness in the corresponding Semmes space after fixing a simple PL representative for each homotopy class. We record this observation in the following lemma.

Lemma 8.2. *Let $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\lambda)$ be a Semmes space and \mathcal{M} collection of meridians on \mathcal{X} that is homotopically finite with respect to $(\mathcal{C}, \mathcal{A}, \mathcal{W})$. Then there exists $L = L(d_\lambda) \geq 1$ so that for every meridian $\alpha \in \mathcal{M}|H$ there exists a meridian β of H homotopic to α in ∂H so that $\beta: \mathbb{S}^1 \rightarrow (\mathbb{R}^3/G, d_\lambda)$ is a $(\lambda^{\text{level}(H)}, L)$ -quasisimilarity.*

We assume as we may meridians in a homotopically finite collection \mathcal{M} are uniformly quasisimilar: there exists $L \geq 1$ so that every $\alpha \in \mathcal{M}|H$ is a $(\lambda^{\text{level}(H)}, L)$ -quasisimilarity.

We record as a lemma the observation that uniformly quasisimilar meridians have a uniformly quasisymmetric collars in the following sense. The claim follows directly from the definition of the metric $d_{\lambda, m}$; the constant ϵ_λ is the constant in Remark 6.7.

Lemma 8.3. *Suppose that $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\lambda)$ is a Semmes space and suppose that \mathcal{M} is a uniformly quasisimilar family of meridians on \mathcal{X} .*

Then there exists $L = L(d_\lambda) \geq 1$ so that for each $k \geq 0$, $H \in \mathcal{C}(X_k)$ and $\alpha \in \mathcal{M}|H$, and each $m \geq 0$, there exists a (λ^k, L) -quasisimilarity

$$\varkappa_\alpha: (\mathbb{B}^{2+m} \times \mathbb{S}^1, \{0\} \times \mathbb{S}^1) \rightarrow (N_{d_{\lambda,m}}(\pi_G(\partial H), \epsilon_\lambda \lambda^k), (\pi_G \circ \alpha) \times \{0\}).$$

In particular, maps in the collection $\{\varkappa_\alpha\}_{\alpha \in \mathcal{M}}$ are uniformly quasisymmetric.

Given a defining sequence $\mathcal{X} = (X_k)$ and a union Y of a nonempty subcollection of handlebodies in $\mathcal{C}(X_k)$, we call

$$(8.5) \quad \Sigma(Y, \mathcal{X}) = \{\sigma \in \Sigma(Y) : |\sigma| \subset X_k \setminus X_{k+1}\}$$

longitudes of Y relative to \mathcal{X} . This subfamily $\Sigma(X_{k'} \cap H, \mathcal{X})$ of $\Sigma(X_{k'} \cap H)$ may be used to determine the circulation $\text{circ}(X_{k'} \cap H, \alpha, H)$.

Lemma 8.4. *Let $k \geq 0$, $H \in \mathcal{C}(X_k)$, and α a meridian of H . Then*

$$\text{circ}(X_{k'} \cap H, \alpha, H) = \min_{\phi \in \mathcal{E}(H, \alpha)} \min_{\sigma \in \Sigma(X_{k'} \cap H, \mathcal{X})} \#(|\sigma| \cap \phi(\mathbb{B}^2))$$

for $k' > k$.

Proof. Let $\sigma = \sigma_1 + \cdots + \sigma_\ell \in \Sigma(X_{k'} \cap H)$ and $\phi \in \mathcal{E}(H, \alpha)$ be chosen so that

$$\#(|\sigma| \cap \phi(\mathbb{B}^2)) = \text{circ}(X_{k'} \cap H, \alpha, H).$$

We claim that there exists a homeomorphism h of $X_{k'} \cap H$, identity on $\partial(X_{k'} \cap H)$, so that $h \circ \sigma = h \circ \sigma_1 + \cdots + h \circ \sigma_\ell \in \Sigma(X_{k'} \cap H, \mathcal{X})$.

For every component H_1, \dots, H_d of $X_{k'} \cap H$ let g_i be the genus of H_i and let $\rho_i: \bigvee^{g_i} \mathbb{S}^1 \rightarrow H_i$ be a core of H_i . Let $\mathcal{R} = \bigcup_i \rho_i(\bigvee^{g_i} \mathbb{S}^1)$. By considering an isotopy of $X_{k'} \cap H$ if necessary, we may assume that $\mathcal{R} \cap |\sigma| = \emptyset$. Then there exists a regular neighborhood X of \mathcal{R} so that $(X_{k'} \cap H) \setminus X$ is homeomorphic to $\partial(X_{k'} \cap H) \times [0, 1)$ and that $|\sigma| \subset (X_{k'} \cap H) \setminus X$. Then there exists a homeomorphism h of $X_{k'} \cap H$, isotopic to the identity, so that $h((X_{k'} \cap H) \setminus X) \cap X_{k'+1} = \emptyset$ and that h is identity on $\partial(X_{k'} \cap H)$. Hence $h \circ \sigma \in \Sigma(X_{k'} \cap H, \mathcal{X})$.

We extend the homeomorphism h by identity on $H \setminus X_{k'}$. Then $h \circ \phi \in \mathcal{E}(H, \alpha)$ and

$$\#(|h \circ \sigma| \cap h(\phi(\mathbb{B}^2))) = \#(|\sigma| \cap \phi(\mathbb{B}^2)) = \text{circ}(X_{k'} \cap H, \alpha, H).$$

The claim follows from $\Sigma(X_{k'} \cap H, \mathcal{X}) \subset \Sigma(X_{k'} \cap H)$. \square

8.3. Intersections in decomposition spaces. When \mathbb{R}^3/G is a manifold factor, circulation of handlebodies in \mathbb{R}^3 can be estimated from above by intersection numbers of longitudes and interior essential maps in the decomposition space \mathbb{R}^3/G , instead of \mathbb{R}^3 . The following proposition deals with this subtle, technical point.

In the following, $\Pi: \mathbb{R}^3/G \times \mathbb{R}^m \rightarrow \mathbb{R}^3/G$ is the projection map $(x, v) \mapsto x$.

Proposition 8.5. *Let $(\mathbb{R}^3/G, \mathcal{X})$ be a decomposition space, $\alpha: \mathbb{S}^1 \rightarrow \partial H$ a meridian of $H \in \mathcal{C}(\mathcal{X})$, and $\zeta: \mathbb{B}^2 \rightarrow \pi_G H$ be a map satisfying $\zeta|_{\partial \mathbb{B}^2} = \pi_G \circ \alpha$. Suppose that $\mathbb{R}^3/G \times \mathbb{R}^m$ is homeomorphic to \mathbb{R}^{3+m} for some $m \geq 0$. Then*

$$\#(\pi_G(|\sigma|) \cap \zeta(\mathbb{B}^2)) \geq \text{circ}(X_{k'} \cap H, \alpha, H)$$

for all $k' > \text{level}(H)$ and every longitude $\sigma \in \Sigma(X_{k'} \cap H, \mathcal{X})$.

The proof is based on the following approximation lemma.

Lemma 8.6. *Under the hypotheses of the proposition, for every $k' > k$ there exists a map $\phi: \mathbb{B}^2 \rightarrow H$ so that $\pi_G \circ \phi|_{\Omega} = \zeta|_{\Omega}$, where Ω is the component of $\zeta^{-1}(\pi_G H \setminus \pi_G(X_{k'}))$ that contains $\partial\mathbb{B}^2$.*

Proof. If $\zeta(\mathbb{B}^2) \cap \pi_G(X_K) = \emptyset$ for some $K > 0$ then we may take $\phi = \pi_G^{-1} \circ \zeta$, since $\pi_G|_{\mathbb{R}^3 \setminus X_K}$ is a homeomorphism. The conclusion follows. Thus we may assume that $\zeta(\mathbb{B}^2) \cap \pi_G(X_K) \neq \emptyset$ for all $K > 0$.

We fix a homeomorphism $f: \mathbb{R}^3/G \times \mathbb{R}^m \rightarrow \mathbb{R}^{3+m}$ and a number $R > 0$ so that $f(\zeta(\mathbb{B}^2)) \subset B^{3+m}(R)$. Let $\mathcal{B}' = B^{3+m}(R+1)$, $\mathcal{B} = B^{3+m}(R+2)$, and

$$\varepsilon = \frac{1}{4} \min\{1, \text{dist}(f(\pi_G(\partial X_{k'+1}) \times \mathbb{R}^m) \cap \mathcal{B}, f(\pi_G(X_{k'+2}) \times \mathbb{R}^m) \cap \mathcal{B})\}.$$

Since ζ and $f|_{f^{-1}\mathcal{B}}$ are uniformly continuous, we may fix $\delta > 0$ so that $|f(\zeta(x)) - f(\zeta(y))| < \varepsilon/5$ for all $x, y \in \mathbb{B}^2$ satisfying $|x - y| < \delta$.

We fix $K > k' + 2$ so that the diameters of components of $\zeta^{-1}(\pi_G(X_K))$ are at most $\delta/2$. Let Ω_K be the component of $\zeta^{-1}(\mathbb{R}^3/G \setminus \pi_G(X_K))$ that contains $\partial\mathbb{B}^2$. Then $\Omega \subset \Omega_K$.

Since π_G is a homeomorphism near the boundary of X_K , we may use the transversality and the PL-structure in \mathbb{R}^3 to modify ζ in a neighborhood of $\pi_G(\partial X_K)$ in $\pi_G(X_{k+2})$ in such a way that the components of $\zeta^{-1}(\pi_G(\partial X_K))$ are topological circles, that $|f(\zeta(x)) - f(\zeta(y))| < \varepsilon/4$ for all $|x - y| < \delta$, and that $f(\zeta(\mathbb{B}^2)) \subset B^{3+m}(R + \varepsilon)$.

For each component C of $\partial\Omega_K$, except for the outermost boundary $\partial\mathbb{B}^2$, we denote by ω the 2-cell in \mathbb{B}^2 enclosed by C , thus $C = \partial\omega$, and define a map $\tilde{\phi}_\omega: \omega \rightarrow \mathbb{R}^{3+m}$ extending $f \circ \zeta|_{\partial\omega}$ as follows.

Let $\tau: \omega \rightarrow \mathbb{B}^2$ be a homeomorphism and fix a point $y_0 \in f(\zeta(\partial\omega))$. Define $\tilde{\phi}_\omega: \omega \rightarrow \mathbb{R}^{3+m}$ so that $\tilde{\phi}_\omega(\tau^{-1}(0)) = y_0$ and

$$\tilde{\phi}_\omega(x) = (1 - |\tau(x)|)y_0 + |\tau(x)|f \circ \zeta \circ \tau^{-1} \left(\frac{\tau(x)}{|\tau(x)|} \right), \quad x \neq \tau^{-1}(0).$$

Then $\tilde{\phi}_\omega|_{\partial\omega} = f \circ \zeta|_{\partial\omega}$. Since $\text{diam } \partial\omega < \delta$ we have $\text{diam } f(\zeta(\partial\omega)) \leq \varepsilon/4$, $\text{diam}(\tilde{\phi}_\omega(\omega)) < \varepsilon$, and $\tilde{\phi}_\omega(\omega) \subset \mathcal{B}'$; since $\zeta(\partial\omega) \subset \pi_G(\partial X_K)$, we have $\tilde{\phi}_\omega(\partial\omega) \subset f(\pi_G(X_{k'+2}) \times \mathbb{R}^m) \cap \mathcal{B}'$. Therefore

$$\begin{aligned} & \text{dist}(\tilde{\phi}_\omega(\omega), f(\pi_G(\partial X_{k'+1}) \times \mathbb{R}^m)) \\ & \geq \text{dist}(\tilde{\phi}_\omega(\partial\omega), f(\pi_G(\partial X_{k'+1}) \times \mathbb{R}^m)) - \text{diam}(\tilde{\phi}_\omega(\omega)) \\ & \geq \min\{1, \text{dist}(\tilde{\phi}_\omega(\partial\omega), f(\pi_G(\partial X_{k'+1}) \times \mathbb{R}^m) \cap \mathcal{B})\} \\ & \quad - \text{diam}(\tilde{\phi}_\omega(\omega)) > 3\varepsilon. \end{aligned}$$

Thus $\tilde{\phi}_\omega(\omega) \subset f(\pi_G(X_{k'+1}) \times \mathbb{R}^m)$.

We define map $\phi: \mathbb{B}^2 \rightarrow \mathbb{R}^3$ by $\phi|_{\Omega_K} = \pi_G^{-1} \circ \zeta|_{\Omega_K}$, and $\phi|_{\omega} = \Pi \circ f^{-1} \circ \tilde{\phi}_\omega$ on every 2-cell ω bounded by a component of $\partial\Omega_K \setminus \partial\mathbb{B}^2$. Since $\pi_G|_{\mathbb{R}^3 \setminus X_K}$ is a homeomorphism and $\tilde{\phi}_\omega|_{\partial\omega} = f \circ \zeta|_{\partial\omega}$, the map is well-defined and continuous.

Since $\phi(\mathbb{B}^2)$ is connected, $\phi(\mathbb{B}^2) \subset H$, and since $\Omega \subset \Omega_K$, $\phi|_{\Omega} = \pi_G^{-1} \circ \zeta|_{\Omega}$. The claim follows. \square

Proof of Proposition 8.5. The map $\phi: \mathbb{B}^2 \rightarrow H$ constructed in Lemma 8.6 belongs to $\mathcal{E}(H; \alpha)$ and satisfies $\pi_G \circ \phi|_{\Omega} = \zeta|_{\Omega}$. Then, by Lemma 8.4,

$$\#(\zeta(\mathbb{B}^2) \cap \pi_G(|\sigma|)) \geq \#(\phi(\mathbb{B}^2) \cap |\sigma|) \geq \text{circ}(X_{k'} \cap H, \alpha, H)$$

for every $\sigma \in \Sigma(X_{k'} \cap H, \mathcal{X})$. The claim follows. \square

8.4. Virtually interior essential components. Let Ω be a 2-manifold with boundary, M an n -manifold with boundary, and $\phi: (\Omega, \partial\Omega) \rightarrow (M, \partial M)$ a map. We say that ϕ is *interior inessential* if there exists a map $\phi': \Omega \rightarrow \partial M$ so that $\phi'|_{\partial\Omega} = \phi|_{\partial\Omega}$; if no such map exists, we say that ϕ is *interior essential*.

If $\phi: (\Omega, \partial\Omega) \rightarrow (M, \partial M)$ is interior essential and Ω is a submanifold of a 2-cell D so that $\partial D \subset \partial\Omega$, we say that ϕ is *virtually interior essential* if there exists a map $\Phi: D \rightarrow M$ so that $\Phi|_{\Omega} = \phi$ and $\Phi(D \setminus \Omega) \subset \partial M$.

Let \mathcal{X} be a defining sequence for a decomposition space. Given $H \in \mathcal{C}(X_k)$, $k' > k$, and a meridian α of H , denote by $\mathcal{E}(H, \alpha; X_{k'})$ the collection of maps $\phi: (\mathbb{B}^2, \partial\mathbb{B}^2) \rightarrow (H, \alpha)$ such that $\phi(\mathbb{B}^2)$ is transverse to $\partial X_{k'}$. Given $\phi \in \mathcal{E}(H, \alpha; X_{k'})$, we say that a component ω of $\phi^{-1}X_{k'}$ is *virtually interior essential with respect to $X_{k'}$* if $\phi|_{\omega}: (\omega, \partial\omega) \rightarrow (X_{k'}, \partial X_{k'})$ is virtually interior essential. We denote by $\Gamma(\phi, X_{k'})$ the set of *virtually interior essential components of $\phi^{-1}X_{k'}$* .

Remark 8.7. *The circulation $\text{circ}(X_{k'} \cap H, \alpha, H)$ is closely related to the minimal number of essential components among all maps in $\mathcal{E}(H, \alpha; X_{k'})$. In fact,*

$$\text{circ}(X_{k'} \cap H, \alpha, H) \geq \min_{\phi \in \mathcal{E}(H, \alpha; X_{k'})} \#\Gamma(\phi, X_{k'}).$$

Indeed, given $\phi \in \mathcal{E}(H, \alpha; X_{k'})$ and $\sigma \in \Sigma(X_{k'} \cap H)$, it follows from the definition of longitudes that $|\sigma| \cap \phi(\omega) \neq \emptyset$ for every $\omega \in \Gamma(\phi, X_{k'})$. Thus $\#(|\sigma| \cap \phi(\mathbb{B}^2)) \geq \#\Gamma(\phi, X_{k'})$.

9. CIRCULATION AND A MODULUS ESTIMATE FOR WALLS

Suppose that $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\lambda)$ is a Semmes space. Let Y be the union of a nonempty subcollection of handlebodies in $\mathcal{C}(X_k)$ and $a > 0$. We call the $(1 + m)$ -chains in

$$\Sigma^m(Y, \mathcal{X}, a) = \{|\sigma| \times [-a, a]^m : \sigma \in \Sigma(Y, \mathcal{X})\}.$$

m-walls over Y of height a relative to \mathcal{X} . Note that these walls do not meet $X_\infty \times \mathbb{R}^m$ and that $\pi_G|_{\mathbb{R}^3 \setminus X_\infty}$ is a homeomorphism. We denote by

$$\hat{\Sigma}^m(Y, \mathcal{X}, a) = (\pi_G \times \text{id})(\Sigma^m(Y, \mathcal{X}, a))$$

the corresponding collection of m -walls in the decomposition space $\mathbb{R}^3/G \times \mathbb{R}^m$.

9.1. An upper estimate of the conformal modulus. The main result of this section is an upper estimate for the conformal modulus of an m -wall family in terms of circulation. This, together with a lower estimate in terms of growth, yields a necessary condition for the existence of quasymmetric parametrization. Our result extends the second part of [12, Proposition 4.5].

Theorem 9.1. *Let $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\lambda)$ be a Semmes space, $m \geq 0$, and let $\alpha: \mathbb{S}^1 \rightarrow \partial H$ be an (λ^k, L) -quasisimilar meridian of $H \in \mathcal{C}(X_k)$. Suppose that $f: (\mathbb{R}^3/G \times \mathbb{R}^m, d_{\lambda, m}) \rightarrow \mathbb{R}^{3+m}$ is η -quasisymmetric. Then there exist $A = A(\eta, d_\lambda, m, L) > 0$ and $C = C(\eta, d_\lambda, m, L)$ so that*

$$\text{Mod}_{\frac{3+m}{1+m}} \left(f(\hat{\Sigma}^m(X_{k'} \cap H, \mathcal{X}, A\lambda^k)) \right) \leq C \left(\frac{1}{\text{circ}(X_{k'} \cap H, \alpha, H)} \right)^{\frac{3+m}{1+m}}$$

for all $k' > k + 1$.

We begin with an intersection lemma which contains the gist of the proof; the constant ϵ_λ in the statement is the fixed constant depending only on d_λ defined in Remark 6.7.

Lemma 9.2. *Suppose $g: (\mathbb{R}^3/G \times \mathbb{R}^m, d_{\lambda, m}) \rightarrow \mathbb{R}^{3+m}$ is η -quasisymmetric, and let $\alpha: \mathbb{S}^1 \rightarrow \partial H$ be a meridian of $H \in \mathcal{C}(\mathcal{X})$. Suppose that $\beta: \mathbb{S}^1 \rightarrow \mathbb{R}^3/G \times \mathbb{R}^m$ is a map homotopic to $\pi_G \circ \alpha$ in $N_{d_{\lambda, m}}(\pi_G(\partial H), \epsilon_\lambda \lambda^{\text{level}(H)})$ and that $g \circ \beta(\mathbb{S}^1) = \partial \mathbb{B}^2 \times \{0\} \subset \mathbb{R}^2 \times \mathbb{R}^{1+m}$. Then there exists $\delta_1 = \delta_1(\eta, d_{\lambda, m}) > 0$ so that*

$$\text{dist}(\partial \mathbb{B}^2, g(\pi_G(X_{k'}) \times \mathbb{R}^m)) \geq \delta_1 > 0$$

for every $k' > \text{level}(H)$; and there exists $A = A(\eta_{g^{-1}}) \geq 1$ so that

$$(9.1) \quad \#(g(w) \cap (\mathbb{B}^2 + j)) \geq \text{circ}(X_{k'} \cap H, \alpha, H)$$

for every $k' > \text{level}(H)$, m -wall $w \in \hat{\Sigma}^m(X_{k'} \cap H, \mathcal{X}, A\lambda^{\text{level}(H)})$, and every $j \in \{0\} \times B^{1+m}(\delta) \subset \mathbb{R}^2 \times \mathbb{R}^{1+m}$, where $\delta = \text{dist}(\partial \mathbb{B}^2, g(\pi_G(X_{k'}) \times \mathbb{R}^m))$.

Proof. Let $k = \text{level}(H)$. We show first that $\delta = \text{dist}(\partial \mathbb{B}^2, g(\pi_G(X_{k'}) \times \mathbb{R}^m))$ is bounded from below by a positive constant depending only on η and λ . Since $\beta(\mathbb{S}^1) \subset N_{d_{\lambda, m}}(\pi_G(\partial H), \epsilon_\lambda \lambda^k) \subset \pi_G(\mathbb{R}^3 \setminus X_{k+1}) \times \mathbb{R}^m$ and $g(\beta(\mathbb{S}^1)) = \partial \mathbb{B}^2$, we may fix $x \in \beta(\mathbb{S}^1)$ and $z \in \pi_G(X_{k'}) \times \mathbb{R}^m$ so that

$$|g(x) - g(z)| = \text{dist}(\partial \mathbb{B}^2, g(\pi_G(X_{k'}) \times \mathbb{R}^m)).$$

Moreover, we may fix $y \in \beta(\mathbb{S}^1)$ so that $d_{\lambda, m}(y, x) = \max_{y' \in \beta(\mathbb{S}^1)} d_\lambda(y', x)$. Since $x, y \in N_{d_{\lambda, m}}(\pi_G(\partial H), \epsilon_\lambda \lambda^k)$ and $\Pi(z) \in \pi_G(X_{k'})$, we have, by quasisymmetry and Remark 6.5,

$$\begin{aligned} |g(x) - g(y)| &\leq \eta \left(\frac{d_{\lambda, m}(x, y)}{d_{\lambda, m}(x, z)} \right) |g(x) - g(z)| \\ &\leq \eta \left(\frac{\text{diam}_{d_{\lambda, m}}(N_{d_{\lambda, m}}(\pi_G(\partial H), \epsilon_\lambda \lambda^k))}{\text{dist}_{d_{\lambda, m}}(\pi_G(\partial X_{k+1}), \pi_G(X_{k'}))} \right) |g(x) - g(z)| \\ &\leq \eta(C(d_{\lambda, m}))\delta. \end{aligned}$$

We fix now $x' \in \beta(\mathbb{S}^1)$ so that $g(x')$ and $g(x)$ are antipodal on $\partial \mathbb{B}^2$. Then, by the choice of y ,

$$|g(x) - g(x')| \leq \eta \left(\frac{d_{\lambda, m}(x, x')}{d_{\lambda, m}(x, y)} \right) |g(x) - g(y)| \leq \eta(1)|g(x) - g(y)|.$$

Thus

$$\delta > \frac{2}{\eta(1)\eta(C(d_{\lambda, m}))}.$$

We prove now (9.1). Let $j \in \{0\} \times B^{1+m}(\delta)$ and define $\phi_j: (\mathbb{B}^2, \partial\mathbb{B}^2) \rightarrow \mathbb{R}^3/G \times \mathbb{R}^m$ to be the map $\phi_j(x) = g^{-1}(x + j)$.

Since $N_{d_\lambda}(\pi_G(\partial H), \epsilon_\lambda \lambda^k) \subset \mathbb{R}^3/G$ is contained in a regular neighborhood of $\pi_G(\partial H)$ by definition of ϵ_λ , $\Pi \circ \phi_j|_{\partial\mathbb{B}^2}$ is homotopic to $\pi_G \circ \alpha$ in $N_{d_\lambda}(\pi_G(\partial H), \epsilon_\lambda \lambda^k) \subset \mathbb{R}^3/G$ and there exists a map $\zeta: (\mathbb{B}^2, \partial\mathbb{B}^2) \rightarrow (\pi_G H, \pi_G \circ \alpha)$ so that $\zeta|_\Omega = \Pi \circ \phi_j|_\Omega$, where $\Omega = (\Pi \circ \phi_j)^{-1}(\pi_G(X_{k'}))$.

We have, by Proposition 8.5, that

$$\#(\pi_G(\sigma) \cap \zeta(\mathbb{B}^2)) \geq \text{circ}(X_{k'} \cap H, \alpha, H)$$

for every $\sigma \in \Sigma(X_{k'} \cap H, \mathcal{X})$. Thus

$$\begin{aligned} \#(g(\pi_G(\sigma) \times \mathbb{R}^m) \cap (\mathbb{B}^2 + j)) &= \#((\pi_G(\sigma) \times \mathbb{R}^m) \cap f^{-1}(\mathbb{B}^2 + j)) \\ &= \#((\pi_G(\sigma) \times \mathbb{R}^m) \cap \phi_j(\mathbb{B}^2)) \\ &\geq \#(\pi_G(\sigma) \cap \Pi\phi_j(\mathbb{B}^2)) \\ &= \#(\pi_G(\sigma) \cap \zeta(\mathbb{B}^2)) \\ &\geq \text{circ}(X_{k'} \cap H, \alpha, H) \end{aligned}$$

for all $\sigma \in \Sigma(X_{k'} \cap H, \mathcal{X})$. This concludes the proof in the case $m = 0$.

Suppose now $m \geq 1$. It suffices to find $A = A(\eta) \geq 1$ so that

$$(9.2) \quad \phi_j(\mathbb{B}^2) \subset \mathbb{R}^3/G \times [-A\lambda^k, A\lambda^k]^m.$$

Let $x \in \partial\mathbb{B}^2$ and $y \in \mathbb{B}^2$. By quasisymmetry of g^{-1} , we have

$$\begin{aligned} |\phi_j(y) - \phi_j(x)| &= |g^{-1}(y + j) - g^{-1}(x + j)| \\ &\leq \eta_{g^{-1}} \left(\frac{|y - x|}{|(-x) - (x + j)|} \right) |g^{-1}(-x) - f^{-1}(x + j)| \\ &\leq \eta_{g^{-1}} \left(\frac{|y - x|}{2 - |j|} \right) \eta_{g^{-1}} \left(\frac{2 + |j|}{2} \right) |g^{-1}(-x) - g^{-1}(x)| \\ &\leq \eta_{g^{-1}}(1) \eta_{g^{-1}}(2) \text{diam}_{d_{\lambda,m}} f^{-1}(\partial\mathbb{B}^2). \end{aligned}$$

Since

$$g^{-1}(\partial\mathbb{B}^2) = |\beta| \subset N_{d_{\lambda,m}}(\pi_G(\partial H), \epsilon_\lambda \lambda^k),$$

(9.2) holds with $A = C(\lambda)\eta_{g^{-1}}(1)\eta_{g^{-1}}(2)$. The claim now follows. \square

The proof of Theorem 9.1 is based on Lemma 9.2 and unknotting properties of quasisymmetric tubes; see Propositions 10.1 and 10.3 in Section 10.

Proof of Theorem 9.1. We first consider the case $m \geq 1$. Let $\alpha: \mathbb{S}^1 \rightarrow \partial H$ be the meridian in the statement and $k' > k + 1$. We assume, as we may, that $\text{circ}(X_{k'} \cap H, \alpha, H) > 0$. So

$$\#(|\sigma| \cap \phi(\mathbb{B}^2)) \geq \text{circ}(X_{k'} \cap H, \alpha, H)$$

for all $\sigma \in \Sigma(X_{k'} \cap H, \mathcal{X})$ and all maps $\phi: (\mathbb{B}^2, \partial\mathbb{B}^2) \rightarrow (H, \alpha)$.

By Lemma 8.3, there exists an η' -quasisymmetric embedding $\varkappa: \mathbb{B}^{2+m} \times \mathbb{S}^1 \rightarrow N_{d_{\lambda,m}}(\pi_G(\partial H), \epsilon_\lambda \lambda^k)$ so that $\varkappa(0, x) = (\pi_G \circ \alpha(x), 0)$ for $x \in \mathbb{S}^1$, where the homeomorphism $\eta': [0, \infty) \rightarrow [0, \infty)$ depends only on the welding structure $(\mathcal{C}, \mathcal{A}, \mathcal{W})$, λ and m . Note that $N_{d_{\lambda,m}}(\pi_G(\partial H), \epsilon_\lambda \lambda^k) \subset \pi_G(\mathbb{R}^3 \setminus X_{k+1}) \times \mathbb{R}^m$.

Set $T = f \circ \varkappa(\mathbb{B}^{2+m} \times \mathbb{S}^1) \subset \mathbb{R}^{3+m}$ and $h = f \circ \varkappa$. By Proposition 10.1, there exist an η'' -quasisymmetric, $\eta'' = \eta''(m, \eta, \eta')$, map $\chi: \mathbb{R}^{3+m} \rightarrow \mathbb{R}^{3+m}$ and a constant $\delta_0 = \delta_0(m, \eta, \eta') > 0$ so that $\chi(T)$ contains the tubular neighborhood $N^{3+m}(\partial\mathbb{B}^2, \delta_0)$ of $\partial\mathbb{B}^2$ in \mathbb{R}^{3+m} and that $\chi \circ f \circ \pi_G \circ \alpha: \mathbb{S}^1 \rightarrow \mathbb{R}^{3+m}$ is homotopic in $\chi(T)$ to the identity map $\text{id}: \partial\mathbb{B}^2 \rightarrow \mathbb{R}^2 \times \mathbb{R}^{1+m}$. Set $\beta = f^{-1} \circ \chi^{-1} \circ \text{id}_{\mathbb{R}^{3+m}}|_{\mathbb{S}^1}: \mathbb{S}^1 \rightarrow \mathbb{R}^3/G \times \mathbb{R}^m$. Thus we have

$$\begin{array}{ccc}
 & \mathbb{B}^{2+m} \times \mathbb{S}^1 & \\
 & \nearrow & \searrow \varkappa \\
 \mathbb{S}^1 & \xrightarrow{\pi_G \circ \alpha} & \mathbb{R}^3/G \times \mathbb{R}^m \\
 & \searrow \beta & \\
 & \mathbb{R}^{3+m} & \longleftarrow \chi \\
 & & \downarrow f \\
 & & \mathbb{R}^{3+m}
 \end{array}$$

where both diagrams commute and maps $\pi_G \circ \alpha$ and β are homotopic in $\varkappa(\mathbb{B}^{2+m} \times \mathbb{S}^1)$.

Note that $\chi \circ f: \mathbb{R}^3/G \times \mathbb{R}^m \rightarrow \mathbb{R}^{3+m}$ is η''' -quasisymmetric for some $\eta''' = \eta'''(\eta, \eta'')$. We set $J = B^{1+m}(\delta_0)$.

Since β is homotopic to $\pi_G \circ \alpha$ in $\varkappa(\mathbb{B}^{2+m} \times \mathbb{S}^1) \subset N_{d_{\lambda,m}}(\pi_G(\partial H), \epsilon_{\lambda}\lambda^k)$ and $\delta_0 < \text{dist}(\partial\mathbb{B}^2, \chi \circ f(\pi_G(X_{k'}) \times \mathbb{R}^m))$, we can find, by applying Lemma 9.2 to $g = \chi \circ f$, a constant $A = A(\eta''', m)$ so that

$$(9.3) \quad \#(\chi \circ f(w) \cap (\mathbb{B}^2 + j)) \geq \text{circ}(X_{k'} \cap H, \alpha, H),$$

where $w \in \hat{\Sigma}^m(X_{k'}, \mathcal{X}, A\lambda^k)$.

Using (9.3) we estimate the conformal modulus of the m -wall family $\chi \circ f(\hat{\Sigma}^m(X_{k'}, \mathcal{X}, A\lambda^k))$ in \mathbb{R}^{3+m} . By the co-area formula, we have, for $w \in \hat{\Sigma}^m(X_{k'}, \mathcal{X}, A\lambda^k)$, that

$$\begin{aligned}
 \mathcal{H}^{1+m}(\chi(f(w))) &\geq \mathcal{H}^{1+m}(\chi(f(w)) \cap (\mathbb{B}^2 \times B^{1+m}(\delta_0))) \\
 &\geq \int_J \#(\chi(f(w)) \cap (\mathbb{B}^2 + j)) \, d\mathcal{H}^{1+m}(j) \\
 &\geq \text{circ}(X_{k'} \cap H, \alpha, H) \mathcal{H}^{1+m}(J).
 \end{aligned}$$

Thus

$$\rho = \frac{1}{\text{circ}(X_{k'} \cap H, \alpha, H)} \frac{1}{\mathcal{H}^{1+m}(J)} \chi_{\mathbb{B}^2 \times J},$$

is an admissible function for the family $\chi(f(\hat{\Sigma}^m(X_{k'}, \mathcal{X}, A\lambda^k)))$, and

$$\begin{aligned}
 \text{Mod}_{\frac{3+m}{1+m}}(\chi(f(\hat{\Sigma}^m(X_{k'}, \mathcal{X}, A\lambda^k)))) &\leq \int_{\mathbb{R}^n} \rho^{\frac{3+m}{1+m}} \, d\mathcal{H}^{3+m} \\
 &= \frac{\mathcal{H}^2(\mathbb{B}^2) \mathcal{H}^{m+1}(J)}{\mathcal{H}^{m+1}(J)^{\frac{3+m}{1+m}}} \left(\frac{1}{\text{circ}(X_{k'} \cap H, \alpha, H)} \right)^{\frac{3+m}{1+m}} \\
 &\leq C(\delta_0, m) \left(\frac{1}{\text{circ}(X_{k'} \cap H, \alpha, H)} \right)^{\frac{3+m}{1+m}}.
 \end{aligned}$$

This concludes the proof for $m \geq 1$.

In the case $m = 0$, we apply Proposition 10.3 to the mapping $h = f \circ \varkappa$ and the 3-manifold $M = f(\pi_G(H))$. Otherwise the proof is the same. \square

10. QUASISYMMETRIC TUBES

In Proposition 10.1 we quantify Zeeman's unknotting theorem to provide a quasisymmetrical unknotting of quasisymmetric tubes in \mathbb{R}^n , $n \geq 4$. In Proposition 10.3 we treat the unknotting in \mathbb{R}^3 with an additional topological criterion.

Proposition 10.1. *Let $m \geq 1$, $h: \mathbb{B}^{2+m} \times \mathbb{S}^1 \rightarrow \mathbb{R}^{3+m}$ be an η -quasisymmetric embedding and $T = h(\mathbb{B}^{2+m} \times \mathbb{S}^1)$. Then there exist an η' -quasisymmetric homeomorphism $\chi: \mathbb{R}^{3+m} \rightarrow \mathbb{R}^{3+m}$, $\eta' = \eta'(m, \eta)$, and a constant $\delta_0 = \delta_0(m, \eta) > 0$ so that*

- (1) $\chi(T)$ contains the tube $N^{3+m}(\partial\mathbb{B}^2, \delta_0)$ in \mathbb{R}^{3+m} , in particular
- (1)' $\partial\mathbb{B}^2 + j \subset \chi(T)$ for $j \in \{0\} \times B^{1+m}(\delta_0) \subset \mathbb{R}^2 \times \mathbb{R}^{1+m}$, and
- (2) $\chi \circ h|(\{0\} \times \mathbb{S}^1)$ is homotopic to the identity map $\text{id}|_{\partial\mathbb{B}^2}$ in $\chi(T)$.

Here $\mathbb{B}^{2+m} \times \mathbb{S}^1$ has the natural Euclidean metric inherited from $\mathbb{R}^{2+m} \times \mathbb{R}^2 = \mathbb{R}^{3+m}$.

We first state a bilipschitz version of Zeeman's theorem on unknotting PL 1-sphere in \mathbb{S}^q for $q \geq 4$. Since the claim follows from [16, Theorem 5.6, Corollary 5.9] almost directly, we omit the details.

For the statement, let $\ell \in \mathbb{Z}_+$, $m \geq 1$. Given $w_1, \dots, w_n \in (1/\ell)\mathbb{Z}^{3+m}$, we set $w = (w_1, \dots, w_n)$, and set γ_w to be the piecewise linear curve $[w_0, w_1] \cup [w_1, w_2] \cup \dots \cup [w_{n-1}, w_0]$ in \mathbb{R}^{3+m} . Given $R > 0$, we also denote by $\mathcal{J}(R, \ell, m; n)$ the collection of Jordan curves in $\{\gamma_w \subset B^{3+m}(R) : w \in ((1/\ell)\mathbb{Z}^{3+m})^n\}$.

Lemma 10.2. *Let $R \geq 1$, $\ell \in \mathbb{Z}_+$, $m \geq 1$, and $n \geq 3$. Then there exists $L_0 = L_0(R, \ell, m, n)$ so that given $\gamma \in \mathcal{J}(R, \ell, m; n)$ there exists an L_0 -bilipschitz map $\chi: \mathbb{R}^{3+m} \rightarrow \mathbb{R}^{3+m}$ satisfying $\chi(\gamma) = \partial B^2(\text{diam } \gamma) \times \{0\} \subset \mathbb{R}^2 \times \mathbb{R}^{1+m}$.*

Proof of Proposition 10.1. Set $\mathbb{S}^1 = \{0\} \times \mathbb{S}^1$. Then, by quasisymmetry,

$$\text{diam } h(\mathbb{S}^1) \leq \eta(5) \text{dist}(h(\mathbb{S}^1), \partial T).$$

Indeed, set $\kappa = \text{dist}(h(\mathbb{S}^1), \partial T)$ and choose $x \in \mathbb{S}^1$ and $y \in \partial(\mathbb{B}^{2+m} \times \mathbb{S}^1)$ be so that $|h(x) - h(y)| = \kappa$. Then

$$|h(x') - h(x)| \leq \eta \left(\frac{|x' - x|}{|y - x|} \right) |h(y) - h(x)| \leq \eta(5)\kappa$$

for all $x' \in \mathbb{S}^1$.

We fix an orientation of $h(\mathbb{S}^1)$ and choose points $z_0, z_1, \dots, z_n = z_0$ on $h(\mathbb{S}^1)$ as follows. Let z_0 be any point on $h(\mathbb{S}^1)$. After z_i has been chosen, let z_{i+1} be the last point z on the subarc of $h(\mathbb{S}^1)$ starting at z_i and ending at z_0 according to the orientation, so that $|z - z_i| = \kappa/100$ if such a point exists; otherwise, we have $|z_0 - z_i| < \kappa/100$ and in this case we remove the already defined value of z_i and set $n = i$ and $z_n = z_0$. We show next that $n \leq n_0$ for some $n_0 = n_0(\eta) > 0$.

Let $s_i = h^{-1}(z_i) \in \mathbb{S}^1$ for $0 \leq i \leq n - 1$. Then there exists an i so that $|s_i - s_{i+1}| \leq 2\pi/n$; for this particular i ,

$$\kappa/100 \leq |z_i - z_{i+1}| \leq \eta \left(\frac{|s_i - s_{i+1}|}{|(-s_i) - s_i|} \right) |h(-s_i) - h(s_i)| \leq \eta(1/n)\eta(5)\kappa.$$

Hence $|s_i - s_{i+1}| \geq C_0$, where C_0 depends on η , and $n \leq 2\pi/C_0$.

We next fix points $w_i \in (\kappa/(1000\sqrt{mn_0}))\mathbb{Z}^{3+m}$ so that $|w_i - z_i| < \kappa/500$ and let γ be the polygonal path $[w_0, w_1] \cup [w_1, w_2] \cup \cdots \cup [w_{n-1}, w_0]$.

By replacing the points w_i with points in $(\kappa/(1000\sqrt{3+mn_0}))\mathbb{Z}^{3+m} \cap B^{3+m}(z_i, \kappa/500)$, we may assume that γ is a Jordan curve. Indeed, if γ is not a Jordan curve, then there exist indices i and j , $i > j$, so that $(w_i, w_{i+1}) \cap (w_j, w_{j+1}) \neq \emptyset$. Since $B^{3+m}(z_i, \kappa/500)$ contains more than n_0^{3+m} points in $(\kappa/(1000\sqrt{3+mn_0}))\mathbb{Z}^{3+m}$ and there are at most $n_0(n_0 - 1)/2$ directions between the points w_1, \dots, w_n , there exists $w' \in B^{3+m}(z_i, \kappa/500)$ so that $(w_i, w') \cap (w_k, w_{k+1}) = \emptyset$ for all $k < i$. We remove all the intersections inductively on i .

Since $\max_{w \in [w_i, w_{i+1}]} \text{dist}(w, z_i) \leq \kappa/40$, we have $\max_{w \in \gamma} \text{dist}(w, h(\mathbb{S}^1)) \leq \kappa/40$. Thus $\text{dist}(\gamma, \partial h(T)) \geq 39\kappa/40$.

Let $\iota: \mathbb{R}^{3+m} \rightarrow \mathbb{R}^{3+m}$ be a linear transformation $\iota(x) = -w_0 + x/\kappa$ that maps γ into $B^{3+m}(\eta(5))$. Then

$$\iota(\gamma) \in \mathcal{J}(2\eta(5), 1/(1000\sqrt{3+mn_0}), m; n).$$

By Lemma 10.2, there exists a L_0 -bilipschitz, therefore η' -quasisymmetric homeomorphism χ' of \mathbb{R}^{3+m} so that $\chi'(\iota(\gamma)) = \partial B^2(\text{diam } \iota(\gamma)) \subset \mathbb{R}^2 \times \mathbb{R}^{1+m}$, where η' depends only on $\eta(5)$, m , and n_0 . Then $\chi = (\text{diam } \iota(\gamma))^{-1} \chi' \circ \iota$ is also η' -quasisymmetric. Since $n \leq n_0$ and n_0 depends only on η , we have that $\eta' = \eta'(m, \eta)$. The existence of the constant δ_0 follows from quasisymmetry of χ and geometry of $\mathbb{B}^{2+m} \times \mathbb{S}^1$. \square

Proposition 10.3. *Let M be a PL 3-manifold with boundary in \mathbb{R}^3 . Suppose $h: \mathbb{B}^2 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ is an η -quasisymmetric embedding with properties that h embeds $\{0\} \times \mathbb{S}^1$ into ∂M and $h|(\{0\} \times \mathbb{S}^1)$ is null-homotopic in M . Let $T = h(\mathbb{B}^2 \times \mathbb{S}^1)$. Then there exist an η' -quasisymmetric homeomorphism $\chi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\eta' = \eta'(\eta)$, and a constant $\delta_0 = \delta_0(\eta) > 0$ so that*

- (1) $\chi(T)$ contains the tube $N^3(\partial \mathbb{B}^2, \delta_0)$ in \mathbb{R}^3 , in particular
- (1)' $\partial \mathbb{B}^2 + j \subset \chi(T)$ for $j \in \{0\} \times [-\delta_0, \delta_0] \subset \mathbb{R}^2 \times \mathbb{R}$, and
- (2) $\chi \circ h|(\{0\} \times \mathbb{S}^1)$ is homotopic to the identity map $\text{id}|_{\partial \mathbb{B}^2}$ in $\chi(T)$.

Proof. Since h embeds $\{0\} \times \mathbb{S}^1$ into ∂M , $\alpha: \partial \mathbb{B}^2 \rightarrow \partial M$, $\alpha(x) = h(0, x)$, is a simple null-homotopic loop in M . Thus there exists an extension $\hat{\alpha}: \mathbb{B}^2 \rightarrow M$ of α . Since M is a PL manifold with boundary, ∂M has a collar in M , we may assume that $\partial \mathbb{B}^2$ has a neighborhood A in \mathbb{B}^2 so that $\hat{\alpha}|_A$ is an embedding and $\hat{\alpha}^{-1}(\hat{\alpha}(A)) = A$. Thus, by Dehn's Lemma (see e.g. [13, Chapter 4]), there exists an embedding $\tau: \mathbb{B}^2 \rightarrow M$ so that $\tau|_{\partial \mathbb{B}^2} = \alpha$.

To unknot quantitatively, we follow the proof of Proposition 10.1 almost verbatim. Let $\kappa = \text{dist}(h(\mathbb{S}^1), \partial T)$ and $n_0(\eta)$ be as in the proof of Proposition 10.1. Then there exists a polygonal Jordan path $\gamma = [w_0, w_1] \cup [w_1, w_2] \cup \cdots \cup [w_{n-1}, w_0]$ with vertices $w_i \in (\kappa/(1000\sqrt{3+mn_0}))\mathbb{Z}^{3+m}$ so that $\max_{w \in \gamma} \text{dist}(w, h(\mathbb{S}^1)) \leq \kappa/20$ and $\text{dist}(\gamma, \partial h(T)) \geq 19\kappa/20$. Therefore γ is PL-isotopic to $h(\{0\} \times \mathbb{S}^1)$ in $h(T)$. We may now fix a scaled $L_0 = L_0(\eta, m, n_0)$ -bilipschitz, therefore η' -quasisymmetric homeomorphism $\chi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ so that $\chi(h(\mathbb{S}^1)) = \partial \mathbb{B}^2$ as in the proof of Proposition 10.1. Conditions (1) and (2) in the statement now follow by quasisymmetry. \square

11. GROWTH AND A MODULUS ESTIMATE FOR WALLS

The main result in this section is a lower estimation of the conformal modulus of a m -wall family, which corresponds partly to the first claim of [12, Proposition 4.5].

Proposition 11.1. *Suppose $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\lambda)$ is a Semmes space. Let $k \geq 0$, and \mathcal{Y} be a collection of handlebodies in $\mathcal{C}(X_k)$ that contains at least one positive genus cube-with-handles, and let Y be their union. Let $m \geq 0$, then the conformal modulus of m -walls*

$$(11.1) \quad \text{Mod}_{\frac{3+m}{1+m}} \left(\hat{\Sigma}^m(Y, \mathcal{X}, a) \right) \geq C \left((\#\mathcal{Y}) \left(\frac{a}{\lambda^k} \right)^m \right)^{1 - \frac{3+m}{1+m}}$$

for every $a > 0$ and a constant $C = C(\mathcal{C}, \mathcal{W}, \mathcal{A}, m) > 0$.

To obtain the estimate, we first fix a collection of cubes-with-handles $\mathcal{H} = \{H_0, H_1, \dots\}$, one for each genus, and a special family of longitudes in each H_g as follows.

Let $Q(x, r) = [x_1 - r, x_1 + r] \times [x_2 - r, x_2 + r] \subset \mathbb{R}^2$ for $x = (x_1, x_2) \in \mathbb{R}^2$ and $r > 0$ and denote the origin of \mathbb{R}^2 by 0 . Set $H_0 = Q(0, 1) \times [0, 1]$.

For each $g > 0$, fix points $\{p_1, \dots, p_g\}$ in $Q(0, 1 - 1/(10g))$ having pairwise distance at least $1/(20g)$. Let $\Omega_g = Q(0, 1) \setminus \cup_i (\text{int}Q(p_i, 1/(100g)))$ and $H_g = \Omega_g \times [0, 1]$. Then H_g is a genus g handlebody. For every $0 \leq t \leq 1/(100g)$ and every $0 < s < 1$, fix a 1-cycle

$$\sigma_{t,s}^g = \left(\partial Q(0, 1-t) \cup \partial Q(p_1, t + \frac{1}{100g}) \cup \dots \cup \partial Q(p_g, t + \frac{1}{100g}) \right) \times \{s\}$$

in H_g .

Lemma 11.2. *Given $g > 0$, the 1-cycles $\sigma_{t,s}^g$ defined above are longitudes of H_g for all $0 \leq t \leq 1/(100g)$ and $0 < s < 1$. Moreover, if ω is a 2-manifold in \mathbb{B}^2 and $\zeta: (\omega, \partial\omega) \rightarrow (H_g, \partial H_g)$ is virtually interior essential, then $\zeta(\omega) \cap \sigma_{t,s}^g \neq \emptyset$.*

Proof. We denote $\Omega = \Omega_g$, $H = H_g = \Omega_g \times [0, 1]$, and $\sigma_{t,s} = \sigma_{t,s}^g$.

To show that $\sigma_{t,s}$ is a longitude, let $\alpha: \mathbb{S}^1 \rightarrow \partial H$ be a meridian of H and $\phi: (\mathbb{B}^2, \partial\mathbb{B}^2) \rightarrow (H, \alpha)$ a map. We claim that $\phi(\mathbb{B}^2) \cap \sigma_{t,s} \neq \emptyset$.

Consider first the case $t = 0$. Suppose toward contradiction that there is an $s \in (0, 1)$ so that $\phi(\mathbb{B}^2) \cap \sigma_{0,s} = \emptyset$. After postcomposing ϕ with a homeomorphism from $H \setminus \sigma_{0,s}$ onto $H \setminus (\partial\Omega \times [0, 1])$, we may assume that $\phi: (\mathbb{B}^2, \partial\mathbb{B}^2) \rightarrow (H, \Omega \times \{0, 1\})$. Suppose that $\phi(\partial\mathbb{B}^2) \subset \Omega \times \{1\}$. Since ϕ is interior essential, $\phi(\partial\mathbb{B}^2)$ is not trivial in $\pi_1(\Omega \times \{1\})$. Hence $\phi(\partial\mathbb{B}^2)$ is not trivial in $\pi_1(\Omega \times [0, 1]) = \pi_1(H)$. Since $\phi(\mathbb{B}^2) \subset H$, this is a contradiction.

To prove the second statement in the lemma for $t = 0$, let $\zeta: (\omega, \partial\omega) \rightarrow (H, \partial H)$ be the map given. Since ζ is virtually interior essential, it has an extension $\zeta': D_\omega \rightarrow H$ satisfying $\zeta'(D_\omega \setminus \omega) \subset \partial H$, where D_ω is the 2-cell in \mathbb{B}^2 with $\partial D_\omega \subset \partial\omega$; see Section 8.4. After applying a homotopy to ζ' which leaves $\zeta'|_{\partial D_\omega}$ fixed, we may assume that $\zeta'(D_\omega) \cap \partial H = \zeta'(\partial D_\omega) \cap \partial H$. Since ζ' is interior essential, $\zeta'(D_\omega) \cap \sigma_{0,s} \neq \emptyset$ for all $s \in (0, 1)$. Since $\zeta'(\partial D_\omega) \subset \zeta'(\partial\omega)$, $\zeta'(\partial\omega) \cap \sigma_{0,s} \neq \emptyset$. Since $\zeta|\partial\omega = \zeta'|\partial\omega$, the claim follows.

To verify the claim in the case $0 < t \leq 1/(100g)$ for a given $s \in (0, 1)$, let $\Omega^{t,s}$ be the planar closed region with boundary $\sigma_{t,0}$, and $H^{t,s} = \Omega_t \times [s/2, 1 -$

$s/2]$ a cube-with- g -handles contained in H . Note that $\sigma_{t,s} \subset \partial H^{t,s}$. Since $H \setminus H^{t,s}$ is a regular neighborhood of ∂H in H , $\phi^{-1}H^{t,s}$ contains a component, say ω' , on which $\phi|_{\omega'}: (\omega', \partial\omega') \rightarrow (H^{t,s}, \partial H^{t,s})$ is virtually interior essential. Then, by the argument above, $\phi(\omega') \cap \sigma_{t,s} \neq \emptyset$ and hence $\phi(\mathbb{B}^2) \cap \sigma_{t,s} \neq \emptyset$. This proves the claim.

The second statement in the case $t > 0$ follows the same argument for $t = 0$. □

Proof of Proposition 11.1. By passing to a bilipschitz equivalent metric if necessary, we may assume that $d_\lambda = d_\theta$, where θ is a λ -modular embedding $\mathbb{R}^3/G \rightarrow \mathbb{R}^n$.

As a preliminary step, we fix for every $\mathbf{c} = (A_{\mathbf{c}}, B_{\mathbf{c}}) \in \mathcal{C}$, a PL-homeomorphism $\xi_{\mathbf{c}}: A_{\mathbf{c}} \rightarrow H_{g_{\mathbf{c}}}$, where $g_{\mathbf{c}}$ is the genus of $A_{\mathbf{c}}$. Since \mathcal{C} is finite, the mappings $\xi_{\mathbf{c}}$ are uniformly bilipschitz, and there exists $t_{\mathcal{C}} \in (0, 1/(100g))$ so that

$$\xi_{\mathbf{c}}(B_{\mathbf{c}}) \cap \sigma_{t,s}^{g_{\mathbf{c}}} = \emptyset$$

for every $0 \leq t \leq t_{\mathcal{C}}$, every $0 < s < 1$, and $\mathbf{c} \in \mathcal{C}$.

We fix a special family of longitudes for each H_g in \mathcal{H} and an induced family of longitudes on \mathcal{X} as follows. For each $g > 0$, let

$$\Sigma(H_g, \mathcal{H}) = \{\sigma_{t,s}^g : 0 \leq t \leq r_{\mathcal{C}}, 0 < s < 1\};$$

and for $g = 0$, define $\Sigma(H_0, \mathcal{H}) = \emptyset$.

By Lemma 11.2, these 1-cycles are longitudes of H_g . Define for every $H \in \mathcal{C}(\mathcal{X})$ an induced family of longitudes of H by

$$\Sigma(H, \mathcal{X}, \mathcal{H}) = \{\varphi_H^{-1} \circ \xi_{c_H}^{-1}(\sigma) : \sigma \in \Sigma(H_{g_H}, \mathcal{H})\},$$

where g_H is the genus of H and $\varphi_H: H^{\text{diff}} \rightarrow \mathbf{c}_H^{\text{diff}}$ is the chart map in \mathcal{A} .

By (8.2), every 1-cycle in Y of the form

$$\tau_{t,s} = \sum_{H \in \mathcal{Y}} \varphi_H^{-1} \circ \xi_{c_H}^{-1}(\sigma_{t,s}^{g_H}),$$

$0 \leq t \leq t_{\mathcal{C}}$ and $0 < s < 1$, is a longitude of Y . Set

$$\Sigma(Y, \mathcal{X}, \mathcal{H}) = \{\tau_{t,s} : 0 \leq t \leq t_{\mathcal{C}} \text{ and } 0 < s < 1\},$$

and

$$\Sigma^m(Y, \mathcal{X}, \mathcal{H}; a) = \{\tau \times [-a, a]^m : \tau \in \Sigma(Y, \mathcal{X}, \mathcal{H})\}$$

the collection of corresponding m -walls over Y of height a .

Since $\Sigma^m(Y, \mathcal{X}, \mathcal{H}; a) \subset \Sigma^m(Y, \mathcal{X}; a)$, it suffices to show that the estimate (11.1) holds for the surface family $\hat{\Sigma}^m(Y, \mathcal{X}, \mathcal{H}; a) = (\pi_G \times \text{id})(\Sigma^m(Y, \mathcal{X}, \mathcal{H}; a))$.

Before continuing, we recall that, since the embedding $\theta: \mathbb{R}^3/G \rightarrow \mathbb{R}^n$ is λ -modular, there exists $L = L(\mathcal{C}, \mathcal{A}, \mathcal{W}) \geq 1$ so that for every $k \geq 0$ and every $H \in \mathcal{C}(X_k)$, the map

$$\zeta_H = \pi_G \circ \varphi_H^{-1} \circ \xi_{c_H}^{-1}|_{\xi_{c_H}(\mathbf{c}_H^{\text{diff}})}: \xi_{c_H}(\mathbf{c}_H^{\text{diff}}) \rightarrow \pi_G(H^{\text{diff}})$$

is a (λ^k, L) -quasisimilarity, and extension $\xi_{c_H}(\mathbf{c}_H^{\text{diff}}) \times \mathbb{R}^m \rightarrow (\pi_G(H) \times \mathbb{R}^m, d_{\lambda,m})$ defined by

$$\zeta_H: (x, z) \mapsto (\pi_G \circ \varphi_H^{-1} \circ \xi_{c_H}^{-1}(x), \lambda^k z)$$

is also a (λ^k, L) -quasisimilarity.

In the following estimation of the modulus of surface families, we denote by \mathcal{H}_δ^β and by \mathcal{H}_e^β the β -dimensional Hausdorff measures with respect to $d_{\lambda,m}$ and the Euclidean metric, respectively.

Suppose that ρ is an admissible Borel function for $\hat{\Sigma}^m(Y, \mathcal{X}, \mathcal{H}; a)$ on $\mathbb{R}^3/G \times \mathbb{R}^m$, that is,

$$\int_{\pi_G(\tau_{t,s}) \times [-a,a]^m} \rho \, d\mathcal{H}_\delta^{1+m} \geq 1$$

for every $\tau_{t,s} \in \Sigma(Y, \mathcal{X}, \mathcal{H})$. We assume as we may that ρ is supported in $\pi_G(Y \setminus X_{k+1}) \times [-a,a]^m$.

We have, for every $0 \leq t \leq t_e$ and every $0 < s < 1$, that

$$\begin{aligned} & \sum_{H \in \mathcal{Y}} (L\lambda^k)^{1+m} \int_{\sigma_{t,s}^{gH} \times [-\lambda^{-k}a, \lambda^{-k}a]^m} \rho \circ \zeta_H \, d\mathcal{H}_e^{1+m} \\ & \geq \sum_{H \in \mathcal{Y}} \int_{\zeta_H(\sigma_{t,s}^{gH} \times [-\lambda^{-k}a, \lambda^{-k}a]^m)} \rho \, d\mathcal{H}_\delta^{1+m} \\ & = \int_{\pi_G(\tau_{t,s}) \times [-a,a]^m} \rho \, d\mathcal{H}_\delta^{1+m} \geq 1. \end{aligned}$$

Thus

$$\begin{aligned} & \sum_{H \in \mathcal{Y}} \int_{H_{gH} \times [-\lambda^{-k}a, \lambda^{-k}a]^m} \rho \circ \zeta_H \, d\mathcal{H}_e^{3+m} \\ (11.2) \quad & \geq C \int_{[0,t_e] \times [0,1]} \left(\sum_{H \in \mathcal{Y}} \int_{\sigma_{t,s}^{gH} \times [-\lambda^{-k}a, \lambda^{-k}a]^m} \rho \circ \zeta_H \, d\mathcal{H}_e^{1+m} \right) d\mathcal{H}_e^2 \\ & \geq Ct_e \lambda^{-k(1+m)}, \end{aligned}$$

where C depends only $(\mathcal{C}, \mathcal{A}, \mathcal{W})$.

Let $p = (3+m)/(1+m)$. Then, by (11.2),

$$\begin{aligned} & \sum_{H \in \mathcal{Y}} \int_{H_{gH} \times [-\lambda^{-k}a, \lambda^{-k}a]^m} (\rho \circ \zeta_H)^p \, d\mathcal{H}_e^{3+m} \\ (11.3) \quad & \geq \left(\mathcal{H}_e^{3+m} \left(\bigcup_{H \in \mathcal{Y}} H_{gH} \times [-\lambda^{-k}a, \lambda^{-k}a]^m \right) \right)^{1-p} \\ & \quad \times \left(\sum_{H \in \mathcal{Y}} \int_{H_{gH} \times [-\lambda^{-k}a, \lambda^{-k}a]^m} \rho \circ \zeta_H \, d\mathcal{H}_e^{3+m} \right)^p \\ & \geq C(\#\mathcal{Y})^{1-p} (\lambda^{-k}a)^{m(1-p)} \lambda^{-k(1+m)p} \\ & = C(\#\mathcal{Y})^{1-p} \lambda^{-k(m+p)} a^{m(1-p)}, \end{aligned}$$

where $C > 0$ depends only on m and $(\mathcal{C}, \mathcal{A}, \mathcal{W})$.

Since ζ_H is a (λ^k, L) -quasisimilarity, ζ_H^{-1} is $L\lambda^{-k}$ -Lipschitz. By the change of variables,

$$\begin{aligned} \int_{\pi_G(H) \times [-a, a]^m} \rho^p d\mathcal{H}_\delta^{3+m} &= \int_{\pi_G(H) \times [-a, a]^m} (\rho \circ \zeta_H)^p \circ \zeta_H^{-1} d\mathcal{H}_\delta^{3+m} \\ &\geq \left(\frac{\lambda^k}{L}\right)^{3+m} \int_{\mathbf{H}_{g_H} \times [-\lambda^{-k}a, \lambda^{-k}a]^m} (\rho \circ \zeta_H)^p d\mathcal{H}_e^{3+m}, \end{aligned}$$

for every $H \in \mathcal{Y}$. Since ρ is supported in $\pi_G(Y \setminus X_{k+1}) \times [-a, a]^m$, we have

$$\begin{aligned} \int_{\mathbb{R}^3/G \times \mathbb{R}^m} \rho^p d\mathcal{H}_\delta^{3+m} &= \int_{\pi_G(Y \setminus X_{k+1}) \times [-a, a]^m} \rho^p d\mathcal{H}_\delta^{3+m} \\ &\geq (\lambda^k/L)^{3+m} \sum_{H \in \mathcal{Y}} \int_{\mathbf{H}_{g_H} \times [-\lambda^{-k}a, \lambda^{-k}a]^m} (\rho \circ \zeta_H)^p d\mathcal{H}_e^{3+m} \\ &\geq C(\#\mathcal{Y})^{1-p} \lambda^{k(3+m)} a^{m(1-p)} \lambda^{-k(m+p)} \\ &= C \left((\#\mathcal{Y})(a/\lambda^k)^m \right)^{1-p}, \end{aligned}$$

where C depends only on m and $(\mathcal{C}, \mathcal{A}, \mathcal{W})$. The claim follows. \square

12. A NECESSARY CONDITION FOR QUASISYMMETRIC PARAMETRIZATION

The existence of quasisymmetric parametrization of $(\mathbb{R}^3/G, d_\lambda)$ by \mathbb{R}^{3+m} requires a balance among the growth, circulation and the scaling factor of the Semmes space. We prove this result in this section.

Theorem 12.1. *Let $(\mathbb{R}^3/G, \mathcal{X}, (\mathcal{C}, \mathcal{A}, \mathcal{W}), \theta, d_\lambda)$ be a Semmes space. Assume that \mathcal{X} has order of growth at most γ and order of circulation at least ω . Let $m \geq 0$. Suppose that there exists a quasisymmetric homeomorphism $(\mathbb{R}^3/G \times \mathbb{R}^m, d_{\lambda, m}) \rightarrow \mathbb{R}^{3+m}$, then*

$$\lambda^m \omega^{\frac{3+m}{2}} \leq \gamma.$$

We obtain now Theorem 1.1 as a corollary.

Proof of Theorem 1.1. Since $\omega_\mathcal{X}^3 > \gamma_\mathcal{X}^2 \geq 1$, we may fix λ so that $\omega_\mathcal{X}^{-1/2} < \lambda < \gamma_\mathcal{X}^{-1/3}$. Then $(\mathbb{R}^3/G \times \mathbb{R}^m, d_{\lambda, m})$ is Ahlfors $(3+m)$ -regular for all $m \geq 0$ and

$$\lambda^m \omega_\mathcal{X}^{\frac{3+m}{2}} > \gamma_\mathcal{X}.$$

So there are no quasisymmetric homeomorphisms $(\mathbb{R}^3/G \times \mathbb{R}^m, d_{\lambda, m}) \rightarrow \mathbb{R}^{3+m}$ for any $m \geq 0$. The linear local contractibility follows from Lemma 3.1 and Proposition 6.9. \square

To combine modulus estimates in Sections 9 and 11, we need a one-sided comparison between the modulus of a wall family and the modulus of a quasisymmetric image of the same family. The proof in [12, Proposition 4.1] for the case of the Whitehead continuum applies almost verbatim also to Semmes spaces $\mathbb{R}^3/G \times \mathbb{R}^m$; we omit the details.

Proposition 12.2. *Suppose $f: \mathbb{R}^3/G \times \mathbb{R}^m \rightarrow \mathbb{R}^{3+m}$ is an η -quasisymmetric homeomorphism, and Y is the union of a nonempty subcollection of handlebodies in $\mathcal{C}(X_k)$ for a $k > 0$. Then there exists $C = C(\eta) > 0$ so that*

$$\text{Mod}_{\frac{3+m}{1+m}}(\hat{\Sigma}^m(Y, \mathcal{X}, a)) \leq C \text{Mod}_{\frac{3+m}{1+m}} f(\hat{\Sigma}^m(Y, \mathcal{X}, a))$$

for $a > 0$.

Proof of Theorem 12.1. By definition of circulation, there exist a welding structure $(\mathcal{C}', \mathcal{A}', \mathcal{W}')$ and a homotopically finite collection of meridians \mathcal{M} of \mathcal{X} with respect to $(\mathcal{C}', \mathcal{A}', \mathcal{W}')$ so that (8.4) holds. In view of Lemma 6.2, we may assume that $(\mathcal{C}', \mathcal{A}', \mathcal{W}')$ is $(\mathcal{C}, \mathcal{A}, \mathcal{W})$. In view of Lemma 8.2, we may assume meridians in \mathcal{M} are uniformly quasisimilar with respect to parameters λ and $L = L(d_\lambda) \geq 1$.

Since order of circulation of \mathcal{X} is at least ω , there exists $C > 0$ so that for each $\ell > 1$, there exist $k, k' \geq 0$, $k' - k \geq \ell$, $H \in \mathcal{C}(X_k)$, and $\alpha \in \mathcal{M}|H$ so that

$$\text{circ}(X_{k'} \cap H, \alpha, H) \geq C\omega^{k'-k}.$$

Let f be an η -quasisymmetric mapping $(\mathbb{R}^3/G \times \mathbb{R}^m, d_{\lambda, m}) \rightarrow \mathbb{R}^{3+m}$. From Theorem 9.1, Proposition 11.1, and Proposition 12.2 it follows that

$$\left(\#\mathcal{C}(X_{k'} \cap H)(A\lambda^k)^m \lambda^{-k'm} \right)^{1-p} \leq C \left(\frac{1}{\text{circ}(X_{k'} \cap H, \alpha, H)} \right)^p,$$

where $p = (3+m)/(1+m)$, $C > 0$ depends only on $(\mathcal{C}, \mathcal{A}, \mathcal{W})$, λ and m , and A is the constant defined in Theorem 9.1. Since the order of growth of \mathcal{X} is at most γ , we have

$$\begin{aligned} \omega^{(k'-k)p} &\leq C (\text{circ}(X_{k'} \cap H, \alpha, H))^p \\ &\leq C \left(\#\mathcal{C}(X_{k'} \cap H)(A\lambda^k)^m \lambda^{-k'm} \right)^{p-1} \\ &\leq C \left(\gamma^{k'-k} \lambda^{km} \lambda^{-k'm} \right)^{p-1} \leq C \lambda^{(k-k')m(p-1)} \gamma^{(k'-k)(p-1)}, \end{aligned}$$

where $C = C(\mathcal{C}, \mathcal{W}, \mathcal{A}, m, \eta) \geq 1$. Thus

$$\lambda^m \omega^{\frac{p}{p-1}} \leq C^{\frac{1}{k'-k}} \gamma \leq C^{1/\ell} \gamma.$$

The claim now follows by letting $\ell \rightarrow \infty$. \square

13. NECKLACES

As an application of Theorem 7.2 we prove the existence of quasisymmetric parametrization for decomposition spaces associated with *Antoine's necklaces* when the chains are long. For the statement, we introduce some terminology.

Let $I \geq 3$, a union $\bigcup_{i=1}^I T_i$ of pair-wise disjoint tori T_1, \dots, T_I in \mathbb{R}^3 is called a *chain* if $T_i \cup T_j$ is a *Hopf link* for $\{i, j\} = \{1, I\}$ and for $|i - j| = 1$, and an *unlink* otherwise.

Suppose T a torus in \mathbb{R}^3 , and $\bigcup_{i=1}^I T_i$ is a torus chain contained in $\text{int}T$ in such a way that there is a homeomorphism $h: T \rightarrow \mathbb{B}^2 \times \mathbb{S}^1$ satisfying $h(\partial T) = \partial \mathbb{B}^2 \times \mathbb{S}^1$ and having the property that arguments of $p(h(T_i))$ are contained in $[\frac{2\pi i}{I}, \frac{2\pi(i+4/3)}{I}]$ for each $i = 1, \dots, I$. Here $p: \mathbb{B}^2 \times \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is

the projection map $(x, s) \mapsto s$. In this case, we say $\bigcup_{i=1}^I T_i$ is a *necklace chain in T* .

Let $\phi_i: U \rightarrow U_i$ be PL-homeomorphisms from a neighborhood U of T onto mutually disjoint neighborhoods U_i of $T_i, 1 \leq i \leq I$, satisfying $T_i \subset U_i \subset T \subset U$. The initial package $(T, T_1, \dots, T_I; \phi_1, \dots, \phi_I)$ yields a defining sequence $\mathcal{X} = (X_k)$ and a decomposition space, called an *Antoine's I -necklace space*, \mathbb{R}^3/G ; see Section 4.2. It is easy to see that the diameters of components of X_k can be arranged to tend to zero. Thus the components of X_∞ are singletons and \mathbb{R}^3/G is homeomorphic to \mathbb{R}^3 .

As discussed in Section 4.2, the initial package induces a welding structure for the I -necklace space \mathbb{R}^3/G , therefore for each $\lambda > 0$, a modular embedding of \mathbb{R}^3/G and a Semmes metric d_λ on \mathbb{R}^3/G . Semmes spaces $(\mathbb{R}^3/G, d_\lambda)$ associated with necklaces are linearly locally contractible because tori T_i 's are contractible in T , and these spaces are Ahlfors 3-regular when $\lambda^3 I < 1$.

The existence of quasisymmetric parametrization is proved in the following.

Theorem 13.1. *For every $I \geq 10$, there exists a Semmes metric d on the decomposition space \mathbb{R}^3/G associated to Antoine's I -necklace so that $(\mathbb{R}^3/G, d)$ is quasisymmetric to \mathbb{R}^3 .*

The proof of Theorem 13.1 relies on the possibility of fitting a necklace chain of length I in a torus, using only tori all similar to the larger one; we find it easier to fit a rectangular chain in a rectangular torus than to fit a round chain in a round torus.

Accepting Proposition 13.2 from the next section for the time being, we give the proof of the theorem.

Proof of Theorem 13.1. Let $I \geq 10$ and T, T_1, \dots, T_I be the tori constructed in Proposition 13.2. Let $\phi_i: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be similarity maps $x \mapsto \lambda_i x + v_i$ so that $\phi_i(T) = T_i$ for $1 \leq i \leq I$. Then the initial package $(T, T_1, \dots, T_I, \phi_1, \dots, \phi_I)$ gives rise to a natural self-similar welding structure in \mathbb{R}^3 as in Section 4.2. The claim now follows from Theorem 7.2. \square

13.0.1. Rectangular necklaces. Let $0 < \lambda < b < a$. We define

$$\begin{aligned} R_+(a, b, \lambda) &= \left[-\frac{\lambda}{2}, a + \frac{\lambda}{2}\right] \times \left[-\frac{\lambda}{2}, b + \frac{\lambda}{2}\right], \\ R_-(a, b, \lambda) &= \left(\frac{\lambda}{2}, a - \frac{\lambda}{2}\right) \times \left(\frac{\lambda}{2}, b - \frac{\lambda}{2}\right), \end{aligned}$$

and

$$T(a, b, \lambda) = (R_+(a, b, \lambda) \setminus R_-(a, b, \lambda)) \times \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right].$$

Let $L(a, b) = \partial([0, a] \times [0, b]) \times \{0\}$ be the 1-dim boundary of the rectangle $[0, a] \times [0, b] \times \{0\}$. We say $T(a, b, \lambda)$ is a torus with *length* $a + \lambda$, *width* $b + \lambda$, *thickness* λ and *core* $L(a, b)$.

Let $T = T(a, b, \lambda)$. We say that components of

$$\partial T \cap (\mathbb{R} \times \{-\lambda/2, b + \lambda/2\} \times \mathbb{R}) \quad \text{and} \quad \partial T \cap (\{-\lambda/2, a + \lambda/2\} \times \mathbb{R}^2)$$

are the *long and short faces* of T , respectively. We call also the components of

$$\partial T \cap (\mathbb{R}^2 \times \{-\lambda/2, \lambda/2\})$$

as the *boundary annuli* of T .

We call the 3-cells $[-\frac{\lambda}{2}, a + \frac{\lambda}{2}] \times [-\frac{\lambda}{2}, \frac{\lambda}{2}] \times [-\frac{\lambda}{2}, \frac{\lambda}{2}]$ and $[-\frac{\lambda}{2}, a + \frac{\lambda}{2}] \times [b - \frac{\lambda}{2}, b + \frac{\lambda}{2}] \times [-\frac{\lambda}{2}, \frac{\lambda}{2}]$ as *two long sides (front and back)* of T , and similarly $[-\frac{\lambda}{2}, \frac{\lambda}{2}] \times [-\frac{\lambda}{2}, b + \frac{\lambda}{2}] \times [-\frac{\lambda}{2}, \frac{\lambda}{2}]$ and $[a - \frac{\lambda}{2}, a + \frac{\lambda}{2}] \times [-\frac{\lambda}{2}, b + \frac{\lambda}{2}] \times [-\frac{\lambda}{2}, \frac{\lambda}{2}]$ *two short sides (left and right)* of T .

We say that a torus T in \mathbb{R}^3 is a *rectangular torus* if there exist a similarity map $g: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $0 < \lambda < b < a$ so that $T = g(T(a, b, \lambda))$. Furthermore, T is (p, q, r) -oriented if $g = h \circ O$, where h is a similarity of the form $x \mapsto \mu x + v$, $\mu > 0$, and O is an orthogonal transformation taking the standard basis (e_1, e_2, e_3) to (e_p, e_q, e_r) . We call the images of the long (resp. short) sides (resp. faces) of $T(a, b, \lambda)$ as the *long* (resp. *short*) sides (resp. faces) of T .

In what follows we use the following three types of *tightly fitted torus pairs*. Let $T = T(A, B, 1)$ and let $T' = g(T(a, b, \lambda))$ be an oriented torus contained in T . We say that T' is *tightly fitted into* T if one of the following conditions hold:

- (1) T' is a $(1, 2, 3)$ -oriented torus contained in a long side of T , so that each long face of T' intersects ∂T ;
- (2) T' is a $(1, 3, 2)$ -oriented torus contained in a long side of T , so that the long faces of T' are contained in the boundary annuli of T ;
- (3) T' is a $(2, 1, 3)$ -oriented torus contained in a short side of T , so that each long face of T' intersects ∂T and the short faces of T' are contained in the long faces of T .

If T' is either $(1, 2, 3)$ - or $(1, 3, 2)$ -oriented torus, we have the following relations

$$(13.1) \quad a + \lambda \leq A + 1, \quad b + \lambda = 1, \quad \text{and} \quad 2\lambda < 1.$$

If T' is $(2, 1, 3)$ -oriented,

$$(13.2) \quad a + \lambda = B + 1, \quad b + \lambda = 1, \quad \text{and} \quad 2\lambda < 1.$$

The main part of the proof of Theorem 13.1 is the following proposition.

Proposition 13.2. *Suppose $I \geq 10$. There exist $A > B > 1$ and $a_i > b_i > \lambda_i$, $(1 \leq i \leq I)$ satisfying*

$$\frac{a_i}{A} = \frac{b_i}{B} = \frac{\lambda_i}{1},$$

and tori T and T_i , $1 \leq i \leq I$, congruent to $T(A, B, 1)$ and $T(a_i, b_i, \lambda_i)$, respectively, such that the union $\bigcup_{1 \leq i \leq I} T_i$ is a necklace chain in T .

Proof. We construct for each $I \geq 10$, a torus $T = T(A, B, 1)$ and a chain $\bigcup_{1 \leq i \leq I} T_i$ which consists of tori all similar to T and is tightly fitted in T .

Since the tori in the chain are pair-wise disjoint, there exist similarity maps $h_i: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $x \mapsto \mu x + v_i$, with $\mu \in (0, 1)$ and $v_i \in \mathbb{R}^3$, so that the new chain $\bigcup_{i=1}^I h_i(T_i)$ is contained in the interior of T . Hence tori $h_1(T_1), \dots, h_I(T_I)$ satisfy the claims of the proposition.

It remains to construct tori T, T_1, \dots, T_I with aforementioned properties.

Case I. Suppose $I = 4k \geq 12$. Then $I = 2K + 2$ for some odd integer $K \geq 5$. We search for $A > B > 2$ and $a > b > 2\lambda$ satisfying

$$(13.3) \quad \frac{a}{A} = \frac{b}{B} = \frac{\lambda}{1},$$

and mutually disjoint tori T_i , $1 \leq i \leq 2K + 2$, which are congruent to $T(a, b, \lambda)$ and contained in $T(A, B, 1)$ so that $\bigcup T_i$ forms a necklace-chain positioned as follows.

Tori T_1 and T_{K+2} are $(2, 1, 3)$ -oriented torus tightly fitted in the two short sides (left and right) of $T(A, B, 1)$ with cores lying on the plane $\{x_3 = 0\}$. Tori T_i are $(1, 3, 2)$ -oriented for even i , and T_i are $(1, 2, 3)$ -oriented for odd indices $i \neq 1, K + 2$.

Tori T_2, T_3, \dots, T_{K+1} are tightly fitted in the front side of $T(A, B, 1)$, with the cores of T_2, T_4, \dots, T_{K+1} lying on the plane $\{x_2 = 0\}$ and the cores of T_3, T_5, \dots, T_K lying on $\{x_3 = 0\}$. Tori $T_{K+3}, T_{K+4}, \dots, T_{2K+2}$ are tightly fitted in the back side of $T(A, B, 1)$, with the cores of $T_{K+3}, T_{K+5}, \dots, T_{2K+1}$ lying on the plane $\{x_2 = B\}$ and the cores of $T_{K+4}, T_{K+4}, \dots, T_{2K}$ lying on the plane $\{x_3 = 0\}$.

Since the necklace-chain $\bigcup_{i=1}^I T_i$ is tightly fitted in $T(A, B, 1)$,

$$(13.4) \quad a + \lambda = B + 1 \quad \text{and} \quad b + \lambda = 1.$$

Since tori T_1 and T_2 , of thickness λ , are linked,

$$(13.5) \quad 3\lambda < 1.$$

In order to fit the linked chain $T_1 \cup T_2 \cup \dots \cup T_{K+2}$ in a long side of $T(A, B, 1)$, we seek for $0 < \epsilon, \delta < 1/10$ so that

$$(13.6) \quad A + 1 = K(a + \lambda) - (K - 1)(2 + \epsilon)\lambda + 2(1 + \delta)\lambda,$$

$$(13.7) \quad a + \lambda > 2(2 + \epsilon)\lambda,$$

and

$$(13.8) \quad 1 > (3 + \delta)\lambda.$$

Note that $K(a + \lambda) - (K - 1)(2 + \epsilon)\lambda$ is the total length of the union $T_2 \cup T_3 \cup \dots \cup T_{K+1}$, with $(K - 1)(2 + \epsilon)\lambda$ measuring the $K - 1$ overlaps and $(1 + \delta)\lambda$ measuring the distance from the chain to either short face of $T(A, B, 1)$. Conditions (13.7) and (13.8) are imposed to allow room for linking between consecutive tori in the union $T_1 \cup T_2 \cup \dots \cup T_{K+2}$.

We now check that (13.3) to (13.8) can be realized with proper choices of $A, B, a, b, \lambda, \epsilon$, and δ . By (13.3) and (13.4), we have relations

$$(13.9) \quad A + 1 = \lambda^{-2} \quad \text{and} \quad B + 1 = a + \lambda = \lambda^{-1}.$$

Furthermore, by (13.6) and (13.9),

$$(13.10) \quad 2(K - 2)\lambda^3 - (2\delta - (K - 1)\epsilon)\lambda^3 - K\lambda + 1 = 0.$$

Let $0 < \epsilon < 1/(5K)$ to be fixed later and fix $\delta = (K - 1)\epsilon/2$. Then (13.10) reads as

$$(13.11) \quad 2(K - 2)\lambda^3 - K\lambda + 1 = 0.$$

It is now easy to check that (13.11) admits a solution $\lambda \in (0, 3/10)$. We now choose ϵ small enough so that (13.7) and (13.8) hold. The parameters A, B, a , and b are now uniquely determined by (13.9) and (13.3).

Case II. Suppose that $I = 4k + 2 \geq 10$. Then $I = 2K + 2$ for an even $K \geq 4$. Again we will fit a necklace-chain $\bigcup_{i=1}^{2K+2} T_i$, consisting of tori all similar to $T(A, B, 1)$, in the torus $T(A, B, 1)$.

Since K is even, the linking condition forces T_1 and T_{K+2} to have different (p, q, r) -orientations and unequal sizes. Let T_1 be a $(2, 1, 3)$ -oriented torus tightly fitted in the left side of $T(A, B, 1)$ with the core lying on the plane $\{x_3 = 0\}$, and let T_{K+2} be a smaller $(2, 3, 1)$ -oriented torus (not tightly fitted) in the right side of $T(A, B, 1)$ with its core lying on the 2-plane $\{x_1 = A\}$.

As in Case I, we choose T_i to be a $(1, 3, 2)$ -oriented when $i \neq K+2$ is even and T_i to be a $(1, 2, 3)$ -oriented when $i \neq 1$ is odd. Tori T_2, T_3, \dots, T_{K+1} shall be tightly fitted in the front side of $T(A, B, 1)$ with the cores of T_2, T_4, \dots, T_K lying on the plane $\{x_2 = 0\}$ and the cores of T_3, T_5, \dots, T_{K+1} lying on $\{x_3 = 0\}$. Tori $T_{K+3}, T_{K+4}, \dots, T_{2K+2}$ shall be tightly fitted in the back side of $T(A, B, 1)$ with the cores of $T_{K+3}, T_{K+5}, \dots, T_{2K+1}$ lying on the plane $\{x_2 = B\}$ and cores of $T_{K+4}, T_{K+6}, \dots, T_{2K+2}$ lying on the plane $\{x_3 = 0\}$. Furthermore, one short face of T_{K+1} and one short face of T_{K+3} are placed in a common short face of $T(A, B, 1)$.

Tori $T_i, 1 \leq i \leq 2K+2$ and $i \neq K+2$, are congruent to $T(a, b, \lambda)$ and torus T_{K+2} is congruent to a smaller $T(a', b', \lambda')$; all are similar to $T(A, B, 1)$.

It is straightforward to check that numbers $A > B > 1, a > b > \lambda > 0$ and $a' > b' > \lambda' > 0$ can be found so that $\bigcup_{i=1}^I T_i$ is a chain tightly fitted in T . We omit the details.

Case III. Suppose that $I \geq 11$ is odd. Then $I = 2K + 3$ for some $K \geq 4$. For K even, there exist, by Case I, numbers A, B, a, b, λ and tightly fitted tori T_1, \dots, T_{2K+2} in $T = T(A, B, 1)$ so that tori T_1, \dots, T_{2K+2} are congruent to $T(a, b, \lambda)$. For K odd, we have, in addition, parameters $a', b',$ and λ' so that tori T_1, \dots, T_{2K+2} are congruent to either $T(a, b, \lambda)$ or $T(a', b', \lambda')$. Let $\epsilon > 0$ and $\delta > 0$ be the parameters appearing in these constructions. We rename the first torus T_1 to T_0 .

The plan is to replace tori T_2, T_3, T_4 congruent to $T(a, b, \lambda)$ by four tori t_1, t_2, t_3, t_4 congruent to a smaller torus $T(a'', b'', \lambda'')$ which is similar to $T(a, b, \lambda)$. The new collection $T_0, t_1, t_2, t_3, t_4, T_5, \dots, T_{2K+2}$ forms the necklace chain for the case $I = 2K + 3$.

Denote by F_θ the rotation in \mathbb{R}^3 about the x_1 -axis by an angle θ , so that $F_\theta(\mathbb{R}^2 \times \{0\}) = P_\theta$, where P_θ is the plane $\{x_3 = x_2 \tan \theta\}$ in \mathbb{R}^3 . Recall that T_0 is a $(2, 1, 3)$ -torus and T_5 is a $(1, 2, 3)$ -torus with cores lying on the plane P_0 .

For $j = 1, \dots, 4$, let t_j be a translate of $F_{2j\pi/5}(T(a'', b'', \lambda''))$ in the direction of x_1 , where the translation will be fixed later. Then the core of t_j lies on the plane $P_{2j\pi/5}$; and the planes containing the cores of two consecutive tori in $\{T_0, t_1, t_2, t_3, t_4, T_5\}$ form an angle $2\pi/5$.

Numbers $A > B > 1, a > b > \lambda > 0$ and $a' > b' > \lambda' > 0$ are retained from the previous cases. To realize the plan, we need to choose $a'' > b'' > \lambda'' > 0$ satisfying $\frac{a''}{A} = \frac{b''}{B} = \frac{\lambda''}{1} < \lambda$, so that the 4 new tori t_1, t_2, t_3, t_4 can be fitted lengthwise into the space vacated by T_2, T_3, T_4 to form a necklace chain. The calculations leading to these choices are routine however tedious, we omit the details. This completes the proof. \square

14. THE BING DOUBLE AND THE WHITEHEAD CONTINUUM REVISITED

The construction of the space \mathbb{R}^3/Bd associated to the Bing double is illustrated and discussed in Daverman's book [6, Example 1, pp. 62-63] and in an article of Freedman and Skora [8]; see the original article [1] or [4] for a highly nontrivial shrinking procedure that leads to a homeomorphism $\mathbb{R}^3/\text{Bd} \approx \mathbb{R}^3$.

We fix an initial package consisting of three tori T, T_1, T_2 in \mathbb{R}^3 so that T_1 and T_2 are linked in T but not in \mathbb{R}^3 as in [6, Figure 9-1], and three homeomorphisms $\phi_i: T \rightarrow T_i$. Denote by $\mathcal{X} = (X_k)$ the defining sequence induced by the initial package as described in Section 4.2, by Bd the (cellular) decomposition, and by \mathbb{R}^3/Bd the decomposition space.

Semmes showed [19, Theorem 1.12c] that \mathbb{R}^3/Bd admits a metric d so that the space $(\mathbb{R}^3/\text{Bd}, d)$ is quasiconvex, Ahlfors 3-regular and linearly locally contractible and it supports certain Sobolev and Poincaré inequalities that are crucial for analysis, but this space is not quasisymmetric to \mathbb{R}^3 . Semmes' construction of the metric d has served as a model for modular metrics defined in Section 6; it is easy to verify that d is bilipschitz equivalent to a modular metric.

The non-existence of a quasisymmetric homeomorphism $(\mathbb{R}^3/\text{Bd}, d) \rightarrow \mathbb{R}^3$ is based on a lemma of Freedman and Skora on essential intersections (Lemma 2.4 in [8]).

We state their lemma in the following.

Lemma 14.1. *Let T_1 and T_2 be two solid tori embedded in $\mathbb{B}^2 \times \mathbb{S}^1$ as in the Bing double construction. Let $(P, \partial P) \subset (\mathbb{B}^2 \times \mathbb{S}^1, \partial \mathbb{B}^2 \times \mathbb{S}^1)$ be an embedded connected planar surface representing the generator of the relative homology group $H_2(\mathbb{B}^2 \times \mathbb{S}^1, \partial, \mathbb{Z})$. Suppose P and $T_1 \cup T_2$ meet in transverse general position. Then for $i = 1$ or 2 , $P \cap T_i$ must contain at least two surfaces which represent generators of $H_2(T_i, \partial, \mathbb{Z})$.*

Using the notion of circulation Lemma 14.1 can be interpreted as follows.

Lemma 14.2. *Let \mathbb{R}^3/Bd the decomposition space associated to the Bing double Bd and $\mathcal{X} = (X_k)$ the defining sequence associated to the initial package $(T, T_1, T_2, \phi_1, \phi_2)$. Then*

$$\text{circ}(X_k, T; \alpha) \geq 2^k$$

for every $k \geq 0$ and every meridian α on T .

The Freedman-Skora lemma yields that the defining sequence \mathcal{X} of the Bing double has order of circulation at least 2; in fact the order of growth of \mathcal{X} is exactly 2. Theorem 1.2 now follows from Theorem 12.1.

In [12] the homological argument of the Freedman-Skora lemma was used to obtain a version of the intersection lemma and to show that the standard defining sequence for the Whitehead continuum has an order of circulation at least 2. We refer to [12] for results on the nonexistence of quasisymmetric parametrization of the decomposition spaces associated with the Whitehead continuum.

15. BING'S DOGBONE

The decomposition space \mathbb{R}^3/Db associated with Bing's dogbone [2] was the first known example of a decomposition space which is not homeomorphic to \mathbb{R}^3 but whose product with a line, $(\mathbb{R}^3/\text{Db}) \times \mathbb{R}$, is homeomorphic to \mathbb{R}^4 ; see [3].

Bing's dogbone space \mathbb{R}^3/Db is constructed as follows. Let A be a PL cube-with-2-handles standardly embedded in \mathbb{R}^3 , and let A_1, A_2, A_3, A_4 be four cubes-with-handles of genus 2 embedded in the interior of A as illustrated in [2, Fig. 1, p.486].

Let $\phi_j: U \rightarrow U_j$ be PL-homeomorphisms from a neighborhood U of A onto mutually disjoint neighborhoods $U_i, 1 \leq i \leq 4$, of A_i satisfying $A_i \subset U_i \subset A \subset U$. The intersection

$$\text{Db} = \bigcap_{l=0}^{\infty} \bigcup_{\alpha \in S_l} \phi_{\alpha}(A)$$

is called *Bing's dogbone*. The decomposition \mathbb{R}^3/Db is topologically different from \mathbb{R}^3 even though each nondegenerate component of Db is a tame arc [2]. On the other hand, $(\mathbb{R}^3/\text{Db}) \times \mathbb{R}$ is \mathbb{R}^4 .

The initial package $(A, A_1, \dots, A_4, \phi_1, \dots, \phi_4)$ yields a defining sequence $\mathcal{X}_{\text{Db}} = (X_k)$: $X_0 = A$ and

$$X_{k+1} = \bigcup_{\alpha=1}^4 \phi_{\alpha}(A)$$

for $k \geq 0$. Recall that $X_k = \bigcup_{\alpha \in \mathcal{S}_k} \phi_{\alpha}(A)$, where $\phi_{\alpha} = \phi_{\alpha_1} \circ \dots \circ \phi_{\alpha_k}$ and $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, 2, 3, 4\}^k$. The initial package induces a welding structure $(\mathcal{C}_{\text{Db}}, \mathcal{A}_{\text{Db}}, \mathcal{W}_{\text{Db}})$ on the defining sequence \mathcal{X}_{Db} ; in particular \mathcal{C} consists of a single condenser $(A, \bigcup_{i=1}^4 A_i)$. See Section 4.2 for details.

Theorem 15.1. *Let $(\mathbb{R}^3/\text{Db}, d_{\lambda})$ be a Semmes space associated to the defining sequence \mathcal{X}_{Db} and the welding structure $(\mathcal{C}_{\text{Db}}, \mathcal{A}_{\text{Db}}, \mathcal{W}_{\text{Db}})$. Suppose $m \geq 1$ and $2^{-\frac{1+m}{m}} < \lambda < 2^{-2/3}$. Then $(\mathbb{R}^3/\text{Db} \times \mathbb{R}^m, d_{\lambda, m})$ is Ahlfors $(3+m)$ -regular and linearly locally contractible, but it is not quasisymmetrically equivalent to \mathbb{R}^{3+m} .*

The Ahlfors regularity follows from Proposition 6.8, since \mathcal{X} has order of growth 4. The linear local contractibility follow from Proposition 6.9, since every A_i is contractible in A .

To show that $(\mathbb{R}^3/\text{Db}) \times \mathbb{R}^m$ is not quasisymmetric to \mathbb{R}^{3+m} , we estimate the order of circulation of \mathcal{X} in A from below.

As in [2, Fig. 1], let C_1 and C_2 be two disjoint 3-cells in A so that handles of $\bigcup A_i$ are sorted into two groups, and each group consists of four pairwise linked handles, one from each A_i , and is contained in one of the 3-cells C_1 or C_2 . Then $C_1 \cup C_2 \cup A_1 \cup A_4$ and $C_1 \cup C_2 \cup A_2 \cup A_3$ is a pair of solid tori in A .

The arrangement of handlebodies $\bigcup A_i$ is understood as follows. We fix essential 2-disks D_1, D_2, D_3 in A as in [2, Fig. 1]. These disks have the property that if $h: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a homeomorphism that is identity outside A then

- (1) $h(A_1) \cup h(A_4)$ and $h(A_2) \cup h(A_3)$ intersect both D_1 and D_2 ,
- (2) $h(A_1) \cup h(A_3)$ and $h(A_2) \cup h(A_4)$ intersect both D_1 and D_3 , and
- (3) $h(A_1) \cup h(A_2)$ and $h(A_3) \cup h(A_4)$ intersect both D_2 and D_3 .

We use topological properties of the initial package to show the following estimate of Freedman-Skora type. This estimate implies that the order of circulation of \mathcal{X} is at least 4. This together with Theorem 12.1 proves Theorem 15.1.

Lemma 15.2. *Let γ be a meridian of A that is isotopic to ∂D_1 on ∂A . Then*

$$(15.1) \quad \text{circ}(X_k, \gamma, A) \geq 4^{k-1}$$

for every $k \geq 1$.

Proof. As a preliminary step, we define tori T^O and T^X as follows

$$T^O = C_1 \cup C_2 \cup A_1 \cup A_4 \quad \text{and} \quad T^X = C_1 \cup C_2 \cup A_2 \cup A_3.$$

Then

$$A_{\alpha 1} \cup A_{\alpha 4} \subset (T^O)_\alpha \subset A_\alpha \quad \text{and} \quad A_{\alpha 2} \cup A_{\alpha 3} \subset (T^X)_\alpha \subset A_\alpha,$$

where $A_\alpha = \phi_\alpha A$, $(T^O)_\alpha = \phi_\alpha(T^O)$, $(T^X)_\alpha = \phi_\alpha(T^X)$, and $\alpha \in \{1, 2, 3, 4\}^k$.

Note that tori $(T^O)_1 \cup (T^O)_4$ are linked in T^O the way that the two first stage tori are linked in the 0-th stage torus as in the construction of the Bing double. Note also that the same can be said about the linking of $(T^X)_1 \cup (T^X)_4$ in T^O , $(T^O)_2 \cup (T^O)_3$ in T^X and $(T^X)_2 \cup (T^X)_3$ in T^X .

Therefore for every $\alpha \in \{1, 2, 3, 4\}^k$, tori $(T^O)_{\alpha 1} \cup (T^O)_{\alpha 4}$ are linked in $(T^O)_\alpha$ the way the first stage tori are linked in the 0-th stage torus as in the Bing double, and the same can be said about the linking of $(T^X)_{\alpha 1} \cup (T^X)_{\alpha 4}$ in $(T^O)_\alpha$, $(T^O)_{\alpha 2} \cup (T^O)_{\alpha 3}$ in $(T^X)_\alpha$ and $(T^X)_{\alpha 2} \cup (T^X)_{\alpha 3}$ in $(T^X)_\alpha$.

This linking property has the following consequences.

(I). If $f: (\mathbb{B}^2, \partial\mathbb{B}^2) \rightarrow (A, \partial D_1)$ is map with the property $f(\partial\mathbb{B}^2) = \partial D_1$, then $f(\mathbb{B}^2)$ intersects both T^O and T^X virtually interior essentially. Indeed, let Q be a 3-cell in \mathbb{R}^3 so that $Q \cap A \subset \partial A$, $Q \cap \partial D_1 = \emptyset$, and that $Q \cup A$ is a torus. We denote $T = Q \cup A$. Since a core of T^O is also a core of T , we have that $f(\mathbb{B}^2)$ intersects T^O virtually interior essentially. The same argument applies also to T^X .

(II). Suppose Ω is a 2-manifold in \mathbb{B}^2 and $f: (\Omega, \partial\Omega) \rightarrow (T^O, \partial T^O)$ is a virtually interior essential map. Then by the standard argument of filling T^O with 2-disks, we have that f has an virtually interior essential intersection with $A_1 \cup A_4$; see e.g. the proof of wildness of Antoine's necklace [6, Prop. 5, pp. 73-74]. The same can be said about T^X and $A_2 \cup A_3$.

The estimate of circulation (15.1) follows the following claim and the relation between the essential intersections and the circulations as stated in Remark 8.7.

Claim. Let $f: (\mathbb{B}^2, \partial\mathbb{B}^2) \rightarrow (A, \partial A)$ be an interior essential map so that $f(\partial\mathbb{B}^2)$ is isotopic to ∂D_1 on ∂A . Then $f(\mathbb{B}^2) \cap X_k$ has at least 4^k virtually interior essential components. It remains to verify the claim.

Let $\varsigma: \{1, 2, 3, 4\} \rightarrow \{O, X\}$ be the map defined as $\varsigma(1) = \varsigma(4) = O$ and $\varsigma(2) = \varsigma(3) = X$, and $\varsigma^k = \varsigma \times \cdots \times \varsigma: \{1, 2, 3, 4\}^k \rightarrow \{O, X\}^k$ be the product

map. Set $\mathcal{S}_k = \{1, 2, 3, 4\}^k$, $\Sigma_k = \{O, X\}^k$, and $s_k(w) = (\zeta^k)^{-1}(w)$. Note that for $w = (w_1, \dots, w_k) \in \Sigma_k$

$$s_k(w) = (\zeta^k)^{-1}(w) = \{(\alpha_1, \dots, \alpha_k) \in \mathcal{S}_k, \alpha_j \in \zeta^{-1}(w_j) \text{ for all } 1 \leq j \leq k\}.$$

So $\mathcal{S}_k = \cup_{w \in \Sigma_k} s_k(w)$ is a disjoint union.

For each $k \geq 1$, we sort the 4^k handlebodies in X_k into 2^k mutually disjoint groups as follows. If $k = 1$, the two groups are $X_1(O) = \{A_1, A_4\}$ and $X_1(X) = \{A_2, A_3\}$. Suppose $k \geq 2$, define for $w \in \Sigma_k$

$$X_k(w) = \{A_\alpha : \alpha \in s_k(w)\}.$$

So $X_k = \cup_{w \in \Sigma_k} X_k(w)$ is a disjoint union of 2^k groups.

Fix a $w \in \Sigma_k$, we will focus on the 2^k handlebodies in $X_k(w)$ and consider a finite defining sequence associated with this particular $w = (w_1, w_2, \dots, w_k)$ as follows. Set

$$Z_0 = A, \quad Z_1 = T^{w_1}, \quad \text{and} \quad Z_j = \cup_{\alpha \in s_{j-1}(w_1, w_2, \dots, w_{j-1})} (T^{w_j})_\alpha$$

for $2 \leq j \leq k$. Note that

$$Z_{j+1} \cap (T^{w_j})_\alpha = (T^{w_{j+1}})_{\alpha i_1} \cup (T^{w_{j+1}})_{\alpha i_2},$$

where $\{i_1, i_2\} = \zeta^{-1}(w_j)$, for every $(T^{w_j})_\alpha$ in Z_j .

Let $f: (\mathbb{B}^2, \partial\mathbb{B}^2) \rightarrow (A, \partial A)$ be an interior essential map so that $f(\partial\mathbb{B}^2)$ is isotopic to ∂D_1 on ∂A . By applying a homotopy near ∂A , we may assume that $f(\mathbb{B}^2) \cap \partial A = \partial D_1$. Then by (I), $f(\mathbb{B}^2)$ intersects both T^O and T^X virtually interior essentially. In particular, $f(\mathbb{B}^2) \cap Z_1$ has at least one virtually interior essential component.

In view of the linking relation (of the Bing double type) between tori in consecutive generations, we may apply the lemma of Freedman and Skora (Lemma 14.1) iteratively to conclude that $f(\mathbb{B}^2) \cap Z_k$ has at least 2^{k-1} virtually interior essential components.

Tori in Z_k are pair-wise disjoint and each torus contains two handlebodies in $X_k(w)$. It follows from (II) above that $f(\mathbb{B}^2) \cap X_k(w)$ has at least 2^{k-1} virtually interior essential components.

The claim follows by summing over all $w \in \Sigma_k$. This completes the proof of the theorem. \square

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