SYMMETRIC STABLE PROCESSES STAY IN THICK SETS

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Abstract. Let $X(t)$ be the symmetric α-stable process in $\mathbb{R}^d (0 < \alpha < 2,\ d \geq 2)$. And let $W(f)$ be the thorn $\{x \in \mathbb{R}^d : 0 < x_1 < 1, (x_2^2 + \cdots + x_d^2)^{1/2} < f(x_1)\}$ where $f : (0,1) \to (0,1)$ is continuous, increasing with $f(0^+) = 0$. Recently Burdzy and Kulczycki gave an exact integral condition on f for the existence of a random time s such that $X(t)$ remains in the the thorn $X(s) + \overline{W(f)}$ for all $t \in [s, s+1)$. We extend their theorem to general open sets W with $0 \in \partial W$. In general, α-processes may stay in sets which are quite lacunary, and are not locally connected at 0.

1. Introduction.

Let $X(t)$ be the symmetric α-stable process in $\mathbb{R}^d (0 < \alpha < 2,\ d \geq 2)$, $f : (0,1) \to (0,\infty)$ be a nondecreasing left continuous function satisfying $f(0^+) = 0$, and $W(f)$ be the thorn $\{x \in \mathbb{R}^d : 0 < x_1 < 1, (x_2^2 + \cdots + x_d^2)^{1/2} < f(x_1)\}$. In $[BK]$, Burdzy and Kulczycki give an exact integral condition on f for the existence of a random time s such that $X(t)$ remains in the thorn $X(s) + \overline{W(f)}$ for all $t \in [s, s+1)$.

In this note we extend their theorem on thorns to general open sets having 0 on the boundary. These sets need not be locally connected at 0 and can be quite lacunary; this is possible due to the jumping property of the symmetric α-stable process.

This line of investigation is motivated by the existence of cone points for Brownian paths. For literature and some unsolved cases, see $[Bu]$.

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Let W be an open set in $\mathbb{R}^d$ that contains 0 on its boundary, (Ω, P) be the probability space on which $X(t)$ is defined, $t_0 > 0$ and

$$A(W) = \{\omega \in \Omega : \exists s = s(\omega) \geq 0 \quad \text{such that} \quad X(t,\omega) \in X(s,\omega) + \overline{W} \text{ for all } t \in [s, s+t_0)\}.$$

We say $\omega \in \Omega$ has a W-point if $\omega \in A(W)$ for some $t_0 > 0$.

Let $I(f) = \int_0^1 \frac{f(r)^{\alpha+d-1}}{r^{\alpha+d}} dr$. The theorem of Burdzy and Kulczycki $[BK]$ says that if $I(f) = \infty$ then symmetric α-stable process has $W(f)$-thorn points a.s., and if $I(f) < \infty$ then α-process has no $W(f)$-thorn points a.s.; precisely,

Theorem A. *For any* $t_0 > 0$,

(i) $P(A(W(f))) = 1$ *if* $I(f) = \infty$,

and

(ii) $P(A(W(f))) = 0$ *if* $I(f) < \infty$.

It is clear that $I(f) < \infty$ if and only if $\sum_{k=1}^{\infty} \frac{f(2^{-k})^{\alpha+d-1}}{(2^{-k})^{\alpha+d-1}} < \infty$.

For an arbitrary open set W with $0 \in \partial W$, we give in Theorem 1, a thickness condition on W under which $P(A(W)) = 1$; and in Theorem 2, a thinness condition on W under which $P(A(W)) = 0$. These are natural extensions of Theorem A, and the proofs follow the same structure. The proof in $[BK]$ uses very precise harmonic measure estimates obtained by comparing sections of thorns with cylinders; here we must rely on very general estimates and make more use of the jumps. Unlike thorns, general sets do not point in a specific direction, and the uncertainty of the starting time $s(\omega)$ gives rise to a problem which can not be solved by shifting the set W along an axis; these complications are handled by putting bands around W.

The conditions in Theorems 1 and 2 do not match and are complicated (see Section 3); however, in the case of thorns and also the examples below, they are sharp.

Example 1. (Lacunary Rings). *Let* $W = \bigcup_{j=1}^{\infty} \{2^{-j} < |x| < 2^{-j}(1+\delta_j)\}$ *with* $0 \le \delta_j < \frac{1}{2}$ *satisfying*

$$\delta_j 2^{-j} < \delta_i 2^{-i} \text{ whenever } \delta_i, \delta_j > 0 \text{ and } j > i.$$

Then

(i) $P(A(W)) = 1$ *if* $\sum \delta_j^{\alpha+1} = \infty$,

and

(ii) $P(A(W)) = 0$ *if* $\sum \delta_j^{\alpha+1} < \infty$.

In this example, we allow δ_j to be 0 infinitely often.

Example 2. (Blocks of Varying Shape). *Let* $m(j)$ *be integers in* $[1, d]$ *and* δ_j *be numbers in* $[0, \frac{1}{2})$ *satisfying*

$$\delta_j 2^{-j} < \delta_i 2^{-i} \text{ whenever } \delta_i, \delta_j > 0 \text{ and } j > i. \tag{1.1}$$

Let Q_j *be a rectangular cube contained in* $\{\frac{5}{8}2^{-j} < |x| < \frac{7}{8}2^{-j}\}$ *obtained by translation and rotation of* $(0, \delta_j 2^{-j-5}/\sqrt{d})^{m(j)} \times (0, 2^{-j-5}/\sqrt{d})^{d-m(j)}$ ($Q_j = \phi$ *when* $\delta_j = 0$)*; and let* $W = \bigcup_{1}^{\infty} Q_j$. *Then*

(i) $P(A(W)) = 1$ *if* $\sum \delta_j^{\alpha+m(j)} = \infty$,

and

(ii) $P(A(W)) = 0$ *if* $\sum \delta_j^{\alpha+m(j)} < \infty$.

In this example, we allow δ_j to be 0 infinitely often.

Example 3. **(Scattered Cubes).** *Let* $\{r_k\}_0^\infty$ *and* $\{\epsilon_k\}_0^\infty$ *be decreasing sequences of positive numbers so that* $r_0 = \epsilon_0 = 1$, $\epsilon_k < \frac{1}{10}$, $(\epsilon_k r_k)^{-1}$ *is a power of* 2, $N_k \equiv \epsilon_{k-1} r_{k-1}/r_k$ *is an odd integer and* $\epsilon_k^{d+\alpha} < N_k^{-\alpha}$, *for any* $k \geq 1$.

All cubes here have edges parallel to the coordinate axes. Let $Q_0 = (-\frac{1}{2}, \frac{1}{2})^d$, $\mathcal{C}_0 = \{Q_0\}$ *and* $\mathcal{C}'_0 = \phi$. *After* $Q_j, \mathcal{C}_j$ *and* $\mathcal{C}'_j$ *have been defined for* $0 \leq j \leq k-1$ *with* $\ell(Q_j) = \epsilon_j r_j$, *we subdivide* Q_k *into a collection* $\mathcal{S}_k$ *of* N_k^d *subcubes of side length* r_k *each.* $\mathcal{C}_k$ *consists of those cubes having side length* $\epsilon_k r_k$ *and concentric to those in* $\mathcal{S}_k$*; let* Q_k *be the cube in* $\mathcal{C}_k$ *that contains the origin* 0, *and* $\mathcal{C}'_k = \mathcal{C}_k \backslash \{Q_k\}$. *For future discussion, we also choose and fix one cube from* $\mathcal{C}'_k$ *that is closest to* Q_k, *call it* Q'_k. *Let*

$$W = \bigcup_{k=1}^{\infty} \bigcup_{Q \in \mathcal{C}'_k} Q.$$

Then

(i) $P(A(W)) = 1$ *if* $\sum \epsilon_k^{\alpha+d} = \infty$,

and

(ii) $P(A(W)) = 0$ *if* $\sum \epsilon_k^{\alpha+d} < \infty$.

Section 2 contains properties of symmetric α-stable processes needed later, Section 3 contains the main theorems; proofs of Theorems 1, 2 and examples are given in Sections 4, 5 and 6 respectively.

2. Preliminaries.

A symmetric α-stable process X on $\mathbb{R}^d$ is a Lévy process (homogeneous independent increments) whose transition density $p(t,x)$ is uniquely determined by its Fourier transform, $\int_{\mathbb{R}^d} e^{ix\cdot\xi} p(t,x)dx = e^{-t|\xi|^\alpha}$. Here α must be in $(0,2]$. When $\alpha = 2$, it is the Brownian motion except for a linear time change. From now on, symmetric α-stable processes are restricted to the case $0 < \alpha < 2$. Denote by (Ω, P) the probability space on which $X(t)$ is defined. Sample paths are discontinuous, and are right continuous with left limits a.s. $([B], [BGR])$.

In the following, $B(x,r)$ is the ball centered at x of radius r, and $|S|$ is the Lebesgue measure (volume) of the set S. We use c (or c') to denote positive constants depending at most on d and α, $c(\cdot)$ to denote positive constants depending on d, α and the variables in the parentheses, and $C_j (j = 1, 2, \dots)$ to denote specific constants depending on d and α only. We write $a \lesssim b$ when $a/b \le c$ for some constant c, and $a \cong b$ when $a \lesssim b$ and $b \lesssim a$.

As usual E^x is the expectation with respect to the process starting from $x \in R^d$. For any open set D in $\mathbb{R}^d$, X^D is the symmetric α-stable process killed upon leaving D and $\tau_D = \inf\{t > 0 : X(t) \notin D\}$ is the first exit time.

For any $x \in D$, the α-harmonic measure $\mu^x(\cdot, D)$ is a measure on D^c defined by

$$\mu^x(A, D) = P^x(X(\tau_D) \in A), \quad A \subseteq D^c;$$

it is monotone in D, i.e.

$$\mu^x(A, D) \le \mu^x(A, \tilde{D}) \quad \text{if} \quad D \subseteq \tilde{D}.$$

In the case of a ball $B = B(0,r)$, it was shown by M. Riesz that

$$d\mu^x(y,B) = k_B(x,y)dy \tag{2.1}$$

where

$$k_B(x,y) = \begin{cases} C_1 \left(\frac{r^2-|x|^2}{|y|^2-r^2}\right)^{\alpha/2} |x-y|^{-d}, & |y| > r, \\ 0, & |y| \le r. \end{cases}$$

Note, from (2.1) and the monotonicity that

$$\mu^x(S,D) = 0 \quad \text{if} \quad S \quad \text{is a sphere in} \quad D^c.$$

Denote by G the Green function of X, i.e.

$$G(x,y) = \int_0^\infty p(t, x-y)dt = C_2|x-y|^{-d+\alpha}.$$

And denote by $G_D(x,y)$ the Green function of X^D, i.e.

$$G_D(x,y) = C_2\left[|x-y|^{-d+\alpha} - \int_{D^c} |y-z|^{-d+\alpha} d\mu^x(z,D)\right] \quad \forall x, y \in D, x \ne y,$$

$G_D(x,x) = \infty$ if $x \in D$ and $G_D(x,y) = 0$ in $(D \times D)^c$. Green function has the scaling property

$$G_D(x,y) = a^{-\alpha+d} G_{aD}(ax, ay),\ a > 0;$$

and for any measurable $f \ge 0$ on D,

$$E^x\left[\int_0^{\tau_D} f(X(s))ds\right] = \int_D G_D(x,y)f(y)dy \quad \forall x \in D.$$

In particular,

$$E^x(\tau_D) = \int_D G_D(x,y)dy \quad \forall x \in D.$$

It is well known that

$$E^x(\tau_{B(x,r)}) = C_3 r^\alpha, \tag{2.2}$$

and

$$E^x(\tau_D) \lesssim |D|^{\alpha/d}. \tag{2.3}$$

For any bounded measurable $\phi \geq 0$ on D^c,

$$E^x[\phi(X(\tau_D)) : X(\tau_D) \neq X(\tau_{D-})] = C_4 \int_{D^c} \int_D \frac{G_D(x,y)}{|y-z|^{d+\alpha}} dy \phi(z) dz, \tag{2.4}$$

where $X(\tau_{D-}) = \lim_{t \uparrow \tau_D} X(t)$ exists a.s. ([IW]). Note from (2.4) and $X(\tau_{D-}) \in \overline{D}$ that for $x \in D$ and $A \subseteq \overline{D}^c$,

$$\mu^x(A, D) = C_4 \int_A \int_D \frac{G_D(x,y)}{|y-z|^{d+\alpha}} dy dz, \tag{2.5}$$

and

$$\mu^x(A, D) \lesssim E^x(\tau_D) \operatorname{dist}(A, D)^{-\alpha-d} |A|. \tag{2.6}$$

When $\max\{\operatorname{diam} D, \operatorname{diam} A\} \leq a \operatorname{dist}(A, D)$, we obtain from (2.5) the following estimates

$$\mu^x(A, D) \cong c(a) E^x(\tau_D) \operatorname{dist}(A, D)^{-\alpha-d} |A|. \tag{2.7}$$

We shall use (2.7) repeatedly for X^D having certain prescribed jumps.

3. Theorems.

Let W be an open set with $0 \in \partial W$.

Theorem 1. *Suppose that*

$$\int_W E^x(\tau_W)|x|^{-\alpha-d}dx = \infty, \tag{3.1}$$

then $P(A(W)) = 1$.

In the case of a thorn $W(f), E^x(\tau_{W(f)}) \cong f(x_1)^\alpha$ for any x satisfying $(x_2^2 + x_3^2 + \cdots + x_n^2)^{1/2} < f(x_1)/2$, hence

$$\int_{W(f)} E^x(\tau_{W(f)})|x|^{-\alpha-d} \cong \int_0^1 \frac{f(r)^{\alpha+d-1}}{r^{\alpha+d}}dr.$$

Therefore for thorns, Theorem 1 is equivalent to Theorem A(i).

For general open sets W, it is unclear whether

$$\int_W E^x(\tau_W)|x|^{-\alpha-d}dx < \infty \tag{3.2}$$

implies $P(A(W)) = 0$.

Before stating the thinness conditions under which $P(A(W)) = 0$, we need a few definitions. For any positive integers j and n, let

$$W(j) = W \cap \{|x| < 2^{-j}\},$$

$$W^*(j) = W \cap \{2^{-j-1} \le |x| < 2^{-j}\},$$

$$p(j) = \max\{i \le j-2 : W^*(i) \ne \phi\},$$

$$W_n = \{x : \mathrm{dist}(x, W) < 2^{-n}\} = W + B(0, 2^{-n}),$$

$$W_n(j) = W_n \cap \{|x| < 2^{-j}\},$$

$$W_n^*(j) = W_n \cap \{2^{-j-2} \le |x| < 2^{-j}\},$$

and

$$p_n(j) = \max\{i \le j-2 : W_n^*(i) \ne \phi\}.$$

For $x \in W(j)$, define

$$\lambda^x(W,j) = \mu^x(W^*(p(j)), W(j-1)) 2^{-p(j)(d+\alpha)} |W^*(p(j))|^{-1} \tag{3.3}$$

and

$$\Lambda(W,j) = \sup\{\lambda^x(W,j) : x \in W^*(j)\}; \tag{3.4}$$

for $x \in W_n(j)$, the expressions $\lambda^x(W_n,j)$ and $\Lambda(W_n,j)$ are defined analogously.

Remark. *The quantity $\lambda^x(W,j)$ is a substitute for $E^x(\tau_W)$, and is comparable to $E^x(\tau_W)$ when $W(j-1)$ and $W^*(p(j))$ are separated by a large ring. In fact,*

$$\lambda^x(W,j) \cong E^x(\tau_{W(j-1)}), \quad \text{if} \quad p(j) < j-2, \tag{3.5}$$

and

$$\lambda^x(W,j) \gtrsim E^x(\tau_{W(j-1)}) \quad \text{if} \quad p(j) = j-2;$$

the equivalence relation in the case $p(j) < j-2$ follows from (2.7) and the fact that $|y-z| \cong 2^{-p(j)}$ for $y \in W(j-1)$ and $z \in W^*(p(j))$. When $p(j) = j-2$ and $W(j-1)$ and $W^*(j-2)$ are separated by a ring $\{a < |x| < b\}$ of width $b-a$ at least $\beta 2^{-j}$, we have

$$\lambda^x(W,j) \cong c(\beta) E^x(\tau_{W(j-1)}). \tag{3.6}$$

Theorem 2. *Let W be an open set with $0 \in \partial W$. Suppose that there is an infinite collection $\mathcal{A}$ of (n,i) with integers $n > i > 0$, satisfying $W^*(i) \neq \phi$,*

$$\mu^0(W_n^*(i), W_n(i+1)) \cong \mu^0(W_n^*(i), B(0, 2^{-n})); \tag{3.7}$$

and for each $\epsilon > 0$, there exists K so that

$$\sum_{j=K}^{i} \Lambda(W_n, j)(2^{-j})^{-d-\alpha}|W_n^*(j)| < \epsilon \quad \forall (n,i) \in \mathcal{A}. \tag{3.8}$$

Then $P(A(W)) = 0$.

Condition (3.8) measures the thinness of W in the manner of (3.2). Condition (3.7) is introduced for technical reasons; it says that the probability of the process landing in $W_n^*(i)$ upon leaving $W_n(i+1)$ is equivalent to that of the process jumping directly from the ball $B(0, 2^{-n})$ to $W_n^*(i)$. It would be desirable to remove (3.7) or to replace it by a geometric condition.

The reason for expanding W to W_n is to surround the path $X(t)$, $t > s(\omega)$, when the initial position $X(s(\omega))$ can only be located to within a ball of radius 2^{-n}. For sets with certain geometric characteristics, e.g. thorns or those in Examples 1, 2 or 3, the enlargement plays a minor role. However, when the set is scattered, W_n can be substantially larger that W. An assumption such as (3.2) does not guarantee the boundedness of $\sum_{j=i}^{n} \Lambda(W_n, j)(2^{-j})^{-\alpha-d}|W_n^*(j)|$; and the series $\sum_{j=n+1}^{\infty} \Lambda(W_n, j)(2^{-j})^{-\alpha-d}|W_n^*(j)|$ is always infinite. For this reason, the portion of W in $\{2^{-n} \leq |x| \leq 2^{-j}\}$ needs to be considered separately using (3.7).

Conditions (3.7) and (3.8) are used for all open sets, therefore they are complicated and the geometrical implications are less apparent. We now examine these conditions on sets having special characteristics.

(A) When volumes $|W_n^*(j)|$ change very regularly:

$$c^{-1} < |W_n^*(j)|/|W_n^*(j+1)| < c \quad \forall n, j \geq 1,$$

note from (3.3) and (3.4) that (3.8) is equivalent to

$$\sum_{j=K}^{i} \sup_{x \in W_n^*(j)} \mu^x(W_n^*(j-2),\ W_n(j-1)) < \epsilon \quad \forall (n,i).$$

(B) For open sets W whose complement $R^d \backslash W$ contains a sequence of uniformly fat rings going to 0, e.g.

$$\mathbb{R}^d \backslash W \supseteq \bigcup_{j=1}^{\infty} \left\{ \frac{3}{4} 2^{-j} < |x| < 2^{-j} \right\},$$

it follows from (3.5) and (3.6) that (3.8) is equivalent to

$$\sum_{j=K}^{i} \left(\sup_{x \in W_n^*(j)} E^x(\tau_{W_n(j-1)}) \right) (2^{-j})^{-\alpha-d} |W_n^*(j)| < \epsilon \quad \forall (n,i).$$

(C) For thorns $W(f)$, $I(f) < \infty$ implies (3.7) and (3.8). Consider only pairs (n,i) satisfying

$$f(2^{-i})/2 \leq 2^{-n} < f(2^{-i}). \tag{3.9}$$

Lemma 4.5 in [BK] yields (3.7). Kulczycki has shown to us that for all thorns, with no assumption on $I(f)$,

$$\mu^x(W^*(j-2),\ W(j-1)) \lesssim E^x(\tau_{W(j-1)})(2^{-j})^{-\alpha-d}|W^*(j-2)| \quad \forall x \in W^*(j).$$

Since $E^x(\tau_{W(j-1)}) \lesssim f(2^{-j+1})^\alpha$ and $|W^*(j-2)| \lesssim f(2^{-j+2})^{d-1}2^{-j}$, we obtain, from $I(f) < \infty$,

$$\sum_{j=1}^{\infty} \Lambda(W,j)(2^{-j})^{-\alpha-d}|W^*(j)| < \infty.$$

Since (3.9) implies $W_n \cap \{|x| \geq 2^{-i}\} \subseteq \{x : (x_2^2 + \cdots + x_n^2)^{1/2} < 3f(x_1)\}$, condition (3.8) holds for all such pairs (n, i).

4. Proof of Theorem 1.

We follow the proof of Theorem A(i) in [BK] and give details at two crucial points for general open sets. The key to the proof is (4.1); roughly it says that when W is thick at 0, in order to travel from $W \cap \{|x| < \epsilon\}(\epsilon > 0$ small) to $W \cap \{|x| > \frac{1}{8}\}$ without leaving W, at least half of the paths must pass W "section by section" without making extremely long jumps. The reasoning which leads to (4.1) for general sets, uses harmonic measure estimates for paths with prescribed jumps (2.7). For $\omega \in \Omega$, the starting time $s(\omega)$ for the path $X(t,\omega)$ to stay in $X(s(\omega)) + \overline{W}$ for a given period of time is chosen as a limit of a sequence; and the continuity (4.6) of X at $s(\omega)$ is essential. Details on the continuity are given for the sake of completeness, since W need not be locally connected at 0.

We assume as we may that $W \subseteq \{|x| < \frac{1}{4}\}$, and let $\{a_n\}$ be a sequence of integers with $a_1 = 2$ and $a_{n+1} > 5 + a_n$. Let

$$W[n] = W \cap \{|x| < 2^{-a_n}\}$$

and

$$W^*[n] = W \cap \{2^{-a_{n+1}} \leq |x| < 2^{-a_n}\}.$$

Note that $W = W[1],\ W[n] = W(a_n)$ and $W^*[n] \neq W^*(a_n)$. Let also $a_0 = 0$, $W^*[0] = \{\frac{1}{2} < |x| < 1\}$.

Define

$$F_1 = \{X_{\tau_{W[1]}} \in W^*[0]\}$$

and

$$F_{n+1} = \{X_{\tau_{W[n+1]}} \in W^*[n]\} \cap \theta^{-1}_{\tau_{W[n+1]}} F_n, \quad n \geq 1,$$

where θ is the shift operator. Note on the set $\{X_{\tau_{W[n+1]}} \in W^*[n]\}$, we have $\theta^{-1}_{\tau_{W[n+1]}}(\{X_{\tau_{W[n]}} \in W^*[n-1]\}) = \{X_{\tau_{W[n]}} \in W^*[n-1]\}$. So $F_{n+1} = \bigcap_{m=1}^{n+1} \{X_{\tau_{W[m]}} \in W^*[m-1]\}$.

Lemma 1. *Under the assumption (3.1), the sequence $\{a_n\}$ can be chosen so that*

$$P^x(F_n) \geq \frac{1}{2} P^x(F_1) \quad \forall n \in \mathbb{N}_+ \quad \textit{and} \quad x \in W[n]. \tag{4.1}$$

Proof. Let $H_n = F_1 \backslash F_n$. Inequality (4.1) follows from the following

$$P^x(H_n) \leq \frac{n}{n+1} P^x(F_n) \quad \forall n \in N_+ \text{ and } x \in W[n]. \tag{4.2}$$

Recall that $a_0 = 0$ and $a_1 = 2$, and that (4.2) holds trivially for $n = 1$. Suppose that a_n's have been selected and (4.2) has been verified for $n = 1, 2, \ldots, m$; we shall choose a_{m+1} and verify (4.2) for $m + 1$. Consider any $a_{m+1} > 5 + a_m$ and $x \in W[m+1]$. Then

$$\begin{aligned} P^x(F_{m+1}) &= \sum_{k=a_m}^{-1+a_{m+1}} E^x(X_{\tau_{W[m+1]}} \in W^*(k); P^{X_{\tau_{W[m+1]}}}(F_m)) \\ &\geq \sum_{k=3+a_m}^{-2+a_{m+1}} E^x(X_{\tau_{W[m+1]}} \in W^*(k); \frac{1}{2} P^{X_{\tau_{W[m+1]}}}(F_1)). \end{aligned}$$

Note from (2.4) that

$$P^x(F_{m+1}) \gtrsim \sum_{k=3+a_m}^{-2+a_{m+1}} \int_{W[m+1]} \int_{W^*(k)} \frac{G_{W[m+1]}(x,y)}{|y-z|^{d+\alpha}} P^z(F_1)\,dz\,dy.$$

Since $\mathrm{dist}(z, W^*[0]) \cong 1$ and $|y-z| \cong |z|$ for $z \in W^*(k)$ and $y \in W[m+1]$, and $P^z(F_1) = \mu^z(W^*[0], W) \cong E^z(\tau_W)$ by (2.7), we have

$$P^x(F_{m+1}) \gtrsim E^x(\tau_{W[m+1]}) \sum_{k=3+a_m}^{-2+a_{m+1}} \int_{W^*(k)} \frac{E^z(\tau_W)}{|z|^{d+\alpha}}\,dz. \tag{4.3}$$

On the other hand, it follows from (2.6) and the induction hypothesis that for any $x \in W[m+1]$,

(4.4)
$$\begin{aligned} P^x(H_{m+1}) &= P^x(F_1, X_{\tau_{W[m+1]}} \in W^*[m], (\theta^{-1}_{\tau_{W[m+1]}} F_m)^c) + P^x(F_1, X_{\tau_{W[m+1]}} \notin W^*[m]) \\ &\le \frac{m}{m+1} P^x(F_{m+1}) + c(2^{-a_m})^{-d-\alpha} E^x(\tau_{W[m+1]}). \end{aligned}$$

The argument is adopted from (3.4) and (3.5) in [BK], where only the boundedness of the thorn is used in the proof. From (4.3), (4.4) and the assumption (3.1), it follows that if a_{m+1} is large enough then

$$P^x(H_{m+1}) \le \frac{m+1}{m+2} P^x(F_{m+1}) \quad \forall x \in W[m+1].$$

This completes the proof of Lemma 1.

Fix $\{a_n\}_0^\infty$ as in Lemma 1, and choose a point y_n in each $W^*[n]$. As in [BK], define for $1 \le k \le n$,

$$S_k^n = \inf\{t \ge 0 : X(t) \notin X(0) - y_n + W[n-k+1]\}.$$

Then $S_1^n \le S_2^n \le \cdots \le S_n^n$. Let R_n be the first S_k^n such that $X(S_k^n) \notin X(0) - y_n + W^*[n-k]$ if it exists; otherwise let $R_n = \inf\{t \ge 0 : X(t) \notin X(0) - y_n + W\}$.

Following the argument of Lemma 3.3 in [BK], and using the Markov property, (2.6) and Lemma 1 above (in place of Lemma 3.2 in [BK]), we obtain

$$E(R_n) \cong E^{y_n}(\tau_W) \lesssim c(W, t_0)\ P(R_n \geq t_0). \tag{4.5}$$

Define for $n \geq 1$, a sequence of stopping times as follows: $T(0, n) = 0$,

$$T(j+1, n) = \begin{cases} T(j, n) + (R_n \wedge t_0) \circ \theta_{T(j,n)}, & \text{if} \quad T(j, n) < t_0, \\ T(j, n) & \text{if} \quad T(j, n) \geq t_0; \end{cases}$$

define also

$$F(j, n) = \{\omega \in \Omega : T(j+1, n) - T(j, n) = t_0\},$$

and

$$H_n = \bigcup_{j=0}^{\infty} F(j, n).$$

Lemma 2. *There exists a positive constant* $c(W, t_0)$ *so that*

$$P(H_n) \geq c(W, t_0) \quad \forall n \geq 1.$$

Proof. Unlike the situation in [BK], condition (3.1) does not imply $E(R_n) \to 0$ as $n \to \infty$. For each $n \geq 1$, we consider two possibilities: $E(R_n) < t_0/10$ or $E(R_n) \geq t_0/10$. In the first case, choose an integer m_n such that $t_0/4 \leq m_n E(R_n) \leq t_0/2$, and then proceed as in [BK]. When $E(R_n) \geq t_0/10$, we note that from (4.5) that

$$P(H_n) \geq P(F(0, n)) = P(T(1, n) = t_0) = P(R_n \geq t_0) \geq c(d, \alpha, W, t_0) E(R_n) \geq c'(d, \alpha, W, t_0).$$

Let

$$H = \limsup_{n \to \infty} H_n$$

and

$$A^0 = \{\omega \in \Omega : \exists s = s(\omega) \in [0, t_0) \text{ such that } X(t, \omega) \in X(s, \omega) + \overline{W} \text{ for all } t \in [s, s + t_0)\}.$$

In view of Lemma 2 and the fact that H_n's are independent, to prove the theorem, it is sufficient to check $H \subseteq A^0$.

Assume that $\omega \in H$. Then there exist sequences $\{j_k\}$ and $\{n_k\}$ (depending on ω) so that $n_k \uparrow \infty$, $\omega \in F(j_k, n_k)$, and $s_k \equiv T(j_k, n_k)$ converges to some $s \in [0, t_0]$. The crucial step in proving $\omega \in A^0$ is to verify the continuity of X at s:

$$\lim_{t \to s} X(t) = X(s). \tag{4.6}$$

After that, $\omega \in A^0$ follows easily.

To this end, we may assume that $\{s_k\}$ is monotone and consider only the case when $\{s_k\}$ is strictly increasing; the decreasing case is analogous and simpler. Since X is right continuous and has left limits, both $X(s) = \lim_{t \downarrow s} X(t)$ and $X(s-) = \lim_{t \uparrow s} X(t)$ exist.

Assume that $X(s) \neq X(s-)$, and choose m so that

$$2^{-a_m} < |X(s) - X(s-)|/8.$$

Choose $\delta \in (0, t_0/2)$ so that

$$|X(t) - X(s-)| < 2^{-a_{m+1}-3} \quad \forall t \in (s - \delta, s);$$

and choose k_0 so that if $k > k_0$ then $s_k \in (s - \delta, s)$, thus

$$|X(s_k) - X(s-)| < 2^{-a_{m+1}-3}.$$

Fix an integer $k > k_0$, with $n_k > m+2$. Since $\omega \in F(j_k, n_k)$, it follows that for $t \in [s_k, s_k + t_0)$,

$$X(t) \in X(s_k) - y_{n_k} + W,$$

and that if $X(t)$ leaves $X(s_k) - y_{n_k} + W[p]$ $(1 \leq p \leq n_k)$, then it goes to $X(s_k) - y_{n_k} + W^*[p-1]$.

Consider $t \in [s_k, s)$, then t is in $(s-\delta, s) \cap [s_k, s_k + t_0)$; therefore

$$|X(t) - X(s_k)| \leq 2^{-a_{m+1}-2}$$

and

$$X(t) \in X(s_k) - y_{n_k} + W.$$

Hence

$$X(t) \in X(s_k) - y_{n_k} + W[m+1] \quad \forall t \in [s_k, s),$$

which implies that

$$X(s) \in X(s_k) - y_{n_k} + W[m].$$

Consequently,

$$\begin{aligned} |X(s) - X(s-)| &\leq |X(s) - X(s_k)| + |X(s_k) - X(s-)| \\ &\leq 2^{-a_m+1} < |X(s) - X(s-)|/2, \end{aligned}$$

which is impossible. Therefore $X(s) = X(s-)$ and the continuity (4.6) follows. This completes the proof of Theorem 1.

5. Proof of Theorem 2.

Again we follow the structure of the proof of Theorem A(ii) in [BK]. The key is Lemma 3; very roughly, it says that when W is thin at 0, the probability of the process starting in $W \cap \{|x| < \epsilon\}(\epsilon > 0$ small), making at least m "forward landings" in $W \cap \{\epsilon \leq |x| \leq \frac{1}{8}\}$ before leaving W, goes down geometrically with respect to m. Methods of estimating harmonic measures for thorns do not apply; we use (2.7) repeatedly. Because W does not point in any specific direction, we need to put a band around W to contain paths with small shifts.

Given $i_0 > 1$ and $X(0) = x \in W(i_0)$. Define a sequence of stopping times $S(m)$ as follows. Let $S(0) = 0$ and

$$S(m+1) = \begin{cases} \tau_{W(i_m - 1)} & \text{if} \quad i_m > 1, \\ S(m) & \text{if} \quad i_m = 0, \end{cases}$$

where $i_m (m \geq 1)$ is the integer > 1 such that $X(S(m)) \in W^*(i_m)$ if it exists, and $i_m = 0$ otherwise. While $i_m (m \geq 1)$ is uniquely determined by induction, the choice of i_0 is not; the specific value of i_0 is important in defining $\{S(m)\}$. Note that $i_{m+1} < i_m - 1$, $0 < S(1) < S(2) < \cdots < S(m)$, and that $\{i_1, i_2, \cdots, i_m\}$ records the forward landings according to the rules given.

For $i < k, m \geq 1$ and $x \in W(i)$, define

$$H(k, i, m, x, W) = \{\omega \in \Omega : i_0 = i, X(0) = x, S(m-1) < S(m), X(S(m)) \in W^*(k)\}$$

to be the collection of paths that start at x, with $i_0 = i$, and end in $W^*(k)$ at time $S(m)$.

Lemma 3. *There exists $C_0 > 0$ so that for $m \geq 1$, $i > k > K$ and $x \in W(i)$, if*

$$\sum_{j=K}^{i-2} \Lambda(W, j)(2^{-j})^{-d-\alpha} |W^*(j)| < C_0^{-1}, \tag{5.1}$$

then

$$P^x(H(k,i,m,x,W)) \le C_0 2^{-m} \lambda^x(W,i)(2^{-k})^{-d-\alpha}|W^*(k)|. \tag{5.2}$$

Proof. We write

$$P^x(H(k,i,m,x,W)) = P^x(S(m-1) < S(m), X(S(m)) \in W^*(k)).$$

In the case $i = k+1$, $X(S(1)) \in W(k)^c$; and (5.2) holds trivially.

Assume from now on, $i \ge k+2$ and $|W^*(k)| > 0$. We shall prove (5.2) by induction on m.

When $m = 1$ and $i = k+2$, note from (3.3) that

$$\begin{aligned}
&P^x(S(0) < S(1), X(S(1)) \in W^*(k)) = \mu^x(W^*(i-2), W(i-1)) \\
&= 2^{1+2(d+\alpha)} 2^{-1} \mu^x(W^*(i-2), W(i-1)) 2^{-i(d+\alpha)} |W^*(i-2)|^{-1} 2^{k(d+\alpha)} |W^*(k)| \\
&= 2^{1+2(d+\alpha)} 2^{-1} \lambda^x(W,i) 2^{k(d+\alpha)} |W^*(k)|.
\end{aligned}$$

When $m = 1$ and $i > k+2$, in view of (2.7),

$$\begin{aligned}
&P^x(S(0) < S(1), X(S(1)) \in W^*(k)) \\
&= \mu^x(W^*(k), W(i-1)) \cong E^x(\tau_{W(i-1)})(2^{-k})^{-d-\alpha}|W^*(k)|.
\end{aligned}$$

Since $E^x(\tau_{W(i-1)}) \lesssim \lambda^x(W,i)$,

$$\begin{aligned}
&P^x(S(0) < S(1), X(S(1)) \in W^*(k)) \\
&\le C_5 2^{-1} \lambda^x(W,i)(2^{-k})^{-d-\alpha}|W^*(k)|
\end{aligned}$$

for some $C_5 > 0$. Let

$$C_0 = \max\{2^{1+2(d+\alpha)}, C_5\}$$

then (5.2) holds for $m = 1$.

Assume that (5.2) has been proved for some $m \geq 1$ and all $i > k > K$ and $x \in W(i)$. Given $i \geq k+2$ and $x \in W(i)$, we have

$$\begin{aligned}
&P^x(S(m) < S(m+1), X(S(m+1)) \in W^*(k)) \\
&= \sum_{j=k+2}^{i-2} E^x(S(m-1) < S(m), X(S(m)) \in W^*(j),\ P^{X(S(m))}(X_{\tau_{W(j-1)}} \in W^*(k)).
\end{aligned}$$

(Note that when $j = k+1$ or $i-1$, the events are void.)

$$\begin{aligned}
&\leq \sum_{j=k+2}^{i-2} P^x(S(m-1) < S(m), X(S(m)) \in W^*(j)) \sup_{y \in W^*(j)} P^y(X_{\tau_{W(j-1)}} \in W^*(k)) \\
&= \sum_{j=k+2}^{i-2} P^x(H(j,i,m,x,W)) \sup_{y \in W^*(j)} P^y(H(k,j,1,y,W)).
\end{aligned}$$

The induction hypothesis yields that

$$\begin{aligned}
&P^x(S(m) < S(m+1), X(S(m+1)) \in W^*(k)) \\
&\leq \sum_{j=k+2}^{i-2} C_0 2^{-m} \lambda^x(W,i) 2^{j(d+\alpha)} |W^*(j)| C_0 2^{-1} \Lambda(W,j) 2^{k(d+\alpha)} |W^*(k)| \\
&\leq C_0^2 2^{-m-1} \lambda^x(W,i) 2^{k(d+\alpha)} |W^*(k)| \sum_{j=K}^{i-2} \Lambda(W,j) 2^{j(d+\alpha)} |W^*(j)| \\
&= C_0 2^{-m-1} \lambda^x(W,i) 2^{k(d+\alpha)} |W^*(k)|.
\end{aligned}$$

Now (5.2) has been proved for all $m \geq 1$.

For each $n > 0$, we define a sequence of stopping times $\{T(j,n)\}$ modelled on those in [BK] by letting $T(0,n) = 0$ and

$$T(j+1,n) = \inf\{s > T(j,n) : X(s) \notin B(X(T(j,n)), 2^{-n})\} \quad \text{for} \quad j \geq 0.$$

Since $\{\tau_{B(0,2^{-n})} \circ \theta_{T(j,n)}\}$ are independent and identically distributed, the proof of Lemma 4.7 in [BK] yields

$$(5.3) \qquad \sum_{j=0}^{\infty} P(T(j,n) \leq N) \leq c(d,\alpha) N / E(\tau_{B(0,2^{-n})}) \cong N 2^{n\alpha},$$

which in turn implies that $P(\{\lim_{j\to\infty} T(j,n) < \infty\}) = 0$.

We assume as we may that all sample paths $t \to X(t,\omega)$, are right continuous with left limits; that for all $n > 0$,

$$\lim_{j\to\infty} T(j,n) = \infty;$$

and that ω does not belong to the following set

$$\Omega_1 = \{\omega \in \Omega : \exists s = s(\omega) \geq 0,\ \exists a = a(\omega) > 0 \ni X(t,\omega) = X(s,\omega) \quad \forall t \in [s, s+a)\}.$$

Let $Q(s,n) = \inf\{t > s, X(t) \notin B(X(s), 2^{-n})\}$. Then for all s, $Q(s,n,\omega) > s$, $\lim_{n\to\infty} Q(s,n,\omega) = s$ and $\lim_{n\to\infty} X(Q(s,n,\omega)) = X(s,\omega)$ by the right continuity of the process. For $a > 0$, let

$$\begin{aligned} Z(s,a,\omega) =& \{\ell \geq 1 : \exists q \geq 1 \text{ such that } Q(s,q,\omega) \in (s, s+a) \quad \text{and} \\ & X(Q(s,q,\omega)) \in B(X(s,\omega), 2^{-\ell}) \backslash B(X(s,\omega), 2^{-\ell-1})\}, \end{aligned}$$

which represents another way to record forward landings. Since $\omega \notin \Omega_1$, $Z(s,a,\omega)$ is an infinite set. For integers $i > k$, let

$$Z(s,a,k,i,\omega) = Z(s,a,\omega) \cap [k,i].$$

For $\Gamma \subseteq [0,\infty)$ and $k > 0$, let

$$\begin{aligned} A(\Gamma,k) = \{\omega \in \Omega : \exists s = s(\omega) \in \Gamma \text{ and } a = a(\omega) > 0 \text{ such that } X(t,\omega) \in X(s,\omega) + \overline{W} \quad \forall t \in [s, s+a), \\ \text{and} \sup_{t\in[s,s+a)} |X(t,\omega) - X(s,\omega)| \in [2^{-k-1}, 2^{-k})\}. \end{aligned}$$

To show $P(A(W)) = 0$, it suffices to prove

$$P(A([0,N],k)) = 0 \quad \forall N, k > 0. \tag{5.4}$$

Fix N and k from now on. For $m \geq 1$ and $i > k$, let

$$A(\Gamma, k, i, m) = \{\omega \in A(\Gamma, k) : \#Z(s(\omega), a(\omega), k, i) \geq m\}.$$

Because $\#Z(s, a, \omega) = \infty$,

$$A([0, N], k) = \bigcup_{i=k+1}^{\infty} A([0, N], k, i, m)$$

for all $m \geq 1$. Since $A([0, N], k, i, m)$ increases as i increases, in order to prove (5.4) it suffices to show that

$$P(A([0, N], k, i, 6m)) \leq c(k)N2^{-m} \tag{5.5}$$

for all $m \geq 1$ and all pairs $(n, i) \in \mathcal{A}$ with $i > k > K$ for some $K > 0$.

Fix $(n, i) \in \mathcal{A}$ with $i > k$, then

$$P(A([0, N], k, i, 6m)) = \bigcup_{j=0}^{\infty} P(A([0, N] \cap [T(j, n), T(j + 1, n)], k, i, 6m). \tag{5.6}$$

Suppose

$$\omega \in A([0, N] \cap [T(j, n), T(j + 1, n)], k, i, 6m), \tag{5.7}$$

then

(a) $T(j, n) \leq N$;

(b) there exist $s = s(\omega) \in [T(j, n), T(j + 1, n))$, and $a = a(\omega) > 0$ such that $X(t, \omega) \in X(s) + \overline{W(k)}$ for all $t \in [s, s + a)$;

(c) $\sup\{|X(t) - X(s)| : s \leq t < s + a\} \in [2^{-k-1}, 2^{-k})$;

and

(d) $\#Z(s(\omega), a(\omega), k, i) \geq 6m$.

Since $|X(s) - X(T(j,n))| < 2^{-n}$, inequalities $2^{-j-1} < |x - X(s)| < 2^{-j}$ $(j \leq n-2)$ imply $2^{-j-2} < |x - T(j,n)| < 2^{-j+1}$. We shift the reference point from $X(s)$ to $X(T(j,n))$, then the path of ω is contained in the enlarged set $\overline{W}_n$ with respect to $X(T(j,n))$. Consequently,

(b') $X(t) \in B(X(T(j,n)), 2^{-n}) + \overline{W(k)} \subseteq X(T(j,n)) + \overline{W_n(k)}$, for all $t \in [T(j,n), s+a)$;

(c') $\sup\{|X(t) - X(T(j,n))| : T(j,n) \leq t < s+a\} \in [2^{-k-2}, 2^{-k+1})$;

and

(d') $\#Z(T(j,n), s(\omega) + a(\omega) - T(j,n), k, i) \geq 2m$.

The decrease from $6m$ in (d) to $2m$ in (d') is due to the shift from $X(s)$ to $X(T(j,n))$. Therefore it follows from (a), (b'), (c') and (d') that

$$\omega \in \{T(j,n) \leq N\} \cap \theta^{-1}_{T(j,n)} \Bigg(\bigcup_{m'=m}^{\infty} \bigcup_{k'=k-1}^{k+1} H(k', i+2, m', 0, W_n) \Bigg). \tag{5.8}$$

The reason for the decrease from $2m$ in (d) to m in (5.8) is the following. In defining $S(m)$, the set $\{i_0, i_1, \ldots, i_m\}$ that records the forward landings, does not contain consecutive integers; on the other hand, $Z(T(j,n,\omega), s(\omega) + a(\omega) - T(j,n,\omega), k, i)$ may contain blocks of consecutive integers. The change from i in (d') to $i+2$ in (5.8) is for the convenience when quoting Lemma 2, the change is insignificant because m is large. From (5.6) $\sim$ (5.8) and the strong Markov property, it follows that

$$P(A([0,N], k, i, 6m)) \leq \sum_{j=0}^{\infty} P(T(j,n) \leq N) \Bigg(\sum_{m'=m}^{\infty} \sum_{k'=k-1}^{k+1} P^0(H(k', i+2, m', 0, W_n)) \Bigg).$$

Apply Lemma 3 to W_n and use (3.8) in place of (5.1), we obtain for $k > K$ (some $K > 0$),

$$P(H(k', i+2, m', 0, W_n)) \leq C_0 2^{-m'} \lambda^0(W_n, i+2)(2^{-k'})^{-d-\alpha} |W_n^*(k')|.$$

It has been stated in (5.3) that $\sum_{j=0}^{\infty} P(T(j,n) \le N) \lesssim N2^{n\alpha}$. Therefore for $k > K$,

$$P(A([0,N],k,i,6m)) \le c(k)N2^{n\alpha}2^{-m}\lambda^0(W_n, i+2).$$

Recall from (3.3) that

$$\lambda^0(W_n, i+2) = \mu^0(W_n^*(i), W(i+1))2^{-i(d+\alpha)}|W_n^*(i)|^{-1}.$$

Finally, condition (3.7) and harmonic measure estimate (2.7) yield

$$\lambda^0(W_n, i+2) \cong \mu^0(W_n^*(i), B(0,2^{-n})2^{-i(d+\alpha)}|W^*(i)|^{-1}) \cong E^0(\tau_{B(0,2^{-n})}) \cong 2^{-n\alpha}.$$

Finally $P(A([0,N],k,i,6m)) \le c(k)N2^{-m}$ for $k > K$, which is (5.5). This proves $P(A(W)) = 0$.

6. On Examples.

First we verify Example 2. The following lemma on expected life time should be known.

Lemma 4. *Let* $S = (0,1) \times (-\infty,\infty)^{d-1}$. *Then* $\sup_{x \in S} E^x(\tau_S) < \infty$.

Proof. Let $T = (-1,1) \times (-\infty,\infty)^{d-1}$. Then

$$a \equiv \sup_{x \in S} P^x(X(t) \in S \quad \forall\ 0 \le t \le 1) \le P^0(X(t) \in T \quad \forall\ 0 \le t \le 1) < 1,$$

and $P^x(X(t) \in S \quad \forall\ 0 \le t \le N) \le a^N$($N$ positive integer) for all $x \in S$. From this, it follows that $E^x(\tau_S) \le (1-a)^{-2}$ for all $x \in S$.

Lemma 5. *Let* $0 < \delta < 1$, m *an integer in* $[1,d]$ *and* $Q = (0,\delta)^m \times (0,1)^{d-m}$. *Then for any* $x \in (\frac{\delta}{4}, \frac{3\delta}{4})^m \times (\frac{1}{4}, \frac{3}{4})^{d-m}$,

$$E^x(\tau_Q) \cong \sup_{x \in Q} E^x(\tau_Q) \cong \delta^\alpha.$$

Proof. Let $T_m = (-1,1)^m \times (-\infty,\infty)^{d-m}$; note from Lemma 4 and the monotonicity that $C_6 \equiv \max_{1\le m\le d} \sup_{x\in T_m} E^x(\tau_{T_m})$ is finite. Again by monotonicity and scaling that $\sup_{x\in Q} E^x(\tau_Q) \lesssim C_6\delta^\alpha$. The fact that $E^x(\tau_Q) \gtrsim \delta^\alpha$ for all $x \in (\frac{\delta}{4}, \frac{3\delta}{4})^m \times (\frac{1}{4}, \frac{3}{4})^{d-m}$ follows from (2.3). This completes the proof.

To check Example 2, we note from Lemma 5 and scaling that

$$\sup_{x\in W(i)} E^x(\tau_{W(i)}) \gtrsim \delta_i^\alpha 2^{-i\alpha}.$$

Therefore $\int_W E^x(\tau_W)|x|^{-d-\alpha}dx \gtrsim \sum \delta_i^{\alpha+m(i)}$; the assertion (i) in Example 2 follows from Theorem 1.

Assume that $\delta_i \neq 0$ for infinitely many i's; otherwise (ii) is trivial. Consider only pairs (n,i) satisfying $\delta_i > 0$ and $\delta_i 2^{-i-1} \le 2^{-n} < \delta_i 2^{-i}$. We claim that

$$E^x(\tau_{W_n(i)}) \lesssim 2^{-n\alpha} \quad \forall x \in W_n(i).$$

Since $E^x(\tau_{W_n(i)})$ is continuous in $W_n(i)$ and goes to 0 as x approaches $\partial W_n(i)$, $\sup\{E^x(\tau_{W_n(i)}) : x \in W_n(i)\}$ is attained at some point $z \in W_n(i)$. Assume that $z \in W_n^*(j)$ for some $j \in [i,n]$. Then

$$\begin{aligned} E^z(\tau_{W_n(i)}) &= E^z(\tau_{W_n^*(j)}) + \int_{W_n(i)\setminus W_n^*(j)} E^y(\tau_{W_n(i)})d\mu^z(y, W_n^*(j)) \\ &\le E^z(\tau_{W_n^*(j)}) + E^z(\tau_{W_n(i)})\mu^z(W_n(i)\setminus W_n^*(j), W_n^*(j)). \end{aligned}$$

Note from the definition of W that $W_n(i)^c$ contains some ball of diameter 2^{-j-1} within a distance 2^{-j+1} from $W_n^*(j)$. Calculations using (2.7) and the monotonicity yield

$$\mu^z(W_n(i)^c, W_n^*(j)) > C_7 > 0;$$

and by Lemma 5,

$$E^z(\tau_{W_n(i)}) \le C_7^{-1} E^z(\tau_{W_n^*(j)}) \lesssim (\delta_j 2^{-j})^\alpha \lesssim 2^{-n\alpha}.$$

This proves the claim.

From the harmonic measure estimate (2.7) and the claim it follows that

$$\begin{aligned}\mu^0(W_n^*(i), W_n(i+1)) &\cong E^0(\tau_{W_n(i+1)})(2^{-i})^{-\alpha-d}|W_n^*(i)| \\ &\lesssim 2^{-n\alpha}(2^{-i})^{-\alpha-d}|W_n^*(i)| \cong \mu^0(W_n^*(i), B(0,2^{-n})).\end{aligned}$$

This proves (3.7) in Theorem 2.

Note from (1.1), (3.6) and Lemma 5 that for $x \in W_n(j)$ and $j \ge i$,

$$\lambda^x(W_n, j) \cong E^x(\tau_{W_n(j-1)}) \lesssim (\delta_j^\alpha + \delta_{j-1}^\alpha)2^{-j\alpha},$$

(the sum $\delta_j^\alpha + \delta_{j-1}^\alpha$ is needed since δ_{j-1} may be zero); and that

$$\sum_{j=1}^{i} \Lambda(W_n, j)(2^{-j})^{-\alpha-d}|W_n^*(j)| \lesssim \sum_{j=1}^{i} \delta_j^{\alpha+m(j)}.$$

This proves (3.8) in Theorem 2 and thus assertion (ii) in Example 2.

Remark. *In Example 2, the requirement in keeping Q_j's uniformly apart is for the convenience of the proof. The conclusions remain if Q_j's are allowed to stay in $\{2^{-j-1} < |x| < 2^{-j}\}$, or are replaced by bilipschitz images of Q_j's with uniformly bounded bilipschitz constants.*

Example 1 is a variation of Example 2 in the case $m(j) = 1$ for all j. It is especially interesting to note that $P(A(W)) = 1$ as long as $\limsup \delta_j > 0$; in particular, W can be very lacunary.

In Example 3, the set is scattered, and we need some harmonic measure estimates. For $x \in \mathbb{R}^d$, let

$$\|x\| = \max\{|x_j| : 1 \le j \le d\}.$$

Lemma 6. *Let* $0 < \epsilon < \frac{1}{10}$, $r > 0$, $\mathcal{L}$ *be the set of lattice points in* $\mathbb{R}^d$, $W = \bigcup_{x \in \mathcal{L}} B(x, \epsilon)$ *and* $W^r = W \cap \{\|x\| < r + \frac{1}{4}\}$. *Then*

$$\mu^{x_0}(W \backslash B(x_0, \epsilon), B(x_0, \epsilon)) \cong \epsilon^{\alpha+d} \quad \forall x_0 \in \mathcal{L}; \tag{6.1}$$

suppose $\epsilon^{\alpha+d} < N^{-\alpha}$ *and* $N > 10$, *then*

$$\mu^x(W \backslash W^N, W^N) \lesssim \epsilon^{\alpha+d} N^{-\alpha} \quad \forall x \in W^{N/2}, \tag{6.2}$$

$$\mu^{x_0}(W \backslash W^N, W^N) \cong \mu^{x_0}(W^{2N} \backslash W^N, B(x_0, \epsilon)) \cong \epsilon^{\alpha+d} N^{-\alpha} \quad \forall x_0 \in \mathcal{L} \text{ with } \|x\| \le \frac{N}{2}, \tag{6.3}$$

and there exists $C_8 > 0$ *so that if* $0 < \epsilon < C_8$ *then*

$$E^x(\tau_W) \lesssim \epsilon^\alpha \quad \forall x \in W. \tag{6.4}$$

Proof. It follows from (2.1) that

$$\mu^0(W \backslash B(0, \epsilon), B(0, \epsilon)) \cong E^0(\tau_{B(0,\epsilon)}) \int_1^\infty t^{-d-\alpha} \epsilon^d t^{d-1} dt \cong \epsilon^{\alpha+d},$$

and (6.1) follows by translation.

Monotonicity and calculation as above yield that if $x \in B(x_0, \epsilon) \subseteq W^N$ then

$$\begin{aligned} \mu^x(W \backslash W^N, W^N) &\le \mu^x(W \backslash B(x_0, \epsilon), B(x_0, \epsilon)) \\ &\le \mu^x(W \backslash B(x_0, \epsilon), B(x, 2\epsilon)) \cong \epsilon^{\alpha+d}. \end{aligned} \tag{6.5}$$

If $x \in B(x_0, \epsilon) \subseteq W^{N/2}$, then (2.1), (2.2) (2.5) and monotonicity yield

$$\begin{aligned}
\mu^x(W\backslash W^N, B(x_0,\epsilon)) &\leq \mu^x(W\backslash W^N, B(x, 2\epsilon)) \\
&\cong E^x(\tau_{B(x,2\epsilon)}) \int_{N/2}^{\infty} t^{-d-\alpha}\epsilon^d t^{d-1} dt \qquad (6.6) \\
&\cong \epsilon^{\alpha+d} N^{-\alpha} \cong \mu^{x_0}(W^{2N}\backslash W^N, B(x_0,\epsilon)).
\end{aligned}$$

Now let $x \in B(x_0, \epsilon) \subseteq W^{N/2}$. Then from the Markov property, (6.5), (6.6) and the assumption $\epsilon^{\alpha+d} < N^{-\alpha}$, it follows that

$$\begin{aligned}
\mu^x(W\backslash W^N, W^N) = \mu^x(W\backslash W^N, B(x_0,\epsilon)) &+ \int_{W^N\backslash B(x_0,\epsilon)} \mu^y(W\backslash W^N, W^N) d\mu^x(y, B(x_0,\epsilon)) \\
&\lesssim \epsilon^{\alpha+d} N^{-\alpha} + \epsilon^{2(\alpha+d)} \lesssim \epsilon^{\alpha+d} N^{-\alpha}.
\end{aligned}$$

This gives (6.2).

The estimate in (6.3) follows from (6.2), (6.6) and the fact that $\mu^{x_0}(W\backslash W^N, W^N) \geq \mu^{x_0}(W\backslash W^N, B(x_0,\epsilon))$.

It is easy to see from the geometry of the set W that $\inf\limits_{x\in W} P^x(X(1) \in W^c) > 0$. Arguing as in Lemma 4 we obtain $\sup\limits_{x\in W} E^x(\tau_W) < \infty$. Since $E^x(\tau_W)$ is continuous in W and approaches 0 uniformly on ∂W, $\sup\limits_{x\in W} E^x(\tau_W)$ is attained in W. Since W is translation invariant we may choose $z \in B(0,\epsilon)$ so that $E^z(\tau_W) = \sup\limits_{x\in W} E^x(\tau_W)$. By Markov property, monotonicity and (6.5),

$$\begin{aligned}
E^z(\tau_W) &= E^z(\tau_{B(0,\epsilon)}) + \int_{W\backslash B(0,\epsilon)} E^x(\tau_W) d\mu^z(x, B(0,\epsilon)) \\
&\leq E^z(\tau_{B(0,\epsilon)}) + E^z(\tau_W)\mu^z(W\backslash B(0,\epsilon), B(0,\epsilon)) \\
&\leq E^z(\tau_{B(0,\epsilon)}) + C_9 E^z(\tau_W)\epsilon^{\alpha+d}.
\end{aligned}$$

Now if $\epsilon^{\alpha+d} < (2C_9)^{-1}$, then

$$E^z(\tau_W) \leq 2E^z(\tau_{B(0,\epsilon)}) \lesssim \epsilon^{\alpha},$$

which gives (6.4).

To verify Example 3, we apply Theorems 1 and 2 in the rectangular settings, i.e. in the definitions of $W(j)$, $W^*(j)$ and $W_n(j)$ and $W_n^*(j)$, we use $\|\cdot\|$ instead of $|\cdot|$, e.g. $W(j) = W \cap \{\|x\| < 2^{-j}\}$.

Assume $\Sigma\epsilon_k^{\alpha+d} = \infty$. Using (2.1) and (2.2), we obtain for $x \in \frac{1}{2}Q \in \mathcal{C}_k'$, $E^x(\tau_W) \gtrsim \epsilon_k^\alpha r_k^\alpha$ and

$$\int_{\bigcup_{\mathcal{C}_k'} Q} |x|^{-d-\alpha} dx \cong \int_{r_k}^{\epsilon_{k-1} r_{k-1}} t^{-d-\alpha} \epsilon_k^d t^{d-1} dt \cong \epsilon_k^d r_k^{-\alpha}.$$

Therefore $\int_W E^x(\tau_W)|x|^{-d-\alpha}dx \gtrsim \Sigma\epsilon_k^{d+\alpha} = \infty$; the conclusion $P(A(W)) = 1$ follows from Theorem 1.

Next we verify part (ii), and let $n(k)$ be the integer satisfying $2^{-n(k)} = \epsilon_k r_k$, $i(k) = n(k-1)$, and $m(k)$ be the smallest integer such that $2^{-m(k)-1} \le r_k - \epsilon_k r_k$; in other words, $\{\|x\| < 2^{-m(k)-1}\}$ is the largest cube of the form $\{\|x\| < 2^{-j}\}$ that does not meet $\cup\{x + Q_k : x \in Q \in \mathcal{C}_k'\}$. Note that $2^{-m(k)} \cong r_k$, $n(k) > m(k) > i(k)$ and that

$$\bigcup\{x + Q_k : x \in Q \in \mathcal{C}_k'\} \subseteq \{2^{-m(k)-1} < \|x\| < 2^{-i(k)}\}$$

and

$$W_{n(k)} \subseteq \{\|x\| < 2^{-n(k)}\} \bigcup \bigcup_{\ell=1}^{k} \{2^{-m(\ell)-1} < |x| < 2^{-i(\ell)}\}$$

for each $k \ge 1$.

We shall check (3.7) and (3.8) for pairs $(n(k),\ i(k))$, $k \ge 1$.

Note from the monotonicity, the assumption $\epsilon_k^{\alpha+d} < N_k^{-\alpha}$ and a scaled version of (6.3) that

$$\mu^0(W_{n(k)}^*(i(k)), W_{n(k)}(i(k)+1)) \cong \mu^0(W_{n(k)}^*(i(k)), Q_k) \cong \epsilon_k^{\alpha+d} N_k^{-\alpha}.$$

This gives (3.7).

To check (3.8), we fix $k \geq 1$; and for simplicity, we use $(n, i), W_n$ for $(n(k), i(k))$ and $W_{n(k)}$ and use $p(j)$ for $\max\{i : i \leq j-2 : W_n^*(i) \neq \phi\}$. We then proceed to estimate $\mu^x(W_n^*(p(j)), W_n(j-1))$ and $\Lambda(W_n, j)$ for $j \in \bigcup_{\ell=1}^{k} [i(\ell), m(\ell)]$ and $x \in W_n^*(j)$.

Let $\ell \in [1, k]$ and consider first $j \in [i(\ell)+2, m(\ell)]$; in this case $p(j) = j-2$, $|W_n^*(p(j))| \cong |W_n^*(j)|$ and there are $\mathcal{N}(k, \ell, j) \cong 2^{-jd} r_\ell^{-d}$ cubes in $\mathcal{C}_k'$ that meet $W_n(j-2)$. Therefore monotonicity and a scaled version of (6.3) imply that for $x \in W_n^*(j)$,

$$\begin{aligned}\mu^x(W_n^*(p(j)), W_n(j-1)) &= \mu^x(W_n^*(j-2), W_n(j-1))\\ &\cong \epsilon_\ell^{d+\alpha} (\mathcal{N}(k, \ell, j)^{\frac{1}{d}})^{-\alpha} \cong \epsilon_\ell^{d+\alpha} r_\ell^\alpha 2^{j\alpha}.\end{aligned}$$

Consequently, it follows from (3.3) and (3.4) that

$$\tag{6.7} \begin{aligned}&\sum_{j=i(\ell)+2}^{m(\ell)} \Lambda(W, j)(2^{-j})^{-d-\alpha} |W_n^*(j)|\\ \cong &\sum_{j=i(\ell)+2}^{m(\ell)} \epsilon_\ell^{d+\alpha} r_\ell^\alpha 2^{j\alpha} \cong \epsilon_\ell^{\alpha+d} r_\ell^\alpha 2^{m(\ell)\alpha} \cong \epsilon_\ell^{\alpha+d}.\end{aligned}$$

For $\ell \in [1, k]$ and $j = i(\ell)$ or $i(\ell)+1$, we have $p(j) = m(\ell-1)$ and $2^{-p(j)} \cong r_{\ell-1}$, and have $W(j-1) = W(i(\ell)) \subseteq \bigcup_{\mathcal{C}_\ell} Q$, $2^{-j} = \epsilon_{\ell-1} r_{\ell-1}$ and $|W_n^*(i(\ell))| \cong |W_n^*(i(\ell)+1)| \cong (\epsilon_{\ell-1} r_{\ell-1})^d \epsilon_\ell^d$. Because there is a thick ring separating $W_n(j-1)$, from $W_n(p(j))$, it follows from (3.6) that

$$\lambda^x(W_n, j) \cong E^x(\tau_{W_n(j-1)}) = E^x(\tau_{W_n(i(\ell))}) \quad \forall x \in W_n^*(j).$$

A scaled version of (6.4) shows that

$$E^x(\tau_{W_n(i(\ell))}) \lesssim \epsilon_\ell^\alpha r_\ell^\alpha \quad \forall x \in W_n^*(j).$$

Therefore when $j = i(\ell)$ or $i(\ell) + 1$,

$$\Lambda(W_n, j)(2^{-j})^{-d-\alpha}|W_n^*(j)| \lesssim \epsilon_\ell^\alpha r_\ell^\alpha \epsilon_{\ell-1}^{-d-\alpha} r_{\ell-1}^{-d-\alpha}(\epsilon_{\ell-1} r_{\ell-1})^d \epsilon_\ell^d \lesssim \epsilon_\ell^{\alpha+d}. \tag{6.8}$$

With $k \geq 1$ still fixed, we obtain from (6.7) and (6.8)

$$\begin{aligned} &\sum_{j=1}^{i(k)} \Lambda(W_{n(k)}(j))|W_n^*(j)|2^{j(d+\alpha)} \\ &\leq \sum_{\ell=1}^{k} \sum_{j=i(\ell)}^{m(\ell)} \Lambda(W_{n(k)}(j))|W_n^*(j)|2^{j(d+\alpha)} \lesssim \sum_{\ell=1}^{k} \epsilon_\ell^{\alpha+d}. \end{aligned}$$

Since $\sum\limits_{\ell=1}^{\infty} \epsilon_\ell^{d+\alpha} < \infty$, it is clear that there exists K so that the condition (3.8) is satisfied for all pairs $(n(k), i(k))$; assertion (ii) in Example 3 follows from Theorem 2.

Remark. *In part (ii) of Example 3,* $\epsilon_k^{\alpha+k} < N_k^{-\alpha}$ *is used to obtain (3.7) and* $\Sigma \epsilon_\ell^{d+\alpha} < \infty$ *is used to obtain (3.8).*

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