Logic Comprehensive Examination — Math 410
January 25, 2003

Do 5 of the 6 problems, **including Problem 6**. Indicate clearly which problems you intend to have graded. In doing part (b) of a problem, you may assume the results of part (a) (if part (a) is not a definition), whether or not you correctly worked part (a).

To receive credit, each of your solutions must be justified.

Notation and terminology: L denotes a countable first order language (with equality, as a logical symbol) and Σ denotes a set of sentences in L. For each Σ, $\mathrm{Th}(\Sigma)$ denotes the set of sentences σ in L such that there is a formal proof of σ from Σ in L (*i.e.*, $\Sigma \vdash_L \sigma$). The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote (respectively) the sets of all natural numbers (including 0), all integers (positive, negative, and zero), all rational numbers, and all real numbers.

Problem 1. Let L have just one nonlogical symbol, a binary relation symbol P. In each part below, determine whether or not there can exist an L-sentence σ with the stated property. If so, give an example and a proof that your example has the claimed property; if not, give a proof of the impossibility.

(a) For every nonempty set A, both σ and $\neg\sigma$ have models whose underlying set is A.

(b) For every nonempty set A, σ has a model whose underlying set is A if and only if A is finite with an even number of elements.

(c) For every nonempty set A, σ has a model whose underlying set is A if and only if A is infinite.

Problem 2. (a) Let φ be an L-formula and x a variable; define what it means for an occurrence of x in φ to be a *free* occurrence.

(b) Let $\varphi(x)$ be an L-formula and t an L-term; let $\varphi(t/x)$ denote the result of replacing every free occurrence of x in φ by t. You may assume without proof that $\varphi(t/x)$ is an L-formula. Give an explicit condition ensuring that the sentence $\forall x\varphi(x) \to \varphi(t/x)$ is valid in every L-structure. Indicate how the correctness of your condition would be proved; you need not give all the details of this argument. (Your condition should be sufficiently general to capture the cases that are needed for other arguments in first-order logic.)

Problem 3. (a) State the Completeness Theorem for L.

(b) State the Compactness Theorem for L.

(c) Explain how the Compactness Theorem follows from the Completeness Theorem.

Please turn over for Problems 4, 5 and 6.

Problem 4. Assume that Σ is finitely satisfiable. (Recall that a set Δ of sentences is *finitely satisfiable* if each finite subset of Δ has a model.) Indicate how to prove that there is a countable language L' containing L and a finitely satisfiable, complete set of L'-sentences Σ' such that

(i) $\Sigma \subseteq \Sigma'$, and

(ii) whenever $\varphi(x)$ is an L' formula such that $\Sigma' \models_{L'} \exists x \varphi(x)$ then there is a constant symbol c of L' such that $\Sigma' \models_{L'} \varphi(c/x)$.

Also indicate why the existence of Σ' is important.

Your argument should not use the Completeness Theorem nor the Compactness Theorem.

Problem 5. Let L have just one nonlogical symbol, a binary relation symbol P. Consider the L-structure $\mathcal{Z} = (\mathbb{Z}, E)$ where E is defined on $\mathbb{Z}$ by

$$E(m, n) \iff m - n \text{ is an integer multiple of } 3.$$

Give a decidable set Σ of L-sentences of which $\mathcal{Z}$ is a model, such that for any L-sentence τ

$$\Sigma \vdash_L \tau \iff \mathcal{Z} \models \tau.$$

Your proof that Σ is decidable can be informal, making use of Church's Thesis; however, the sentences in Σ should be given explicitly.

Remember that you must include a solution to the following problem:

Problem 6. Let L contain the constant symbol 0, the unary function symbol S, and the binary relation symbol $<$. Suppose Σ contains every L-sentence σ such that σ only contains the nonlogical symbols $0, S, <$ and σ is true in the structure $(\mathbb{N}, 0, S, <)$.

(i) Define what it means for a relation $R \subseteq \mathbb{N}^k$ to be *representable* in Σ.

(ii) Define what it means for a function $f \colon \mathbb{N}^k \to \mathbb{N}$ to be *representable* in Σ.

(iii) For each $A \subseteq \mathbb{N}$, show that A is representable in Σ if and only if the characteristic function of A is representable in Σ.

(iv) Assume $R, S \subseteq \mathbb{N}^2$ are representable in Σ, and that $A \subseteq \mathbb{N}$ satisfies

$$A = \{m \in \mathbb{N} \mid \text{for some } n \in \mathbb{N}, R(m, n)\} = \{m \in \mathbb{N} \mid \text{for all } n \in \mathbb{N}, S(m, n)\}.$$

Show that A is representable in Σ.