Logic Comprehensive Examination
Math 570 (formerly Math 410)
February 5, 2005

Do 5 of the 6 problems, **including Problem 6**. Indicate clearly which problem you have decided to omit. In doing a part of any problem, you may assume the results of any earlier part of the same problem, whether or not you correctly worked it.

Each problem is worth 20 points, for a total of 100 points. To receive credit, each of your solutions must be justified.

Notation and terminology: L denotes a countable first order language (with equality, as a logical symbol) and Σ denotes a set of sentences in L. For each Σ, $\mathrm{Th}(\Sigma)$ denotes the set of sentences σ in L such that there is a formal proof of σ from Σ in L (*i.e.*, $\Sigma \vdash_L \sigma$). The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote (respectively) the sets of all natural numbers (including 0), all integers (positive, negative, and zero), all rational numbers, and all real numbers. A set S is *countable* if there is a bijection between S and a subset of $\mathbb{N}$; in particular, all finite sets are countable.

Problem 1. Let t and u be syntactically distinct terms in L. Show that the formula $\neg t = u$ is satisfiable.

Problem 2. Determine whether or not the following sentence is valid, where R is a 3-ary predicate symbol:

$$\forall x \forall y \exists z\, R(x,y,z) \;\rightarrow\; \forall x \exists z \forall y\, R(x,y,z)$$

Problem 3. Let L' be obtained from L by adding a new unary predicate symbol P. Let σ be an L-sentence. Show that the following two conditions are equivalent:
(1) $\exists x \exists y\, (\neg x = y) \;\vdash_L\; \sigma$
(2) $\exists x \exists y\, (P(x) \wedge \neg P(y)) \;\vdash_{L'}\; \sigma$

Problem 4. Suppose the nonlogical symbols of L are the constant symbols c_n for $n \in \mathbb{N}$, and let Σ be the set of all L-sentences of the form $\neg c_m = c_n$ where $m, n \in \mathbb{N}$ are distinct. If $\mathcal{A}$ is an L-structure with underlying set A, recall that $S \subseteq A$ is definable in $\mathcal{A}$ if there is an L-formula $\varphi(x)$ for which $S = \{a \in A \mid \mathcal{A} \models \varphi[a]\}$.
(a) Show that if $\mathcal{A}$ is a model of Σ with underlying set A and $S \subseteq A$ is definable in $\mathcal{A}$, then S or $A \setminus S$ must be finite.
(b) Show that there exists a model $\mathcal{A}$ of Σ with underlying set A such that some finite subset of A is not definable in $\mathcal{A}$.

Problem 5. Suppose L has a finite number of nonlogical symbols. Let Σ be a consistent set of L-sentences such that $\mathrm{Th}(\Sigma)$ is decidable. Show that there exists a set Δ of L-sentences such that $\Sigma \subseteq \Delta$, Δ is maximal consistent, and Δ is decidable. (Your proof that Δ is decidable can be informal.)

Please turn over for Problem 6.

Remember that you must do the following problem.

Problem 6. (Each part is worth 5 points.) Let L contain the constant symbol 0 and the unary function symbol S.

(a) Define what it means for a relation $R \subseteq \mathbb{N}^k$ to be *representable* in Σ.

(b) Show that if $A, B \subseteq \mathbb{N}$ are representable in Σ, then so is the union $A \cup B$.

(c) Define what it means for a function $F\colon \mathbb{N}^k \to \mathbb{N}$ to be representable (*as a function*) in Σ.

(d) Let $R \subseteq \mathbb{N}$ be representable in Σ. Show that the characteristic function χ_R of R is representable in Σ as a function. ($\chi_R(n) = 1$ if $n \in R$ and $\chi_R(n) = 0$ if $n \notin R$, for all $n \in \mathbb{N}$.)