Logic Comprehensive Examination — Math 410
May 31, 2003

Do 5 of the 6 problems, **including Problem 6**. Indicate clearly which problems you intend to have graded. In doing part (b) of a problem, you may assume the results of part (a) (if part (a) is not a definition), whether or not you correctly worked part (a).

To receive credit, each of your solutions must be justified.

Notation and terminology: L denotes a countable first order language (with equality, as a logical symbol) and Σ denotes a set of sentences in L. For each Σ, $\mathrm{Cn}(\Sigma)$ denotes the set of sentences σ in L such that there is a formal proof of σ from Σ in L (*i.e.*, $\Sigma \vdash_L \sigma$). The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote (respectively) the sets of all natural numbers (including 0), all integers (positive, negative, and zero), all rational numbers, and all real numbers.

Problem 1. Suppose L contains binary relation symbols P and Q, and no other non-logical symbol. Let σ and τ be sentences in L. Assume that P does not occur in σ and that Q does not occur in τ; of course σ and τ can have atomic formulas such as $x = y$ in common. Determine whether the following statement *must* hold in every case:

If σ and τ each have an infinite model, then $\sigma \wedge \tau$ is satisfiable.

Problem 2. Let Γ be a set of sentences in L. Assume that for each L-structure $\mathcal{M}$, at least one sentence from Γ is true in $\mathcal{M}$. Show that there exist finitely many sentences $\sigma_1, \ldots, \sigma_n$ in Γ such that $\sigma_1 \vee \cdots \vee \sigma_n$ is logically valid.

Problem 3. Suppose L contains a unary function symbol f and no other non-logical symbol. Let $\mathcal{C}$ be the class of all L-structures $\mathcal{M}$ such that $f^{\mathcal{M}}$ is 1-1 and onto, and has no finite cycles. (A *finite cycle* for a function $F\colon A \to A$ is a sequence $a_0, \ldots, a_n$ of distinct elements of A such that $n \geq 0$, $F(a_i) = a_{i+1}$ when $0 \leq i < n$ and $F(a_n) = a_0$.)
(a) Describe the countable structures in $\mathcal{C}$ up to isomorphism.
(b) Show that any two structures in $\mathcal{C}$ whose underlying sets have the same uncountable cardinality are isomorphic.
(c) Show that any two structures in $\mathcal{C}$ are elementarily equivalent.

Problem 4. Recall that a set Δ of sentences is *finitely satisfiable* if each finite subset of Δ has a model. Let Σ be a finitely satisfiable set of sentences in L satisfying the following two conditions:
(i) Σ is maximal among finitely satisfiable sets of sentences in L;
(ii) whenever $\phi(x)$ is an L-formula such that $\exists x \phi(x)$ is in Σ, then there is a constant symbol c of L such that $\phi(c/x)$ is in Σ.
Give an explicit construction of an L-structure $\mathcal{M}$ that is a model of Σ. Be sure to give a careful proof that your structure is well defined and that it is a model of Σ. Do not use the Completeness Theorem or the Compactness Theorem in your solution to this problem.

Please turn over for Problems 5 and 6.

Problem 5. Let L contain the constant symbol 0 and the unary function symbol S, and suppose that L has only finitely many non-logical symbols. Let Σ be a finite set of L-sentences. Let A be a subset of $\mathbb{N}$ that is representable in Σ. Describe, without giving all details, a proof that A is computable.

Remember that you must do the following problem.

Problem 6. Let L contain the constant symbol 0 and the unary function symbol S.
(i) Define what it means for a relation $R \subseteq \mathbb{N}^k$ to be *representable* in Σ.
(ii) Show that if $A, B \subseteq \mathbb{N}$ are representable in Σ, then so are $A \cup B$ and $A \setminus B$.