Logic Comprehensive Examination
Math 570 (formerly Math 410)
May 25, 2005

There are five problems and you should do all of them. In doing a part of any problem, you may assume the results of any earlier part of the same problem, whether or not you correctly worked it.

Each problem is worth 20 points, for a total of 100 points. To receive credit, each of your solutions must be justified.

Notation and terminology: L denotes a countable first order language (with equality, as a logical symbol) and Σ denotes a set of L-sentences. For each Σ, $\mathrm{Th}(\Sigma)$ denotes the set of L-sentences σ such that there is a formal proof of σ from Σ in L (*i.e.*, $\Sigma \vdash_L \sigma$). The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote (respectively) the sets of all natural numbers (including 0), all integers (positive, negative, and zero), all rational numbers, and all real numbers. A set S is *countable* if there is a bijection between S and a subset of $\mathbb{N}$; in particular, all finite sets are countable.

Problem 1. Recall that an L-formula is *existential* if it has the form $\exists x_1 \dots \exists x_n \varphi$ where φ is quantifier free. Assume that every L-formula of the form $\forall x \psi$, in which ψ is quantifier free, is Σ-equivalent to an existential L-formula.

(a) (10 points) Show that every L-formula is Σ-equivalent to an existential L-formula.

(b) (10 points) Show that if $\mathcal{M}$ and $\mathcal{N}$ are models of Σ with $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} \preceq \mathcal{N}$; that is, for each $L(M)$-sentence σ, show that $\mathcal{M} \models \sigma$ iff $\mathcal{N} \models \sigma$.

Problem 2. Let L be the language of abelian groups; so L has binary function symbols $+$ and $-$ for addition and subtraction, respectively, and a constant symbol 0 for the identity element. Consider the additive abelian groups $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ as L-structures.

(a) (10 points) Show that the set $\{(a, a) \mid a \in \mathbb{Z}\}$ is not definable in $\mathbb{Z} \times \mathbb{Z}$ by any L-formula $\varphi(x)$.

(b) (10 points) Find an L-sentence that is true in $\mathbb{Z}$ but false in $\mathbb{Z} \times \mathbb{Z}$. (Hint: think modulo 2.)

Problem 3. Specify a first order language L and L-structures $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent, $\mathcal{M}$ has exactly one automorphism, and $\mathcal{N}$ has infinitely many automorphisms.

Please turn over for the remaining problems.

Problem 4. Let L be the first order language whose nonlogical symbols consist of two unary predicate symbols P, Q. Let Σ consist of the sentence $\forall x(Px \to Qx)$ and the infinite family of sentences

$$\exists x_1 \dots \exists x_n (\bigwedge_i Px_i \wedge \bigwedge_{i<j} x_i \neq x_j)$$

where n ranges over the integers ≥ 1.

(a) (8 points) Show that there does not exist any finite set Δ of L-sentences such that $\mathrm{Th}(\Delta) = \mathrm{Th}(\Sigma)$.

(b) (6 points) Classify the countable models of Σ up to isomorphism.

(c) (6 points) Classify the models of Σ up to elementary equivalence. (That is, determine the complete extensions of Σ.)

Problem 5. Let L be the first order language whose nonlogical symbols consist of a constant symbol 0 for the number zero, a unary function symbol S for the successor function, a binary predicate symbol $<$ for the ordering relation, and two binary function symbols $+$ and $\times$ for addition and multiplication. Assume that $\Sigma \vdash_L \sigma$ whenever σ is a quantifier free L-sentence that is true in the standard model of arithmetic $(\mathbb{N}, 0, S, <, +, \times)$.

(a) (5 points) Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function on $\mathbb{N}$. Define what it means for f to be *representable (as a function) in* Σ.

(b) (15 points) Suppose g and h are functions on $\mathbb{N}$ that are representable in Σ and f is the function defined on $\mathbb{N}$ by $f(n) = g(n) \times h(n)$ for all n. Show that f is representable in Σ.