**Comprehensive Exam, Math 570, August 27, 2005.**

Each of the five problems counts for 20 points. Explain your answers.

**Conventions**: The equality symbol $=$ is considered as a logical symbol, $L$ is a language, $\mathbf{N}$ is the set of natural numbers (including 0), $\mathbf{Z}$ is the set of integers, and $\mathbf{R}$ is the set of real numbers. "Recursive" means the same as "computable."

1. Let $L$ have just a unary function symbol $f$.

(i) Find $L$-sentences $\sigma_1$ and $\sigma_2$ such that

$$\not\models \sigma_1, \quad \not\models \sigma_2, \quad \sigma_1 \not\models \sigma_2, \text{ and } \sigma_2 \models \sigma_1.$$

(ii) Find a consistent $L$-sentence all whose models are infinite.

2. (i) Indicate (without proof) all automorphisms of $(\mathbf{Z}, 0, +)$.

(ii) Is the element 1 definable in the structure $(\mathbf{Z}, 0, +)$? "Yes" means that there exists a formula $\phi(x)$ in the language $\{0, +\}$ such that for all integers $k$:

$$(\mathbf{Z}, 0, +) \models \phi(k) \text{ if and only if } k = 1.$$

(iii) Let $L$ consist of the constant symbol 0, the binary relation symbol $<$ and the binary function symbol $f$. Indicate an $L$-formula $\psi(x)$ with the following property: for each function $f : \mathbf{R}^2 \to \mathbf{R}$ the formula $\psi(x)$ defines in the structure $(\mathbf{R}, 0, <, f)$ the set of all $a \in \mathbf{R}$ such that $\lim_{y \to +\infty} f(a, y) = 0$.

3. Let $L$ consist of one unary relation symbol $U$. Let $M$ be an $L$-structure such that both $U^M$ and $M \setminus U^M$ are infinite.

(i) Let $\sigma_n$ be the $L$-sentence

$$[\exists x_1 \cdots \exists x_n (\bigwedge_{i<j} x_i \neq x_j \wedge \bigwedge_i U(x_i))] \wedge [\exists x_1 \cdots \exists x_n (\bigwedge_{i<j} x_i \neq x_j \wedge \bigwedge_i \neg U(x_i))].$$

(Note that $\sigma_n \in \mathrm{Th}(M)$.) Prove that $\{\sigma_n : n \in \mathbf{N}\} \vdash \sigma$ for all $\sigma \in \mathrm{Th}(M)$.

(ii) Prove that there is no $\theta \in \mathrm{Th}(M)$ such that $\theta \vdash \sigma$ for all $\sigma \in \mathrm{Th}(M)$.

4. Let $L$ be countable, and let $\Sigma_1$ and $\Sigma_2$ be sets of $L$-sentences such that no nonlogical symbol occurs in both $\Sigma_1$ and $\Sigma_2$. Suppose $\Sigma_1$ and $\Sigma_2$ are consistent but have no finite model. Show that $\Sigma_1 \cup \Sigma_2$ is consistent.

5. Recall that a set $A \subseteq \mathbf{N}$ is *recursively enumerable* if there is a recursive set $R \subseteq \mathbf{N}^2$ such that for all $a \in \mathbf{N}$: $a \in A$ iff $(a, b) \in R$ for some $b \in \mathbf{N}$. Using this definition, show:

(i) The intersection of two recursively enumerable subsets of $\mathbf{N}$ is recursively enumerable.

(ii) If $A$ is a nonempty recursively enumerable subset of $\mathbf{N}$, then $A = f(\mathbf{N})$ for some recursive function $f : \mathbf{N} \to \mathbf{N}$.