Logic Comprehensive Exam (Math 570),
August 2007

There are 5 problems. Each problem is worth 20 points, for a total of 100 points. To receive credit, each of your solutions must be justified.

Convention
In the exercises, $L$ will be a language and $=$ is considered a logical symbol. Any model-theoretic structure is by convention non-empty.
Another word for ‘recursive’ is ‘computable’.

1 Exercise
Two sets $A, B \subseteq \mathbb{N}$ are said to be recursively isomorphic if there is a recursive bijection $h: \mathbb{N} \to \mathbb{N}$ such that $h[A] = B$.

(a) Show that if $A$ and $B$ are infinite recursive subsets of $\mathbb{N}$ with infinite complements $\mathbb{N} \setminus A$ and $\mathbb{N} \setminus B$, then $A$ and $B$ are recursively isomorphic.

(b) Describe all recursive isomorphism classes of subsets of $\mathbb{N}$.

2 Exercise
Let $L$ be a language with only finitely many non-logical symbols and let $T$ be a decidable $L$-theory. Show that there is a complete decidable $L$-theory $T' \supseteq T$. 

1
3 Exercise

Let $L$ be the language whose non-logical symbols are a binary predicate symbol $<$ and a unary predicate symbol $P$. Let $\mathcal{Q}$ be $L$-structure $\mathcal{Q} = (\mathbb{Q}, <^\mathcal{Q}, P^\mathcal{Q})$, where $<^\mathcal{Q}$ denotes the usual strict ordering on $\mathbb{Q}$ and $P^\mathcal{Q} = \{ q \in \mathbb{Q} | q < 0 \}$.

(a) Is there an $L$-formula $\phi(x)$ defining the set $\{1\}$ in $\mathcal{Q}$?

(b) Is there an $L$-formula $\psi(x)$ defining the set $\{0\}$ in $\mathcal{Q}$?

(c) Indicate a finite set $\Sigma$ of $L$-sentences such that for all $L$-sentences $\sigma$ we have $\Sigma \vdash \sigma \iff \mathcal{Q} \models \sigma$.

4 Exercise

Let $L$ be the language whose only non-logical symbol is a binary relation symbol $<$ and let $\sigma$ be an $L$-sentence. Suppose that for all $n$ there is a model $\mathcal{M} = (M, <^\mathcal{M})$ of $\sigma$ such that $<^\mathcal{M}$ linearly orders $M$ and $|M| \geq n$. Show that there is a model $\mathcal{M} = (M, <^\mathcal{M})$ of $\sigma$, linearly ordered by $<^\mathcal{M}$, with distinct elements $a_0, a_1, a_2, \ldots$ such that $\ldots <^\mathcal{M}_a_2 <^\mathcal{M}_a_1 <^\mathcal{M}_a_0$.

5 Exercise

Let $\Sigma$ be a finite consistent set of sentences in a language $\mathcal{L}$ with only finitely many non-logical symbols, including a constant symbol $0$ and a unary function symbol $S$.

(a) Define what it means for a set $A \subseteq \mathbb{N}$ to be representable in $\Sigma$.

(b) If $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$ are representable in $\Sigma$, does it follow that the difference set $A \setminus B$ is representable in $\Sigma$?