Linkages with Modular Constraint

Guantao Chen

Georgia State University

Joint work with
Shuhong Gao and Zhiquan Hu

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Theorem (Bollobás)
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Every k-connected graph contains a cycle C such that $|C| \equiv 0 \bmod k$.

Theorem (C. and Saito)
Every graph with minimum degree $\delta(G) \geq 3$ contains a cycle of length divisible by 3.

Definition

A graph G is k-linked, if G has at least $2k$ vertices, and for every sequence $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ of $2k$ distinct vertices, G contains k pairwise vertex-disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i joins x_i and y_i for $i = 1, 2, \ldots, k$.

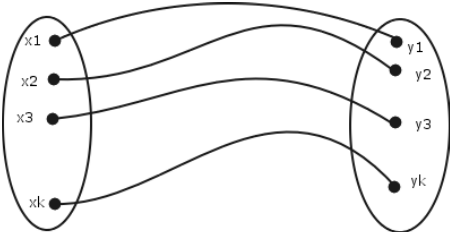

Figure: A k-linked graph

Connection between linkages and connectivity:

- Every k-linked graph is -connected.

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- (Kawarabayashi, Kostochka and Yu) $f(k) \leq 12k$.
- (Thomas and Wollan) $f(k) \leq 2k$ if $||G|| \geq 5|G|$.

Definition

A graph G is said to be k-linked modulo $(m_1, m_2, \ldots, m_k)$ if G is k-linked and, in addition, for any k-tuple $(d_1, d_2, \ldots, d_k)$ of natural numbers, the paths $P_1, P_2, \ldots, P_k$ can be chosen such that P_i has length d_i mod m_i for $i = 1, 2, \ldots, k$.

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Theorem (Thomassen)

For any natural numbers k, p there exists a natural number $\gamma(k, p)$ such that any $\gamma(k, p)$-connected graph G has the following property: For any k-tuple $(m_1, m_2, \ldots, m_k)$ of natural numbers (such that each m_i is odd and less than p) G is k-linked modulo $(m_1, m_2, \ldots, m_k)$.

Thomassen's proof

- A function $\eta(s, t)$ is chosen such that (for some s sufficiently larger than both k and p and $t = p!$) each graph of minimum degree at least $\eta(s, t)$ contains a subdivision H of $K_{s,t}$ such that each edge of $K_{s,t}$ corresponds a path of length 1 mod t in H.

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- The upper bound given in the proof is that

$$\underbrace{\xi 2^{\xi} \cdot 2^{\xi 2^{\xi}} \cdot 2^{2^{\xi 2^{\xi}}} \cdots}_{s^2(2t+1)^{2t+1}-s^2 \text{ times}}$$

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- $\xi := \xi(s^2(2t+1)^t)$ guarantees that a graph with minimum degree at least ξ contains a subdivision of $K_{s^2(2t+1)^t, s^2(2t+1)^t}$ satisfying the parity condition on the path lengths.

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- The upper bound of $\gamma(k,p)$ is enormous.

The Methodology

Instead of building one powerful subgraph as Thomassen did, we establish a path systems such that there exist a number segments in each path can be replaced by paths with difference residue and establish a number theory result on distributions of co-sets of integers to ensure the desired path system exists.

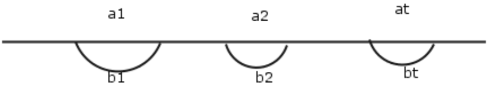

Figure: $a_i \not\equiv b_i \mod m$

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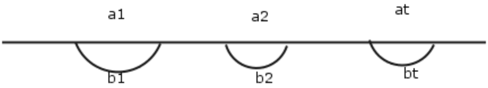

Figure: $a_i \not\equiv b_i \mod m$

Question: How many such special segments do we need?

sumsets of integers

Definition

Let $m \geq 2$ and $k \geq 1$ be two integers and let $A_1, A_2, \ldots, A_k$ be k sets of integers such that $0 \in A_i$. Then,

$$\begin{aligned} A_1 + A_2 + \cdots + A_k &= \{a_1 + \cdots + a_k \mid a_i \in A_i\} \quad \text{and}, \\ A_1 + A_2 + \cdots + A_k \bmod m &= \{a_1 + \cdots + a_k \bmod m \mid a_i \in A_i\}. \end{aligned}$$

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Conditions such that $A_1 + A_2 + \cdots + A_k$ mod $m = \mathbb{Z}/(m)$

Three Classic Results

Theorem (Cauchy and Davenport)
Let p be a prime and let A and B be two subsets of $\mathbb{Z}_p$. Then $|A+B| \geq \min\{p, |A|+|B|-1\}$.

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Let p be a prime and let $a_1, a_2, \cdots, a_k$ be k integers which are nonzero and distinct modulo p. If $k \geq 3\sqrt{6p}$, then $\sum_{i=1}^{k}\{0, a_i\} \text{mod} = \mathbb{Z}_p$.

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Theorem (Olson)
Let p be a prime and let $a_1, \cdots, a_{p-1}$ be $p-1$ nonzero integers (not necessarily distinct) such that $a_i \not\equiv 0 \bmod p$ for each $i \in [1, p-1]$. $\sum_{i=1}^{p-1}\{0, a_i\} \bmod p = \mathbb{Z}_p$.

A Lemma

Lemma

Let $m \geq 2$ be a positive integer and let $A \subset \mathbb{Z}$ such that for each prime factor p of m there is $a \in A$ such that $p \nmid a$. Then, for any $S \subset \mathbb{Z}$, if $S + A \equiv S \bmod m$ then $S \equiv \{0, 1, ..., m-1\} \mod m$.

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- Let $H = < a >$ be the subgroup of $\mathbb{Z}_m$ generated by a. Then $p^e \mid |H|$, where p^e is the highest power of p in m;
- For each $r \in S$, applying $S + A = S$, we have $a + r \in S$, $2a + r \in S$, $\cdots$, $ma + r \in S$, i.e. $r + H \subseteq S$, so S is a union of cosets of H. Then, $p^e \mid |H| \mid |S|$.

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- Therefore, $|S| = m$, so follows the lemma.

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- *If for each prime factor* p *of* m *and for each* i *there is an* $a_i \in A_i$ *which is not divisible by* p, *then*
- $A_1 + \cdots + A_{m-1} \bmod m = \mathbb{Z}_m$.

Proof

- Let $S_i = A_1 + \cdots + A_i$ for each $1 \leq i \leq m-1$. Then

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{m-1} \subseteq \mathbb{Z}_m$$

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- Otherwise, $|S_{m-1}| \geq m$, which in turn shows $S_{m-1} = \mathbb{Z}_m$.

Small Cycle Adjusters for m odd

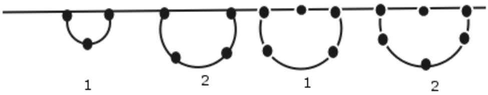


Figure: Cycles of length ≤ 6

Using Linkages Building Adjusters

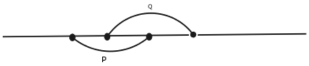
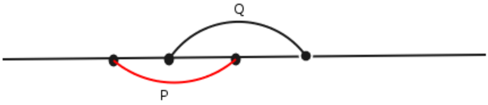


Figure: two path adjustment

Using Linkages Building Adjusters



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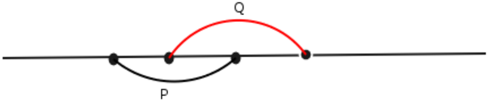

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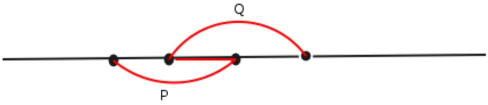

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For any integer $m > 2$, either

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- $|Q| \not\equiv 2 \bmod m$ or
- $|P| + |Q| \not\equiv 2 \bmod m$.

no small cycles

Theorem (Mader)

There is a constant g such that every $2k$-connected graph with $g(G) \geq g$ is k-linked.

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Theorem (Kawarabayashi)
Let G be a $2k$-connected graph. If $g(G) \geq 7$ and $k \geq 21$, then G is k-linked.

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- *Then every $f(m_1, m_2, \ldots, m_k)$-connected graph is k-linked modulo $(m_1, m_2, \ldots, m_k)$.*

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A graph that is k-linked modulo $(2, 2, \ldots, 2)$ is called *a parity k-linked graph*.

Theorem (Thomassen)

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Theorem (Kawarabayashi and Reed)

For every natural number k, every $50k$-connected graph G is k-parity-linked if $bi(G) \geq 4k-3$.

A General Result

Theorem

Let $(m_1, m_2, \ldots, m_k)$ be a k-tuple of natural numbers such that m_i is odd for each $i \in [\ell+1, k]$. If G is $45(m_1 + \cdots + m_k)$-connected and $bi(G) \geq 2k + 2\ell - 3$, then G is k-linked modulo $(m'_1, m'_2, \ldots, m'_k)$ where

$$m'_i := \begin{cases} 2m_i & \text{if } 1 \leq i \leq \ell, \text{ and} \\ m_i & \text{if } \ell + 1 \leq i \leq k. \end{cases}$$

Theorem

For every natural number k, every $45k$-connected graph G is k-parity-linked if $bi(G) \geq 4k - 3$.