On-line Ramsey Theory

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- Builder wins if Painter creates a monochromatic copy of K_s^t;
- Otherwise Painter wins when she has colored all $\binom{|V|}{s}$ edges.

On-line f-Ramsey Numbers

- For $c, s, t \in \mathbb{N}$ the *on-line* f-Ramsey number $f\text{-Ram}_c^s(t)$ is defined to be the least n such that Builder can win the (c, s, t)-Ramsey game while building a graph G with $f(G) \leq n$.

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- Trivially

$$f(K_s^t) \leq f\text{-oRam}_c^s(t) \leq f\text{-Ram}_c^s(t) \leq f(K_s^n),$$

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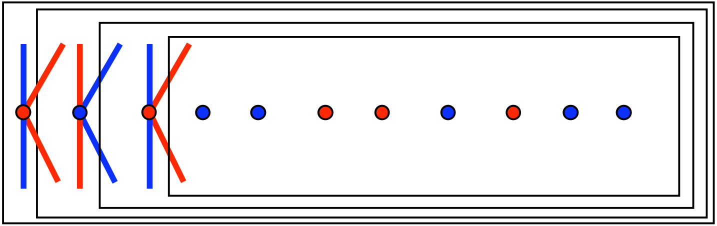

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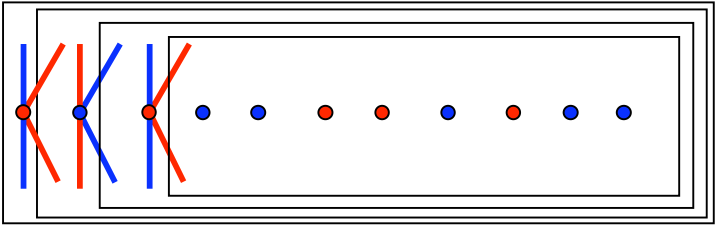

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$$\text{col-oRam}(t) = \text{col}(K^t) = t \text{ ?}$$

Partial Result

Theorem (Grytczuk, Hałuszczak and HK)

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Main Theorem

Theorem (HK and Konjevod)

For all $c, s, t \in \mathbb{N}$, the on-line coloring Ramsey number satisfies the trivial lower bound

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Question

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- We define

$$\operatorname{col}(H) = 1 + \min_{\prec} \max_{v} d^+(v)$$

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- **Proof.** Color the vertices using First-Fit in an order that witnesses the coloring number of H.
- **Fact**

$$\chi(K_s^t) = \left\lceil \frac{t}{s-1} \right\rceil = \left\lfloor \frac{t-1}{s-1} \right\rfloor + 1 = \mathrm{col}(K_s^t)$$

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- Then iterate the procedure on S.
- We need the following auxiliary result.

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  - $E_i := (E_{i-1} \cup \{X_i\}) - \{X_j \in E_{i-1} : X_j \nsubseteq S_i\}$.
- Presenter wins if H_i contains a copy of K_s^t for some i.

(p,s,t)-Survival Game

- The (p,s,t)-*survival game* is played by *Presenter* and *Chooser*.
- First Presenter fixes an s-graph $H_0 = (S_0, E_0)$, $E_0 = \emptyset$.
- The i-th round starts with $H_{i-1} = (S_{i-1}, E_{i-1})$.
- During the i-th round $H_i = (S_i, E_i)$ is constructed by:
  - Presenter plays a p-subset $P_i \subseteq S_{i-1}$.
  - Chooser selects an s-set $X_i \subseteq P_i$.
  - $S_i := S_{i-1} - (P_i - X_i)$
  - $E_i := (E_{i-1} \cup \{X_i\}) - \{X_j \in E_{i-1} : X_j \not\subseteq S_i\}$.
- Presenter wins if H_i contains a copy of K_s^t for some i.
- Otherwise Chooser wins when $|S_i| < t$

Key Technical Result

Theorem (HK-Konjevod)

For all positive integers p, s, t, with $s \leq p$, Presenter has a winning strategy in the (p, s, t)-survival game.

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- The winning strategy for Presenter requires more than $A(2^s - 1, t)$ starting vertices, where A is the Ackermann function.

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