IMPROVING SOME "GOOD" RAMSEY RESULTS

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The purpose of the talk is to present this approach.

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FACT: p-good graphs are poor expanders.

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Question **Is the wheel $K_1 + C_n$ a p-good graph for fixed $p > 3$ and n large?**

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Question **Fix $p \geq 3$, $l \geq 1$, $r \geq 1$, and a connected graph G.**
Is $K_l + G_1^r$ a p-good graph for every large graph G_1 homeomorphic to G?

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Question Suppose $p \geq 3$, $K \geq 1$, T_n is a tree of order n, and $\mathbf{k} = (k_1, ..., k_n)$ is a vector of integers with $0 < k_i \leq K$ for all $i \in [n]$. Is $T_n^{\mathbf{k}}$ a p-good graph for p, K fixed and n large?

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Question **Is the cube Q^n a p-good graph for fixed $p \geq 3$ and n large?**

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(i) R contains a p-joint of size cn^{p-2};

(ii) B contains every q-degenerate (γ, η)-splittable graph G of order n.

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If $p \geq 3$, the graphs $B_p(t)$ in the above theorem may be replaced with other graphs like, e.g., wheels.

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- three involved embedding algorithms for degenerate splittable graphs.

THANK YOU