

## 4. Monochromatic hypergraph cycles

(ongoing research with Györfás,  
Lehel and Schelp)

$K_n^{(r)}$  is the complete  $r$ -uniform hypergraph.

There are several natural definitions for a hypergraph cycle.

Loose cycles in  $K_n^{(3)}$

$C_m$  is a loose cycle in  $K_n^{(3)}$ , if it has vertices  $\{v_1, \dots, v_m\}$  and edges  $\{v_1 v_2 v_3, v_3 v_4 v_5, v_5 v_6 v_7, \dots, v_{m-1} v_m v_1\}$  (so  $m$  is even)

If  $\mathcal{H}$  is a 3-uniform hypergraph:

$$N(x, y) = \{z \mid (x, y, z) \in E(\mathcal{H})\}$$

Minimum degree:

$$\delta(\mathcal{H}) = \min_{x, y} |N(x, y)|$$

Density theorem:

Kühn, Osthus '06

If  $\mathcal{H}$  is a 3-uniform hypergraph with  $n \geq n_0$  vertices and  $\delta(\mathcal{H}) \geq \frac{n}{4} + \varepsilon n$ , then  $\mathcal{H}$  contains a loose Hamiltonian cycle.

Coloring theorem:

Haxell, Łuczak, Peng, Rödl, Ruciński, Simonovits, Skokan '06

Every 2-coloring (edge) of  $K_n^{(3)}$  with  $n \geq n_0$  contains a monochromatic loose  $C_m$  with  $m \sim \frac{4}{5}n$ .

Tight cycles in  $K_n^{(3)}$

$C_m$  is a tight cycle in  $K_n^{(3)}$ , if it has vertices  $\{v_1, \dots, v_m\}$  and edges

$\{v_1 v_2 v_3, v_2 v_3 v_4, v_3 v_4 v_5, \dots, v_m v_1 v_2\}$ .

Density theorem:

Rödl, Ruciński, Szemerédi '05

If  $\mathcal{H}$  is a 3-uniform hypergraph with  $n \geq n_0$  vertices and  $\delta(\mathcal{H}) \geq \frac{1}{2} + \varepsilon n$ , then  $\mathcal{H}$  contains a tight Hamiltonian cycle.

Coloring theorem:

HLPRRSS '06

Berge-cycles in  $K_n^{(r)}$

$C_m = (v_1, E_1, v_2, E_2, \dots, v_m, E_m, v_1)$  is a Berge-cycle in  $K_n^{(r)}$  if

1.)  $v_1, \dots, v_m$  are all distinct vertices

2.)  $E_1, \dots, E_m$  are all distinct edges

3.)  $v_k, v_{k+1} \in E_k$  for  $k=1, \dots, m$

where  $v_{m+1} = v_1$

## Coloring theorems:

Theorem (Gyárfás, Lehel, G.S., Schelp '06)

Every 2-coloring of  $K_n^{(3)}$  with  $n \geq 5$  contains a monochromatic Hamiltonian Berge-cycle.

Conjecture

Every  $(r-1)$ -coloring of  $K_n^{(r)}$  with  $n \geq n_0$  contains a m.c. Hamiltonian Berge-cycle.

Theorem 3. (GYLSS '06)

Every 3-coloring of  $K_n^{(3)}$  with  $n \geq n_0$  contains a m.c. Berge-cycle  $C_m$  with  $m \sim \frac{4}{5}n$ .

Conjecture

Every  $r$ -coloring of  $K_n^{(r)}$  with  $n \geq n_0$  contains a m.c. Berge-cycle  $C_m$  with

$$m \sim \frac{2r-2}{2r-1}n.$$

## 5. Proofs: A coloring adaptation of the Regularity Lemma-Blow-up Lemma method

Step 1: Apply the  $r$ -color version of the Reg. Lemma.

Step 2: Define the reduced graph on the clusters.

Step 3: Find a "nice" monochromatic object (connected matching) in the reduced graph.

Step 4: Blow it back up into the original graph with the Blow-up Lemma.