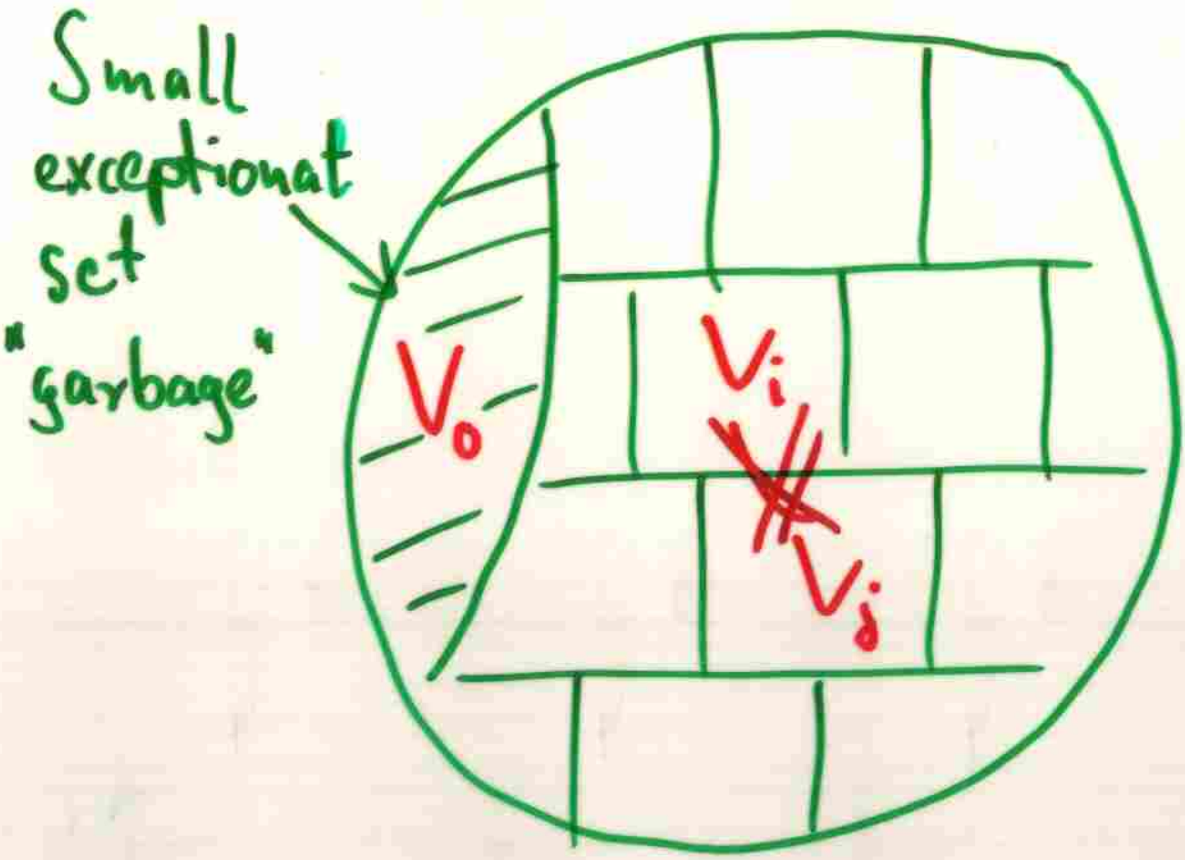
First step is the Regularity Lemma



Most of the pairs (V_i, V_j) are ε-regular in every color!

r-color version of the Regularity lemma (Szemerédi)

For every $\varepsilon > 0$ and positive integers m, r there are positive integers M, N with the following property:

For all graphs $G_1, G_2, \dots, G_r$ with $V(G_1) = \dots = V(G_r) = V$, $|V| = n \geq N$, there is a partition of the vertex set into $\ell + 1$ clusters

$$V = V_0 \uplus V_1 \uplus \dots \uplus V_\ell$$

such that

1) $m \leq \ell \leq M$
2) $|V_1| = \dots = |V_\ell|$
3) $|V_0| < \varepsilon n$
4) apart from at most $\varepsilon \binom{\ell}{2}$ exceptional pairs, the pairs (V_i, V_j) are (ε, G_s)-regular for $s = 1, \dots, r$.

One of the most powerful tools in discrete mathematics (e.g. Rödl, Gowers, Tao)

But sometimes (eg. in $R(P_n, P_n, P_n)$) we have to be more careful, we use multi-colorings: (P_i, P_j) is colored with s if $|E_{G_s}(V_i, V_j)| \geq \delta |V_i||V_j|$.

Then mostly red in G^R

$\Updownarrow$

mostly red in the original coloring.

For the hypergraph results we use the r-color version of the Hypergraph Regularity lemma (Chung). G^R is a $(1-\varepsilon)$-dense hypergraph now.

We use majority coloring.

We take the shadow graph, $\Gamma(G^R)$, i.e. $(x,y) \in E(\Gamma(G^R))$ iff $\exists\ E \in E(G^R)$ such that $\{x,y\} \subset E$ and (x,y) gets the color of E. Multi-coloring!

We define the so-called reduced graph G^R: the vertices correspond to the clusters, $P_1, \dots, P_\ell$ and we have an edge between P_i and P_j if the pair (V_i, V_j) is non-exceptional, so (ε, G_s)-regular for $s = 1, \dots, r$.

One-to-one correspondance:

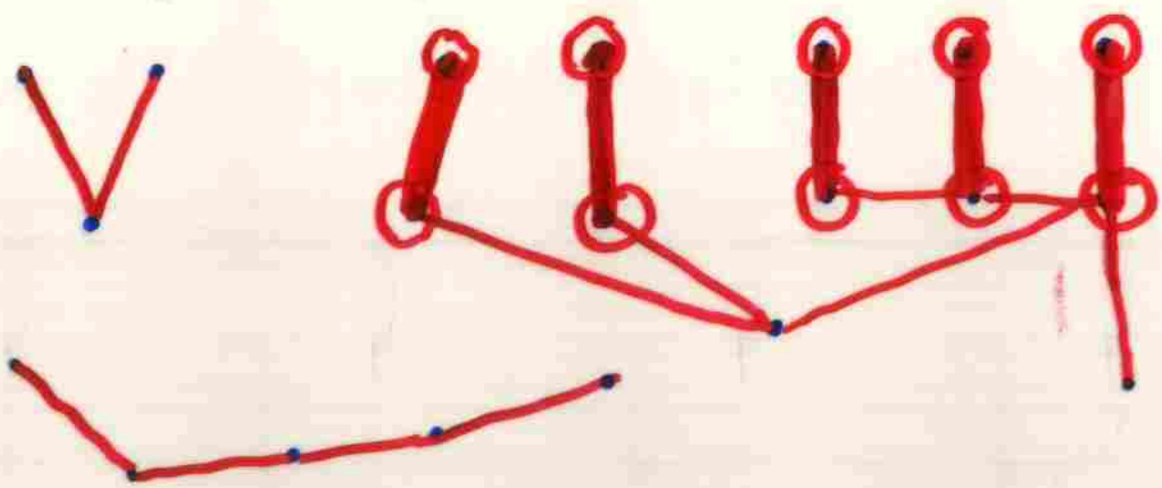
$$\begin{array}{cccc} P_1 & \cdot & & P_\ell \\ \updownarrow & \updownarrow & \cdots & \updownarrow \\ V_1 & \cdot & & V_\ell \end{array} \qquad f : P_i \to V_i$$

Coloring in G^R

Usually it's just majority, so the edge (P_i, P_j) is colored with a color s that contains the most edges from $K(V_i, V_j)$.

Main idea of the 3 proofs

Look for large monochromatic connected matchings (suggested first by Łuczak) in G^R (or in $\Gamma(G^R)$ for hypergraphs).



Red connected matching M in G^R

$\Downarrow$

There is a red path P covering most of the vertices of $f(M)$.