Cycles and Stability

by

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<u>Def.</u> A graph G is <u>Ramsey unsaturated</u> if $\exists e \in E\overline{G} \ni r(G+e) = r(G)$, and <u>Ramsey saturated</u> otherwise.

We've shown that cycles of order at least five are unsaturated.

Question: How many edges can added to C_n (n odd and large) without increasing its Ramsey number?

<u>Theorem 1</u> Let $0 < \alpha < 10^{-5}$, $0 \le \beta < 10^{-5}$, and $n \geq \frac{\alpha^{-1}}{2}$. If G is a graph of order n with $e(G) > (\frac{1}{4} - \beta)n^2$, then one of the following hold.

(i) $c(G) \geq (\frac{1}{2} + \alpha)n$

(ii) $\exists$ a set $M \subseteq V(G)$ with

<u>Theorem 2</u>. Let $0 < r < 10^{-5}$. If $G = G(n)$ is a graph of order n with $e(G) > n^2/4$, then one of the following hold.

(i) $C_t \subset G \quad \forall t \in [3, \lceil (\frac{1}{2}+r)n \rceil]$.

(ii) $\exists$ a vertex $v \ni G - v = H_1 \cup H_2$ are vertex disjoint graphs satisfying

$$\left(\tfrac{1}{2} - 900r\right)n < |H_1| \leq |H_2| < \left(\tfrac{1}{2} + 900r\right)n.$$

Proof Ideas of Theorem 2 (using Thm 1)

(1) Known if $e(G) > \frac{n^2}{4}$, then $C_t \subseteq G\ \forall\ 3 \le t \le c(G)$. Therefore if (i) fails $c(G) \le \lfloor (\frac{1}{2}+r)n \rfloor$.

(2) If $n < \frac{r^{-1}}{2}$ and $n = 2k$, then by Erdős-Gallai

$$k < c(G) \le \lfloor k + rn \rfloor, \quad (*),$$

since $n < r^{-1}/2$. Similar for n odd.

(3) Using the given degree conditions find two components G_1 and G_2 in $G-M$, $|G_1| \leq |G_2|$, $\ni$ G_1 and G_2 are Hamiltonian connected.

(4) Done if two vertex disjoint paths from G_1 to G_2. If not, then select v separating G_1 and G_2. Since $V(G_1) - v$ induces a connected subgraph in $G - v$, let H_1 be the component containing $G_1 - v$ and let H_2 be the union of the remaining components

Theorem 3 Let $0 < \alpha < 5 \times 10^{-6}$, $0 \le \beta < \alpha/25$, $n \ge \alpha^{-1}/2$. If $G = G(n)$ is a graph with $e(G) > (\frac{1}{4} - \beta)n^2$, then one of the following hold.

(i) $C_t \subset G$ $\forall\, t \in [3, \lceil(\frac{1}{2}+\alpha)n\rceil]$.

(ii) $\exists$ a partition $V(G) = V_0 \cup V_1 \cup V_2 \ni$

$$|V_0| \le 2000\alpha n,$$

$$(\tfrac{1}{2} - 10\sqrt{\alpha+\beta})n < |V_1| \le |V_2| < (\tfrac{1}{2} + 10\sqrt{\alpha+\beta})n,$$

$\delta(G - V_0) \ge 2n/5$, and either

$$E(G - V_0) \subseteq V_1^{(2)} \cup V_2^{(2)} \text{ or}$$

$$E(G - V_0) \subseteq V_1 \times V_2$$

<u>Theorem 4</u> Let the edges of K_{2n-1} be three colored, i.e. let $E(K_{2n-1}) = E(R) \cup E(B) \cup E(Y)$, where R, B, Y are graphs with $V(R) = V(B) = V(Y) = 2n-1$. Let the minimum degree $\delta(R \cup B) > (2-10^{-6})n$. Then if n is sufficiently large either $C_t \subset R \ \forall t \in [3, n]$ or $C_t \subset B \ \forall t \in [3, n]$.

<u>Theorem 5.</u> Let $n > 4$, $0 < \beta \leq \lfloor n/2 \rfloor / n$, and $E(K_{\lfloor (2-\beta)n \rfloor}) = E(R) \cup E(B)$ be a 2-edge coloring $\ni C_n \not\subseteq R$ and $C_n \not\subseteq B$. Then $\exists\, u \in V(K_{\lfloor (2-\beta)n \rfloor})$ and a partition $V(K_{\lfloor (2-\beta)n \rfloor}) = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \{u\}$ with $(1-\beta)n - 1 < |\mathcal{U}_1| \leq |\mathcal{U}_2| < n$ satisfying up to color

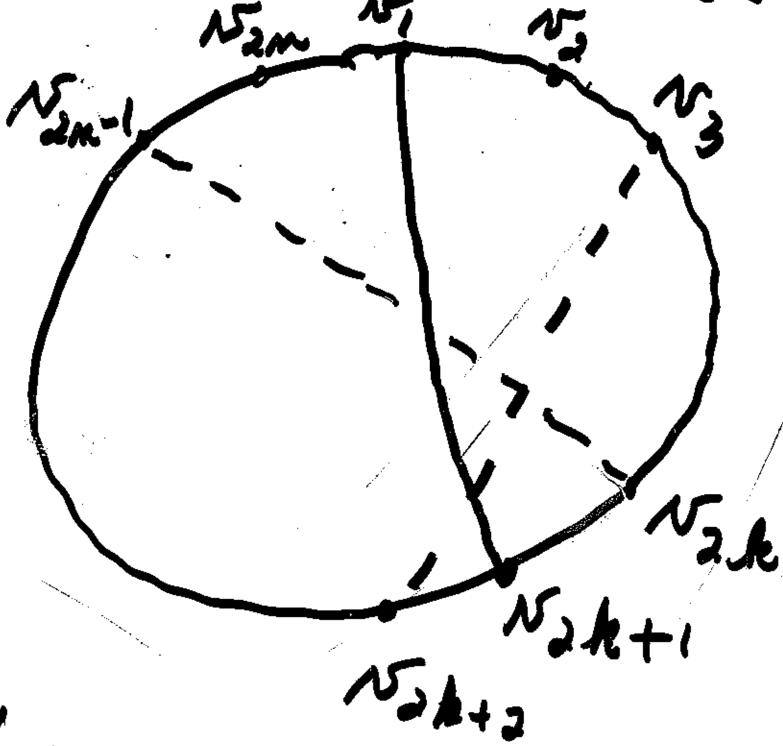
$$E(R-u) = \mathcal{U}_1^{(2)} \cup \mathcal{U}_2^{(2)} \text{ and}$$

$$E(B-u) = \mathcal{U}_1 \times \mathcal{U}_2.$$

Lemma. Let G be a Hamiltonian graph of order $2n \ni C_{2n-1} \not\subseteq G$ and $C_{2n-1} \not\subseteq \overline{G}$. Then $\exists$ a partition $V(G) = \mathcal{U}_1 \cup \mathcal{U}_2 \ni |\mathcal{U}_1| = |\mathcal{U}_2| = n$ and $\mathcal{U}_1, \mathcal{U}_2$ are both independent sets. Moreover $\exists\, u \in V(G) \ni G - u = K_{n,n-1}$.

Proof of Lemma. Assume $(v_1, v_2, \cdots, v_{2n}, v_1)$ is Hamiltonian cycle. Note $(v_1, v_3, \cdots, v_{2n-1}, v_1)$ and $(v_2, v_4, \cdots, v_{2n}, v_2)$ are n-cycles in $\overline{G}$.

We show $U_1 = \{v_1, v_3, \cdots, v_{2n-1}\}$ and $U_2 = \{v_2, v_4, \cdots, v_{2n}\}$ are both independent sets. Suppose not.



Suppose $v_1 v_{2k+1} \in E$

$\Longrightarrow (v_3, v_5, \cdots, v_{2n-1}, v_{2k}, v_{2k-2}, \cdots, v_{2k+2}, v_3)$ is a C_{2n-1} in $\overline{G}$, ⊛ Therefore $\overline{G}[U_1]$ and $\overline{G}[U_2]$ are complete in $\overline{G}$.

$\Longrightarrow E_{\overline{G}}(U_1, U_2)$ contains no pair of disjoint edges.

Proof of Thm 5. (Idea)

For n even, $n = 2k$, lemma gives theorem since $\beta \le \frac{1}{2}$, $\lfloor (2-\beta)2k \rfloor \ge 3k \ge 3k-1$ and $r(C_{2k}) = 3k-1$. for $k > 2$, so the theorem holds for n even.

If n is odd, $n = 2k+1$, we have $\lfloor (2-\beta)(2k+1) \rfloor \ge 3k+2$, so up to color, $C_{2k+2} \subset B$. But by assumption no $C_{2k+1} \subset R$ and no $C_{2k+1} \subset B$
$\Rightarrow$ by lemma complete graphs W_1 and W_2 (in R) $\ni |W_1| = k$, $|W_2| = k+1$.

Conjectures for <u>Ramsey unsaturated</u> and <u>Ramsey saturated</u> graphs.

1. $K_n - e$ is Ramsey saturated for $n \geq 3$, i.e. $r(K_n - e) < r(K_n)$.
More generally if a fixed number of edges are deleted from K_n, n large, then the resulting graph is Ramsey saturated.

2. Almost all graphs are Ramsey unsaturated.

3. If T_n is a nonstar tree of order $n \geq 4$, then $r(T_n + e) = r(T_n)$ for each edge $\ni T_n + e$ is bipartite.

4. There exists constants $c > 0$ and $\epsilon > 0$ such that at least $cn^{1+\epsilon}$ of the n vertex graphs are Ramsey saturated.

5. The graph formed by joining a new vertex to a constant number of vertices of K_n with n large is unsaturated.

6. $r(C_n + k\text{-chord}) = r(C_n)$ provided not both n and k are even.