

Partition regular sets of matrices

József Solymosi

University of British Columbia

Introduction

A $u \times v$ matrix A is image partition regular if for any colouring of \mathbb{N} using finitely many colours, there is some $\vec{x} \in \mathbb{N}^v$ with all entries of $A\vec{x}$ monochrome.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \end{pmatrix}$$

Image partition regular matrices
were used by W. Denker ((m, p, c) -sets)
were characterized by N. Hindman
and I. Leader '93 and by
N. Hindman, I. Leader, and D. Strauss
'02

There are extensions to infinite
matrices and to character fields.

The notion of image partition regularity is more natural than kernel partition regularity

A $u \times v$ matrix A is kernel partition regular if whenever $IN = \bigcup_{i=1}^r D_i$, there exist $i \in [r]$ and $(\vec{x} \in IN^v$ such that $A\vec{x} \in D_i^u)$

$\vec{x} \in D_i^v$ such that $A\vec{x} = \vec{0}$

R. Rado obtained a combinatorial characterization of kernel partition regular matrices. (columns condition)

A set of $u \times v$ matrices A_1, \dots, A_k is image partition regular if for any colouring of \mathbb{N}^u using finitely many colours, there is some $\vec{x} \in \mathbb{N}^v$ with all vectors $A_i \vec{x}$ monochrome.

Example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

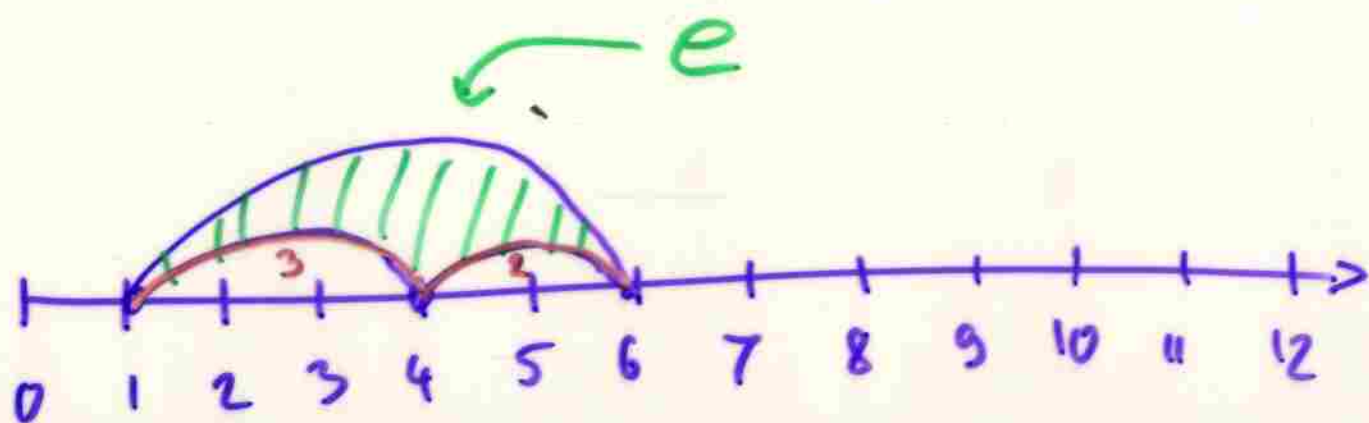
$$\begin{pmatrix} a \\ L \\ a \end{pmatrix}$$

Definition: The identity subspace of a set of $u \times v$ matrices A_1, \dots, A_k

$$IS(A_1, \dots, A_k) = \{ \vec{x} \in \mathbb{R}^v : A_i \vec{x} = A_j \vec{x} \}$$

Notation: $\dim(A_1, \dots, A_k) = \dim(IS)$

Definition: A set of $u \times v$ matrices A_1, \dots, A_k is image partition regular if for any finite colouring of \mathbb{N}^v there is some $\vec{x} \in \mathbb{N}^v$, $\vec{x} \notin IS(A_1, \dots, A_k)$ with all vectors $A_i \vec{x}$ monochrome.

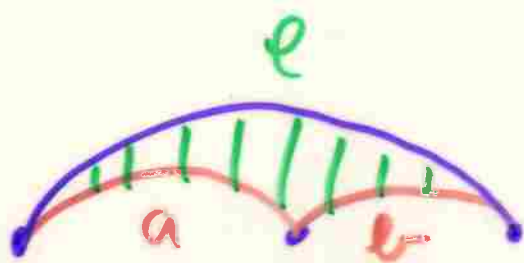


Let's take the complete
3-uniform hypergraph on $[n]$.

χ is a finite colouring of \mathbb{N}^2 .

$$\chi(e) := \chi \begin{pmatrix} f(3,2) \\ g(3,2) \end{pmatrix}$$

$$f, g : \mathbb{N}^2 \rightarrow \mathbb{N}$$

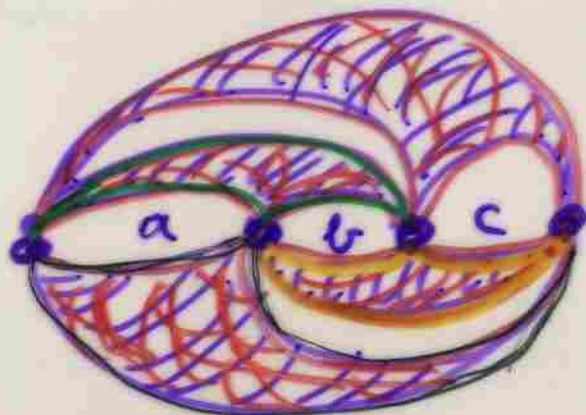


The colour of e is one of the colours of the vectors

- | | | | |
|--|--|--|--|
| 1. | 2. | 3. | 4. |
| $\begin{bmatrix} a \\ b \end{bmatrix}$ | $\begin{bmatrix} b \\ a \end{bmatrix}$ | $\begin{bmatrix} a \\ a+b \end{bmatrix}$ | $\begin{bmatrix} b \\ a+b \end{bmatrix}$ |

For every edge choose the vector independently at random. ($p = \frac{1}{4}$)

$$\begin{bmatrix} a \\ a+b \end{bmatrix} 3.$$



$$\begin{bmatrix} c \\ a+b \end{bmatrix} 2.$$

$$\begin{bmatrix} a \\ c+b \end{bmatrix} 1.$$

$$\begin{bmatrix} c \\ c+b \end{bmatrix} 4.$$