Lab III — Forced Oscillations*

Vadim Zharnitsky and Jeremy T. Tyson

September 3, 2004

Goal of the project

We consider the equations of the form

$$mx'' + cx' + kx = f(t)$$

and we are going to investigate graphically the phenomena of: *resonance, beating, transient behavior, and steady state oscillations.*

1 Getting started

First, start IODE and click on `Second order linear ODEs` in the main menu. You will get an interface very similar to the one for direction fields (which we used in an earlier lab).

Go to the right side of the screen and click on "Solution Method". It is normally set to `Euler`. Change it to `Runge--Kutta`. (This is a more accurate numerical scheme; the Euler method is far too crude for what we are going to do in this lab.)

To ensure that you have the correct set up, try entering the following differential equation

$$x'' + x = 0.$$

Note: *When you enter differential equations put "0" for any coefficients that are missing. Do not leave empty spaces; IODE will not understand that.*

Now plot the solution with initial condition

$$(x(0), x'(0)) = (1, 0).$$

Then plot the solution with initial condition

$$(x(0), x'(0)) = (0, 1).$$

You should see on the screen the graphs of $\cos(t)$ and $\sin(t)$.

Specifying initial conditions with the mouse

You can also plot solutions by pressing down the left mouse button at the desired initial point $(t_0, x(t_0))$ and then dragging the mouse at the desired slope a short distance. When you release the mouse, IODE will plot the solution passing through this point with this slope.[1]

[1] If you just press down the left mouse button and release it without dragging, then the initial slope will be taken as 1 (*i.e.* $x'(t_0) = 1$).

2 Questions

1. **Beating (undamped, forced).** Set the display parameters to be $-100 < t < 100$ and $-50 < x < 50$. Plot the solution of the equation

$$x'' + x = \cos(1.1t)$$

with initial condition $(x(0), x'(0)) = (0, 0)$. You should see beating. Graphically estimate the period of the beating, which is defined to be twice the distance between the neighboring troughs. Can you explain why it has this value?[2]

Graphical estimate of beating period (attach your solution plot):

Theoretical explanation:

What happens to the period of beating as you change the forcing frequency from $\omega = 1.1$ to $\omega = 1.05$? What happens to the amplitude of the beating?

Observations (attach your solution plot):

Theoretical explanation:

[2]**Hint:** *The period of beating equals* 2π *divided by the frequency of beating. The trig identity* $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$ *may also be useful.*

2. **Resonance (undamped, forced).** Plot the solution of the equation

$$x'' + x = \cos(t)$$

with initial condition $(x(0), x'(0)) = (0, 0)$. Note the forcing frequency $\omega = 1$ here is equal to the natural frequency of the system. You should see an oscillating solution with growing amplitude. The solution should look like the graph of a function of the form $ct \sin t$. Graphically estimate the value of c. Check your answer by using the `Plot arbitrary function` command to plot the amplitude of the envelope curves.

Estimate of c (attach your solution plot showing amplitude envelope):

Considering the graphs and solutions which you found in problem 1, in what sense (if any) can you say that the solution to the initial value problem

$$x'' + x = \cos(\omega t), \qquad x(0) = 0, \quad x'(0) = 0,$$

approaches the solution to the initial value problem

$$x'' + x = \cos(t), \qquad x(0) = 0, \quad x'(0) = 0$$

as $\omega \to 1$? (Hint: Think about how the period and amplitude of the beating change as ω is decreased from 1.1 to 1.05.)

Answer:

3. **Transient behavior (damped, forced).** For this example, use display parameters $0 < t < 20, -10 < x < 10$. Plot solutions of the equation

$$x'' + x' + x = \sin(0.5t)$$

with a variety of initial conditions. Choose the initial conditions by clicking and dragging the mouse: all initial conditions should be taken near $t = 0$, with $x(0)$ and $x'(0)$ not too large. You should see that solutions with different initial conditions approach the same solution after some time τ. We say "transient" behavior occurs during the time interval $[0, \tau]$. After that time, all different solutions appear to coincide on the graph, and you observe only steady oscillations. Graphically estimate τ.

Estimate of τ (attach your plot):

Can you explain why τ has this value?

Theoretical explanation:

In the above example, we have $c = 1$. What happens (experimentally) to the transient time τ as c decreases? Try $c = 0.5, 0.25$. What happens as c increases? Try $c = 2, 4, 8$.

Answer (attach some graphs):

Can you explain intuitively the observed dependence of τ on the friction coefficient c? Can you explain it theoretically?

Explanations:

Now return to the equation $x'' + x' + x = \sin(0.5t)$ and try varying the forcing frequency. Replace $\omega = 0.5$ with $\omega = 0.3, 0.7, 0.9, 1.1, 1.3$. What happens (experimentally) to the amplitude $C(\omega)$ of the resulting wave? For what value of ω does it appear that $C(\omega)$ is largest? [3]

Answer (attach some graphs):

Can you explain intuitively the observed dependence of $C(\omega)$ on the forcing frequency ω? Can you explain it theoretically?

Explanations:

[3]**Hint:** *In order to get an accurate estimate for* $C(\omega)$*, it may be useful to restrict the viewing window in the* x*-variable. Try the display parameters* $0 < t < 50$, $0.5 < x < 1.5$.