

Math 385 Section D2 Fall 2001

Supplementary material on the method of undetermined coefficients

Review of homogeneous equations

The homogeneous constant coefficient linear equation $a_n y^{(n)} + \dots + a_1 y' + a_0 y = 0$ has the characteristic polynomial $a_n r^n + \dots + a_1 r + a_0 = 0$. From the roots of the polynomial we can find the solutions. We can also rewrite the equation in a weird-looking but useful way.

Examples:

equation: $y'' - 5y' + 6y = 0$.

polynomial: $r^2 - 5r + 6 = 0$. (factored): $(r - 2)(r - 3) = 0$. roots: 2, 3

weird-looking form of equation: $(D - 2)(D - 3)y = 0$ or $(D^2 - 5D + 6)y = 0$.

linearly independent solutions: $y_1 = e^{2x}, y_2 = e^{3x}$.

general solution: $y = c_1 e^{2x} + c_2 e^{3x}$.

equation: $y'' + 10y' + 25y = 0$.

polynomial: $r^2 + 10r + 25 = 0$. (factored): $(r + 5)^2 = 0$. roots: $-5, -5$

weird-looking form of equation: $(D + 5)^2 y = 0$ or $(D^2 + 10D + 25)y = 0$.

linearly independent solutions: $y_1 = e^{-5x}, y_2 = x e^{-5x}$.

general solution: $y = c_1 e^{-5x} + c_2 x e^{-5x}$.

equation: $y'' - 4y' + 8y = 0$.

polynomial: $r^2 - 4r + 8 = 0$. (factored): $(r - 2 - 2i)(r - 2 + 2i) = 0$.

roots: $2 + 2i, 2 - 2i$

w-l. f. of equation: $(D - 2 - 2i)(D - 2 + 2i)y = 0$ or $(D^2 - 4D + 8)y = 0$.

linearly independent solutions: $y_1 = e^{2x} \cos 2x, y_2 = e^{2x} \sin 2x$.

general solution: $y = e^{2x}(c_1 \cos 2x + c_2 \sin 2x)$.

equation: (already in a weird-looking form) $(D^2 + 1)^2(D - 1)^3 y = 0$.

polynomial: $(r^2 + 1)^2(r - 1)^3 y = 0$. roots: $i, i, -i, -i, 1, 1, 1$.

general solution: $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x + (c_5 + c_6 x + c_7 x^2) e^x$.

Annihilators

If f is a function, the **annihilator** of f is a "differential operator" $\tilde{L} = a_n D^n + \dots + a_1 D + a_0$ with the property that $\tilde{L}f = 0$.

By reversing our thought process for homogeneous differential equations, we can easily find the annihilator for lots of functions:

Examples

function: $f(x) = e^x$

annihilator: $\tilde{L} = (D - 1)$

check: $(D - 1)f = D e^x - e^x = \frac{d}{dx} e^x - e^x = 0$.

function: $f(x) = x^2 e^{-7x}$

annihilator: $\tilde{L} = (D - 7)^3$

function: $f(x) = e^{-x} \cos 4x$.

corresponding "root of polynomial": $-1 \pm 4i$.

polynomial with this root: $(r + 1 + 4i)(r + 1 - 4i) = r^2 + 2r + 5$.

annihilator: $\tilde{L} = D^2 + 2D + 5$.

function: $f(x) = x^k e^{\alpha x}$.

corresponding “root of polynomial”: α (with multiplicity $k + 1$)

corresponding polynomial: $(r - \alpha)^{k+1}$.

annihilator: $\tilde{L} = (D - \alpha)^{k+1}$.

function: $f(x) = x^k e^{ax} \sin bx$.

corresponding “root of polynomial”: $a \pm bi$ (with multiplicity $(k + 1)$)

polynomial with this root: $((r - a - bi)(r - a + bi))^{k+1} = (r^2 - 2ar + (a^2 + b^2))^{k+1}$.

annihilator: $\tilde{L} = (D^2 - 2aD + (a^2 + b^2))^{k+1}$.

function: $e^x + \cos 2x$.

To do this, note that if \tilde{L}_1 is the annihilator of f_1 and \tilde{L}_2 is the annihilator of f_2 , then $\tilde{L}_1 \tilde{L}_2$ is the annihilator of $f_1 + f_2$. This is true because

$$\tilde{L}_1 \tilde{L}_2 (f_1 + f_2) = \tilde{L}_1 \tilde{L}_2 f_1 + \tilde{L}_1 \tilde{L}_2 f_2 = \tilde{L}_1 \tilde{L}_2 f_1 + 0 = \tilde{L}_2 \tilde{L}_1 f_1 = 0.$$

So the annihilator is $(D - 1)(D^2 + 4)$.

(you can also check directly that this works.)

The Method of Undetermined Coefficients

To find a particular solution of the equation $Ly = f$, we want to guess the “correct form” of y_p and then substitute and adjust the coefficients to get a solution. How do we determine the correct form?

To see this, consider the following:

Let \tilde{L} denote the annihilator of f . By definition $\tilde{L}f = 0$. If y is a solution, then

$$0 = \tilde{L}f = \tilde{L}(Ly).$$

So y is a solution of the homogeneous equation $\tilde{L}Ly = 0$. We can call this the “annihilator equation”. We know how to find all solutions, and this will tell us the correct form for our guess.

This is the basis for the following procedure for determining y_p :

To solve $Ly = f$:

Step 1. Find the annihilator for f . Call it \tilde{L} .

Step 2. Let y_a denote the general solution of the annihilator equation $\tilde{L}Ly_a = 0$.

Find y_a

Step 3 Find the general solution y_c of the complementary homogeneous equation $Ly_c = 0$.

Step 4 Our guess y_p will consist of the general solution y_a of the annihilator equation, with all terms y_c from the complementary equation omitted.

Remark

Suppose I have an equation with a complicated “right-hand side” consisting of more than one term, which I can write symbolically as

$$Ly = f_1 + f_2.$$

One way to deal with this is to consider each of the two terms on the right-hand side separately. In other words, suppose I can find functions y_1 and y_2 that solve

$$Ly_1 = f_1, \quad Ly_2 = f_2.$$

Then let $y = y_1 + y_2$. I claim that y solves our original problem. This is easy to check, since

$$Ly = L(y_1 + y_2) = L(y_1) + L(y_2) = f_1 + f_2.$$

For example, equation 4 below is $y^{(5)} - y''' = e^x + 2x^2 + 5$. I could break the right-hand side up into two pieces: $f_1 = e^x$, $f_2 = 2x^2 + 5$. Then by going through my procedure, I would find

$$(\text{guess for } f_1) : y_{p,1} = Axe^x, \quad (\text{guess for } f_2) : y_{p,2} = Bx^3 + Cx^4 + Dx^5$$

and so

$$(\text{guess for whole problem}) y_p = Axe^x + Bx^3 + Cx^4 + Dx^5.$$

Examples

equation 1: $y'' + 4y' + 4y = 2e^x$, equivalently $(D + 2)^2 y = 2e^x$, or $Ly = 2e^x$ for $L = (D^2 + 4D + 4)$.

Step 1: annihilator of right-hand side: $\tilde{L} = (D - 1)$

Step 2 If we apply the operator $\tilde{L} = D - 1$ to both sides of the equation $y'' + 4y' + 4y = 2e^x$, we get

$$(D - 1)(D^2 + 4D + 4)y = (D - 1)(2e^x) = \frac{d}{dx}(2e^x) - 1 \cdot 2e^x = 0.$$

So the annihilator equation is $(D - 1)(D - 2)^2 y_a = 0$.

We could have found this by just using the general expression for the annihilator equation: $\tilde{L}Ly_a = 0$.

The general solution of the annihilator equation is $y_a = c_1 e^x + (c_2 + c_3 x)e^{-2x}$.

Step 3 general solution of complementary equation is $y_c = (c_2 + c_3 x)e^{-2x}$.

Step 4. So we guess $y_p = c_1 e^x$.

If I substitute $y_p = c_1 e^x$ into the equation I find

$$Ly_p = y_p'' + 4y_p' + 4y_p = 9c_1 e^x.$$

So to solve $Ly_p = 2e^x$ I have to take $c_1 = 2/9$. Then I get the solution $y = \frac{2}{9}e^x$.

What happens if instead I substitute $y = y_a = c_1 e^x + (c_2 + c_3 x)e^{-2x}$ into the equation? If I do this and do all the calculations, I'll find that

$$Ly = y'' + 4y' + 4y = 9c_1 e^x.$$

So to solve $Ly = 2e^x$ I have to take $c_1 = 2/9$. But there are no conditions on c_2, c_3 , so they can be anything. So this gives me the general solution of the inhomogeneous equation:

$$y = \frac{2}{9}e^x + (c_2 + c_3 x)e^{-2x}.$$

So we see: if I substitute y_p into the equation and solve for the undetermined coefficients I get a particular solution. If I substitute y_a into the equation and try to solve, I get the general solution. However, in practice it is much better to substitute y_p , because the algebra is much simpler.

equation 2: $y'' + 4y' + 4y = e^{-2x}$, equivalently $(D + 2)^2 y = e^{-2x}$, or $Ly = e^{-2x}$ for $L = (D^2 + 4D + 4)$.

Step 1: annihilator of right-hand side: $\tilde{L} = (D + 2)$

Step 2 If we apply the operator $\tilde{L} = D + 2$ to both sides of the equation $y'' + 4y' + 4y = e^{-2x}$, we get

$$(D + 2)(D + 2)^2 y = (D + 2)(e^{-2x}) = \frac{d}{dx}(e^{-2x}) + 2 \cdot e^{-2x} = 0.$$

So the annihilator equation is $(D - 2)^3 y_a = 0$.

Again we could have found this just from the formula $\tilde{L}Ly_a = 0$.

The general solution of the annihilator equation is $y_a = (c_1 + c_2x + c_3x^2)e^{2x}$.

Step 3 general solution of complementary equation is $y_c = (c_2 + c_3x)e^{2x}$.

Step 4. So we guess $y_p = c_1x^2e^{2x}$.

As above: if I substitute y_p into the equation and solve for the undetermined coefficients I get a particular solution. If I substitute y_a into the equation and try to solve, I get the general solution. However, in practice it is much better to substitute y_p , because the algebra is much simpler.

equation 3: $y''' + 9y' = x^2 \sin 3x$, equivalently $(D + 3i)(D - 3i)Dy = x^2 \sin 3x$, or $Ly = x^2 \sin 3x$ for $L = (D^2 + 9)D$.

Step 1: annihilator of right-hand side: $\tilde{L} = (D^2 + 9)^3$

Step 2 annihilator equation is $\tilde{L}Ly_a = 0$, ie $(D^2 + 9)^4 Dy_a = 0$, ie $(D - 2)^4 Dy_a = 0$. General solution is $y_a = (A + Bx + Cx^2 + Dx^3) \cos 3x + (E + Fx + Gx^2 + Hx^3) \sin 3x + K$

Step 3 general solution of complementary equation is $y_c = A \cos 3x + B \sin 3x$.

Step 4. So we guess $y_p = (Bx + Cx^2 + Dx^3) \cos 3x + (Fx + Gx^2 + Hx^3) \sin 3x$.

equation 4: $y^{(5)} - y''' = e^x + 2x^2 + 5$. We can write the left-hand side as $(D^5 - D^3)y = D^3(D - 1)(D + 1)y$.

Step 1: annihilator of right-hand side: $\tilde{L} = (D - 1)D^3$.

Step 2 annihilator equation is $\tilde{L}Ly_a = 0$, ie $D^6(D - 1)^2(D + 1)y_a = 0$. General solution is $y_a = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + (G + Hx)e^x + Ie^{-x}$.

Step 3 general solution of complementary equation is $y_c = A + Bx + Cx^2 + De^x + Ee^{-x}$

Step 4. So we guess $y_p = Ax^3 + Bx^4 + Cx^5 + Dxe^x$.

equation 5: $y'' - 6y' + 13y = xe^{3x} \sin 2x$. We can write the left-hand side as $(D - 3 + 2i)(D - 3 - 2i)y$ or as $(D^2 - 6D + 13)y$.

Step 1: annihilator of right-hand side: the function $xe^{3x} \sin 2x$ corresponds to the root $3 \pm 2i$, repeated twice (because of the factor of x). So $\tilde{L} = ((D - 3 + 2i)(D - 3 - 2i))^2 = (D^2 - 6D + 13)^2$.

Step 2 annihilator equation is $\tilde{L}Ly_a = 0$, ie $(D^2 - 6D + 13)^3 y_a = 0$. General solution is $y_a = (A + Bx + Cx^2)e^{3x} \cos 2x + (D + Ex + Fx^2)e^{3x} \sin 2x$.

Step 3 general solution of complementary equation is $y_c = Ae^{3x} \cos 2x + Be^{3x} \sin 2x$.

Step 4. So we guess $y_p = (Bx + Cx^2)e^{3x} \cos 2x + (Ex + Fx^2)e^{3x} \sin 2x$.

equation 6: $y^{(4)} - 2y'' + y = x^2 \cos x$. We can write the left-hand side as $(D^2 - 1)^2 y$ or as $(D - 1)^2(D + 1)^2 y$.

Step 1: annihilator of right-hand side: the function $x^2 \cos x$ corresponds to the root i , repeated three times (because of the factor of x^2). So $\tilde{L} = ((D - i)(D + i))^3 = (D^2 + 1)^3$.

Step 2 annihilator equation is $\tilde{L}Ly_a = 0$, ie $((D - i)(D + i))^3(D - 1)^2(D + 1)^2 y_a = 0$. General solution is $y_a = (A + Bx + Cx^2) \cos x + (D + Ex + Fx^2) \sin x + (G + Hx)e^x + (I + Jx)e^{-x}$.

Step 3 general solution of complementary equation is $y_c = (A + Bx)e^x + (C + Dx)e^{-x}$.

Step 4. So we guess $y_p = (A + Bx + Cx^2) \cos x + (D + Ex + Fx^2) \sin x$.