

3.8 (SUPPLEMENT) — ORTHOGONALITY OF EIGENFUNCTIONS

We now develop some properties of eigenfunctions, to be used in Chapter 9 for Fourier Series and Partial Differential Equations.

1. Definition of Orthogonality

We say functions $f(x)$ and $g(x)$ are *orthogonal* on $a < x < b$ if $\int_a^b f(x)g(x) dx = 0$.

[Motivation: Let's approximate the integral with a Riemann sum, as follows. Take a large integer N , put $h = (b - a)/N$ and partition the interval $a < x < b$ by defining $x_1 = a + h, x_2 = a + 2h, \dots, x_N = a + Nh = b$. Then

$$\begin{aligned} \int_a^b f(x)g(x) dx &\approx f(x_1)g(x_1)h + \dots + f(x_N)g(x_N)h \\ &= (u_N \cdot v_N)h \end{aligned}$$

where $u_N = (f(x_1), \dots, f(x_N))$ and $v_N = (g(x_1), \dots, g(x_N))$ are vectors containing the values of f and g . The vectors u_N and v_N are said to be orthogonal (or perpendicular) if their dot product equals zero ($u_N \cdot v_N = 0$), and so when we let $N \rightarrow \infty$ in the above formula it makes sense to say the functions f and g are orthogonal when the integral $\int_a^b f(x)g(x) dx$ equals zero.]

Example. $\sin x$ and $\cos x$ are orthogonal on $-\pi < x < \pi$, since $\int_{-\pi}^{\pi} \sin x \cos x dx = \frac{1}{2} \sin^2 x \Big|_{-\pi}^{\pi} = 0$.

2. Integration Lemma

Suppose functions $X_n(x)$ and $X_m(x)$ satisfy the differential equations

$$\begin{aligned} X_n'' + \lambda_n X_n &= 0, & a < x < b, \\ X_m'' + \lambda_m X_m &= 0, & a < x < b, \end{aligned}$$

for some numbers λ_n, λ_m . Then

$$(\lambda_n - \lambda_m) \int_a^b X_n(x)X_m(x) dx = [X_n(x)X_m'(x) - X_n'(x)X_m(x)]_a^b.$$

Proof.

$$\begin{aligned} \text{LHS} &= \int_a^b [(\lambda_n X_n)X_m - X_n(\lambda_m X_m)] dx && \text{by taking the } \lambda \text{'s inside the integral} \\ &= \int_a^b [-X_n'' X_m + X_n X_m''] dx && \text{since } \lambda_n X_n = -X_n'' \text{ and } \lambda_m X_m = -X_m'' \\ &= \int_a^b [X_n X_m' - X_n' X_m]' dx && \text{as you can check by differentiating!} \\ &= \text{RHS} && \text{by the Fundamental Theorem of Calculus.} \end{aligned}$$

□

3. Boundary Conditions

Consider a function $X(x)$ for $a < x < b$. Four boundary condition (“BC”) types are:

Dirichlet BC	$X(a) = X(b) = 0$
Neumann BC	$X'(a) = X'(b) = 0$
Mixed BC	$X(a) = X'(b) = 0$
Mixed BC	$X'(a) = X(b) = 0$
Periodic BC	$X(a) = X(b), \quad X'(a) = X'(b)$

(The two varieties of Mixed BC are very similar, differing only as to which endpoint has $X = 0$ and which has $X' = 0$.)

4. Orthogonality of Eigenfunctions Theorem

Suppose $X_n'' + \lambda_n X_n = 0$ and $X_m'' + \lambda_m X_m = 0$ on $a < x < b$, and that X_n and X_m both satisfy the same type of BC. If $\lambda_n \neq \lambda_m$ then X_n and X_m are orthogonal, so that

$$\int_a^b X_n(x)X_m(x) dx = 0.$$

Proof. By the Integration Lemma, we have

$$\begin{aligned} & \int_a^b X_n(x)X_m(x) dx \\ &= \frac{1}{\lambda_n - \lambda_m} [X_n(x)X_m'(x) - X_n'(x)X_m(x)]_a^b \\ &= 0 \quad \text{under Dirichlet BCs, because } X_n(a) = X_n(b) = 0 \text{ and } X_m(a) = X_m(b) = 0 \\ &= 0 \quad \text{under Neumann BCs, because } X_n'(a) = X_n'(b) = 0 \text{ and } X_m'(a) = X_m'(b) = 0 \\ &= 0 \quad \text{under Mixed BCs, for similar reasons.} \end{aligned}$$

For periodic BCs, we use that $X_n(b) = X_n(a)$ and $X_m'(b) = X_m'(a)$ and so on, to see

$$\begin{aligned} [X_n(x)X_m'(x) - X_n'(x)X_m(x)]_a^b &= [X_n(b)X_m'(b) - X_n'(b)X_m(b)] \\ &\quad - [X_n(a)X_m'(a) - X_n'(a)X_m(a)] = 0. \end{aligned}$$

□

5. Example

Show that $\sin x$ and $\sin 2x$ are orthogonal for $0 < x < \pi$.

Solution. We could just show $\int_0^\pi \sin(x)\sin(2x) dx = 0$ by using trigonometric identities to evaluate the integral. But it is easier to notice that both $X_1(x) = \sin(x)$, with $\lambda_1 = 1^2$, and $X_2(x) = \sin(2x)$, with $\lambda_2 = 2^2$, are eigenfunctions for

$$\begin{aligned} X'' + \lambda X &= 0, & 0 < x < \pi, \\ X(0) = X(\pi) &= 0 & \text{(Dirichlet BC).} \end{aligned}$$

Since $\lambda_1 \neq \lambda_2$, the Orthogonality Theorem implies $\int_0^\pi X_1 X_2 dx = 0$, so that $\sin(x)$ and $\sin(2x)$ are orthogonal for $0 < x < \pi$.

