Project I — Direction Fields*

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Goal of the project

Consider the differential equation

$$\frac{dy}{dx} = f(x, y).$$

We aim to understand graphically how properties of the function $f(x, y)$ affect the direction field. In particular, we consider functions f that are everywhere positive (or negative), or that depend only on x (or only on y), or functions f that are periodic in x (or in y).

Tips

Use menu item `Enter differential equation`, in the Direction Fields module of Iode, to enter a differential equation. Use menu item `Change display parameters` to enter the domain and range of the plot, and then left-click on the graph to plot a solution curve.

Instructions

Answer Problems 1–6 on the Answer Sheet attached at the end. Do Problems 8–13 on a different sheet of paper, and for each problem include both an Answer and an Explanation.

Part I — Matching direction fields

Let k, A, B, C be constants. Consider the 6 differential equation types:

1. $\frac{dy}{dx} = ky$, where $k > 0$
2. $\frac{dy}{dx} = ky$, where $k < 0$
3. $\frac{dy}{dx} = 2x(A - x)$
4. $\frac{dy}{dx} = y(B - y)$
5. $\frac{dy}{dx} = x^2 + \exp(x^2)\cos^2(3y)$
6. $\frac{dy}{dx} = -e^y - C(1 - \cos(x))$

Match equations and plots on Answer Sheet at end. □

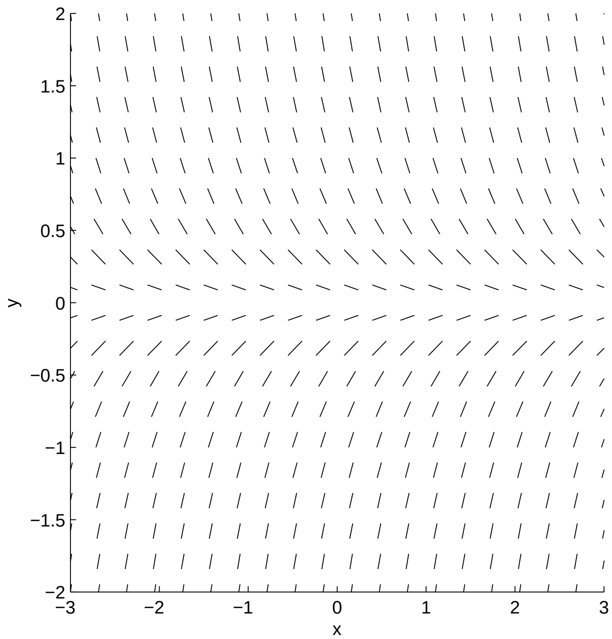
Your task is to fill in the Answer Sheet at the end. This involves matching up each equation with its direction field plot chosen from the plots on the following pages, and briefly describing the most visually distinctive feature of the direction field.

i. To accompany each answer, you should create and print a direction field plot of your own, using Iode. Your direction field plots do not have to match the given ones exactly, just qualitatively. Each plot should also **show a solution curve** (you can use any initial condition you like). And use the `Enter caption` menu item to add your **last name and the problem number**, to each plot you print.

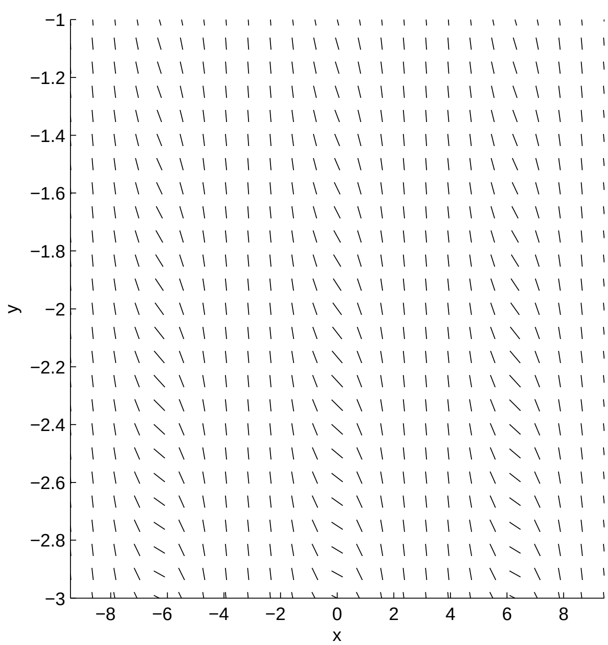
ii. For example, the correct answer 1C could be justified by plotting the direction field for $\frac{dy}{dx} = 2y$, because this differential equation fits the type of equation 1 (for the choice of constant $k = 2$), and its direction field plot (which you can create using Iode) looks very much like Fig. C. The distinctive feature could be described as "horizontal at $y = 0$".

iii. To identify distinctive features, it might help to blur your vision a little when looking at the direction field. Then check that your "distinctive feature" can be justified from the form of the differential equation. For example, check that the differential equation $\frac{dy}{dx} = ky$ does imply "slope= $\frac{dy}{dx} = 0$ at $y = 0$".

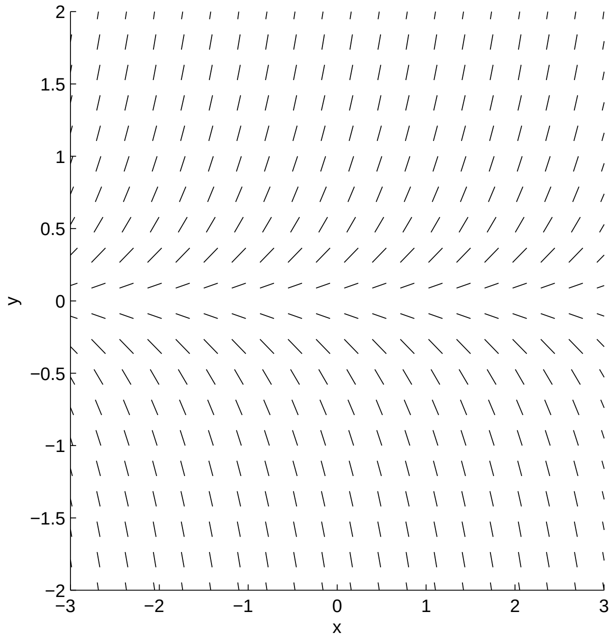
The given plots are:



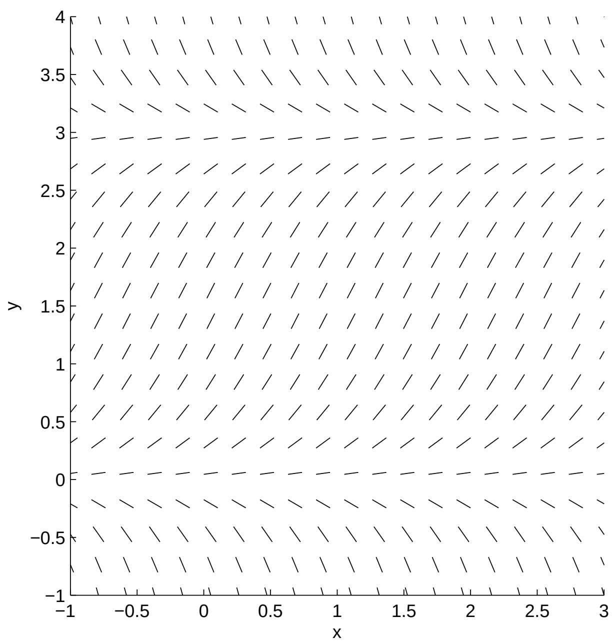
A.



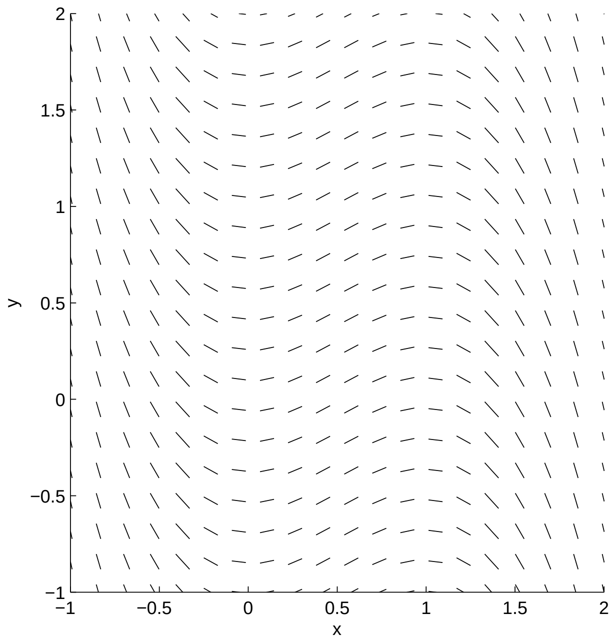
B.


2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−3
−2
−1
0
1
2
3
x
y

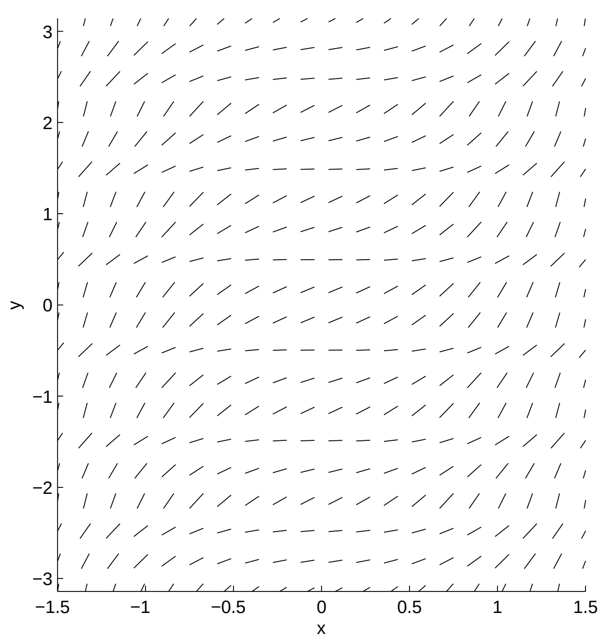
C.


4
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1
1.5
2
2.5
3
x
y

D.



E.



F.

Part II — Qualitative properties

For the next lot of questions, we need the concept of periodicity. A function $g(x)$ is called **periodic** if for some number $P > 0$ one has $g(x) = g(x + P)$ for all x. Then P is called a *period* of the function g.

Example 1. $g(x) = \sin(x)$ is periodic with $P = 2\pi$, since $\sin(x) = \sin(x + 2\pi)$ for all x. Notice that the graph of $\sin x$ repeats itself every 2π units, which we expect from the periodicity.

Example 2. $g(x) = \sin(x) - 1$ is periodic with $P = 2\pi$, since $\sin(x) - 1 = \sin(x + 2\pi) - 1$ for all x.

Example 3. The function $g(x)$ defined piecewise by the rule

$$g(x) = \begin{cases} 1, & \text{if } \ldots -2 \le x < -1, \text{ or } 0 \le x < 1, \text{ or } 2 \le x < 3, \ldots \\ 2, & \text{if } \ldots -1 \le x < 0, \text{ or } 1 \le x < 2, \text{ or } 3 \le x < 4, \ldots. \end{cases}$$

is periodic with $P = 2$. (Exercise: sketch a graph of this function!) This example shows periodic functions do not have to be trigonometric functions.

Example 4. $g(x, y) = 3^{\cos 2x} + y^2$. This function of two variables is periodic in the x-variable with period $P = \pi$, because $g(x, y) = g(x + \pi, y)$ for all x and y, by the previous example. But $g(x, y)$ is not periodic in the y-variable, because no matter what $P > 0$ you try, you can check that $g(x, 0) \neq g(x, P)$ for example.

Now we come to the questions. For full credit, you must give both an Answer and an Explanation. A sample solution is provided for the first question, in order to show the level of detail required. Notice the Explanation makes reference to the most relevant 3 features of the equations and direction fields in questions #1–6. You should do the same wherever possible.

But sometimes this will not be possible, and then you should use Iode to **create a few examples** of relevant direction fields and solution curves, to help give you ideas. (Include these plots with your solution, if you feel they help.) **Experimentation with Examples 1–4 is essential** for answering some of these questions!

7. The solution curves of $\frac{dy}{dx} = f(x, y)$ are all increasing from left to right if [fill in the blanks].

   Answer: ...if $f(x, y) > 0$ for all x and y, in other words if f is positive everywhere.

   Explanation: If f is positive then $\frac{dy}{dx}$ is positive, which means the slope is positive and so the curve is increasing from left to right. You can see this happening in Fig. F; check that the formula corresponding to Fig. F has $f(x, y) > 0$ for all x, y.

8. Suppose $y = y(x)$ solves $\frac{dy}{dx} = f(x)$.

   If $y(x)$ is periodic then $f(x)$ is periodic. (True/False, and Explain)

9. Suppose $y = y(x)$ solves $\frac{dy}{dx} = f(x)$.

   If $f(x)$ is periodic then $y(x)$ is periodic. (True/False, and Explain)

10. The plot of the direction field for $\frac{dy}{dx} = f(x, y)$ shows a vertically repeating pattern (e.g. Fig. F) if [fill in the blanks, and Explain].

11. Suppose $y = y(x)$ solves $\frac{dy}{dx} = f(x)$.

    If $y(x) \to \infty$ as $x \to \infty$ then $f(x) \to \infty$ as $x \to \infty$. (True/False, and Explain)

12. Consider $\frac{dy}{dx} = f(y)$, and suppose $f(2) = 0$. What feature do you observe in the direction field, at height 2?

    (*Note.* Here f does not depend on x.)

13. Find a function $f(y)$ with the property that: for each solution $y(x)$ of $\frac{dy}{dx} = f(y)$, the limiting value $\lim_{x \to \infty} y(x)$ equals 3 if $y(0) > 0$. [Your explanation can consist of a direction field with illustrative solution curves plotted on it.]

Write down an Answer and an Explanation for each question. $\square$

Reference: mathematical expressions in Matlab, Octave

For simple expressions, we use the usual keyboard characters:

`2*x`	means $2x$,
`(x^3-1)/6`	means $(x^3 - 1)/6$,
`pi`	means π.

Built-in functions

`exp(x)`	exponential, e^x
`log(x)`	natural logarithm, $\ln x$
`log10(x)`	base 10 logarithm, $\log_{10} x$
`abs(x)`	absolute value, $\lvert x\rvert$
`sqrt(x)`	square root, $\sqrt{x}$
`sign(x)`	signum function, which equals $\begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

`sin(x)`		`sinh(x)`	
`cos(x)`	trigonometric	`cosh(x)`	hyperbolic
`tan(x)`	functions	`tanh(x)`	trigonometric
`cot(x)`	(x in radians)	`coth(x)`	functions
`sec(x)`		`sech(x)`	
`csc(x)`		`csch(x)`	
`asin(x)`		`asinh(x)`	
`acos(x)`	inverse	`acosh(x)`	inverse
`atan(x)`	trigonometric	`atanh(x)`	hyperbolic
`acot(x)`	functions	`acoth(x)`	trigonometric
`asec(x)`		`asech(x)`	functions
`acsc(x)`		`acsch(x)`	

Example 1.

`sin(exp(y))^4`	means $\sin^4(e^y)$,
`acos(exp(1)^(-1))`	means $\arccos(e^{-1})$.

More Matlab and Octave commands are explained at `www.octave.org/docs.html`.

Project I — Direction Fields — Answer Sheet for #1–6

For each answer 1–6, print off and attach a plot of the corresponding direction field. (Your direction fields do not have to match the given ones exactly, just qualitatively.)

Each plot should also **show a solution curve** (you can use any initial condition you like). And use the `Enter caption` menu item to add your **last name and the problem number**, to each plot you print.

Equation	*Plot*	*Distinctive feature of direction field*
1		
2		
3		
4		
5		
6		