Project III — Growth and Decay for Second order linear ODEs*

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A complete solution for this project consists of: **written answers** for problems 1ac, 2ac, 3ac, 4ab, 5a–f, and **Iode plots** for problems 1b, 2b, 3b.

1 Iode Preliminaries

Start by launching Iode and spending some time getting acquainted with the `Second order linear ODEs` module, as follows.

When you start the module, it is initialized to the equation $(1)x''(t) + (0.5)x'(t) + (1)x(t) = \cos 2t$. If you want to plot the "forcing function" $f(t) = \cos 2t$, then just go to the `Options` menu and choose `Show forcing function`. Choosing this option again will remove the forcing function graph.

For now we are studying homogeneous equations, and so you should set the forcing function to 0: choose `Enter differential equation` in the `Equation` menu, and keep the current values of m, c and k, then for $f(t)$, just enter 0.

Plot some numerical solutions with each of the following three methods.

Method 1. Enter the values $t_0, x(t_0)$ and $x'(t_0)$ into the `Initial conditions` boxes, and then click on the `Plot solution` button.

Method 2. Enter the desired initial slope $x'(t_0)$ into the relevant `Initial conditions` box, and then click on the graph where you want the solution to start. Iode will plot a solution through the click point whose initial slope $x'(t_0)$ is the value that you entered in the box.

Method 3. Press down the left mouse button at the desired initial point in the graph, and then drag the mouse a short distance at the desired slope. When you release the mouse button, Iode will plot a solution starting at the point where you first pressed the button, with the initial slope of the solution being given by the line going from where you first pressed the button to where you finally released it. Play around and see how it works!

By now you will have plotted a lot of solutions on your graph. Clear them away with `Clear plot` in the `Options` menu.

Tip. If you ever want more information, browse the Iode manual online.

2 Growth and Decay Concepts

We want to distinguish four different kinds of growth/decay behavior. We will say that a solution

- **decays** if it converges to 0 for large values of t,
- **grows** if it tends to $+\infty$ for large values of t, or if it tends to $-\infty$ for large values of t,
- **grows while oscillating** if it keeps changing sign for large values of t and the amplitude of the oscillation tends to infinity,
- **decays while oscillating** if it keeps changing sign for large values of t and the amplitude of the oscillation tends to zero. (This is a special case of decay.)

Example 1.
e^{-t} decays,
e^t grows,
$-e^t$ grows,
$e^t \sin t$ grows while oscillating,
$e^{-t} \sin t$ decays while oscillating,
$\cos t$ falls into none of the growth/decay categories; it just **oscillates**.

Aside. A subtle feature of our definition is that a function that "grows while oscillating" does not "grow", because it keeps changing sign and so it neither tends to $+\infty$ nor to $-\infty$.

3 Problems

First work through the previous two sections.

Use `Change display parameters` in the `Equation` menu to set the display parameters to $0 < t < 6, -5 < x < 5$, for all of the following numerical experiments. *Note.* It is essentially impossible to find display parameters that will give good pictures for both decaying and growing solutions, but the above choice is adequate.

1. a With pencil and paper, find the general solution of the equation

   $$(1)x'' + (1.5)x' + (.5)x = 0.$$

   b Use Iode to plot five numerical solutions of this equation, with the initial data being "random" (meaning you should use Method 3 above to press-and-drag at various initial points, with some of the drag-slopes being positive and some negative). But:

   To keep matters manageable, take all your **initial points** in the part of the window where $0 < t < 1, -1 < x < 1$, and take all your **initial slopes** between -1 and $+1$.

   Add to your plot a caption containing your last name and the problem number, and then print the plot. For each solution on the plot, write whether it decays, or grows, or grows while oscillating.

   c Based on the randomly chosen initial data from part (b), state what percentage of solutions decay, what percentage grow, and what percentage grow while oscillating.

   Then use the general solution you have found in part (a) to explain these percentages.

2. Repeat Problem 1 for the equation

   $$(1)x'' - (\sqrt{3})x' - (.25)x = 0.$$

   Your explanation in part (c) should be *carefully* done.

   (*Note.* $\sqrt{3}$ is typed as `sqrt(3)`, in Matlab.)

3. Repeat Problem 1 for the equation

   $$(1)x'' - (.5)x' + (.5)x = 0.$$

   Again, your explanation in part (c) should be *carefully* done.

4. *Introduction.* Consider the equation $x'' + 2x' + 10x = 0$. The general solution of this is $x(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$. From this it is easy to see that all solutions decay while oscillating as $t \to \infty$. (Right?) Similarly, for the equation $x'' - 2x' + 10x = 0$, the general solution is $x(t) = e^{t}(c_3 \cos 3t + c_4 \sin 3t)$, and from this it is easy to see that all solutions grow while oscillating as $t \to \infty$. (*Note.* If you want to look at these two equations using Iode, you will have to adjust the display parameters to get a good picture.)

   The problem. Next is a harder example in the same spirit, for a fourth order equation. Iode does not have a module for 4^{th} order equations and so you have to do the problem analytically, rather than numerically.

   a With pencil and paper, find the general solution of the equation

   $$x^{(4)} + 16x'' + 100x = 0$$

   (*Hint.* $r^4 + 16r^2 + 100 = (r^2 + 2r + 10)(r^2 - 2r + 10)$. Now use comments in the previous paragraph.)

   b Predict what percentage of solutions with randomly chosen initial data will grow while oscillating, and what percentage will decay while oscillating. Explain.

5. *Conclusions.* Consider a second order linear constant coefficient homogeneous equation having roots r_1 and r_2 for its characteristic equation. Summarize your conclusions about the behavior of the solution $x(t)$, for randomly chosen initial data, when:

   a $r_1 < r_2 < 0$,

   b $r_1 < 0 < r_2$,

   c $0 < r_1 < r_2$,

   d $r_1, r_2 = \alpha \pm \beta i$ and $\alpha < 0$,

   e $r_1, r_2 = \alpha \pm \beta i$ and $\alpha = 0$,

   f $r_1, r_2 = \alpha \pm \beta i$ and $\alpha > 0$.

   Your answer can consist of just a short phrase, such as “grows” or “decays while oscillating”.