Mini-Project VI — Heat and Wave Equations*

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1 Preliminaries

Start Matlab and Iode, and get into the `Partial differential equations` module of Iode.

Note. Iode solves the heat and wave equations on an interval using separation of variables and Fourier series. See the Iode manual for details, at the Iode website.

2 Questions

1 **The heat equation.** We study the following initial value problem for the heat equation:

$$\begin{aligned} u_t &= (1.2)u_{xx}, \qquad\qquad 0 < x < 30, \quad t > 0, \\ u(x,0) &= x, \\ u_x(0,t) &= u_x(30,t) = 0. \end{aligned}$$

To examine this with Iode, use the `Enter equation and boundary conditions` menu item to enter the heat equation with Neumann boundary conditions, and then use `Enter parameters and initial data` to enter $k = 1.2, L = 30, T = 60$ and $u(x,0) = x$. (The duration $T = 60$ tells Iode to compute the solution for $0 < t < 60$.)

Hint. The upper graph will now show the 3D-plot of the solution for $0 < x < 30$ and $0 < t < 60$. Try rotating this plot by pressing the left mouse button over it and dragging the mouse around.

(a) What is the temperature at $x = 5$ after 60 seconds? To answer this, first find an appropriate *snapshot* of the solution (using the lower graph in the Iode window), then use your plot to estimate the value of $u(5,60)$. Print your snapshot. Then find an appropriate *section* of the solution (again using the lower graph), and use your plot to estimate the value of $u(5,60)$. The snapshot and the section should give the same answer for $u(5,60)$!

Check. Because of the Neumann boundary conditions, your solution should quickly flatten out at $x = 0$ and $x = L$, for $t > 0$. Further, as $t \to \infty$ the value of u should approach 15 (the average value of the initial function $u(x,0) = x$) at every point x. Does this seem to be happening?

(b) After approximately how long will the temperature at $x = 5$ be 12.5 degrees? For this, first enter a much larger value of the duration T. Then plot an appropriate section of the solution; you want a plot of $u(5,t)$ as a function of t. Print off your plot and use it to estimate the time at which $u(5,t) = 12.5$.

2 **The wave equation.** Use the Partial differential equations module to solve the following two problems.

| Wave Problem D. | Enter the wave equation with Dirichlet boundary conditions. Enter $c = 2, L = 4, T = 4$, and for the intitial displacement $u(x,0)$ enter `triangle(x,2,3)` and for the initial velocity $u_t(x,0)$ just enter 0. (*Note.* `triangle(x,a,b)` is an Iode function that has a triangular graph for $a < x < b$ and equals zero elsewhere.)

The 3D-plot of the solution should show that as t increases, the triangle of height 1 at $t = 0$ splits into two overlapping triangles of height $1/2$, one moving left and the other moving right. This is exactly as predicted by the D'Alembert formula.

(i) Look at the snapshots of the solution, starting with $t = 0$. How far away is the edge of the right-moving triangle from the endpoint at $x = 4$, and how long does it take to get to that endpoint? Then relate your answer to the interpretation of c as the wavespeed.

(ii) In what way do the triangles change after they hit the endpoints? Illustrate with a snapshot, say from time $t = 1.5$.

(iii) Explain how your conclusions are affected (if at all) when you re-do (i) and (ii) with initial displacement `bump(x,2,3)`, and with initial velocity still 0. (*Optional.* You can also play around using `hat(x,2,3)` as the initial displacement, but you'll want to increase the number of plot points and the "top harmonic" value in the `Options --> Change resolution` menu item, because the `hat` function has jumps and so Iode needs to plot a lot of points and compute a lot of terms in the Fourier series, to get a good approximation.)

| Wave Problem N. | Repeat Wave Problem D but with Neumann boundary conditions.