

Example 2: Show that the set $M_{2 \times 3}$ of all 2×3 matrices together with the usual operations of matrix addition and scalar multiplication is a vector space.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

Closure: $A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}$
is a 2×3 matrix, so $A + B$ is in $M_{2 \times 3}$

$kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{bmatrix}$ is a 2×3 matrix
 $M_{2 \times 3}$ is closed under scalar multiplication

Zero: $A + \mathbf{0} = A = \mathbf{0} + A$ need $\mathbf{0}$

Define $\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ in $M_{2 \times 3}$
additive identity exists

Additive inverse: need $-A$ so that $-A + A = \mathbf{0}$

Define $-A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$ negative element exists

From matrix theory

$$A + B = B + A$$

$$A + (B + C) = (A + B) + C$$

$$k(A + B) = kA + kB$$

$$(k_1 + k_2)A = k_1A + k_2A$$

$$k_1(k_2A) = (k_1k_2)A$$

We have the following:

$$\boxed{k_1, k_2 \in \mathbb{R}}$$

$$1 \cdot A = \begin{bmatrix} 1 \cdot a_{11} & 1 \cdot a_{12} & 1 \cdot a_{13} \\ 1 \cdot a_{21} & 1 \cdot a_{22} & 1 \cdot a_{23} \end{bmatrix} = A$$

We showed that 10 properties hold, so $(M_{2 \times 3}, +, \cdot)$

is a vector space.

Remarks:

- 1) When working with matrices we use the standard notation $+$ and \bullet for addition and scalar multiplication and reserve the notations \oplus and \odot for when we are talking about vector addition and scalar multiplication.
- 2) Using the same methods as in Ex#2 we can show that $M_{m \times n}$ (the set of all $m \times n$ matrices) is a vector space. The operations of $+$ and \bullet hold, and we use the notation $(M_{m \times n}, +, \bullet)$ to represent that.
- 3) Are 2×3 matrices vectors? Yes, as elements of a vector space.

How does this relate to the old definition of a vector was an n -tuple of numbers?

* ANY VECTOR in any FINITE DIMENSIONAL vector space can be written as an ordered list of numbers (even if you naturally would not do so).

In Ex #2 we could have written A as $(a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23})$ a 1×6 vector or as a 2×3 matrix. The operations would still hold, as would the definition of a vector space.

In fact $(M_{2 \times 3}, +, \bullet)$ is equivalent to $(\mathbf{R}^6, +, \bullet)$

* This is an extremely important fact/result in linear algebra.
Any finite dimensional vector space can be represented as $(\mathbf{R}^n, +, \bullet)$ with an appropriate n .